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On the Convergence of a Single Step Third order Method for Solving Equations

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On the Convergence of a Single Step Third Order Method for Solving Equations

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I. Introduction

Let $F : D \subset S \longrightarrow S$ be a differentiable function, where $S = \mathbb{R}$ or $S = \mathbb{C}$ and D is an open nonempty set.

We are interested in computing a solution x^* of equation

$$F(x) = 0. \tag{1.1}$$

The point x^* is needed in closed form. But this form is attained only in special cases. That explains why most solution methods for (1.1) are iterative. There is a plethora of local convergence results for high convergent iterative methods based on Taylor expansions requiring the existence of higher than one derivatives not present on these methods. But there is very little work on the semi-local convergence of these methods or the local convergence using only the derivative of the operator appearing on these methods. We address these issues using a method by S. Kumar defined by

$$x_0 \in D,\ x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.2}$$

where $A_n = F'(x_n) - \gamma F(x_n)$, $\gamma \in S$. It was shown in [6] that the order of this method is three and for $e_n = x_n - x^*$,

$$e_{n+1} = (\gamma - a_2)e_n^2 + O(e_n^3), \tag{1.3}$$

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Ref

6. Kumar, S, Kanwar, V, Tomar, S. K, Singh, S., Geometrically constructed families of Newton's method for unconstarined optimization and nonlinear equations, Int. J. Math. Math. Sci., 2011, (2011), 1-9.

where $a_m = \frac{1}{m!}\frac{F^{(m)}(x^*)}{F'(x^*)}$, $m = 2, 3, \ldots$. It follows that the convergence requires the existence of F', F'', F''' but F'', F''' do not appear on method (1.2). So, these assumptions limit the applicability of the method. Moreover, no computable error bounds on $|x_n - x^*|$ or uniqueness of the solution results are given.

For example [1]: Let $E = E_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Define λ on D by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\lambda'''(t)$ is not bounded on D. So, the convergence of scheme (1.2) is not guaranteed by the previous analyses in [6]. We address all these concerns by using conditions only on F' in both the local and semi-local case that appears on method (1.2). This way we expand the applicability of this method. Our technique is very general so it can be used to extend the applicability of other methods along the same lines [2–5, 7–10]. Throughout this paper $U(x, r) = \{y : |x - y| < r\}$ and $U[x, r] = \{y : |x - y| \leq r\}$ for $x \in S$ and $r > 0$.

The rest of the paper is set up as follows: In Section 2 we present the semi-local analysis, where in Section 3 we present local analysis. The numerical experiments are presented in Section 4.

II. Semi-Local Analysis

Let L_0, L, γ, δ be given positive parameters and $\eta \geq 0$. Define scalar sequence $\{t_n\}$ by

$$\begin{aligned} t_0 &= 0, t_1 = \eta, \\ t_{n+2} &= t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2|\gamma|\delta(t_n + \eta)(t_{n+1} - t_n)}{2(1 - (L_0 + |\gamma|\delta)t_{n+1}))}. \end{aligned} \tag{2.1}$$

Next, we shall prove that this equence is majorizing for method (1.2). But first we need to define more parameters and scalar functions:

$$\alpha_0 = \frac{Lt_1}{2(1 - (L_0 + |\gamma|\delta)t_1)},$$

$$\triangle = L^2 - 8(L_0 + |\gamma|\delta)(1|\gamma|\delta - L),$$

functions $f : [0, 1) \longrightarrow \mathbb{R}$, $g : [0, 1) \longrightarrow \mathbb{R}$ by

$$f(t) = \frac{2|\gamma|\delta\eta}{1 - t} + \frac{2t(L_0 + |\gamma|\delta)\eta}{1 - t} + 2|\gamma|\delta\eta - 2t,$$

$$g(t) = 2(L_0 + |\gamma|\delta)t^2 + Lt + 2|\gamma|\delta - L$$

and sequences of polynomials $f_n : [0, 1) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} f_n(t) &= Lt^n\eta + 2|\gamma|\delta(1 + t + \ldots + t^{n-1})\eta + \eta \\ &\quad +2t(L_0 + |\gamma|\delta)(1 + t + \ldots + t^n)\eta - 2t. \end{aligned}$$

Ref

6. Kumar, S, Kanwar, V, Tomar, S. K, Singh, S., Geometrically constructed families of Newton's method for unconstrained optimization and nonlinear equations, Int. J. Math. Math. Sci., 2011, (2011), 1-9.

Notes

Notice that $\triangle$ is the discriminant of g. Consider that any of these conditions hold:

(C1) There exists minimal $\beta \in (0,1)$ such that $f(\beta) = 0$ and $\triangle \leq 0$. Then, suppose $\alpha_0 \leq \beta$.

(C2) There exists minimal $\beta \in (0,1)$ such that $f(\beta) = 0$, $\alpha \in (0,1) : f(\alpha) = 0$, and $\triangle > 0$. Then, suppose $\alpha_0 \leq \alpha \leq \beta$.

(C3) $f(t) \neq 0$ for all $t \in [0,1)$ and $\triangle \leq 0$.

(C4) $f(t) \neq 0$ and $\triangle > 0$. Notice that g has two solutions: $0 \leq s_1 < s_2 < 1$. Suppose $\alpha_0 \leq s \in (s_1, s_2]$ and $f(s) \leq 0$.

Let us denote these conditions by (C).

Next, we present convergence results for sequence (2.1).

Lemma 2.1 *Suppose:*

$$(L_0 + |\gamma|\delta)t_{n+1} < 1. \tag{2.2}$$

Then, the following assertions hold

$$0 \leq t_n \leq t_{n+1} \tag{2.3}$$

and

$$t^* = \lim_{n \longrightarrow \infty} t_n \leq \frac{1}{L_0 + |\gamma|\delta}. \tag{2.4}$$

Proof. Assertions follow from (2.1) and (2.2), where t^* is the unique least upper bound of sequence $\{t_n\}$.

The next result is shown under stronger conditions but which are easier to verify than (2.2).

Lemma 2.2 *Suppose: conditions (C) hold. Then, assertions (2.3) and (2.4) hold too.*

Proof. Mathematical induction on m is used to show

$$(I_m): \quad \frac{L(t_{m+1} - t_m) + 2|\gamma|\delta(t_m + \eta)}{2(1 - (L_0 + |\gamma|\delta)t_{m+1}} \leq \alpha. \tag{2.5}$$

This estimate hols for $m = 0$ by the definition of α_0 and conditions (C). Then, we get $0 \leq t_2 - t_1 \leq \alpha(t_1 - t_0) = \alpha\eta$ and $t_2 \leq t_1 + \alpha\eta = \frac{1-\alpha^2}{1-\alpha}\eta < \frac{\eta}{1-\alpha}$. Suppose $0 \leq t_{m+1} - t_m \leq \alpha^m\eta$ and $t_m \leq \frac{1-\alpha^m}{1-\alpha}\eta$. Then, (2.5) holds if

$$\begin{aligned} &L\alpha^m\eta + 2|\gamma|\delta((1 + \alpha + \ldots + \alpha^{m-1})\eta + \eta) \\ &+2\alpha(L_0 + |\gamma|\delta)(1 + \alpha + \ldots + \alpha^m)\eta - 2\alpha \leq 0 \end{aligned} \tag{2.6}$$

or

$$f_m(t) \leq 0 \ \text{ at } \ t = \alpha. \tag{2.7}$$

We need a relationship between two consecutive polynomials f_m :

$$\begin{aligned} f_{m+1}(t) &= f_{m+1}(t) - f_m(t) + f_m(t) \\ &= Lt^{m+1}\eta + 2|\gamma|\delta(1 + t + \ldots + t^m)\eta + 2|\gamma|\delta\eta \\ &\quad +2t(L_0 + |\gamma|\delta)(1 + t + \ldots + t^{m+1})\eta + f_m(t) \end{aligned}$$

$$
\begin{aligned}
& -Lt^m\eta - 2|\gamma|\delta(1+t+\ldots+t^{m-1})\eta - 2|\gamma|\delta\eta \\
& -2t(L_0+|\gamma|\delta)(1+t+\ldots+t^{m-1})\eta - 2|\gamma|\delta\eta \\
& -2t(L_0+|\gamma|\delta)(1+t+\ldots+t^{m})\eta + 2t \\
= \;& f_m(t) + g(t)t^m\eta,
\end{aligned}
$$

so

$$f_{m+1}(t) = f_m(t) + g(t)t^m\eta. \tag{2.8}$$

Define function $f_\infty : [0,1) \longrightarrow \mathbb{R}$ by

$$f_\infty(t) = \lim_{m\longrightarrow\infty} f_m(t). \tag{2.9}$$

It then follows from (2.6) and (2.9) that

$$f_\infty(t) = f(t). \tag{2.10}$$

Case (C1) We have by (2.8) that

$$f_m(t) \le f_{m+1}(t). \tag{2.11}$$

So, (2.7) holds if

$$f_\infty(t) \le 0, \tag{2.12}$$

which is true by the choice of β.

Case(C2) Then, again (2.11) and (2.12) hold by the choice of α and β.

Case(C4) We have

$$f_{m+1}(t) \le f_m(t),$$

so (2.7) holds if $f_1(\alpha) \le 0$, which is true by (C4).

The induction for items (2.5) so the induction for (2.3) is completed too leading again to the verification of the assertions for $\frac{1}{L_0+|\gamma|\delta}$ in (2.4) replaced by $\frac{\eta}{1-\alpha}$.

Next, we introduce the conditions (A) to be used in the semi-local convergence of method (1.2).

Suppose:

(A1) There exist $x_0 \in D$, $\eta \ge 0$ such that $A_0 \ne 0$ and $\|A_0^{-1}F(x_0)\| \le \eta$.

(A2) There exists $L_0 > 0$ such that $\|A_0^{-1}(F'(v) - F'(x_0))\| \le L_0\|v - x_0\|$ for all $v \in D$. Set $D_0 = U(x_0, \frac{1}{L_0}) \cap D$.

(A3) There exist $L > 0, \delta > 0$ such that $\|A_0^{-1}(F'(v) - F'(w))\| \le L\|v - w\|$ and

$$\|A_0^{-1}(F(v) - F(x_0))\| \le \delta\|v - x_0\|,$$

for all $v, w \in D_0$.

(A4) Conditions of Lemma 2.1 or Lemma 2.2 hold and

(A5) $U[x_0, t^*] \subset D$.

Next, we show the semi-local convergence of method (1.2) under the conditions (A).

Notes

Theorem 2.3 *Suppose that conditions (A) hold. Then, sequence $\{x_n\}$ generated by method (1.2) is well defined remains in $U(x_0, t^*)$ and converges to a solution of equation (1.1) such that $x^* \in U[x_0, t^*]$.*

Proof. Mathematical induction is used to show

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n. \tag{2.13}$$

This estiamte holds by (A1) and (1.2) for $n = 0$. Indeed, we have

$$\|x_1 - x_0\| = \|A_0^{-1} F(x_0)\| = \eta = t_1 - t_0 < t^*,$$

so $x_1 \in U(x_0, t^*)$. Suppose (2.13) holds for all values of m smaller or equal to $n-1$. Next, we show $A_{m+1} \neq 0$. Using the definition of A_{m+1}, (A2), (A3) we get in turn that

$$\begin{aligned}
\|A_0^{-1}(A_{m+1} - A_0)\| &= \|A_0^{-1}(F'(x_{n+1}) - \gamma F(x_{m+1}) - F'(x_0) + \gamma F(x_0))\| \\
&\leq \|A_0^{-1}(F'(x_{m+1}) - F'(x_0))\| + |\gamma| \|A_0^{-1}(F(x_{m+1}) - F(x_0))\| \\
&\leq L_0 \|x_{m+1} - x_0\| + |\gamma| \delta \|x_{m+1} - x_0\| \\
&\leq L_0 (t_{m+1} - t_0) + |\gamma| \delta (t_{m+1} - t_0) \\
&= (L_0 + |\gamma| \delta) t_{m+1} < 1, \qquad (2.14)
\end{aligned}$$

where we also used by the induction hypotheses that

$$\begin{aligned}
\|x_{m+1} - x_0\| &\leq \|x_{m+1} - x_m\| + \|x_m - x_{m-1}\| + \ldots + \|x_1 - x_0\| \\
&\leq t_{m+1} - t_0 = t_{m+1} < t^*,
\end{aligned}$$

so $x_{m+1} \in U(x_0, t^*)$. It also follows from (2.14) that $A_{m+1} \neq 0$ and

$$\|A_{m+1}^{-1} A_0\| \leq \frac{1}{1 - (L_0 + |\gamma| \delta_{m+1})} \tag{2.15}$$

by the Banach lemma on inverses of functions [8]. Moreover, we can write by method (1.2):

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - F'(x_m)(x_{m+1} - x_m) + \gamma F'(x_m)(x_{m+1} - x_m), \tag{2.16}$$

since $F(x_m) = -(F'(x_m) - \gamma F(x_m))(x_{m+1} - x_m)$. By (A3) and (2.16), we obtain in turn

$$\begin{aligned}
\|A_0^{-1} F(x_{m+1})\| &\leq \| \int_0^1 A_0^{-1}(F'(x_m + \theta(x_{m+1} - x_m)) \\
&\quad -F'(x_m)) d\theta (x_{m+1} - x_m)\| \\
&\quad + \|A_0^{-1} F'(x_m)\| \|x_{m+1} - x_m\| \\
&\leq \frac{L}{2} \|x_{m+1} - x_m\|^2 \\
&\quad + |\gamma| (\|A_0^{-1}(F(x_{m+1}) - F(x_0)\| + \|A_0^{-1} F(x_0)\|) \|x_{m+1} - x_m\|
\end{aligned}$$

Ref

8. Ortega, L.M., Rheinboldt, W.C., Iterative Solution of Nonlinear Equations in Several Variables, Academic press, New York, (2000).

$$\leq \frac{L}{2}(t_{m+1}-t_m)^2+|\gamma|(\delta\|x_{m+1}-x_0\|+\eta)(t_{m+1}-t_m)$$

$$\leq \frac{L}{2}(t_{m+1}-t_m)^2$$

$$+|\gamma|(\delta t_{m+1}+\eta)(t_{m+1}-t_m). \tag{2.17}$$

Notes

It then follows from (1.2), (2.15) and (2.17) that

$$|x_{m+2}-x_{m+1}\| \leq \|A_{m+1}^{-1}A_0\|\|A_0^{-1}F(x_{m+1})\| \leq t_{m+2}-t_{m+1}, \tag{2.18}$$

and

$$\|x_{m+2}-x_0\| \leq \|x_{m+2}-x_{m+1}\|+\|x_{m+1}-x_0\|$$

$$\leq t_{m+2}-t_{m+1}+t_{m+1}-t_0=t_{m+2}<t^*. \tag{2.19}$$

But sequence $\{t_m\}$ is fundamental. So, sequence $\{x_m\}$ is fundamental too (by (2.18)), so it converges to some $x^* \in U[x_0,t^*]$. By letting $m \longrightarrow \infty$ in (2.17), we deduce that $F(x^*)=0$.

Next, we present a uniqueness of the solution result for equation (1.1).

Proposition 2.4 Suppose

(1) There exists $x_0 \in D, K>0$ such that $F'(x_0) \neq 0$ and

$$\|F'(x_0)^{-1}(F'(v)-F'(x_0))\| \leq K\|v-x_0\| \tag{2.20}$$

for all $v \in D$.

(2) The point $x^* \in U[x_0,a] \subseteq D$ is a simple solution of equation $F(x)=0$ for some $a>0$.

(3) There exists $b \geq a$ such that

$$K(a+b)<2.$$

Set $B=U[x_0,b]\cap D$. Then, the only solution of equation $F(x)=0$ in B is x^*.

Proof. Set $M=\int_0^1 F'(z^*+\theta(x^*-z^*))d$ for some $z \in B$ with $F(z^*)=0$. Then, in view of (2.20)

$$\|F'(x_0)^{-1}(M-F'(x_0))\| \leq K\int_0^1((1-\theta)\|x_0-x^*\|+\theta\|x_0-z^*\|)d\theta$$

$$\leq \frac{K}{2}(a+b)<1,$$

so, $z^*=x^*$ follows from $M\neq 0$ and $M(z^*-x^*)=F(z^*)-F(x^*)=0-0=0.$

Notes

III. Local Convergence

Let β_0, β and β_1 be positive parameters. Set

$$\beta_2 = \beta_0 + |\delta|\beta_1.$$

Define function $h : [0, \frac{1}{\beta_2}) \longrightarrow \mathbb{R}$ by

$$h(t) = \frac{\beta t}{2(1-\beta_0 t)} + \frac{|\gamma|\beta_1^2 t}{(1-\beta_0 t)(1-\beta_2 t)}.$$

Suppose this function has a minimal zero $\rho \in (0, \frac{1}{\beta_2})$. We shall use conditions (H). Suppose:

(H1) The point $x^* \in D$ is a simple solution of equation (1.1).

(H2) There exists $\beta_0 > 0$ such that

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq \beta_0 \|v - x^*\|$$

for all $v \in D$. Set $D_1 = U(x^*, \frac{1}{\beta_0}) \cap D$.

(H3) There exist $\beta > 0, \beta_1 > 0$ such that

$$\|F'(x^*)^{-1}(F'(v) - F'(w))\| \leq \beta \|v - w\|$$

and

$$\|F'(x^*)^{-1}(F(v) - F(x^*))\| \leq \beta_1 \|v - x^*\|$$

for all $v, w \in D_1$.

(H4) Function $h(t) - 1$ has a minimal solution $\rho \in (0, 1)$.
and

(H5) $U[x^*, \rho] \subset D$.

Notice that $A(x^*) = F'(x^*)$. Then, we get the estimates

$$\begin{aligned}
&\|F'(x^*)^{-1}(F(x_n) - \gamma F(x_n) - F'(x^*) + \gamma F(x^*)\| \\
\leq\ & \|F'(x^*)^{-1}(F'(x_m) - F'(x^*))\| + |\gamma| \|F'(x^*)^{-1}(F(x_m) - F(x^*))\| \\
\leq\ & \beta_0 \|x_m - x^*\| + |\gamma|\beta_1 \|x_m - x^*\| \\
=\ & \beta_2 \|x_m - x^*\| < \beta_2 \rho < 1,
\end{aligned}$$

$$\begin{aligned}
\|F'(x^*)^{-1}(A_m - F'(x_m))\| &= \|F'(x^*)^{-1}(F'(x_m) - \gamma F(x_m) - F'(x_m))\| \\
&= |\gamma| \|F'(x^*)^{-1} F(x_m)\| \\
&\leq |\gamma|\beta_1 \|x_m - x^*\|,
\end{aligned}$$

$$\begin{aligned}
x_{m+1} - x^* &= x_m - x^* - F'(x_m)^{-1}F(x_m) + F'(x_m)^{-1}F(x_m) - A_m^{-1}F(x_m) \\
&= (x_m - x^* - F'(x_m)^{-1}F(x_m)) + (F'(x_m)^{-1} - A_m^{-1})F(x_m) \\
&= (x_m - x^* - F'(x_m)^{-1}F(x_m)) \\
&\quad + F'(x_m)^{-1}(A_m - F'(x_m))A_m^{-1}F(x_m),
\end{aligned}$$

leading to

$$\begin{aligned}\|x_{m+1}-x^*\| &\leq \frac{\beta\|x_m-x^*\|^2}{2(1-\beta_0\|x_m-x^*\|)} \\ &\quad + \frac{|\gamma|\beta_1^2\|x_m-x^*\|^2}{(1-\beta_0\|x_m-x^*\|)(1-\beta_2\|x_m-x^*\|)} \\ &< h(\rho)\|x_m-x^*\| = \|x_m-x^*\| < \rho.\end{aligned}$$

Notes

So, we get

$$\|x_{m+1}-x^*\| \leq p(\|x_m-x^*\|) < \rho, \;\; p = h(\|x_0-x^*\|) \tag{3.1}$$

and $x_{m+1} \in U(x^*, \rho)$. Hence, we conclude by (3.1) that $\lim_{m\longrightarrow\infty} x_m = x^*$. Therefore, we arrive at the local convergence result for method (1.2).

Theorem 3.1 *Under conditions (H) further suppose that* $x_0 \in U(x^*, \rho)$. *Then, sequence* $\{x_n\}$ *generated by method (1.2) is well defined in* $U(x_0, \rho)$, *remains in* $U(x_0, \rho)$ *and converges to* x^*.

Next, we present a uniqueness of the solution result for equation (1.2).

Proposition 3.2 *Suppose*

(1) The point x^* *is a simple solution of equation* $F(x) = 0$ *in* $U(x^*, \tau) \subset D$ *for some* $\tau > 0$.

(2) Condition (H2) holds.

(3) There exists $\tau^* \geq \tau$ *such that*

$$\beta_0\tau^* < 2.$$

Set $B_1 = U[x_0, \tau^*] \cap D$. *Then, the only solution of equation (1.1) in* B_1 *is* x^*.

Proof. Set $M_1 = \int_0^1 F'(z^* + \theta(x^* - z^*))d$ for some $z^* \in B_1$ with $F(z^*) = 0$. Then, using (H2), we get in turn that

$$\begin{aligned}\|F'(x^*)^{-1}(M_1 - F'(x^*))\| &\leq \int_0^1 (1-\theta)\|z^* - x^*\|d\theta \\ &\leq \frac{\beta_0}{2}\tau^* < 1,\end{aligned}$$

so, $z^* = x^*$ follows from $M_1 \neq 0$ and $M_1(z^* - x^*) = F(z^*) - F(x^*) = 0 - 0 = 0$.

IV. Numerical Example

We verify convergence criteria using method (1.2) for $\gamma = 0$, so $\delta = 0$.

Example 4.1 *(Semi-local case) Let us consider a scalar function* F *defined on the set* $D = U[x_0, 1-q]$ *for* $q \in (0, \frac{1}{2})$ *by*

$$F(x) = x^3 - q.$$

Choose $x_0 = 1$. *Then, we obtain the estimates* $\eta = \frac{1-q}{3}$,

$$\begin{aligned}|F'(x_0)^{-1}(F'(x) - F'(x_0))| &= |x^2 - x_0^2| \\ &\leq |x + x_0||x - x_0| \leq (|x - x_0| + 2|x_0|)|x - x_0|\end{aligned}$$

Global Journal of Science Frontier Research (F) Volume XXII Issue I Version I Year 2022

$$= \quad (1-q+2)|x-x_0| = (3-q)|x-x_0|,$$

for all $x \in D$, *so* $L_0 = 3-q$, $D_0 = U(x_0, \frac{1}{L_0}) \cap D = U(x_0, \frac{1}{L_0})$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(y)-F'(x)| &= |y^2-x^2| \\ &\leq |y+x||y-x| \leq (|y-x_0+x-x_0+2x_0)||y-x| \\ &= (|y-x_0|+|x-x_0|+2|x_0|)|y-x| \\ &\leq (\frac{1}{L_0}+\frac{1}{L_0}+2)|y-x| = 2(1+\frac{1}{L_0})|y-x|, \end{aligned}$$

for all $x, y \in D$ *and so* $L = 2(1+\frac{1}{L_0})$.

Next, set $y = x - F'(x)^{-1}F(x), x \in D$. *Then, we have*

$$y + x = x - F'(x)^{-1}F(x) + x = \frac{5x^3+q}{3x^2}.$$

Define fundtion $\bar{F}$ *on the interval* $D = [q, 2-q]$ *by*

$$\bar{F}(x) = \frac{5x^3+q}{3x^2}.$$

Then, we get by this definition that

$$\begin{aligned} \bar{F}'(x) &= \frac{15x^4-6xq}{9x^4} \\ &= \frac{5(x-q)(x^2+xq+q^2)}{3x^3}, \end{aligned}$$

where $p = \sqrt[3]{\frac{2q}{5}}$ *is the critical point of function* $\bar{F}$. *Notice that* $q < p < 2-q$. *It follows that this function is decreasing on the interval* (q, p) *and increasing on the interval* $(q, 2-q)$, *since* $x^2+xq+q^2 > 0$ *and* $x^3 > 0$. *So, we can set*

$$K_2 = \frac{5(2-q)^2+q}{9(2-q)^2}$$

and

$$K_2 < L_0.$$

But if $x \in D_0 = [1-\frac{1}{L_0}, 1+\frac{1}{L_0}]$, *then*

$$L = \frac{5\varrho^3+q}{9\varrho^2},$$

where $\varrho = \frac{4-q}{3-q}$ *and* $K < K_1$ *for all* $q \in (0, \frac{1}{2})$. *For* $q = 0.45$, *we have*

n	*1*	*2*	*3*	*4*	*5*	
t_n	*0.1833*	*0.2712*	*0.3061*	*0.3138*	*0.3142*	*0.3142*
$(L_0+\lvert\gamma\rvert\delta)t_{n+1}$	*0.4675*	*0.6916*	*0.7804*	*0.8001*	*0.8011*	*0.8011*

Thus condition (2.2) satisfied.

Notes

Example 4.2 *Let* $F : [-1,1] \longrightarrow \mathbb{R}$ *be defined by*

$$F(x) = e^x - 1$$

Then, we have for $x^* = 0, \beta_0 = e - 1, \beta = e^{\frac{1}{e-1}}$ *and* $\beta_1 = 0$. *So, we have* $\rho = 0.3827$.

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