Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 22 Issue 2 Version 1.0 Year 2022
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

By Joachim Moussounda Mouanda, Jean Raoul Tsiba & Kinvi Kangni
Blessington Christian University

Abstract- We introduce Mouanda's choice function for matrices which allows us to construct the galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of the galaxies of circulant matrices with positive integers as entries. The characterization of the matrix solutions of the equation $X^2+Y^2=Z^2$ allows us to show that the equation $X^{2n}+X^{2n}=Z^{2n}(n\geq 2)$ has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation $X^n+Y^n=Z^n(n\geq 3)$ has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we show Fermat's Last Theorem for complex polynomials over $\mathbb{D}$ associated to circulant matrices.

Keywords: *Fermat's equation, polynomials, model theory, circulant matrices, Mouanda's choise function, galaxy, Toeplitz matrices.*

GJSFR-F Classification: *MSC 2010: 11D41, 11C08, 03C95.*

ONFERMATSLASTTHEOREMMATRIXVERSIONANDGALAXIESOFSEQUENCESOFCIRCULANTMATRICESWITHPOSITIVEINTEGERSASENTRIES

Strictly as per the compliance and regulations of:

Notes

On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

Joachim Moussounda Mouanda [α], Jean Raoul Tsiba [σ] & Kinvi Kangni [ρ]

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatum in duos ejusdem nominis fas est dividere: cujes rei demonstrationem mirabilem sane detexi. Hane marginis exiguitas non caperet.

-Pierre de Fermat (1637).

Abstract- We introduce Mouanda's choice function for matrices which allows us to construct the galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of the galaxies of circulant matrices with positive integers as entries. The characterization of the matrix solutions of the equation $X^2 + Y^2 = Z^2$ allows us to show that the equation $X^{2n}+X^{2n}=Z^{2n}(n\geq 2)$ has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation $X^n+Y^n= Z^n(n\geq 3)$ has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we show Fermat's Last Theorem for complex polynomials over $\mathbb{D}$ associated to circulant matrices.

Keywords: *Fermat's equation, polynomials, model theory, circulant matrices, Mouanda's choise function, galaxy, Toeplitz matrices.*

I. Introduction and Main Result

It is well known that there are many solutions in integers to the equation $x^2 + y^2 = z^2$, for instance (3,4,5); (5,12,13). Around 1500 B.C, the Babylonians were aware of the solution (4961, 6480, 8161) and the Egyptians knew the solutions (148, 2736, 2740) and (514, 66048, 66050). Also Greek mathematicians were attracted to the solutions of this equation. We notice that this equation has sequences of complex number solutions

$$(1 + 2i \times a^k, 2i \times a^k - 2 \times a^{2k}, 1 + 2i \times a^k - 2 \times a^{2k}), a \in \mathbb{C}, k \in \mathbb{N}$$

and matrix solutions

Author α σ: Blessington Christian University, Mathematics Department Nkayi, Republic of Congo.
e-mails: mmoussounda@yahoo.fr, jrtsiba@hotmail.com
Author ρ: Felix Houphouet Boigny University, Mathematics Department Abidjan, Ivory Coast
e-mail: kinvi.kangni@univ-fhb.edu.ci

$$\left(\begin{pmatrix} -2 & 2i & 0 \\ 0 & -2 & 2i \\ 2i & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 2i & 1 \\ 1 & 0 & 2i \\ 2i & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 2i & 1 \\ 1 & -2 & 2i \\ 2i & 1 & -2 \end{pmatrix}\right).$$

In 1637, Pierre de Fermat wrote a note in the margin of his copy of Diophantus Arithmetica [1] stating that the equation

$$x^n + y^n = z^n, n \in \mathbb{N}(n > 2), xyz \neq 0 \tag{1.1}$$

has no integer solutions. This is the Fermat Last Theorem. He claimed that he had found the proof of this Theorem. The only case Fermat actually wrote down a proof is the case $n = 4$. In his proof, Fermat introduced the idea of infinite descent which is still one the main tools in the study of Diophantine equations. He proved that the equation $x^4 + y^4 = z^2$ has no solutions in relatively prime integers with $xyz \neq 0$. Solutions to this equation correspond to rational points on the elliptic curve $v^2 = u^3 - 4u$. The proof of the case $n = 3$ was given first by Karl Gauss. In 1753, Leonhard Euler gave a different prove of Fermat's Last Theorem for $n = 3$[2, 3]. In 1823, Sophie Germain proved that if l is a prime and $2l + 1$ is also prime, the equation $x^l + y^l = z^l$ has no solutions (x,y,z) with $xyz \neq 0(modl)$. The case $n = 5$ was proved simultaneously by Adrien Marie Legendre in 1825 [4, 5] and Peter Lejeune Dirichlet [6] in 1832. In 1839, Gabriel Lame proved the case $n = 7$ [7, 8, 9, 10]. Between 1840 -1843, V. A. Lebesque worked on Fermat's Last Theorem [11, 12]. Between 1847 and 1853, Ernst Eduard Kummer published some masterful papers about this Theorem. Fermat's Last Theorem attracted the attention of many researchers and many studies have been developed around this Theorem. For example the work of Arthur Wieferich (1909), Andre Weil (1940), John Tate (1950), Gerhard Frey (1986), who was the first to suggest that the existence of a solution of the Fermat equation might contradict the modality conjecture of Taniyama, Shimura and Weil [29]; Jean Pierre Serre (1985 - 1986) [14, 15, 16], who gave an interested formulation and (with J. F. Mestre) tested numerically a precise conjecture about modular forms and Galois representations mod p and proved how a small piece of this conjecture the so called epsilon conjecture together Modularity Conjecture would imply Fermat's Last Theorem; Kennedy Ribet (1986) [17], who proved Serre's epsilon conjecture, thus reducing the proof of Fermat's Last Theorem; Barry Mazur (1986), who introduced a significant piece of work on the deformation of Galois representations [18, 19]. However, no final proof was given to this Theorem. This Theorem was unsolved for nearly 350 years. In 1995, using Mazur's deformation theory of Galois representations, recent results on Serre's conjecture on the modularity of Galois representations, and deep arithmetical properties of Hecke algebras, Andrew Wiles with Richard Taylor succeeded in proving that all semi-stable elliptic curves defined over the rational numbers are modular. This result is less than the full Shimura-Taniyama conjecture. This result does imply that the elliptic curve given above is modular. Therefore, proving Fermat's Last Theorem [20, 21]. Many mathematicians are still heavenly involved on studying Fermat's Last Theorem [22, 23, 24]. In 2021, Nag introduced an elementary proof of Fermat's

Ref

1. Diophantus of Alexandria, Arithmetica of Diophantus, 1637.

Last Theorem for epsilons[25]. In 2022, Mouanda constructed the galaxies of sequences of triples of positive integers solutions of the equation $x^2+y^2=z^2$. The unique characterization of the solutions of this equation allowed him to provide an elementary analytic proof of Fermat's Last Theorem [26]. The Fermat Last Theorem for positive integers has been extended over some number fields. In 1966, Domiaty proved that the equation $X^4+Y^4=Z^4$ is solvable in $M_2(\mathbb{Z})$ [27]. Let $GL_n(\mathbb{Z})$ be the group of units of ring $M_n(\mathbb{Z})$. Denote by

$$SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : detA = 1\}. \quad (1.2)$$

In 1989, Vaserstein investigated the question of the solvability of the equation

$$X^n + Y^n = Z^n, n \geq 2, n \in \mathbb{N}, \quad (1.3)$$

for matrices of the group $GL_2(\mathbb{Z})$ [28]. In 1993, Frejman studied the solvability of the equation (1.3) in the set of positive integer powers of a matrix A with elements $a_{11}=0, a_{12}=a_{21}=a_{22}=1$ [29]. In 1995, the same case was studied by Grytczuk [30]. The same year, Khazanov proved that in $GL_3(\mathbb{Z})$ solutions of the equation (1.3 do not exist if n is a multiple of either 21 or 96, and in $SL_3(\mathbb{Z})$ solutions do not exist if n is a multiple of 48 [31]. In 1996, Qin gave another proof of Khazanov's result on the solvability of the equation (1.3) in $SL_2(\mathbb{Z})$ [32]. In 2002, Patay and Szakacs described the periodic elements in $GL_2(\mathbb{Z})$ and gave the answer to some problems concerning the equation (1.3) in matrix groups and in irreducible elements of matrix rings [33]. In 2021, Mao-Ting and Jie proved that Fermat's matrix equation has many solutions in a set of 2-by-2 positive semi-definite integral matrices, and has no nontrivial solutions in some classes including 2-by-2 symmetric rational and stochastic quadratic field matrices [34]. Fermat's Last Theorem has been extended to the field of complex polynomials of one variable [35].

This Theorem has many applications in Cryptography.

In this paper, we are mainly concerned with Fermat's Last Theorem for circulant matrices with positive integers as entries. Firstly, we focus our attention on the construction of the galaxies of sequences of triples of circulant matrices with positive integers as entries solutions of the equation $X^2+Y^2=Z^2$. In particular, Mouanda's matrix choice function allows us to construct practical examples of such galaxies. The elementary characterization of these matrix solutions allows us to prove Fermat's Last Theorem for circulant matrices with positive integers as entries.

Theorem 1.1. The equation

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no circulant matrix with positive integers as entries solutions.

We construct a galaxy of sequences of eigenvalues of circulant matrices and we prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we construct a galaxy of sequences of complex polynomials over the unit disk $\mathbb{D}$ associated to circulant matrices and we prove Fermat's Last Theorem for complex polynomials over $\mathbb{D}$.

Ref

25. Nag. B.B, An elementary proof of Fermat's Last Theorem for epsilons, Advances in Pures Mathematics, 11(2021), 735-740.

II. Preliminaries

Definition 2.1. Let $\mathbb{A}$ be a unital Banach algebra. We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set

$$Inv(\mathbb{A}) = \{a \in \mathbb{A} : \exists b \in \mathbb{A}, ab = ba = 1\}$$

is a group under multiplication. If a is an element of $\mathbb{A}$, the spectrum of a is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda.1 \notin Inv(\mathbb{A})\},$$

and its spectral radius is defined to be

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Let $V = \{a_0, a_1, \ldots, a_{m-1}\} \subset \mathbb{C}$ be a subset of the set of complex numbers, denote by C_V the following Toeplitz matrix:

$$C_V = \begin{pmatrix} a_0 & a_1 & \iddots & a_{m-1} \\ a_{m-1} & a_0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & \ldots & a_{m-1} & a_0 \end{pmatrix}.$$

This matrix is called a $m\times m$-complex circulant matrix or a complex circulant matrix of order m. Denote by $C_m(\mathbb{C})$ the commutative algebra of $m \times m$-complex circulant matrices. Let $\epsilon = e^{\frac{2\pi i}{m}}$ be a primitive m-th root of unity. Let us denote by U the following matrix:

$$U = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \ldots & \ldots & \epsilon^{(m-3)} & \epsilon^{m-2} & \epsilon^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \epsilon^{m-3} & \ldots & \ldots & \epsilon^{(m-3)^2} & \epsilon^{(m-3)(m-2)} & \epsilon^{(m-1)(m-3)} \\ 1 & \epsilon^{m-2} & \ldots & \ldots & \epsilon^{(m-2)(m-3)} & \epsilon^{(m-2)^2} & \epsilon^{(m-1)(m-2)} \\ 1 & \epsilon^{m-1} & \ldots & \ldots & \epsilon^{(m-1)(m-3)} & \epsilon^{(m-1)(m-2)} & \epsilon^{(m-1)^2} \end{pmatrix}.$$

This matrix is called Vandermonde matrix. It is well known that this matrix has the following properties:

$$det(U) = \frac{1}{m^{\frac{m}{2}}} \prod_{i,j=0}^{m-1} (\epsilon^j - \epsilon^i) \neq 0,$$

Notes

Notes

U is non-singular, unitary, $U^{-1} = \overline{U}^T, U^T = U$ and $U^{-1} = \overline{U} = U^*$. It is well known that all the elements of $C_m(\mathbb{C})$ are simultaneously diagonalized by the same unitary matrix U, that is, for A in $C_m(\mathbb{C})$, one has

$$U^* A U = D_A$$

with D_A is a diagonal matrix with diagonal entries given by the ordered eigenvalues of A: $\lambda_1^A, \lambda_2^A, \ldots, \lambda_m^A$. The factorization $U^* A U = D_A$ is called the spectral factorization of A [36, 37, 38, 39]. It is possible to write the matrix C_V as one variable complex polynomial. Indeed, let P be the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

It is simple to see that

$$C_V = \sum_{k=0}^{m-1} a_k P^k.$$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk. The complex polynomial

$$f_V(z) = \sum_{k=0}^{m-1} a_k z^k$$

over $\mathbb{D}$ is called the associated complex polynomial of the matrix $C_V = f_V(P)$. It follows that if

$$X = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

is a $m \times m$-complex matrix, then

$$f_V(X)=\sum_{k=0}^{m-1}a_kX^k=\begin{pmatrix} a_0 & a_1 & \ddots & a_{m-1}\\ 0 & a_0 & \ddots & \ddots\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \dots & 0 & a_0\end{pmatrix}$$

is a $m\times m$- upper complex triangular Toeplitz matrix. The complex polynomial

$$f_V(z)=\sum_{k=0}^{m-1}a_kz^k$$

is also called the associated complex polynomial of the matrix $f_V(X)$.

III. The Universe of an Algebra

Definition 3.1. Let $x,y,z\in\mathbb{C}$ be complex numbers. Denote by

$$(x,y,z)^n=(x^n,y^n,z^n)\,,n=\frac{p}{q},p,q\in\mathbb{N},q\neq 0.$$

The triple (x^n,y^n,z^n) is called the triple (x,y,z) to the power n.

Definition 3.2. Let $x,y,z\in\mathbb{C}$ be complex numbers. Denote by

$$\alpha\,(x,y,z)=(\alpha x,\alpha y,\alpha z)\,,(x,y,z)+(a,b,c)=(x+a,y+b,z+c)\,.$$

Definition 3.3. A universe of degree $\frac{p}{q}$ of the algebra $\mathcal{B}$ is the set $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$ of triples (x,y,z) of elements of $\mathcal{B}$ which satisfy the law of stability

$$x^{\frac{p}{q}}+y^{\frac{p}{q}}=z^{\frac{p}{q}},xyz\neq 0,p,q\in\mathbb{N},q\neq 0.$$

The element (x,y,z) is called a star (or a planet) of the universe $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$. Every sequence $(x_k,y_k,z_k)_{n\geq 0}$ of elements of the universe $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$ is called a planet system of elements of $\mathcal{B}$.

The set

$$\mathbb{F}_{\frac{p}{q}}(C_m(\mathbb{C}))=\left\{(X,Y,Z)\in\mathbb{C}^3:X^{\frac{p}{q}}+Y^{\frac{p}{q}}=Z^{\frac{p}{q}},XYZ\neq 0\right\},p,q\in\mathbb{N},q\neq 0,$$

is called the complex ciculant universe of degree $\frac{p}{q}$. In particular, the set

$$\mathbb{F}_n(C_m(\mathbb{N}))=\left\{(X,Y,Z)\in C_m(\mathbb{N})^3:X^n+Y^n=Z^n,XYZ\neq 0\right\},n\in\mathbb{N},n\geq 2,$$

Notes

is called the natural circulant universe of degree n. We are going to show that the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ is not empty. Fermat's Last Theorem for circulant matrices is equivalent to say that

$$\mathbb{F}_n(C_m(\mathbb{N})) = \{\} = \phi, n \geq 3.$$

In other words, there are matrix complex universes which don't have triples of matrices of positive integers as entries elements.

IV. Mouanda's Choice Function for Matrices

Denote by $C_*(C_m(\mathbb{C})) = \{h/h : C_m(\mathbb{C}) \longrightarrow C_m(\mathbb{C})\}$, the set of complex functions over $\mathbb{C}$. Let

$$\Omega(\mathbb{F}_2(C_m(\mathbb{C}))) = \{P : P \subseteq \mathbb{F}_2(C_m(\mathbb{C}))\}$$

be the set of all subsets of $\mathbb{F}_2(C_m(\mathbb{C}))$. Theorem 2.5 of [26] allows us to claim that the appropriate choice of the values of $m_0(k)$ and $n_0(k)$ such that

$$\frac{2(m_0(k) - n_0(k)) \pm \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} \in C_m(\mathbb{C})$$

leads to the construction of sequences of triples of circulant matrices with positive (or negative) integers as entries which satisfy the equation

$$X^2 + Y^2 = Z^2.$$

Let $f_{\mathbb{M}} : C_*(C_m(\mathbb{C})) \times C_*(C_m(\mathbb{C})) \longrightarrow \Omega(\mathbb{F}_2(C_m(\mathbb{C})))$ be the function defined by

$$f_{\mathbb{M}}(m_0(k), n_0(k)) = \begin{bmatrix} m_0(k) = a^{\beta(k)}, k, a, \beta(k) \in C_m(\mathbb{C}), \beta \in C_*(C_m(\mathbb{C})) \\ m_0(k) - n_0(k) \in C_m(\mathbb{C}) \\ X_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} \in C_m(\mathbb{C}) \\ Y_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + n_0(k) \\ Z_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + m_0(k) \end{bmatrix},$$

This type of function is called Mouanda's choice function for matrices. Mouanda's choice function for matrices is a galaxy valued function. This function allows us to construct galaxies of sequences of matrices.

V. A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries

All the galaxies defined in this section have been deduced from the galaxies already introduced in [26].

Ref

26. J. Moussounda Mouanda, On Fermat's Last Theorem and Galaxies of sequences of positive integers, American Journal of Computational Mathematics, 12(2022), 162-189.

Definition 5.1. A multi-galaxy is a galaxy which contains other galaxies. The order of a galaxy is the number of variables of the galaxy.

Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

be a $m \times m-$matrix. Denote by $T_k = P^k, k = 1, 2, ..., m-1$. Let

$$U_m = \{I_m, T_1, T_2, ..., T_{m-1}\}$$

be a finite set of unitary circulant matrices. The elements of the set U_m satisfy the following:

$$T_i T_j = T_{i+j}, T_i T_{m-i} = I_m, T_{m+i} = T_i, T_k = P^k.$$

Denote by

$$\mathcal{P}^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.$$

Let

$$f(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. The Toeplitz matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} = f(P)$$

is called the circulant matrix with complex numbers as entries. The polynomial $f(z)$ is called associated polynomial of the matrix $f(P)$. Recall that the

Notes

set $C_m(\mathbb{C})$ is the commutative algebra of $m \times m$-complex circulant matrices. In other words,

$$C_m(\mathbb{C}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{C} \right\}.$$

It follows that

$$C_m(\mathbb{C}) = \left\{ f(P) : f \in \mathcal{P}^{(m)}(\mathbb{C}) \right\}$$

and

$$C_m(\mathbb{N}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{N} \right\}.$$

It is quiet clear that

$$C_m(\mathbb{N}) \subset C_m(\mathbb{Z}) \subset C_m(\mathbb{Q}) \subset C_m(\mathbb{R}) \subset C_m(\mathbb{C}).$$

Mouanda's matrix choice function $f_{\mathbb{M}}$ allows us to construct galaxies of circulant matrices. For instance, if we choose $m_0 = A^{2k} \times 2, m_0 - n_0 = \alpha I_m, A \in C_m(\mathbb{N})$ the model

$$Gala(\mathbb{N}I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k \\ Y_k(\alpha I_m, A) = 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ Z_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ \alpha = r^2, k, r \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{bmatrix}$$

is called the galaxy of sequences of circulant matrices with positive integers as entries of order 2. For α_0 and A fixed, the triple

$$(X_0(\alpha_0 I_m, A), Y_0(\alpha_0 I_m, A), Z_0(\alpha_0 I_m, A))$$

is called the origin of the galaxy $Gala(\alpha_0 I_m, C_m(\mathbb{N}))$. The elements of $Gala(\alpha_0 I_m, C_m(\mathbb{N}))$ satisfy

$$X_k^2(\alpha_0 I_m, A) + Y_k^2(\alpha_0 I_m, A) = Z_k^2(\alpha_0 I_m, A), k \in \mathbb{N}, A \in C_m(\mathbb{N})$$

and

$$(X_k(\alpha_0 I_m, A), Y_k(\alpha_0 I_m, A), Z_k(\alpha_0 I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), A, B, C, D \in C_m(\mathbb{N})$$

with $k, p, q \in \mathbb{N}, q \neq 0$.

Example 5.2. **A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries**

- The model

$$Gala(4I_m, U_m) = \begin{bmatrix} X_k(4I_m, A) = 4I_m + 4 \times A^k \\ Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\ Z_k(4I_m, A) = 4I_m + 4 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in U_m \\ (X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m) \end{bmatrix}$$

is called the finite galaxy of sequences of circulant matrices with positive integers as entries of order 1. The triple $(X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A))$ is called the origin of the galaxy $Gala(4I_m, U_m)$. The triple

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A))$$

satisfies

$$X_k^2(4I_m, A) + Y_k^2(4I_m, A) = Z_k^2(4I_m, A), k \in \mathbb{N},$$

$$X_0(4I_m, A) + Y_0(4I_m, A) + Z_0(4I_m, A) = 24I_m$$

and

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), k, p, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The finite galaxy $Gala(4I_m, U_m)$ allows the construction of the infinite galaxy

$$Gala(4I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(4I_m, A) = 4I_m + 4 \times A^k \\ Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\ Z_k(4I_m, A) = 4I_m + 4 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m) \end{bmatrix}$$

which has the same origin and stability law than $Gala(4I_m, U_m)$. Therefore, we can say that the galaxy $Gala(\mathbb{N}I_m, C_m(\mathbb{N}))$is a multi-galaxy.

Notes

- Assume that $m = 5, U_5 = \{I_5, T_1, T_2, T_3, T_4\}$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = T_2 \in U_5.$$

Notes

The triples $(X_k(4I_5, A), Y_k(4I_5, A), Z_k(4I_5, A))$ of the galaxy

$$\left[\begin{array}{c} X_k(4I_5, A) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^k \\ Y_k(4I_5, A) = 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^k + 2 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^{2k} \\ Z_k(4I_5, A) = 4I_5 + 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^k + 2 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^{2k} \\ k \in \mathbb{N} \\ (X_0(4I_5, A), Y_0(4I_5, A), Z_0(4I_5, A)) = (8I_5, 6I_5, 10I_5) \\ X_1(4I_5, A) = 4I_5 + 4T_2 \\ Y_1(4I_5, A) = 4T_2 + 2T_4 \\ Z_1(4I_5, A) = 4I_5 + 4T_2 + 2T_4 \end{array}\right]$$

satisfy the equation

$$X_k^2(4I_5, A) + Y_k^2(4I_5, A) = Z_k^2(4I_5, A), k \in \mathbb{N}$$

and

$$(X_k(4I_5, A), Y_k(4I_5, A), Z_k(4I_5, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), D, B, C \in U_n, k, q, p \in \mathbb{N}, q \neq 0.$$

- The elements of the galaxy

$$Gala(9I_m, U_m) = \left[\begin{array}{c} X_k(9I_m, A) = 9I_m + 6 \times A^k \\ Y_k(9I_m, A) = 6 \times A^k + 2 \times A^{2k} \\ Z_k(9I_m, A) = 9I_m + 6 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in U_m \\ (X_0(9I_m, A), Y_0(9I_m, A), Z_0(9I_m, A)) = (15I_m, 8I_m, 17I_m) \end{array}\right]$$

satisfy

$$X_k^2(9I_m, A) + Y_k^2(9I_m, A) = Z_k^2(9I_m, A), k \in \mathbb{N}$$

and

$$(X_k(9I_m, A), Y_k(9I_m, A), Z_k(9I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. Again, from the galaxy $Gala(9I_m, U_m)$, we can construct a galaxy which has an infinite number of elements. Indeed, the galaxy

$$Gala(9I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(9I_m, A) = 9I_m + 6 \times A^k \\ Y_k(9I_m, A) = 6 \times A^k + 2 \times A^{2k} \\ Z_k(9I_m, A) = 9I_m + 6 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(9I_m, A), Y_0(9I_m, A), Z_0(9I_m, A)) = (15I_m, 8I_m, 17I_m) \end{bmatrix}$$

has the same origin and stability law than the galaxy $Gala(9I_m, U_m)$. This galaxy has an infinite number of elements.

- The elements of the galaxy

$$Gala(16I_m, U_m) = \begin{bmatrix} X_k(16I_m, A) = 16I_m + 8 \times A^k \\ Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k} \\ Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in U_m \\ (X_0(16I_m, A), Y_0(16I_m, A), Z_0(16I_m, A)) = (24I_m, 10I_m, 26I_m) \end{bmatrix}$$

satisfy

$$X_k^2(16I_m, A) + Y_k^2(16I_m, A) = Z_k^2(16I_m, A), k \in \mathbb{N}$$

and

$$(X_k(16I_m, A), Y_k(16I_m, A), Z_k(16I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, k, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The galaxy $Gala(16, U_m)$ has a finite number of elements (or planets). However, the galaxy

$$Gala(16I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(16I_m, A) = 16I_m + 8 \times A^k \\ Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k} \\ Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(16I_m, A), Y_0(16I_m, A), Z_0(16I_m, A)) = (24I_m, 10I_m, 26I_m) \end{bmatrix}$$

has an infinite number of elements (or planets).

Notes

Example 5.3. *Assume that* $m_0 - n_0 = 2I_m, m_0 = A^{2k}, A \in C_m(\mathbb{N})$. *We can define the galaxy*

$$\Delta(2I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(2I_m, A) = 2I_m + 2 \times A^k \\ Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{bmatrix}$$

Notes

in which the triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)), k \in \mathbb{N}$, *satisfy*

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N},$$

$$X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m$$

and

$$(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in C_m(\mathbb{N})$.

Example 5.4. ***A Finite Galaxy***

The triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)), k \in \mathbb{N}$, *of the galaxy*

$$\Delta(2I_m, U_m) = \begin{bmatrix} X_k(2I_m, A) = 2I_m + 2 \times A^k \\ Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}$$

satisfy

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N},$$

$$X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m$$

and

$$(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in U_m$.

Example 5.5. *Assume that* $m = 10, U_{10} = \{I_{10}, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9\}$ *with*

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The triples $(X_k(2I_{10},T_1),Y_k(2I_{10},T_1),Z_k(2I_{10},T_1))$ *of the galaxy*

$$\Delta(2I_{10},T_1)=\begin{bmatrix} X_k(2I_{10},T_1)=2I_{10}+2\times T_1^k \\ Y_k(2I_{10},T_1)=2\times T_1^k+T_1^{2k} \\ Z_k(2I_{10},T_1)=2I_{10}+2\times T_1^k+T_1^{2k} \\ k\in\mathbb{N} \end{bmatrix}$$

satisfy

$$X_k^2(2I_{10},T_1)+Y_k^2(2I_{10},T_1)=Z_k^2(2I_{10},T_1),k\in\mathbb{N}$$

and

$$(X_k(2I_{10},T_1),Y_k(2I_{10},T_1),Z_k(2I_{10},T_1))\neq\left(D^{\frac{p}{q}},B^{\frac{p}{q}},C^{\frac{p}{q}}\right),p,q,k\in\mathbb{N},q\neq 0,$$

$D,B,C\in U_{10}$.

VI. Σ-Model

The triples $(X_k(2I_m,A),Y_k(2I_m,A),Z_k(2I_m,A))$ of the galaxy

$$\Sigma(2I_m,U_m)=\begin{bmatrix} X_k(2I_m,A)=2I_m+2\times A^{4k} \\ Y_k(2I_m,A)=2\times A^{4k}+A^{8k} \\ Z_k(2I_m,A)=2I_m+2\times A^{4k}+A^{8k} \\ k\in\mathbb{N},A\in U_m \end{bmatrix}$$

satisfy

$$X_k^2(2I_m,A)+Y_k^2(2I_m,A)=Z_k^2(2I_m,A),k\in\mathbb{N},$$

$$X_0(2I_m,A)+Y_0(2I_m,A)+Z_0(2I_m,A)=12I_m$$

and

$$(X_k(2I_m,A),Y_k(2I_m,A),Z_k(2I_m,A))\neq\left(D^{\frac{p}{q}},B^{\frac{p}{q}},C^{\frac{p}{q}}\right),p,q,k\in\mathbb{N},q\neq 0,$$

$C,D,B\in U_m$. The multi-galaxy $\Sigma(2I_m,C_m(\mathbb{N}))$ has an infinite number of planets.

Example 6.1. The triples $(X_k(2I_m,T_4),Y_k(2I_m,T_4),Z_k(2I_m,T_4))$ *of the galaxy*

$$\Sigma(2I_m,T_4)=\begin{bmatrix} X_k(2I_m,T_4)=2I_m+2\times T_4^{4k} \\ Y_k(2I_m,T_4)=2\times T_4^{4k}+T_4^{8k} \\ Z_k(2I_m,T_4)=2I_m+2\times T_4^{4k}+T_4^{8k} \\ k\in\mathbb{N}, \end{bmatrix}$$

Notes

satisfy

$$X_k^2(2I_m, T_4) + Y_k^2(2I_m, T_4) = Z_k^2(2I_m, T_4), k \in \mathbb{N},$$

$$X_0(2I_m, T_4) + Y_0(2I_m, T_4) + Z_0(2I_m, T_4) = 12I_m$$

and

Notes

$$(X_k(2I_m, T_4), Y_k(2I_m, T_4), Z_k(2I_m, T_4)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C, T_4 \in U_m$.

Example 6.2. *The triples* $(X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3))$ *of the sequence*

$$\Sigma(2I_m, T_3) = \left[\begin{array}{c} X_k(2I_m, T_3) = 2I_m + 2 \times T_3^{4k} \\ Y_k(2I_m, T_3) = 2 \times T_3^{4k} + T_3^{8k} \\ Z_k(2I_m, T_3) = 2I_m + 2 \times T_3^{4k} + T_3^{8k} \\ k \in \mathbb{N} \\ (X_0(2I_m, T_3), Y_0(2I_m, T_3), Z_0(2I_m, T_3)) = (4I_m, 3I_m, 5I_m) \end{array}\right]$$

satisfy

$$X_k^2(2I_m, T_3) + Y_k^2(2I_m, T_3) = Z_k^2(2I_m, T_3), k \in \mathbb{N},$$

$$X_0(2I_m, T_3) + Y_0(2I_m, T_3) + Z_0(2I_m, T_3) = 12I_m$$

and

$$(X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C, T_3 \in U_m, \Sigma(2I_m, T_3) \subset \Sigma(2I_m, U_m) \subset \Sigma(2I_m, C_m(\mathbb{N}))$.

VII. Power Models of Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries of Order 3

A model of a galaxy is a power model if the power of the lead of the model is a power. For example, if we choose $m_0 = A^{2\lambda^k}$ and $m_0 - n_0 = 2 \times \alpha^2 \times \lambda^2 I_m$, the model

$$URS(\mathbb{N}I_m, U_m, \mathbb{N}I_m) = \left[\begin{array}{c} X_k(\alpha I_m, A, \lambda I_m) = 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda^k} \\ Y_k(\alpha I_m, A, \lambda I_m) = 2\alpha \times \lambda \times A^{\lambda^k} + A^{2\lambda^k} \\ Z_k(\alpha I_m, A, \lambda I_m) = 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda^k} + A^{2\lambda^k} \\ k, \alpha, a, \lambda \in \mathbb{N}, \alpha \neq 0, a \neq 0, \lambda \neq 0, A \in U_m \end{array}\right]$$

Global Journal of Science Frontier Research (F) Volume XXII Issue II Version I Year 2022

is a power model. The elements of the model $URS(U_m, U_m, U_m)$ satisfy

$$X_k^2(\alpha I_m, A, \lambda I_m) + Y_k^2(\alpha I_m, A, \lambda I_m) = Z_k^2(\alpha I_m, A, \lambda I_m), k \in \mathbb{N}$$

and

$$(X_k(\alpha I_m, A, \lambda I_m), Y_k(\alpha I_m, A, \lambda I_m), Z_k(\alpha I_m, A, \lambda I_m)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), k, p, q \in \mathbb{N},$$

$, q \neq 0, D, B, C \in U_m, URS(U_m, U_m, U_m) \subset URS(C_m(\mathbb{N}), C_m(\mathbb{N}), C_m(\mathbb{N}))$.

Example 7.1. The elements of the galaxy

$$URS(2I_m, U_m, 2I_m) = \begin{bmatrix} X_k(2I_m, A, 2I_m) = 32I_m + 8 \times A^{2^k} \\ Y_k(2I_m, A, 2I_m) = 8 \times A^{2^k} + A^{2^{k+1}} \\ Z_k(2I_m, A, 2I_m) = 32I_m + 8 \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}$$

satisfy

$$X_k^2(2I_m, A, 2I_m) + Y_k^2(2I_m, A, 2I_m) = Z_k^2(2I_m, A, 2I_m), k \in \mathbb{N}$$

and

$$(X_k(2I_m, A, 2I_m), Y_k(2I_m, A, 2I_m), Z_k(2I_m, A, 2I_m)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, D, B, C \in U_m$.

Example 7.2. If we choose $m_0 = A^{2^{k+1}}$ and $m_0 - n_0 = 2 \times \alpha^2 I_m$, we could construct the galaxy

$$\Omega(\alpha I_m, U_m) = \begin{bmatrix} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}, \alpha \in \mathbb{N}.$$

The elements of the galaxy $\Omega(\alpha I_m, U_m)$ satisfy

$$X_k^2(\alpha I_m, A) + Y_k^2(\alpha I_m, A) = Z_k^2(\alpha I_m, A), k \in \mathbb{N}$$

and

$$(X_k(\alpha I_m, A), Y_k(\alpha I_m, A), Z_k(\alpha I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, k, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The galaxy

Global Journal of Science Frontier Research (F) Volume XXII Issue II Version I Year 2022

Notes

$$\Omega(\alpha I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N},$$

has an infinite number of planet systems.

Notes

The characterization of the elements of the set $\mathbb{F}_2(C_m(\mathbb{N}))$ is completely the same as the characterization of the elements of the set $\mathbb{F}_2(\mathbb{N})$ [26].

Remark 7.3. *Let (X, Y, Z) and (X_1, Y_1, Z_1) be two elements of $\mathbb{F}_2(C_m(\mathbb{N}))$. Then*

$$(X, Y, Z) \neq \left(A^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), (X_1, Y_1, Z_1) \neq \left(A_1^{\frac{p}{q}}, B_1^{\frac{p}{q}}, C_1^{\frac{p}{q}}\right), (X, Y, Z) \neq (X_1, Y_1, Z_1),$$

$A, B, C, A_1, B_1, C_1 \in \mathbb{F}_2(C_m(\mathbb{N}))$.

Let us observe that the characterization of one element of the set $\mathbb{F}_2(C_m(\mathbb{N}))$ allows us to deduce the characterization of the elements of the set $\mathbb{F}_2(C_m(\mathbb{N}))$. In other words, the set $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. Remark 7.3 allows us to prove the following result:

Theorem 7.4. *The equation*

$$X^{2n} + Y^{2n} = Z^{2n}, XYZ \neq 0, n \in \mathbb{N}(n \geq 2)$$

has no circulant matrix with positive integers as entries solutions.

Proof. Assume that there exist $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^{2n} + Y^{2n} = Z^{2n}, n \geq 2, n \in \mathbb{N}.$$

This means that

$$(X^n)^2 + (Y^n)^2 = (Z^n)^2.$$

Therefore,

$$(X^n, Y^n, Z^n) \in \mathbb{F}_2(C_m(\mathbb{N})) = \left\{(A, B, C) \in C_m(\mathbb{N})^3 : A^2 + B^2 = C^2\right\}.$$

Remark 7.3 allows us to claim that we have a contradiction because the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. Finally, there exist no circulant matrices with positive integers as entries $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^{2n} + Y^{2n} = Z^{2n}, n \in \mathbb{N}, n \geq 2.$$

This result allows to claim that the equation

$$\left(X^2\right)^n + \left(Y^2\right)^n = \left(Z^2\right)^n, n \geq 2,$$

has no solution in $C_m(\mathbb{N})$. We can now prove our main result.

Proof of Theorem 1.1

We just need to show that if $(X,Y,Z)\in\mathbb{F}_n(C_m(\mathbb{C})), n\in\mathbb{N}, n\geq 3$, then $(X,Y,Z)\notin C_m(\mathbb{N})$. Let (X,Y,Z) be an element of the universe $\mathbb{F}_n(C_m(\mathbb{C})), n\geq 3$. Then

$$X^n+Y^n=Z^n.$$

This implies that

$$(\sqrt{X})^{2n}+(\sqrt{Y})^{2n}=(\sqrt{Z})^{2n}\Longleftrightarrow (X^{2n})^{\frac{1}{2}}+(Y^{2n})^{\frac{1}{2}}=(Z^{2n})^{\frac{1}{2}}.$$

and

$$(X^{2n})^{\frac{1}{2}}+(Y^{2n})^{\frac{1}{2}}=(Z^{2n})^{\frac{1}{2}}\Longleftrightarrow (X^{\frac{n}{2}})^2+(Y^{\frac{n}{2}})^2=(Z^{\frac{n}{2}})^2.$$

Theorem 7.4 and Remark 7.3 allow us to claim that

$$(\sqrt{X},\sqrt{Y},\sqrt{Z})\notin\mathbb{F}_2(C_m(\mathbb{N})),(X^{2n},Y^{2n},Z^{2n})\notin\mathbb{F}_2(C_m(\mathbb{N}))$$

and

$$(X^{\frac{n}{2}},Y^{\frac{n}{2}},Z^{\frac{n}{2}})\notin\mathbb{F}_2(C_m(\mathbb{N})),n\geq 3,$$

since $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. In other words,

$$(\sqrt{X},\sqrt{Y},\sqrt{Z})\notin C_m(\mathbb{N}),(X^{2n},Y^{2n},Z^{2n})\notin C_m(\mathbb{N})$$

and

$$(X^{\frac{n}{2}},Y^{\frac{n}{2}},Z^{\frac{n}{2}})\notin C_m(\mathbb{N}),n\geq 3.$$

The fact that

$$(X^{2n},Y^{2n},Z^{2n})\notin C_m(\mathbb{N}),n\geq 3$$

implies that

$$(X,Y,Z)\notin C_m(\mathbb{N}).$$

VIII. Eigenvalues of Circulant Matrices

It is well known that if $A=C(\Omega)$, where Ω is a compact Hausdorff space, then $\sigma(f)=f(\Omega)$ for all $f\in A$. Let

$$\varphi(z)=\sum_{k=0}^{m-1}a_kz^k$$

be a complex polynomial over $\mathbb{D}$. Then $\sigma(\varphi)=\varphi(\mathbb{D})$.

Notes

Global Journal of Science Frontier Research (F) Volume XXII Issue II Version I Year 2022

The Spectral Mapping Theorem 1. *[40]. Let $T \in B(H)$ be a normal bounded linear operator on the Hilbert space H and let $f : \sigma(T) \longrightarrow \mathbb{C}$ be a continuous function on $\sigma(T)$. Then $\sigma(f(T)) = f(\sigma(T))$.*

Let us introduce the well known spectrum of circulant matrices associated to complex polynomials over $\mathbb{D}$. Let

Notes

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

be an $m \times m$- matrix. The matrix P is normal. Indeed, $PP^* = P^*P = I_m$. Assume that

$$A_0 = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix}.$$

A simple calculation shows that

$$A_0 = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.$$

The matrix A_0 is considered as a polynomial of one variable. Let us compute the spectrum of the normal matrix P. Let

$$f(\lambda) = det(P - \lambda I_m) = \begin{vmatrix} -\lambda & 1 & 0 & \ddots & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -\lambda & 1 \\ 1 & 0 & \dots & 0 & 0 & -\lambda \end{vmatrix} = 1 - \lambda^m.$$

be the characteristic polynomial of P. Thus, λ is a primitive m-th root of unity. Therefore,

$$\sigma(P) = \left\{ \lambda_k^P = e^{\frac{2\pi k i}{m}} : k = 0, 1, 2, ..., m-1 \right\}.$$

In other words,

$$\sigma(P) = \left\{ 1, e^{\frac{2\pi i}{m}}, e^{\frac{4\pi i}{m}}, e^{\frac{6\pi i}{m}}, e^{\frac{8\pi i}{m}}, e^{\frac{10\pi i}{m}}, e^{\frac{12\pi i}{m}}, e^{\frac{14\pi i}{m}}, ..., e^{\frac{2(m-1)\pi i}{m}} \right\}.$$

Finally,

$$\sigma(P) = \left\{ \lambda_0^P, \lambda_1^P, ..., \lambda_{m-1}^P \right\} \subset \mathbb{D}.$$

The spectral mapping Theorem allows us to claim that

$$\varphi(\sigma(P)) = \sigma(\varphi(P)) = \sigma(A_0).$$

Therefore,

$$\sigma(A_0) = \left\{ \varphi(\lambda_0^P), \varphi(\lambda_1^P), ..., \varphi(\lambda_{m-1}^P) \right\}.$$

IX. Galaxy of Sequences of Eigenvalues of Circulant Matrices

In this section, we construct galaxies of sequences of eigenvalues of circulant matrices.

Theorem 9.1. *Let $X, Y, Z \in C_m(\mathbb{C})$ be three circulant matrices with complex numbers as entries such that*

$$X^n + Y^n = Z^n.$$

Then

$$(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, ..., m-1.$$

In other words, the triples $(\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m-1$. *Thatis, the planet system*

$$M(X,Y,Z) = \begin{bmatrix} \lambda_k^X \\ \lambda_k^Y \\ \lambda_k^Z \\ k = 0,1,2,...,m-1 \end{bmatrix} \subset \mathbb{F}_n(\mathbb{C}).$$

Notes

Proof: Let $(X,Y,Z) \in C_m(\mathbb{C}), n \in \mathbb{N}, n \geq 3$, be an element of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$. The spectral factorization of the matrices X, Y, Z [36, 37, 38] allows us to claim that there exists a unitary matrix U such that

$$X = U \begin{pmatrix} \lambda_0^X & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^X & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^X & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^X \end{pmatrix} U^* = UD_XU^*,$$

$$Y = U \begin{pmatrix} \lambda_0^Y & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^Y & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^Y & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^Y \end{pmatrix} U^* = UD_YU^*$$

and

$$Z = U \begin{pmatrix} \lambda_0^Z & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^Z & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^Z & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^Z \end{pmatrix} U^* = UD_ZU^*.$$

The equation $X^n + Y^n = Z^n$ implies

$$UD_X^nU^* + UD_Y^nU^* = UD_Z^nU^*.$$

It follows that

$$U\left[D_X^n + D_Y^n\right]U^* = UD_Z^nU^*.$$

We can claim that

$$D_X^n + D_Y^n = D_Z^n.$$

Finally,

$$(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, ..., m-1.$$

In other words, the triples $(\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m-1$. That is, the planet system

$$M(X,Y,Z) = \left[\begin{array}{c} \lambda_k^X \\ \lambda_k^Y \\ \lambda_k^Z \\ k = 0, 1, 2, ..., m-1 \end{array}\right] \subset \mathbb{F}_n(\mathbb{C}).$$

Every triple (X,Y,Z) of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$ generates a planet system M(X,Y,Z) which has exactly m elements of the universe $\mathbb{F}_n(\mathbb{C})$. We can say that the galaxies of sequences of circulant matrices are linked to the galaxies of sequences of eigenvalues of circulant matrices. Let us consider the galaxies

$$\Omega(\alpha I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array}\right], \alpha \in \mathbb{N},$$

of circulant matrices. We can construct the galaxies of sequences of eigenvalues of the triples of circulant matrices of the galaxies $\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}$. For example, the galaxies

$$\Omega(\alpha, \sigma(C_m(\mathbb{N}))) = \left[\begin{array}{c} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in C_m(\mathbb{N}) \end{array}\right] \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N},$$

are galaxies of sequences of eigenvalues of triples of circulant matrices of the galaxies $\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}$. As we can see that the galaxies

$$\Omega(\alpha, \sigma(A)) = \left[\begin{array}{c} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ k \in \mathbb{N}, \lambda \in \sigma(A) \end{array}\right] \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N}, A \in C_m(\mathbb{N}),$$

Notes

Notes

have each a finite number of planet systems. In our case, each galaxy has m planet systems. Every galaxy of the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ generates a new galaxy of eigenvalues of elements of $C_m(\mathbb{N})$. Let us consider the galaxy

$$\Sigma(2I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(2I_m, A) = 2I_m + 2 \times A^{4k} \\ Y_k(2I_m, A) = 2 \times A^{4k} + A^{8k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^{4k} + A^{8k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{bmatrix}.$$

We know that the triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A))$ of the galaxy $\Sigma(2I_m, C_m(\mathbb{N}))$ satisfy

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N}.$$

Define the galaxy

$$\Sigma(2, \sigma(C_m(\mathbb{N}))) = \begin{bmatrix} X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\ Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\ Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in C_m(\mathbb{N}) \end{bmatrix}.$$

The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$ of the galaxy $\Sigma(2, \sigma(C_m(\mathbb{N})))$ satisfy

$$X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.$$

We can deduce the galaxies

$$\Sigma(2, \sigma(A)) = \begin{bmatrix} X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\ Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\ Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), \end{bmatrix}, A \in C_m(\mathbb{N}),$$

which have a finite number of planet systems. The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$ of the galaxy $\Sigma(2, \sigma(A))$ also satisfy

$$X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.$$

The first eigenvalue of every matrix of $C_m(\mathbb{N})$ is a positive integer.

Theorem 9.2. Let $A \in C_m(\mathbb{N})$ *be a circulant matrix with positive integers as entries. Then the first eigenvalue* λ_0^A *of* A *is a positive integer. In other words,* $\lambda_0^A \in \mathbb{N}$.

Proof. Let

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} \in C_m(\mathbb{N})$$

be a circulant matrix with positive integers as entries. Then there exists a polynomial

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k, a_k \in \mathbb{N}, z \in \mathbb{D},$$

such that

$$A = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.$$

We know that

$$\sigma(A) = \left\{\varphi(\lambda_0^P), \varphi(\lambda_1^P), ..., \varphi(\lambda_{m-1}^P)\right\}$$

with

$$\left\{\lambda_0^P, \lambda_1^P, ..., \lambda_{m-1}^P\right\} = \left\{1, e^{\frac{2\pi i}{m}}, e^{\frac{4\pi i}{m}}, e^{\frac{6\pi i}{m}}, e^{\frac{8\pi i}{m}}, e^{\frac{10\pi i}{m}}, e^{\frac{12\pi i}{m}}, ..., e^{\frac{2(m-1)\pi i}{m}}\right\}.$$

Therefore,

$$\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in \mathbb{N}.$$

Remark 9.3. Let $\mathcal{A}$ be an algebra and let $A \in C_m(\mathcal{A})$. Then

$$\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in \mathcal{A}.$$

Theorem 9.1 and Theorem 9.2 allow us to provide another proof of our main result.

Notes

Notes

Second Proof of Theorem 1.1

Assume that there exist $X, Y, Z \in C_m(\mathbb{N}), n \in \mathbb{N}, n \geq 3$, three circulant matrices with positive integers as entries such that

$$X^n + Y^n = Z^n.$$

Theorem 9.1 and Theorem 9.2 allow us to claim that

$$(\lambda_0^X)^n + (\lambda_0^Y)^n = (\lambda_0^Z)^n, n \geq 3.$$

This implies that the equation $x^n + y^n = z^n, n \geq 3$ has positive integer solutions. We have a contradiction. Therefore, the equation

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no circulant matrix with positive integers as entries solutions.

Let $\mathcal{A}$ be an algebra and let

$$\mathcal{P}^{(m)}(\mathcal{A}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathcal{A}, z \in \mathbb{D} \right\}$$

be the algebra of polynomials over $\mathbb{D}$. Complex polynomials of the algebra $\mathcal{P}^{(m)}(\mathbb{N})$ allow us to provide Fermat's Last Theorem for eigenvalues of circulant matrices.

Theorem 9.4. The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no positive integer eigenvalues of circulant matrices solutions.

Proof. Assume that there exists a triple (λ, η, μ) of positive integer eigenvalues of circulant matrices X,Y and Z of $C_m(\mathbb{N})$ such that

$$\lambda^n + \eta^n = \mu^n, \lambda\eta\mu \neq 0, n \in \mathbb{N}, n \geq 3, \lambda \in \sigma(X), \eta \in \sigma(Y), \mu \in \sigma(Z).$$

Therefore, there exist three f, g, h complex polynoimals of $\mathcal{P}^{(m)}(\mathbb{N})$ such that

$$f(z)^n + g(z)^n = h(z)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}.$$

In particular,

$$f(P)^n + g(P)^n = h(P)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}$$

with P the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

In other words,

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}, n \geq 3.$$

We have a contradiction. Finally, The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no positive integer eigenvalues of circulant matrices solutions.

X. Fermat's Last Theorem for Complex Polynomials associated to Circulant Matrices

We can now construct models of galaxies of complex polynomials associated to circulant matrices. Recall that

$$\mathcal{P}^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.$$

The galaxies of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$ generate the galaxies of the universe $\mathbb{F}_n(\mathcal{P}^{(m)}(\mathbb{C}))$. For example, from the galaxy $\Sigma(2I_m, C_m(\mathbb{C}))$, we can construct the galaxy

$$\Sigma(2, \mathcal{P}^{(m)}(\mathbb{C})) = \left[\begin{array}{c} X_k(f) = 2 + 2 \times f^{4k} \\ Y_k(f) = 2 \times f^{4k} + f^{8k} \\ Z_k(f) = 2 + 2 \times f^{4k} + f^{8k} \\ k \in \mathbb{N}, f \in \mathcal{P}^{(m)}(\mathbb{C}) \end{array} \right] \subset \mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C})).$$

We can continue doing the same identification process with the remaining galaxies of $\mathbb{F}_2(C_m(\mathbb{C}))$. This process will lead to the construction of the universe $\mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C}))$. Now, we are able to provide Fermat's Last Theorem for complex polynomials over the unit disk $\mathbb{D}$ associated to circulant matrices of the set $C_m(\mathbb{N})$.

Notes

Notes

Theorem 10.1. The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no solutions in $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0.$

Proof. Assume that there exists a triple (f, g, h) of complex polynomials of the set $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0$, such that

$$f(z)^n + g(z)^n = h(z)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}.$$

This implies that

$$f(P)^n + g(P)^n = h(P)^n, n \in \mathbb{N}, n \geq 3.$$

with P the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

In other words, there exist $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}, n \geq 3.$$

We have a contradiction. Finally, The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no solutions in $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0.$

Theorem 1.1, Theorem 9.4 and Theorem 10.1 are equivalent.

REFERENCES RÉFÉRENCES REFERENCIAS

1. Diophantus of Alexandria, Arithmetica of Diophantus, 1637.
2. L. Euler, “Theorematum quorundam arithmeticorum demonstrationes”, Novi Commentarii academiae scientiarum Petropolitanae, 10(1738), 125-146.
3. L. Euler, ”Vollstandige Anleitung zur Algebra”, Roy. Acad. Sci., St. Petersburg (1770).

4. A. M. Legendre, "Recherches sur quelques objets d'analyse indeterminee, et particulierement sur the theoreme de Fermat", Memoires del'academie royal des sciences, (1823) 1-60.
5. A. M. Legendre, "Theorie des nombres", Volume II, 3rd edition, Paris Firmin Didot Frere, 1830.
6. P.G.L. Dirichlet, "Demonstration du theoreme de Fermat pour le cas des 14th puissance", Journal fur die reine und angewandte Mathematik, 9(1832), 390-393.
7. G. Lame, "Memoire sur le dernier theoreme de Fermat", Comptes rendus hebdomadaires des seances de l'Academie des Sciences, 9(1839)45-46.
8. G. Lame, "Memoire d'analyse indeterminee demontrant que l'equation $x^7 + y^7 = z^7$ est impossible en nombres entiers", Journal de Mathematiques Pures et Appliquees, 5(1840) 195-211.
9. G. Lame, "Memoire sur la resolution en nombres complexe de l'equation $A^5 + B^5 + C^5 = 0$", Journal de Mathematiques Pures et Appliquees, 12(1847) 137- 171.
10. G. Lame, "Etude des binomes cubiques $x^3 \pm y^3$", Comptes rendus hebdomadaires des seances de l'Academie des Sciences, 61(1865)921-924, 961-965.
11. V. A. Lebesque, "Demonstration de l'impossibilite de resoudre l'equation $x^7 + y^7 + z^7 = 0$ est impossible en nombres entiers", Journal de Mathematiques Pures et Appliquees, 5(1840) 276-279, 348-349.
12. V. A. Lebesque, "Theoremes nouveaux sur l'equation indeterminee $x^5+y^5 = az^5$", Journal de Mathematiques Pures et Appliquees, 8(1843) 49-70.
13. G. Frey, "Links between stable elliptic curves and certain Diophantine equations", Universitatis Saraviensis. Series Mathematicae, 1 (1986) 1-40.
14. J. P. Serre, "Proprietes Galoisiennes des points d'ordre fini des courbes elliptiques", Invent. Math. 15(1972) 259-331.
15. J. P. Serre, "Sur les representations modulaires de degre 2 de Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)", Duke Math. J. 54(1987), 170-230.
16. J. P. Serre, "Travaux de Wiles (et Taylor)", Partie I. Asterisque 237(1996), 319- 332.
17. K. Ribet, "On modular representations of Gal($\mathbb{Q}/\mathbb{Q}$) arising from modular forms", Inventiones Mathematicae, 100(2),(1990), 431-476.
18. B. Mazur, "Deforming Galois representations", In Galois groups over $\mathbb{Q}$: proceedings, of a work shop held March 23-27, 1987, Y. Ihara, K. Ribet, J. P. Serre, editors. Mathematical Sciences Research Institute publications 16, Springer- Verlag, New York, 1989, pp. 385 - 437.
19. B. Mazur, "Modular curves and the Eisenstein ideal", Publ. Math. I.H.E.S. 47(1977) 33-186.
20. R. Taylor and A. Wiles, "Ring theoretic properties of certain Hecke algebra", Annals of Mathematics. 141(3)(1995), 553-572.

Notes

Notes

21. A. Wiles, “Modular elliptic curves and Fermat’s Last Theorem”, Annals of Mathematics. 141(3)(1995), 443-551.
22. C. Breuil, B. Conrad, F. Diamond, R. Taylor, “On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercices”, Journal of the American Mathematical Society, 14(4),(2001), 843-939.
23. B. Conrad, F. Diamond, R. Taylor, “Modularity of certain potentially Barsotti-Tate Galois representations”, Journal of the American Mathematical Society, 12(2), (1999), 521-567.
24. F. Diamond, On deformation rings and Hecke rings, Annals of Mathematics. Second Series, 144(1),(1996), 137-166.
25. Nag. B.B, An elementary proof of Fermat’s Last Theorem for epsilons, Advances in Pures Mathematics, 11(2021), 735-740.
26. J. Moussounda Mouanda, On Fermat’s Last Theorem and Galaxies of sequences of positive integers, American Journal of Computational Mathematics, 12(2022), 162-189.
27. R. Z. Domiaty, Solution of X4 + Y 4 = Z4 in integral matrices, Amer. Math. Monthly 73(1966), 631.
28. L.N. Veserstein, Non-commutative Number Theory, Contemp. Math. 83(1989), 445-449.
29. D. Frejman, On Fermat’s in the set of Fibonacci matrices, Discuss. Math. 13(1993), 61-64.
30. A. Grytczuk, On Fermat’s equation in the set integral 2 x 2 matrices, Period. Math. Hung. 30(1995), 79-84.
31. A. Khazanov, Fermat’s equation in matrices, Serdica Math. J. 21(1995), 19-40.
32. H. Qin, Fermat’s problem and Goldbach problem over Mn($\mathbb{Z}$), Linear Algebra Appl. 236(1996), 131-135.
33. Z. Patay and A. Szakacs, On Fermat’s problem in matrix ring and groups, Publ. Math. Debrecen, 61/3-4(2002), 487-494.
34. Mao-Ting Chien and Jie Meng, Fermat’s equation over 2-by-2 matrices, Bull. Korean Math. Soc. 58(2021), No.3, 609-616.
35. Enrico Laeng, On Fermat’s Last Theorem for Polynomials, Parabola 35(1)(1999).
36. P.J. Davis, Circulant Matrices, John Wiley and Sons, 1979.
37. P.J. Davis, Circulant Matrices, AMS Chelsea Publishing, 1994.
38. P.J. Davis, Circulant Matrices, Proc. Amer. Math. Soc., No. 2(2013).
39. D. Geller, I. Kra, S. Popescu and S. Simanca, On circulant matrices, Preprint 2002 (pdf at www.math.stonybrook.edu).
40. W. Rudin, Fourier Analysis on groups, Interscience, New York, 1962.