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Highlights

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Fractional Order Riemann Curvature

Discovering Thoughts, Inventing Future

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Modeling Supply Chains by Critical Paths and Leontief Input-Output Table

By Gregory L. Light
Providence College

Abstract- We formulate a supply-chain problem by:(1)casting it in the model of critical-path analysis as defined by predecessor/successor relations, and (2)allowing for mutual dependency among the activity nodes and applying Leontief's input-output structure.

Keywords: scheduling bottleneck, network disruption, pandemic disequilibria.

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5. Wei, J. and Krajewski, L., A model for comparing supply chain schedule integration approaches, Int. J. Prod. Res., 38 (9), 2000, 2099-2123.

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Keywords: scheduling bottleneck, network disruption, pandemic disequilibria.

I. Introduction

Recent global supply-chain problems have received acute attention from every corner on the planet. While ad hoc ex-post analyses are perhaps timely[1], [2], preventative ex-ante treatments are more fundamental. Standard topics of supply chain management include: the bullwhip effect, vertical integration, and point-by-point statistical control[3];this paper seeks to add to the list the critical paths method(CPM)[4], [5] (for applications of CPM, cf. [6], [7]), which bears the common construct of a partially ordered set. In addition, we extend the scope of a directed tree of uni-directional edges/paths connecting all the vertices/nodes[8], [9], to a network of mutually dependent economic agents, which then readily leads to Leontief's input-output table for the gross domestic product (GDP) [10].As such, Section 2 below will connect CPM to a supply-chain problem, and Section 3 will show how an input-output analysis can address a global disruption over an economy, where we will incorporate the apparatus of elasticities of substitution, proportional changes in the ratio of two factors due to a change in the ratio of their prices. Section 4 will draw a summary.

II. Supply Chain by CPM

Let $\mathrm{A} \equiv \left\{ a_i \middle| i = 1, 2, \cdots, n \in \mathrm{N} - \{1\} \right\}$ be a set of activities with strict partial order relations $\prec$, where "$a_j \prec a_k$" denotes a_j being a predecessor of a_k, and denote the set of all the largest elements of A by L, which have no successors. By Hausdorff maximum principle [11], there exists a maximal simply ordered subset B_m of A that has its largest element $a_m \in \mathrm{L}$; i.e., B_m is a critical path.

Next, collect all these critical paths $\{\mathrm{B}_m\}$ and conduct the usual CPM analysis for each B_m[6]; then one arrives at a complete set of optimal solutions for a supply-chain problem.

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III. Supply Chain by Input-Output Analysis

Let $\mathbf{A}=\left(a_{ij}\right)_{n\times n}$ be a matrix of the trading values of economic agent i sold to agent j in GDP. Obtain the row sums $\sum_{j=1}^{n} a_{ij}, \forall i=1,2,\cdots,n;$ analogously, obtain the column sums $\sum_{i=1}^{n} a_{ij}, \forall j=1,2,\cdots,n.$ Measure the economic value of agent$k$ by $\left(\sum_{j=1}^{n} a_{kj}+\sum_{i=1}^{n} a_{ik}\right)\equiv v_k, k=1,2,\cdots,n,$ which represents the total trade value of k, analogous to the sum of exports and imports of an economy.

Next, let $\Sigma_{inputs}{}^{output\ k}\equiv\left(\sigma_{ij}\right)_{n\times n\to k}$ be a matrix of elasticities of substitution between inputs i and j for producer k, $k=1,2,\cdots,n,$ where $\sigma_{ii;k}=1$ and $\sigma_{ij;k}=\sigma_{ji;k}$. Analogously, let $S_{outputs}{}^{consumer\ k}\equiv\left(s_{ij}\right)_{n\times n\to k}$ be a matrix of elasticities of substitution between outputs i and j for consumer k, $k=1,2,\cdots,n,$ where $s_{ii;k}=1$ and $s_{ij;k}=s_{ji;k}$.

Fix k as a producer; measure its inputs substitutability by the product of the elements of the upper triangular sub-matrix of $\Sigma_{inputs}{}^{output\ k}$, or $\Pi_{i<j}\sigma_{ij;k}$; analogously, measure the outputs substitutability for consumer k by $\Pi_{i<j}s_{ij;k}$; now combine these two measures of substitutability for k as a producer and as a consumer by $\left(\Pi_{i<j}\sigma_{ij;k}\cdot\Pi_{i<j}s_{ij;k}\right)\equiv\xi_k$. Finally, measure the economic significance of k by $v_k/\xi_k\equiv\gamma_k$, and establish a decreasing sequence $\left\langle\gamma_{k_j}\right\rangle_{j=1,2,\cdots,n}$, by which one can address an economy-wide supply chain problem more *à propos*, in particular, paying special attention to the case of $\gamma_k=\infty$.

IV. Summary

As has been observed in recent years, there are mainly three kinds of situations that disrupt a general economy: that due to labor shortage for certain specific tasks, that due to the lack of special components in a production process, and that due to disparate outlet distributions when the same commodity does not have a universal availability; all these problems can be alleviated or even prevented by an ex-ante detailed CPM analysis. The Covid-19 pandemic has inflicted losses of GDP's across countries of different degrees; here we contend that by conducting an input-output analysis as outlined above, the public sectors of an economy may act to balance supply and demand strategically over the markets. In summary, this paper has presented two methodologies for dealing with a supply-chain problem. Future studies might pursue Gantt Charts for all the commonly experienced supply chain problems as well as an estimation of the elasticities of substitution in the frame of the above conducted Leontief input-output analyses.

References Références Referencias

1. Bieck, C. and Marshall, A., Redirecting resources to promote post-pandemic growth, Strategy and Leadership, 48 (6), 2020, 45-50.
2. Tsiamas, K. and Rahimifard, S., A simulation-based decision support system to improve the resilience of the food supply chain, Int. J. Comp. Integrated Manufacturing, 34 (9), 2021, 996-1010.
3. Heizer, J., Render, B., and Munson, C., *Operations Management*, Boston, Pearson (2020).

Notes

Notes

4. Hanaka, T., Kagawa, S., Ono, H., and Kanemoto, K., Finding environmentally critical transmission sectors, transactions, and paths in global supply chain networks, Ener. Econ., 68 (1), 2017, 44-52.
5. Wei, J. and Krajewski, L., A model for comparing supply chain schedule integration approaches, Int. J. Prod. Res., 38 (9), 2000, 2099-2123.
6. Light, G., A note on identifying critical activities in project scheduling via linear programming on spreadsheets, with incidental pedagogical remarks, Global J. Sci. Frontier Res., 21 (1), 2021.
7. Light, G., A note on simulating predecessor-successor relationships in critical path models, Global J. Sci. Frontier Res., 21 (2), 2021.
8. Gravner, J., Paguyo, J., and Slivken, E., Maximal spanning time for neighborhood growth on the Hamming plane, SIAM J. Discrete Math., 33(2), 2019, 976–993.
9. Sholander, M., Trees, lattices, order, and betweenness, Proc. Amer. Math. Soc. 3, 1952, 369-381.
10. Dynes, S., Johnson, E., Andrijcic, E., and Horowitz, B., Economic costs of firm-level information infrastructure failures: Estimates from field studies in manufacturing supply chains, Int. J. Logis. Manag., 18 (3), 2007, 420-442.
11. Hausdorff, F., Hausdorff on Ordered Sets, Providence, Am. Math. Soc. (2005).

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Bianchi Type-I Anisotropic String Cosmological Models in the Chassis of Normal Gauge for Lyra's Manifold with Variable Deceleration Parameter

By Shilpi Singhal & Vandna Srivastava

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Abstract- Present paper is the study of the anisotropic spatially Bianchi type-I (B-I) homgeneous cosmological models in the chassis of normal gauge for Lyra's manifold. A deterministic solution has been obtained by taking de- celeration parameter to be dependent on time which results in average scale factor $a(t) = [sinh(\alpha t)]^{\frac{1}{n}}$ has been obtained. Modified field equations given by Einstein for the homogeneous Binachi Type I metric are solved. Our models are in accelerating phase which is consistent to the recent observations. It has been found that the displacement vector β behaves like cosmological term Λ in the normal gauge treatment and the solutions are consistent with recent observations of SNe Ia. It has been found that massive strings dominate in the decelerating universe whereas strings dominate in the accelerating universe. The strings dominate in the early universe and eventually disappear from the universe for sufficiently large times. This is in consistent with the current observations. Some physical and geometric behaviour of these models are also discussed.

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8. Singh, T., Singh, G.P.: J. Math. Phys. 32, 2456 (1991).

Bianchi Type-I Anisotropic String Cosmological Models in the Chassis of Normal Gauge for Lyra's Manifold with Variable Deceleration Parameter

Shilpi Singhal [α] & Vandna Srivastava [σ]

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I. Introduction

Binachi type-I cosmological models are analogous and anisotropic in nature and also which gives physically and geometrically better structure and inturn helps in describingand understanding the physical properties of early universe. Binachi-I give rise to an ellipsoidal structure of the universe inspite of inflation.It has been evident by many theories that the universe is accelerating [1-8]. Recent observations of large scale distribution of galaxies and supernovae , a negative-gravity like substance known as dark energy seems to dominate the Universe [8-18], which accelerates the rate at which the universe is expanding. As the universe expands the dark energy clusters more weakly than matter and also dilutes more slowly than matter. Present time rate of expansion is given by Hubble parameter H whereas expansion of current observable universe is speeding up is given by deceleration parameter.These are vital observational parameters for analysing various properties of any cosmological structures.The time dependence of deceleration parameter and its effect on various cosmologi-

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cal models has also been analysed by Pradhan et al. [19]. Time dependence of deceleration parameter q is in concurrence with the present study of escalting universe

II. Field Equations and Metric

Considering totally anisotropic B-I metric,

$$ds^2 = A^2dx^2 + B^2dy^2 + C^2dz^2 - dt^2, \tag{1}$$

Here A, B and C are the metric potentials.
The energy-momentum tensor for massive string jn perfect fluid is

$$T_i^j = -\lambda x_i x^j + p g_i^j + (\rho + p) v_i v^j, \tag{2}$$

Isotropic pressure is given by p; Rest energy density for the strings is given by ρ ; is tension density of string is given by λ; x^i represents unit space-like vector and v^i is the particle's four-velocity where $x^2 = 0 = x^3 = x^4$ and $x^1 \neq 0$. Also

$$v_i v^i = -x_i x^i = -1, \;\; v^i x_i = 0. \tag{3}$$

let

$$x^i = (A^{-1}, 0, 0, 0). \tag{4}$$

and

$$\rho = \lambda + \rho_p. \tag{5}$$

where represents particle density is being represented by ρ_p In normal gauge, the field equations has been obtained by Sen [4] and given as

$$R_i^j - \frac{1}{2}g_i^j R + \frac{3}{2}\phi_i\phi^j - \frac{3}{4}g_i^j\phi_k\phi^k = -8\pi T_i^j, \tag{6}$$

The field equation given by Einstein (6) with (2) for the metric (1)will result in:

$$\frac{\dot{B}\dot{C}}{BC} + \frac{3}{4}\beta^2 + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + p - \lambda = 0, \tag{7}$$

$$\frac{\dot{C}\dot{A}}{CA} + \frac{3}{4}\beta^2 + \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + p = 0, \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + p = 0, \tag{9}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{3}{4}\beta^2 = \rho. \tag{10}$$

$T^i_{i;j} = 0$ will give

$$(\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \dot{\rho} = -\lambda\frac{\dot{A}}{A}, \tag{11}$$

Ref

19. Reddy, D.R.K.: Astrophys. Space Sci. 300, 381 (2005).

simultaneously Right hand side of Eq. (6) gives

$$\left(R_i^j - \frac{1}{2}g_i^j R\right)_{;j} + \frac{3}{2}\left(\phi_i\phi^j\right)_{;j} - \frac{3}{4}\left(g_i^j\phi_k\phi^k\right)_{;j} = 0. \tag{12}$$

Equation (12)gives

$$\phi_i\left[\frac{\partial\phi^j}{\partial x^j} + \phi^l\Gamma^j_{lj}\right] + \phi^j\left[\frac{\partial\phi_i}{\partial x^j} - \phi_l\Gamma^l_{ij}\right] - \frac{1}{2}g_i^j\phi_k\left[\frac{\partial\phi^k}{\partial x^j} + \phi^l\Gamma^k_{lj}\right] -$$

$$\frac{1}{2}g_i^j\phi^k\left[\frac{\partial\phi_k}{\partial x^j} - \phi_l\Gamma^l_{kj}\right] = 0. \tag{13}$$

Eq. (13) reduces to

$$\beta\left[\frac{\partial(g^{44}\phi_4)}{\partial x^4} + \phi^4\Gamma^4_{44}\right] + g^{44}\phi_4\left[\frac{\partial\phi_4}{\partial t} - \phi_4\Gamma^4_{44}\right] - \frac{1}{2}g_4^4\phi_4\left[\frac{\partial\phi^4}{\partial x^4} + \phi^4\Gamma^4_{44}\right] -$$

$$\frac{1}{2}g_4^4 g^{44}\phi^4\left[\frac{\partial\phi_4}{\partial t} - \phi^4\Gamma^4_{44}\right] = 0. \tag{14}$$

which gives

$$\beta^2 \quad \left.\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = -\beta\dot{\beta}. \tag{15}$$

III. Solutions of the Field Equations

There are five equations from (7)-(10) and (15)whereλ, p, ρ, β which are the cosmological parameters and A, B C metric potentials are the unknowns. For finding its explicit solution Firstly we assumed that the expansion scalar $(\theta)$of cosmological constant is directly proportionate to the integrant ${\sigma^1}_1$ of the shear tensor ${\sigma^j}_i$ which will give following relation:

$$A^{\frac{1}{m}} = BC, \tag{16}$$

where m is any positive value which is well explained by Thorne [79]. Also

$$\sigma \leq 0.3H$$

For spatially homogeneous metric, Collins et al. [82] have found out,that the ratio of σ which gives normal congruence of any cosmological model to the θ which gives homogeneous expansion of the model is constant i.e. $\frac{\sigma}{\theta}$ is constant.

The Hubble parameter is

$$3H = \quad \left.\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \tag{17}$$

Next, we take decelaration parameter to be time dependent

$$q = \frac{-\ddot{a}a}{\dot{a}^2} = b(t), \tag{18}$$

which results in

$$a = [\sinh(\alpha t)]^{1/n} \tag{19}$$

Ref

79. Thorne, K.S.: Astrophys. J. 148, 51 (1967).

also we know

$$a^3 = V = ABC \tag{20}$$

Subtracting (8) from (9), and taking integral twice, we obtain

$$\frac{B}{C} = k_1 exp[k_2 \int (ABC)^{-1} dt], \tag{21}$$

where k_1 and k_2 are the constants.

Solving the equations (16), (19) and (21), we get

$$A = \sinh(\alpha t)^{\frac{3m}{n(m+1)}};, \tag{22}$$

$$B = \sqrt{k_1} \sinh(\alpha t)^{\frac{3}{2n(m+1)}} \exp[\frac{k_2}{2}]F(t) , \tag{23}$$

$$C = \frac{1}{\sqrt{k_1}} \sinh(\alpha t)^{\frac{3}{2n(m+1)}} \exp[\frac{-k_2}{2}]F(t) \tag{24}$$

Hence the model (1) will result in

$$ds^2 = \sinh(\alpha t)^{\frac{6m}{n(m+1)}} dx^2 + k_1 \sinh(\alpha - dt^2 t)^{\frac{6}{2n(m+1)}} \exp[k_2 F(t)] dy^2$$

$$+ \frac{1}{k_1} \sinh(\alpha t)^{\frac{6}{2n(m+1)}} \exp[-k_2 F(t)] dz^2. \tag{25}$$

Now solving Eq. (15)results $\beta = 0$ or $\dot{\beta}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0.$ which results in

$$\dot{\beta} + \beta \quad \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Bigg) = 0, \tag{26}$$

which leads to

$$\frac{\dot{\beta}}{\beta} + \frac{3\alpha}{n} \coth(\alpha t) = 0. \tag{27}$$

Integrating Eq. (27), we obtain

$$\beta = [\sinh(\alpha t)]^{-\frac{3}{n}}, \tag{28}$$

where k is an constant.

$$V = (AB^2) = a^3(t) = [sinh(\alpha t)]^{3/n} \tag{29}$$

The Hubble parameter for our derived model is given by

$$H = \frac{\alpha}{n} coth(\alpha t) \tag{30}$$

Also,

$$q + 1 = n(1 - tanh^2 \alpha t) \tag{31}$$

Notes

here n and α are positive. The value of q which is a time dependent parameter defines the nature of the universe. From Eq.[31] we can see that $q > 0$ when $tanh(\alpha t) < (\frac{(n-1)}{n})^{\frac{1}{2}}$ and $q < 0$ when $tanh(\alpha t) > (\frac{(n-1)}{n})^{\frac{1}{2}}$. Current studies, suggests that the present universe is escalating and value of deceleration parameter lies between the interval of $-1 < q < 0$.Currently $(t_0 = 12.36 Gyr)$ with $(q_0 = -0.52)$ (Amirshashchi et al.)(Yu,Ratna and Wang) based on OHD+JLA Data, we have the following equation for α and n.

$$\alpha t_0 = \tanh^{-1}\left[1 - (\frac{1+q_0}{n})\right]^{\frac{1}{2}} \tag{32}$$

here the current value of H is denoted by H_0 and the present age of universe by t_0 . We consider the three cases based on different data:

$$\alpha = \frac{1}{12.36} tanh^{-1}\left[1 - \left(\frac{0.48}{n}\right)\right]^{\frac{1}{2}} \tag{33}$$

For the present Universe,it is quite clear that the model holds good for $n > 0.48$.

It has been observed by Halford [4] that the cosmological constant Λ and displacement field ϕ_i in Lyra's manifold behaves in same manner . From Eq. (28), $\beta(T)$ which is the displacement vector reduces with the rise of cosmic time when $kand$n are positive and at large times it attains a very small positive value. Recent observations of cosmological bodies Riess et al. [99, 100]; SNe Ia (Garnavich et al. [94, 95];Schmidt et al. [101]); Perlmutter et al. [96]−[98]; suggest that cosmological constant is positive Λ having value $\Lambda(G\hbar/c^3) \approx 10^{-123}$. $\beta(T)$ so obtained in our derived cosmological model is in consensus with recent observations.

The equations for various parameters (p), (ρ),(λ) and (ρ_p) for our model (**??**) are given by

$$p_1 = \frac{-3\sinh(\alpha t)^{-2(3+n)/n}(-(1+m)^2n^2 - (-3+4n+4m^2(-3+2n)+6m(-1+2n))\alpha^2/sinh(\alpha t)^{6/n})}{8(1+m)^2n^2} \tag{34}$$

$$p_2 = \frac{-3\sinh(\alpha t)^{-2(3+n)/n} cosh(2\alpha t)((1+m)^2n^2 + 3(1+2m+4m^2)\alpha^2\sinh(\alpha t)6/n)}{8(1+m)^2n^2} \tag{35}$$

$$p = p_1 + p_2$$

$$\rho_{p1} = \frac{3sinh(\alpha t)^{-2(3+n)/n}(-(1+m)^2n^2 - (3-4n+2m(-9+2n)+4m^2(-3+2n))\alpha^2\sinh(\alpha t))}{8(1+m)^2n^2} \tag{36}$$

$$\rho_{p2} = \frac{3sinh(\alpha t)^{-2(3+n)/n} cosh(2\alpha t)((1+m)^2n^2 + 3(-1+6m+4m^2)\alpha^2\sinh(\alpha t))}{8(1+m)^2n^2} \tag{37}$$

$$\rho_p = \rho_{p1} + \rho_{p2}$$

$$\lambda = -\frac{3(2m-1)\alpha^2(3-2n+3\cosh(2\alpha t)cosh^2(\alpha t))}{4(1+m)n^2} \tag{38}$$

$$\rho = \frac{3}{4}(\frac{3(1+4m)\alpha^2\coth^2(\alpha t)}{(1+m)^2n^2} + \sinh(\alpha t)^{6/n}) \tag{39}$$

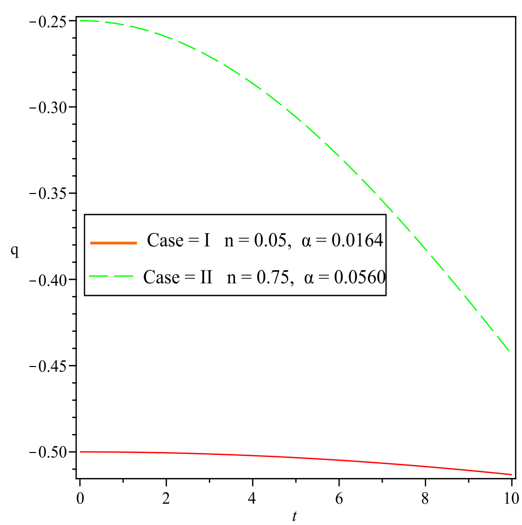
Ref

4. Halford, W.D.: Austr. J. Phys. 23, 863 (1970).

Under the appropriate choice of constants the energy density and particle density satisfies the energy conditions. We can see that at $T = 0$ all the cosmological parameters diverge which implies that the derived model has a uniqueness at initial time .Uniqueness of this type is a Point Type and is explained by (MacCallum [102]). The cosmological parametersλ,ρ, p, , ρ_p and starts with very large values. For $m < 1$ these parameters decreases with the extension of the current universe.In the starting of universe the values of ρ_p and λ were large implying that strings were dominating the beginning of the universe i.e. at initial times.At extremely large values of times, the cosmological parameters ρ_p and λ approaches zero which implies that for extremely large values of times the strings vanishes and because of this the strings are not being detectable in the present time.

IV. Result and Discussion

We can see from Fiq.1 that for $n \leq 1$,our derived model is progressing in escalating phase whereas for $n > 1$, the model is progressing from early de-escalating phase to present escalated phase. It can be seen that our model is evolving only in an escalted phase ($q < 0$) for assuming $n = 0.5$ and $\alpha = 0.0164$ (case I) and in (case II) $n = 0.75$ and $\alpha = .0560$.this is the value of joint OHD+JLA dataset used (Amirhashchi et al. [56, 57]).



(a)

Figure 1: (a)Plot of q to t. Here (a) Case I $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$

Figure.2(a) corresponding to the Eq.30 , Plot of Hubble parameter (H) to t. H decreases with increase of t. Figure.2(b)depicts the behaviour of spatial volume V with respect to t. Spatial V increases as cosmic time tends to infinity and becomes zero at $t = 0$. From Eq. (36) and **fig.3**, we can see that isotropic pressure p increases with the increase of time and p approches to zero for t> 0, n> .48.

From Eq.(38), We can see that the particle density denoted by ρ_p decreases with the increase of cosmic time and remains positive i.e. $\rho_p > 0$ for all time. **Fig.4** is the plot of particle density with respect to time . Here it is to be noted that ρ_p approaches to zero at large value of times in both cases. It is worth mentioning that ρ_p is decreasing fastly in case 1 in comparison to case 2.

From Eq.(39), we can also see that the tension density λ increases with the increase of time and it is always $\lambda < 0$. **fig.5** is the plot of string tension density with respect to time. It can be seen that the λ remains negative in both cases.

Ref

102. MacCallum, M.A.H.: Comm. Math. Phys. 20, 57 (1971).

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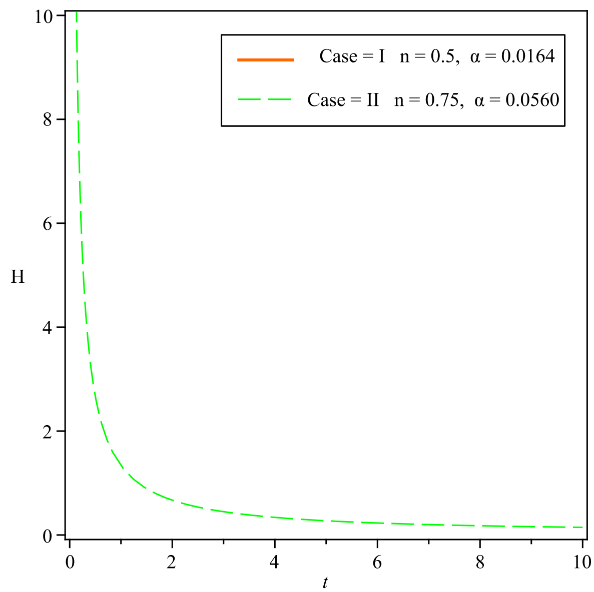
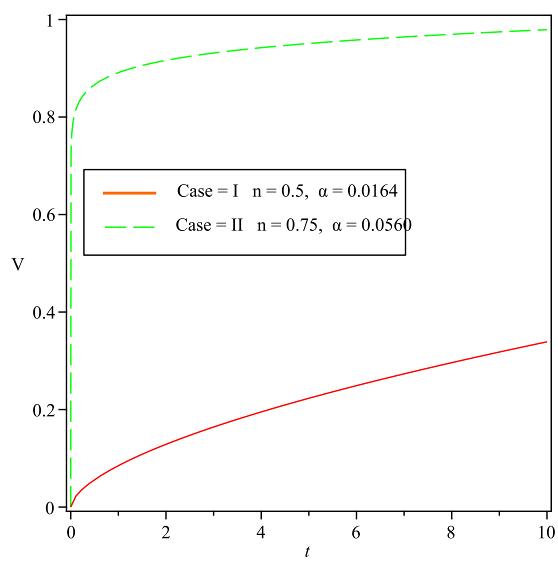
However, it tends towards infinity for t> 0. **fig.6** is the plot of ρ with respect to cosmic time.We can see that energy density decreases more sharply in case 1 than 2 with increase of time t and tends to zero as time increases.



(a)



(b)

Figure 2: Plot of Hubble Parameter and Volume to t Here (a) CaseI $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$.

V. Conclusion

In this paper,Anisotropic spatially homogeneous Bianchi-I cosmological models with perfect fluid within the chassis of normal gauge in Lyra's manifold consid-

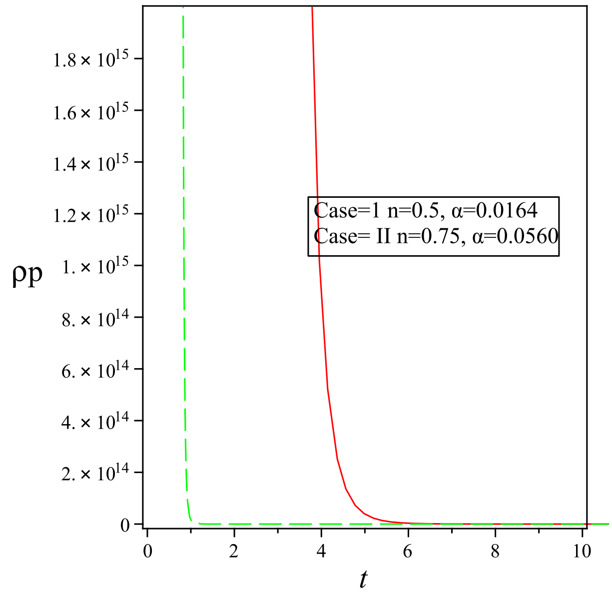
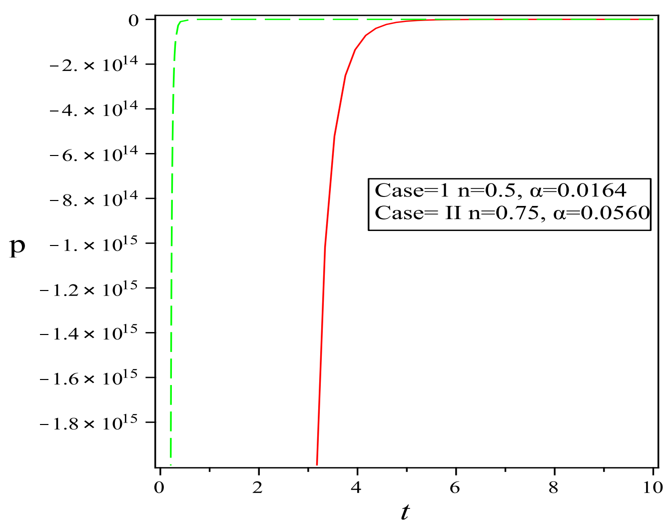


Figure 3: Plot of particle density ρ_p to time t

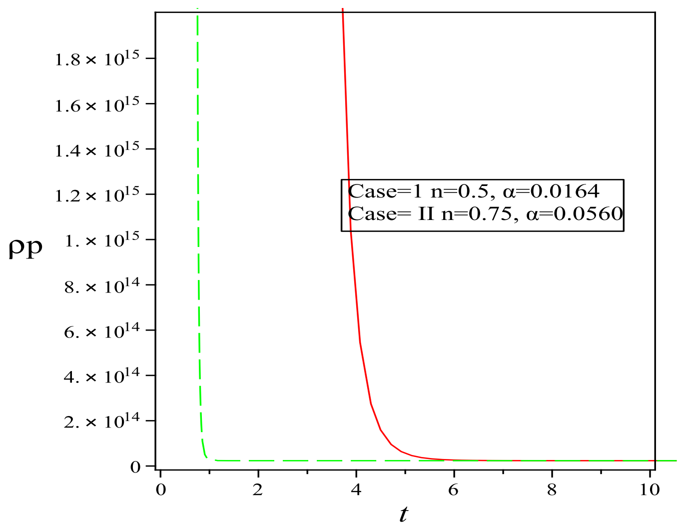


(a)

Figure 4: Plot of Isotropic Pressure p to t Here (a) CaseI $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$.

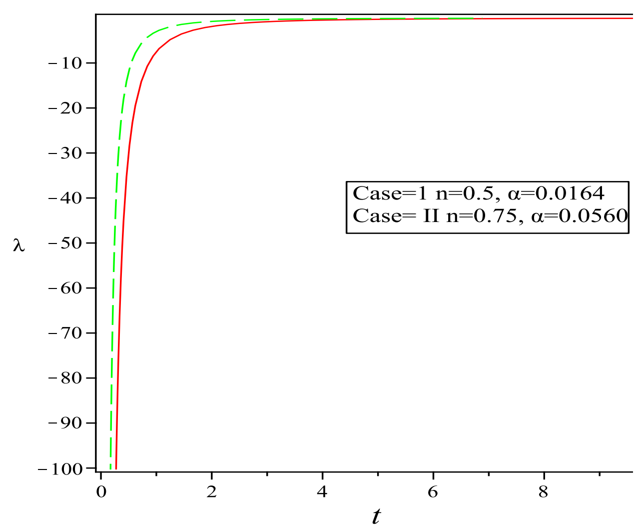
ering deceleration parameter to be time dependent has been studied.

Notes



(a)

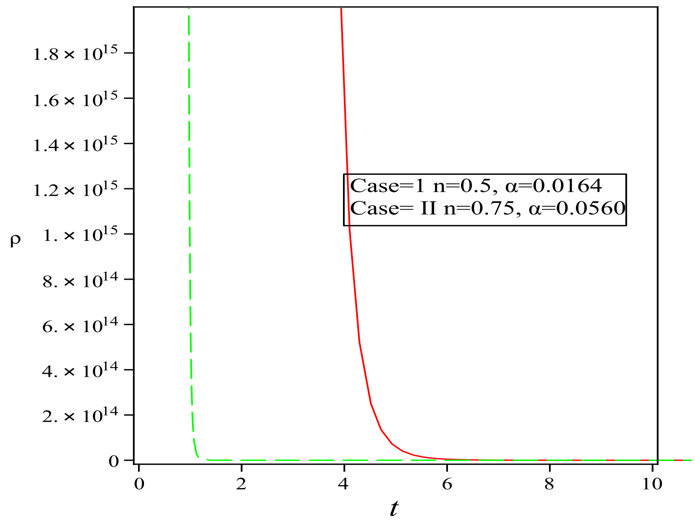
Figure 5: Plot of Particle Density ρ_p to t Here (a) CaseI $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$.



(a)

Figure 6: Plot of String Tension λ to t Here (a) CaseI $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$.

We have taken that the normal congruence of the model to the homogeneous expansion to be constant. i.e.$\frac{\sigma}{\theta}$ = constant. All the physical quantities are



(a)

Figure 7: Plot of energy density ρ to t Here (a) CaseI $n = 0.5, \alpha = 0.0164$ (b) CaseII $n = 0.75, \alpha = 0.0560$.

extremely large at initial time and becomes zero as $T \to \infty$ which is a Point Type uniqueness (MacCallum 1971) at initial time in the derived model.

The derived cosmological model presents the dynamics of strings for various values of $(n, \alpha) = (0.5, 0.0164)$ and $(0.75, 0.0560)$ for various modes of advancement of universe . In the beginning of the universe the strings dominates and at extremely large values of times vanishes which is in concurrence with the current observations.

Further parameters Isotropic Pressure p, Particle Density ρ_p, energy density ρ has been analysed to study their impact with increase in time. The particle density reduces with the increase of cosmic time and becomes negligible at extremely large value of times whereas isotropic pressure and is always negative and at late times it also follows the same pattern and becomes negligible.This negative sign for the pressure (repulsive force)can be explained as a source of the escalation at initial time.

The investigation of such cosmological models in the chassis of Lyra's manifold gives rise to new mode for theoretical formulation for relativistic gravitation and a new prospect for further analysis in astrophysics and cosmology.

References Références Referencias

1. Lyra, G.: Math. Z. 54, 52 (1951)
2. Sen, D.K., Z. Phys. 149, 311 (1957)
3. Sen, D.K., Dunn, K.A.: J. Math. Phys. 12, 578 (1971)
4. Halford, W.D.: Austr. J. Phys. 23, 863 (1970)
5. Halford, W.D.: J. Math. Phys. 13, 1699 (1972)
6. Sen, D.K., Vanstone, J.R.: J. Math. Phys. 13, 990 (1972)
7. Soleng, H.H.: Gen. Rel. Gravit. 19, 1213 (1987)
8. Singh, T., Singh, G.P.: J. Math. Phys. 32, 2456 (1991)
9. Singh, T., Singh, G.P.: Il. Nuovo Cimento B 106, 617 (1991)
10. Singh, T., Singh, G.P.: Int. J. Theor. Phys. 31, 1433 (1992)
11. Singh, T., Singh, G.P.: Fortschr. Phys. 41, 737 (1993)

Notes

12. Singh, G.P., Desikan, K.: Pramana-Journal of Physics 49, 205 (1997)
13. Pradhan, A., Vishwakarma, A.K.: J. Geom. Phys. 49, 332 (2004)
14. Pradhan, A., Yadav, V.K., Chakrabarty, I.: Int. J. Mod. Phys. D 10, 339 (2000).
15. Pradhan, A., Yadav, L., Yadav, A.K.: Astrophys. Space Sci. 299, 31 (2005)
16. Rahaman, F., Chakraborty, S., Kalam, M.: Int. J. Mod. Phys. D 10, 735 (2001)
17. Bhowmik, B.B., Rajput, A.: Pramana-Journal of Physics 62, 1187 (2004)
18. Matyjasek, J.: Int. J. Theor. Phys. 33, 967 (1994) 967.
19. Reddy, D.R.K.: Astrophys. Space Sci. 300, 381 (2005)
20. Casama, R., Melo, C.A.M.de, Pimentel, B.M.: Astrophys. Space Sci. 305, 125 (2006)
21. Rahaman, F., P. Ghosh, P.: Fizika B 13, 719 (2004)
22. Rahaman, F., Bhui, B., Bag, G.: Astrophys. Space Sci. 295, 507 (2005)
23. Rahaman, F.: Int. J. Mod. Phys. D 9, 775 (2000) 775
24. Rahaman, F.: Int. J. Mod. Phys. D 10, 579 (2001)
25. Rahaman, F.: Astrophys. Space Sci. 281, 595 (2002)
26. Shanthi, K., Rao, V.U.M.: Astrophys. Space Sci. 179, 147 (1991)
27. Venkateswarlu, R., Reddy, D.R.K.: Astrophys. Space Sci. 182, 97 (1991)
28. Hoyle, F.: Mon. Not. Roy. Astron. Soc. 108, 372 (1948)
29. Hoyle, F., Narlikar, J.V.: Proc. Roy. Soc. London Ser. A 277, 1 (1964)
30. Hoyle, F., Narlikar, J.V.: Proc. Roy. Soc. London Ser. A 278, 465 (1964)
31. Kibble, T.W.B.: J. Phys. A: Math. Gen. 9, 1387 (1976)
32. Zel'dovich, Ya.B., Kobzarev, I. Yu., Okun, L. B.: Sov. Phys.-JETP 40, 1(1975)
33. Kibble, T.W.B.: Phys. Rep. 67, 183 (1980)
34. Everett, A.E.: Phys. Rev. 24, 858 (1981)
35. Vilenkin, A.: Phys. Rev. D 24, 2082 (1981)
36. Zel'dovich, Ya. B.: Mon. Not. R. Astron. Soc. 192, 663 (1980)
37. Letelier, P.S.: Phys. Rev. D 20, 1294 (1979)
38. Letelier, P.S.: Phys. Rev. D 28, 2414 (1983)
39. Stachel, J.: Phys. Rev. D 21, 2171 (1980)
40. Bali, R., Dave, S.: Astrophys. Space Sci. 288, 503 (2003)
41. Bali, R., Singh, D.K.: Astrophys. Space Sci. 300, 387 (2005)
42. Bali, R., Anjali: Astrophys. Space Sci. 302, 201 (2006)
43. Bali, R., Pareek, U.K., Pradhan, A.: Chin. Phys. Lett. 24, 2455 (2007)

44. Bali, R., Pradhan, A.: Chin. Phys. Lett. 24, 585 (2007)
45. Yadav, M.K., Pradhan, A., Singh, S.K.: Astrophys. Space Sci. 311, 423 (2007).
46. Wang, X.X.: Chin. Phys. Lett. 22, 29 (2005)
47. Wang, X.X.: Chin. Phys. Lett. 23, 1702 (2006)
48. Saha, B., Visinescu, M.: Astrophys. Space Sci. 315, 99 (2008)
49. Saha, B., Rikhvitsky, V., Visinescu, M.: arXiv:0812.1443 (2008)
50. Yadav, M.K., Pradhan, A., Rai, A.: Int. J. Theor. Phys. 46, 2677 (2007)
51. Reddy, D.R.K.: Astrophys. Space Sci. 300, 381 (2005)
52. Reddy, D.R.K., Rao, S., Rao, M.V.: Astrophys. Space Sci. 305, 183 (2005)
53. Rao, V.U.M., Vinutha, T., Sireesha, K.V.S.: Astrophys. Space Sci. 323, 401 (2009)
54. Rao, V.U.M., Vinutha, T.: Astrophys. Space Sci. 325, 59 (2010)
55. Pradhan, A.: Fizika B 16, 205 (2007)
56. H. Amirhashchi, S. Amirhashchi, A, Pradhan and A Beesham, arXiv: 1802.04251v3, (astro-ph.CO) (2018)
57. H.Yu, B,Ratna, F.-Y.Wang, arxiv:1711.03437 [astro-ph.CO] (2018).
58. Pradhan, A., Mathur, P.: Astrophys. Space Sci. 318, 255 (2008)
59. Pradhan, A., Singh, R., Shahi, J.P.: Elect. J. Theor. Phys. 7, 197 (2010)
60. Tripathi, S.K., Behera, D., Routray, T.R.: Astrophys. Space Sci. 325, 93(2010)
61. Misner, C.W.: Astrophys. J. 151, 431 (1968)
62. Saha, B., Visinescu, M.: Int. J. Theor. Phys. 49, 1411 (2010)
63. Saha, B., Rikhvitsky, V., Visinescu, M.: Cent. Eur. J. Phys. 8, 113 (2010)
64. Pradhan, A., Aotemshi, I., Singh, G.P.: Astrophys. Space Sci. 288, 315(2003)
65. Pradhan, A., Rai, V., Otarod, S.: Fizika B, 15, 23 (2006)
66. Pradhan, A., Rai, K.K., Yadav, A.K.: Braz. J. Phys. 37, 1084 (2007)
67. Pradhan, A.: Jour. Math. Sci., 50, 022501 (2009)
68. Pradhan, A., Kumhar, S.S.: Astrophys. Space Sci. 321, 137 (2009)
69. Pradhan, A., Mathur, P.: Fizika B, 18, 243 (2009)
70. Pradhan, A., Ram, P.: Int. Jour. Theor. Phys. 48, 3188 (2009)
71. Casama, R., Melo, C., Pimentel, B.: Astrophys. Space Sci. 305, 125 (2006)

Notes

72. Bali, R., Chandnani, N.K.: J. Math. Phys. 49, 032502 (2008)
73. Bali, R., Chandnani, N.K.: Fizika B, 18, 227 (2009)
74. Kumar, S., Singh, C.P.: Int. Mod. Phys. A 23, 813 (2008)
75. Ram, S., Zeyauddin, M., Singh, C.P.: Int. J. Mod. Phys, A 23, 4991 (2008)
76. Singh, J.K.: Astrophys. Space Sci. 314, 361 (2008)
77. Rao, V.U.M., Vinutha, T., Santhi, M.V.: Astrophys. Space Sci. 314, 213 (2008)
78. Pradhan, A., Chouhan, D.S.: Astrophys. Space Sci. DOI: 10.1007/s10509-010-0478-8 (2010).
79. Thorne, K.S.: Astrophys. J. 148, 51 (1967)
80. Kantowski, R., Sachs, R.K.: J. Math. Phys. 7, 433 (1966)
81. Kristian, J., Sachs, R.K.: Astrophys. J. 143, 379 (1966)
82. Collins, C.B., Glass, E.N., Wilkinson, D.A.: Gen. Rel. Grav. 12, 805 (1980)
83. Berman, M.S.: Il Nuovo Cim. B 74, 182 (1983)
84. Berman M.S., Gomide F.M.: Gen. Rel. Grav. 20, 191 (1988)
85. Saha, B., Rikhvitsky, V.: Physica D 219, 168 (2006)
86. Saha, B.: Astrophys. Space Sci. 302, 83 (2006)
87. Singh, C.P., Kumar, S.: Int. J. Mod. Phys. D 15, 419 (2006)
88. Singh, T., Chaubey, R: Pramana - J. Phys. 67, 415 (2006)
89. Singh, T., Chaubey, R: Pramana - J. Phys. 68, 721 (2007)
90. Zeyauddin, M., Ram, S.: Fizika B 18, 87 (2009)
91. Singh, J.P., Baghel, P.S.: Int. J. Theor. Phys. 48, 449 (2009)
92. Pradhan, A., Jotania, K.: Int. J. Theor. Phys. 49, 1719 (2010)
93. Vishwakarma, R.G.: Class. Quant. Grav. 17, 3833 (2000)
94. Garnavich, P.M.: Astrophys. J. 493, L53 (1998)
95. Garnavich, P.M.: Astrophys. J. 509, 74 (1998)
96. Perlmutter, S. et al.: Astrophys. J. 483, 565 (1997)
97. Perlmutter, S. et al.: Nature 391, 51 (1998)
98. Perlmutter, S. et al.: Astrophys. J. 517, 565 (1999)
99. Reiss, A.G. et al.: Astron. J. 116, 1009 (1998)
100. Reiss, A.G. et al.: Astron. J. 607, 665 (2004)
101. Schmidt, B.P.: Astrophys. J. 507, 46 (1998)
102. MacCallum, M.A.H.: Comm. Math. Phys. 20, 57 (1971)

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On the Convergence of a Single Step Third order Method for Solving Equations

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Abstract- S. Kumar provided the local convergence of a third convergent order method for solving equations defined on the real line. We study the semi-local convergence of this method defined on the real line or complex plain. The local convergence is also provided but under weaker conditions.

Keywords and Phrases: local, semi-local convergence, weak conditions, third convergence order method.

GJSFR-F Classification: *MSC 2010: 49M15, 65J15, 65G99*

ONTHECONVERGENCEOFASINGLESTEPTHIRDORDERMETHODFORSOLVINGEQUATIONS

Strictly as per the compliance and regulations of:

6. Kumar, S, Kanwar, V, Tomar, S. K, Singh, S., Geometrically constructed families of Newton's method for unconstarined optimization and nonlinear equations, Int. J. Math. Math. Sci., 2011, (2011), 1-9.

On the Convergence of a Single Step Third Order Method for Solving Equations

Samundra Regmi [α], Ioannis K. Argyros [σ], Santhosh George [ρ] & Christopher I. Argyros [ω]

Abstract- S. Kumar provided the local convergence of a third convergent order method for solving equations defined on the real line. We study the semi-local convergence of this method defined on the real line or complex plain. The local convergence is also provided but under weaker conditions.

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I. Introduction

Let $F : D \subset S \longrightarrow S$ be a differentiable function, where $S = \mathbb{R}$ or $S = \mathbb{C}$ and D is an open nonempty set.

We are interested in computing a solution x^* of equation

$$F(x) = 0. \tag{1.1}$$

The point x^* is needed in closed form. But this form is attained only in special cases. That explains why most solution methods for (1.1) are iterative. There is a plethora of local convergence results for high convergent iterative methods based on Taylor expansions requiring the existence of higher than one derivatives not present on these methods. But there is very little work on the semi-local convergence of these methods or the local convergence using only the derivative of the operator appearing on these methods. We address these issues using a method by S. Kumar defined by

$$x_0 \in D, \; x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.2}$$

where $A_n = F'(x_n) - \gamma F(x_n)$, $\gamma \in S$. It was shown in [6] that the order of this method is three and for $e_n = x_n - x^*$,

$$e_{n+1} = (\gamma - a_2)e_n^2 + O(e_n^3), \tag{1.3}$$

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where $a_m = \frac{1}{m!}\frac{F^{(m)}(x^*)}{F'(x^*)}$, $m = 2, 3, \ldots$. It follows that the convergence requires the existence of F', F'', F''' but F'', F''' do not appear on method (1.2). So, these assumptions limit the applicability of the method. Moreover, no computable error bounds on $|x_n - x^*|$ or uniqueness of the solution results are given.

For example [1]: Let $E = E_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Define λ on D by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\lambda'''(t)$ is not bounded on D. So, the convergence of scheme (1.2) is not guaranteed by the previous analyses in [6]. We address all these concerns by using conditions only on F' in both the local and semi-local case that appears on method (1.2). This way we expand the applicability of this method. Our technique is very general so it can be used to extend the applicability of other methods along the same lines [2–5, 7–10]. Throughout this paper $U(x, r) = \{y : |x - y| < r\}$ and $U[x, r] = \{y : |x - y| \leq r\}$ for $x \in S$ and $r > 0$.

The rest of the paper is set up as follows: In Section 2 we present the semi-local analysis, where in Section 3 we present local analysis. The numerical experiments are presented in Section 4.

II. Semi-Local Analysis

Let L_0, L, γ, δ be given positive parameters and $\eta \geq 0$. Define scalar sequence $\{t_n\}$ by

$$\begin{aligned} t_0 &= 0, t_1 = \eta, \\ t_{n+2} &= t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2|\gamma|\delta(t_n + \eta)(t_{n+1} - t_n)}{2(1 - (L_0 + |\gamma|\delta)t_{n+1}))}. \end{aligned} \tag{2.1}$$

Next, we shall prove that this equence is majorizing for method (1.2). But first we need to define more parameters and scalar functions:

$$\alpha_0 = \frac{Lt_1}{2(1 - (L_0 + |\gamma|\delta)t_1)},$$

$$\triangle = L^2 - 8(L_0 + |\gamma|\delta)(1|\gamma|\delta - L),$$

functions $f : [0, 1) \longrightarrow \mathbb{R}$, $g : [0, 1) \longrightarrow \mathbb{R}$ by

$$f(t) = \frac{2|\gamma|\delta\eta}{1 - t} + \frac{2t(L_0 + |\gamma|\delta)\eta}{1 - t} + 2|\gamma|\delta\eta - 2t,$$

$$g(t) = 2(L_0 + |\gamma|\delta)t^2 + Lt + 2|\gamma|\delta - L$$

and sequences of polynomials $f_n : [0, 1) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} f_n(t) &= Lt^n\eta + 2|\gamma|\delta(1 + t + \ldots + t^{n-1})\eta + \eta \\ &\quad + 2t(L_0 + |\gamma|\delta)(1 + t + \ldots + t^n)\eta - 2t. \end{aligned}$$

Ref

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Notes

Notice that $\triangle$ is the discriminant of g. Consider that any of these conditions hold:

(C1) There exists minimal $\beta \in (0,1)$ such that $f(\beta) = 0$ and $\triangle \leq 0$. Then, suppose $\alpha_0 \leq \beta$.

(C2) There exists minimal $\beta \in (0,1)$ such that $f(\beta) = 0$, $\alpha \in (0,1) : f(\alpha) = 0$, and $\triangle > 0$. Then, suppose $\alpha_0 \leq \alpha \leq \beta$.

(C3) $f(t) \neq 0$ for all $t \in [0,1)$ and $\triangle \leq 0$.

(C4) $f(t) \neq 0$ and $\triangle > 0$. Notice that g has two solutions: $0 \leq s_1 < s_2 < 1$. Suppose $\alpha_0 \leq s \in (s_1, s_2]$ and $f(s) \leq 0$.

Let us denote these conditions by (C).
Next, we present convergence results for sequence (2.1).

Lemma 2.1 *Suppose:*

$$(L_0 + |\gamma|\delta)t_{n+1} < 1. \tag{2.2}$$

Then, the following assertions hold

$$0 \leq t_n \leq t_{n+1} \tag{2.3}$$

and

$$t^* = \lim_{n\longrightarrow\infty} t_n \leq \frac{1}{L_0 + |\gamma|\delta}. \tag{2.4}$$

Proof. Assertions follow from (2.1) and (2.2), where t^* is the unique least upper bound of sequence $\{t_n\}$.

The next result is shown under stronger conditions but which are easier to verify than (2.2).

Lemma 2.2 *Suppose: conditions (C) hold. Then, assertions (2.3) and (2.4) hold too.*

Proof. Mathematical induction on m is used to show

$$(I_m) : \quad \frac{L(t_{m+1} - t_m) + 2|\gamma|\delta(t_m + \eta)}{2(1 - (L_0 + |\gamma|\delta)t_{m+1}} \leq \alpha. \tag{2.5}$$

This estimate hols for $m = 0$ by the definition of α_0 and conditions (C). Then, we get $0 \leq t_2 - t_1 \leq \alpha(t_1 - t_0) = \alpha\eta$ and $t_2 \leq t_1 + \alpha\eta = \frac{1-\alpha^2}{1-\alpha}\eta < \frac{\eta}{1-\alpha}$. Suppose $0 \leq t_{m+1} - t_m \leq \alpha^m\eta$ and $t_m \leq \frac{1-\alpha^m}{1-\alpha}\eta$. Then, (2.5) holds if

$$\begin{aligned} &L\alpha^m\eta + 2|\gamma|\delta((1 + \alpha + \ldots + \alpha^{m-1})\eta + \eta) \\ &+2\alpha(L_0 + |\gamma|\delta)(1 + \alpha + \ldots + \alpha^m)\eta - 2\alpha \leq 0 \end{aligned} \tag{2.6}$$

or

$$f_m(t) \leq 0 \ \text{ at } \ t = \alpha. \tag{2.7}$$

We need a relationship between two consecutive polynomials f_m :

$$\begin{aligned} f_{m+1}(t) &= f_{m+1}(t) - f_m(t) + f_m(t) \\ &= Lt^{m+1}\eta + 2|\gamma|\delta(1 + t + \ldots + t^m)\eta + 2|\gamma|\delta\eta \\ &\quad +2t(L_0 + |\gamma|\delta)(1 + t + \ldots + t^{m+1})\eta + f_m(t) \end{aligned}$$

$$-Lt^m\eta - 2|\gamma|\delta(1+t+\ldots+t^{m-1})\eta - 2|\gamma|\delta\eta$$

$$-2t(L_0+|\gamma|\delta)(1+t+\ldots+t^{m-1})\eta - 2|\gamma|\delta\eta$$

$$-2t(L_0+|\gamma|\delta)(1+t+\ldots+t^{m})\eta + 2t$$

$$= f_m(t) + g(t)t^m\eta,$$

so

$$f_{m+1}(t) = f_m(t) + g(t)t^m\eta. \tag{2.8}$$

Define function $f_\infty : [0,1) \longrightarrow \mathbb{R}$ by

$$f_\infty(t) = \lim_{m\longrightarrow\infty} f_m(t). \tag{2.9}$$

It then follows from (2.6) and (2.9) that

$$f_\infty(t) = f(t). \tag{2.10}$$

Case (C1) We have by (2.8) that

$$f_m(t) \leq f_{m+1}(t). \tag{2.11}$$

So, (2.7) holds if

$$f_\infty(t) \leq 0, \tag{2.12}$$

which is true by the choice of β.

Case(C2) Then, again (2.11) and (2.12) hold by the choice of α and β.

Case(C4) We have

$$f_{m+1}(t) \leq f_m(t),$$

so (2.7) holds if $f_1(\alpha) \leq 0$, which is true by (C4).

The induction for items (2.5) so the induction for (2.3) is completed too leading again to the verification of the assertions for $\frac{1}{L_0+|\gamma|\delta}$ in (2.4) replaced by $\frac{\eta}{1-\alpha}$.

Next, we introduce the conditions (A) to be used in the semi-local convergence of method (1.2).

Suppose:

(A1) There exist $x_0 \in D$, $\eta \geq 0$ such that $A_0 \neq 0$ and $\|A_0^{-1}F(x_0)\| \leq \eta$.

(A2) There exists $L_0 > 0$ such that $\|A_0^{-1}(F'(v) - F'(x_0))\| \leq L_0\|v - x_0\|$ for all $v \in D$. Set $D_0 = U(x_0, \frac{1}{L_0}) \cap D$.

(A3) There exist $L > 0, \delta > 0$ such that $\|A_0^{-1}(F'(v) - F'(w))\| \leq L\|v - w\|$ and

$$\|A_0^{-1}(F(v) - F(x_0))\| \leq \delta\|v - x_0\|,$$

for all $v, w \in D_0$.

(A4) Conditions of Lemma 2.1 or Lemma 2.2 hold and

(A5) $U[x_0, t^*] \subset D$.

Next, we show the semi-local convergence of method (1.2) under the conditions (A).

Notes

Theorem 2.3 *Suppose that conditions (A) hold. Then, sequence* $\{x_n\}$ *generated by method (1.2) is well defined remains in* $U(x_0, t^*)$ *and converges to a solution of equation (1.1) such that* $x^* \in U[x_0, t^*]$.

Proof. Mathematical induction is used to show

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n. \tag{2.13}$$

This estiamte holds by (A1) and (1.2) for $n = 0$. Indeed, we have

$$\|x_1 - x_0\| = \|A_0^{-1}F(x_0)\| = \eta = t_1 - t_0 < t^*,$$

so $x_1 \in U(x_0, t^*)$. Suppose (2.13) holds for all values of m smaller or equal to $n-1$. Next, we show $A_{m+1} \neq 0$. Using the definition of A_{m+1}, (A2), (A3) we get in turn that

$$\begin{aligned}
\|A_0^{-1}(A_{m+1} - A_0)\| &= \|A_0^{-1}(F'(x_{n+1}) - \gamma F(x_{m+1}) - F'(x_0) + \gamma F(x_0))\| \\
&\leq \|A_0^{-1}(F'(x_{m+1}) - F'(x_0))\| + |\gamma|\|A_0^{-1}(F(x_{m+1}) - F(x_0))\| \\
&\leq L_0\|x_{m+1} - x_0\| + |\gamma|\delta\|x_{m+1} - x_0\| \\
&\leq L_0(t_{m+1} - t_0) + |\gamma|\delta(t_{m+1} - t_0) \\
&= (L_0 + |\gamma|\delta)t_{m+1} < 1, \qquad (2.14)
\end{aligned}$$

where we also used by the induction hypotheses that

$$\begin{aligned}
\|x_{m+1} - x_0\| &\leq \|x_{m+1} - x_m\| + \|x_m - x_{m-1}\| + \ldots + \|x_1 - x_0\| \\
&\leq t_{m+1} - t_0 = t_{m+1} < t^*,
\end{aligned}$$

so $x_{m+1} \in U(x_0, t^*)$. It also follows from (2.14) that $A_{m+1} \neq 0$ and

$$\|A_{m+1}^{-1}A_0\| \leq \frac{1}{1 - (L_0 + |\gamma|\delta_{m+1})} \tag{2.15}$$

by the Banach lemma on inverses of functions [8]. Moreover, we can write by method (1.2):

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - F'(x_m)(x_{m+1} - x_m) + \gamma F'(x_m)(x_{m+1} - x_m), \tag{2.16}$$

since $F(x_m) = -(F'(x_m) - \gamma F(x_m))(x_{m+1} - x_m)$. By (A3) and (2.16), we obtain in turn

$$\begin{aligned}
\|A_0^{-1}F(x_{m+1})\| &\leq \|\int_0^1 A_0^{-1}(F'(x_m + \theta(x_{m+1} - x_m)) \\
&\quad -F'(x_m))d\theta(x_{m+1} - x_m)\| \\
&\quad +\|A_0^{-1}F'(x_m)\|\|x_{m+1} - x_m\| \\
&\leq \frac{L}{2}\|x_{m+1} - x_m\|^2 \\
&\quad +|\gamma|(\|A_0^{-1}(F(x_{m+1}) - F(x_0)\| + \|A_0^{-1}F(x_0)\|)\|x_{m+1} - x_m\|
\end{aligned}$$

Ref

8. Ortega, L.M., Rheinboldt, W.C., Iterative Solution of Nonlinear Equations in Several Variables, Academic press, New York, (2000).

$$\leq \frac{L}{2}(t_{m+1}-t_m)^2 + |\gamma|(\delta\|x_{m+1}-x_0\|+\eta)(t_{m+1}-t_m)$$

$$\leq \frac{L}{2}(t_{m+1}-t_m)^2$$

$$+|\gamma|(\delta t_{m+1}+\eta)(t_{m+1}-t_m). \quad (2.17)$$

Notes

It then follows from (1.2), (2.15) and (2.17) that

$$|x_{m+2}-x_{m+1}\| \leq \|A_{m+1}^{-1}A_0\|\|A_0^{-1}F(x_{m+1})\| \leq t_{m+2}-t_{m+1}, \quad (2.18)$$

and

$$\|x_{m+2}-x_0\| \leq \|x_{m+2}-x_{m+1}\|+\|x_{m+1}-x_0\|$$

$$\leq t_{m+2}-t_{m+1}+t_{m+1}-t_0 = t_{m+2} < t^*. \quad (2.19)$$

But sequence $\{t_m\}$ is fundamental. So, sequence $\{x_m\}$ is fundamental too (by (2.18)), so it converges to some $x^* \in U[x_0,t^*]$. By letting $m \longrightarrow \infty$ in (2.17), we deduce that $F(x^*)=0$.

Next, we present a uniqueness of the solution result for equation (1.1).

Proposition 2.4 Suppose

(1) There exists $x_0 \in D, K>0$ such that $F'(x_0)\neq 0$ and

$$\|F'(x_0)^{-1}(F'(v)-F'(x_0))\| \leq K\|v-x_0\| \quad (2.20)$$

for all $v \in D$.

(2) The point $x^* \in U[x_0,a] \subseteq D$ is a simple solution of equation $F(x)=0$ for some $a>0$.

(3) There exists $b \geq a$ such that

$$K(a+b)<2.$$

Set $B=U[x_0,b]\cap D$. Then, the only solution of equation $F(x)=0$ in B is x^*.

Proof. Set $M=\int_0^1 F'(z^*+\theta(x^*-z^*))d$ for some $z\in B$ with $F(z^*)=0$. Then, in view of (2.20)

$$\|F'(x_0)^{-1}(M-F'(x_0))\| \leq K\int_0^1((1-\theta)\|x_0-x^*\|+\theta\|x_0-z^*\|)d\theta$$

$$\leq \frac{K}{2}(a+b)<1,$$

so, $z^*=x^*$ follows from $M\neq 0$ and $M(z^*-x^*)=F(z^*)-F(x^*)=0-0=0$.

Notes

III. Local Convergence

Let β_0, β and β_1 be positive parameters. Set

$$\beta_2 = \beta_0 + |\delta|\beta_1.$$

Define function $h : [0, \frac{1}{\beta_2}) \longrightarrow \mathbb{R}$ by

$$h(t) = \frac{\beta t}{2(1-\beta_0 t)} + \frac{|\gamma|\beta_1^2 t}{(1-\beta_0 t)(1-\beta_2 t)}.$$

Suppose this function has a minimal zero $\rho \in (0, \frac{1}{\beta_2})$. We shall use conditions (H). Suppose:

(H1) The point $x^* \in D$ is a simple solution of equation (1.1).

(H2) There exists $\beta_0 > 0$ such that

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq \beta_0 \|v - x^*\|$$

for all $v \in D$. Set $D_1 = U(x^*, \frac{1}{\beta_0}) \cap D$.

(H3) There exist $\beta > 0, \beta_1 > 0$ such that

$$\|F'(x^*)^{-1}(F'(v) - F'(w))\| \leq \beta \|v - w\|$$

and

$$\|F'(x^*)^{-1}(F(v) - F(x^*))\| \leq \beta_1 \|v - x^*\|$$

for all $v, w \in D_1$.

(H4) Function $h(t) - 1$ has a minimal solution $\rho \in (0, 1)$.
and

(H5) $U[x^*, \rho] \subset D$.

Notice that $A(x^*) = F'(x^*)$. Then, we get the estimates

$$\begin{aligned}
& \|F'(x^*)^{-1}(F(x_n) - \gamma F(x_n) - F'(x^*) + \gamma F(x^*)\| \\
\leq\ & \|F'(x^*)^{-1}(F'(x_m) - F'(x^*))\| + |\gamma| \|F'(x^*)^{-1}(F(x_m) - F(x^*))\| \\
\leq\ & \beta_0 \|x_m - x^*\| + |\gamma|\beta_1 \|x_m - x^*\| \\
=\ & \beta_2 \|x_m - x^*\| < \beta_2 \rho < 1,
\end{aligned}$$

$$\begin{aligned}
\|F'(x^*)^{-1}(A_m - F'(x_m))\| &= \|F'(x^*)^{-1}(F'(x_m) - \gamma F(x_m) - F'(x_m))\| \\
&= |\gamma| \|F'(x^*)^{-1} F(x_m)\| \\
&\leq |\gamma|\beta_1 \|x_m - x^*\|,
\end{aligned}$$

$$\begin{aligned}
x_{m+1} - x^* &= x_m - x^* - F'(x_m)^{-1}F(x_m) + F'(x_m)^{-1}F(x_m) - A_m^{-1}F(x_m) \\
&= (x_m - x^* - F'(x_m)^{-1}F(x_m)) + (F'(x_m)^{-1} - A_m^{-1})F(x_m) \\
&= (x_m - x^* - F'(x_m)^{-1}F(x_m)) \\
&\quad + F'(x_m)^{-1}(A_m - F'(x_m))A_m^{-1}F(x_m),
\end{aligned}$$

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leading to

$$\begin{aligned}\|x_{m+1}-x^*\| &\leq \frac{\beta\|x_m-x^*\|^2}{2(1-\beta_0\|x_m-x^*\|)} \\ &\quad+\frac{|\gamma|\beta_1^2\|x_m-x^*\|^2}{(1-\beta_0\|x_m-x^*\|)(1-\beta_2\|x_m-x^*\|)} \\ &< h(\rho)\|x_m-x^*\| = \|x_m-x^*\| < \rho.\end{aligned}$$

So, we get

$$\|x_{m+1}-x^*\| \leq p(\|x_m-x^*\|) < \rho,\ \ p = h(\|x_0-x^*\|) \tag{3.1}$$

and $x_{m+1} \in U(x^*,\rho)$. Hence, we conclude by (3.1) that $\lim_{m\longrightarrow\infty} x_m = x^*$. Therefore, we arrive at the local convergence result for method (1.2).

Theorem 3.1 *Under conditions (H) further suppose that* $x_0 \in U(x^*,\rho)$. *Then, sequence* $\{x_n\}$ *generated by method (1.2) is well defined in* $U(x_0,\rho)$, *remains in* $U(x_0,\rho)$ *and converges to* x^*.

Next, we present a uniqueness of the solution result for equation (1.2).

Proposition 3.2 *Suppose*

(1) The point x^* *is a simple solution of equation* $F(x)=0$ *in* $U(x^*,\tau)\subset D$ *for some* $\tau>0$.

(2) Condition (H2) holds.

(3) There exists $\tau^*\geq\tau$ *such that*

$$\beta_0\tau^* < 2.$$

Set $B_1 = U[x_0,\tau^*]\cap D$. *Then, the only solution of equation (1.1) in* B_1 *is* x^*.

Proof. Set $M_1 = \int_0^1 F'(z^*+\theta(x^*-z^*))d$ for some $z^*\in B_1$ with $F(z^*)=0$. Then, using (H2), we get in turn that

$$\begin{aligned}\|F'(x^*)^{-1}(M_1-F'(x^*))\| &\leq \int_0^1(1-\theta)\|z^*-x^*\|d\theta \\ &\leq \frac{\beta_0}{2}\tau^* < 1,\end{aligned}$$

so, $z^*=x^*$ follows from $M_1\neq 0$ and $M_1(z^*-x^*) = F(z^*)-F(x^*) = 0-0 = 0$.

IV. Numerical Example

We verify convergence criteria using method (1.2) for $\gamma=0$, so $\delta=0$.

Example 4.1 *(Semi-local case) Let us consider a scalar function* F *defined on the set* $D=U[x_0,1-q]$ *for* $q\in(0,\frac{1}{2})$ *by*

$$F(x) = x^3 - q.$$

Choose $x_0=1$. *Then, we obtain the estimates* $\eta=\frac{1-q}{3}$,

$$\begin{aligned}|F'(x_0)^{-1}(F'(x)-F'(x_0))| &= |x^2-x_0^2| \\ &\leq |x+x_0||x-x_0| \leq (|x-x_0|+2|x_0|)|x-x_0|\end{aligned}$$

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$$= \quad (1-q+2)|x-x_0| = (3-q)|x-x_0|,$$

for all $x \in D$, *so* $L_0 = 3-q$, $D_0 = U(x_0, \frac{1}{L_0}) \cap D = U(x_0, \frac{1}{L_0})$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(y)-F'(x)| &= |y^2-x^2| \\ &\leq |y+x||y-x| \leq (|y-x_0+x-x_0+2x_0)||y-x| \\ &= (|y-x_0|+|x-x_0|+2|x_0|)|y-x| \\ &\leq (\frac{1}{L_0}+\frac{1}{L_0}+2)|y-x| = 2(1+\frac{1}{L_0})|y-x|, \end{aligned}$$

for all $x, y \in D$ *and so* $L = 2(1+\frac{1}{L_0})$.

Next, set $y = x - F'(x)^{-1}F(x), x \in D$. *Then, we have*

$$y + x = x - F'(x)^{-1}F(x) + x = \frac{5x^3+q}{3x^2}.$$

Define fundtion $\bar{F}$ *on the interval* $D = [q, 2-q]$ *by*

$$\bar{F}(x) = \frac{5x^3+q}{3x^2}.$$

Then, we get by this definition that

$$\begin{aligned} \bar{F}'(x) &= \frac{15x^4-6xq}{9x^4} \\ &= \frac{5(x-q)(x^2+xq+q^2)}{3x^3}, \end{aligned}$$

where $p = \sqrt[3]{\frac{2q}{5}}$ *is the critical point of function* $\bar{F}$. *Notice that* $q < p < 2-q$. *It follows that this function is decreasing on the interval* (q, p) *and increasing on the interval* $(q, 2-q)$, *since* $x^2+xq+q^2 > 0$ *and* $x^3 > 0$. *So, we can set*

$$K_2 = \frac{5(2-q)^2+q}{9(2-q)^2}$$

and

$$K_2 < L_0.$$

But if $x \in D_0 = [1-\frac{1}{L_0}, 1+\frac{1}{L_0}]$, *then*

$$L = \frac{5\varrho^3+q}{9\varrho^2},$$

where $\varrho = \frac{4-q}{3-q}$ *and* $K < K_1$ *for all* $q \in (0, \frac{1}{2})$. *For* $q = 0.45$, *we have*

n	*1*	*2*	*3*	*4*	*5*	
t_n	*0.1833*	*0.2712*	*0.3061*	*0.3138*	*0.3142*	*0.3142*
$(L_0+\vert\gamma\vert\delta)t_{n+1}$	*0.4675*	*0.6916*	*0.7804*	*0.8001*	*0.8011*	*0.8011*

Thus condition (2.2) satisfied.

Example 4.2 *Let* $F : [-1,1] \longrightarrow \mathbb{R}$ *be defined by*

$$F(x) = e^x - 1$$

Then, we have for $x^* = 0, \beta_0 = e - 1, \beta = e^{\frac{1}{e-1}}$ *and* $\beta_1 = 0$. *So, we have* $\rho = 0.3827$.

References Références Referencias

1. Argyros I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, Mathematics, 9(16), 2021, 1942, https://doi.org/10.3390/math9161942.
2. Argyros, I.K.: Convergence and applications of Newton-type iterations. Springer, New York (2008)
3. Argyros, I. K., George, S., Mathematical modeling for the solution of equations and systems of equations with applications, Volume-IV, Nova Publisher, NY, 2021.
4. Behl, R., Kanwar, V., New highly efficient families of higher-order methods for simple roots, permitting $f'(x_n) = 0$, Int. J. Math. Math. Sci., 2014, (2014), 1-12.
5. Kou, J., Li, Y., Wang, X., A family of fifth-order iterations composed of Newton and third-order methods, Appl. Math. Comput., 186, 2,(2007), 1258-1262.
6. Kumar, S, Kanwar, V, Tomar, S. K, Singh, S., Geometrically constructed families of Newton's method for unconstarined optimization and nonlinear equations, Int. J. Math. Math. Sci., 2011, (2011), 1-9.
7. Noor, M. A, Noor, K. I, Fifth order iterative methods for solving nonlinear equations, Appl. Math. Comput., 188, 1,(2007), 406-410.
8. Ortega, L.M., Rheinboldt, W.C., Iterative Solution of Nonlinear Equations in Several Variables, Academic press, New York, (2000).
9. Traub, J. F., Iterative schemes for the solution of equations prentice Hall, New Jersey, U.S.A, (1964).
10. Petkovic, M.S., Neta, B., Petkovic, J. D., Dzunice, J., Multipoint point methods for solving nonlinear equations, Academic Press, Elsevier, USA, 2013.

Notes

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Fractional order Riemann Curvature Tensor in Differential Geometry

By Wedad Saleh
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Abstract- This study discussed some interesting aspects and features of fractional curvature in the differential manifold. In particular, Riemannian fractional curvature tensor, Livi-Civita fractional connection and Bianchi fractional identity are presented.

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FRACTIONALORDERRIEMANNCURVATURETENSORINDIFFERENTIALGEOMETRY

Strictly as per the compliance and regulations of:

Fractional Order Riemann Curvature Tensor in Differential Geometry

Wedad Saleh

Abstract- This study discussed some interesting aspects and features of fractional curvature in the differential manifold. In particular, Riemannian fractional curvature tensor, Livi-Civita fractional connection and Bianchi fractional identity are presented.

I. Introduction

In mathematics, several special functions appear in many applications such as the Gamma function that plays some significant roles in the theory of integral differential equations in particular fractional calculus. Thus, we begin with some definitions, for the details we refer to [1, 15, 8].

The Gamma function of a positive integer η is again a positive integer, while the gamma function $\Gamma(-\eta)$ of a negative integer changes to infinities. The Gamma function any positive η value is defined as follows:

$$\Gamma(\eta) = \int_0^\infty t^{\eta-1} e^{-t} dt.$$

The Gamma function $\Gamma(\eta)$ is considered as a generalization of the factorial and $\Gamma(\eta)$ is defined for $\eta > 0$ by the integral

$$\Gamma(\eta) = \int_0^\infty t^{\eta-1} e^{-t}\, dt.$$

In the classical sense since $\Gamma(0) = \dfrac{\Gamma(1)}{0}$, then it follows that $\Gamma(\eta)$ is not defined for integers $\eta \leq 0$. However, the extension formula gives finite values for $\Gamma(\eta)$, for $\Re(\eta) \leq 0$ since $\Gamma(\eta)$ is analytic everywhere except at $\eta = 0, -1, -2, ...,$ and the residue at $\eta = k$ is given by

$$\mathrm{Res}_{\eta=k}\Gamma(\eta) = \frac{(-1)^k}{k!}.$$

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Ref

15. I. N. Sneddon. Special functions of mathematical physics and chemistry. Oliver & Boyd, (1956).

Now, if $\eta > 0$, then

$$\Gamma(\eta+1) = \eta\Gamma(\eta). \tag{1.1}$$

Equation (1.1) can be used to define $\Gamma(\eta)$ for $\eta < 0$ and $\eta \neq -1, -2, \ldots$ and further, this is one of the most important formulas that were satisfied by the Gamma function.

Even though the Gamma function is defined as a locally summable function on the real line by [17]

$$\mathit{\Gamma}(\eta) = \int_0^\infty t^{\eta-1} e^{-t} dt, \qquad \eta > 0. \tag{1.2}$$

In the classical sense, $\Gamma(\eta)$ function was not defined for the negative integer thus, there was an open problem to give a satisfactory definition. However, by using the neutral limit, it has been shown in [21] that the Gamma function (1.2) is defined as follows:

$$\mathit{\Gamma}(\eta) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{\eta-1} e^{-t} dt$$

for $\eta \neq 0, -1, -2, ...,$ and this function can be defined by neutral limit such as

$$\mathit{\Gamma}(-n) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-n-1} e^{-t} dt$$

$$= \int_1^\infty t^{-n-1} e^{-t} dt$$

$$+ \int_0^1 t^{-n-1} \left[e^{-t} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} t^i \right] dt - \sum_{i=0}^{n-1} \frac{(-1)^i}{i!(n-i)}, n \in \mathbb{N}.$$

It was also proven in [20] the existence of r the derivative of the Gamma function and defined it by equation

$$\mathit{\Gamma}^{(r)}(0) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-1} \ln^r t e^{-t} dt$$

$$= \int_1^\infty t^{-1} \ln^r t e^{-t} dt + \int_0^1 t^{-1} \ln^r t \left[e^{-t} - 1 \right] dt$$

$$\mathit{\Gamma}^{(r)}(-n) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-n-1} \ln^r t e^{-t} dt$$

Ref

21. B. Fisher, A. Kili, cman and D. Nicholas. On the beta function and the neutrix product of distributions. Integral Transforms Spec. Funct., 7 1-2(1998), 35–42.

$$= \int_1^\infty t^{-n-1} \ln^r t e^{-t} dt$$

$$+ \int_0^1 t^{-n-1} \ln^r t \left[e^{-t} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} t^i \right] dt$$

$$- \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} r!(n-i)^{-r-1}$$

for $r \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Also,

$$\Gamma(-r) = \frac{(-1)^r}{r!} \quad (r) - \frac{(-1)^r}{r!} \gamma$$

for $r = 1, 2, \ldots,$ where

$$(r) = \sum_{i=1}^{r} \frac{1}{i}.$$

Thus, the definition can be extended to the whole real line where,

$$\Gamma(0) = \Gamma'(1) = -\gamma,$$

where γ denotes Euler's constant, see [22].

For a function $f \colon V \subset \mathbb{R} \to \mathbb{R}$ with $0 \in V$, the fractional derivative of order α is defined by:

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{f(s) - f(0)}{(t-s)^{1+\alpha}} ds, \quad \alpha < 0 \tag{1.3}$$

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s) - f(0)}{(t-s)^{\alpha-n+1}} ds, \quad \alpha > 0 \tag{1.4}$$

where n is the first integer greater than or equal to α.
The relation (1.3) gives a fractional integral and (1.4) gives a fractional derivative.

We express some of the operators of fractional derivatives, see for example, [4, 7, 9, 10, 12, 16].

1. $\frac{d^\alpha}{dt^\alpha} t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\gamma-\alpha}$, $\alpha \in \mathbb{R}$ or $(\alpha \in \mathbb{C})$ and $1+\gamma \neq 0, -1, \ldots, -n,$

Ref

22. B. Fisher and A. Kılı, cman. Results on the Gamma function. Applied Mathematics & Information Sciences (AMIS), 62 (2012), 173–176.

2. $\dfrac{d^n}{dt^n}\dfrac{d^\alpha}{dt^\alpha}f(t) = \dfrac{d^{n+\alpha}}{dt^{n+\alpha}}f(t), \quad n \in \mathbb{N},$

3. $\dfrac{d^\alpha}{dt^\alpha}(f_1(t) + f_2(t)) = \dfrac{d^\alpha}{dt^\alpha}f_1(t) + \dfrac{d^\alpha}{dt^\alpha}f_2(t),$

4. $\dfrac{d^\alpha}{dt^\alpha}(Cf(t)) = C\dfrac{d^\alpha}{dt^\alpha}f(t)$, where C is a constant,

5. $\dfrac{d^\alpha}{dt^\alpha}f(\beta t) = \beta^\alpha \dfrac{d^\alpha}{[d(\beta t)]^\alpha}f(\beta t).$

It is well known that fractional calculus is an essential and advantageous branch of mathematics, having a broad range of applications at almost every department of sciences. Techniques of fractional calculus have been employed in the modeling of many different phenomena in engineering, physics , and mathematics.The problem in fractional calculus is not only essential but also quite challenging ,which usually involves complicated mathematical solution techniques. However, a general solution theory for almost every issue in this area has yet to be established. Each application has developed its approaches and implementations. Consequently, a single standard method for the problems in fractional calculus has not emerged yet. Therefore, funding reliable and efficient solution techniques along with fast implementation methods are significantly essential and still active research areas.

Further, it is also realized that the operators of fractional integration and derivation have physical and geometric interpretations, which streamline along with their utilization for related issues in various fields of science(see [2, 8, 10, 11, 12, 14, 18, 19]). Moreover, the fractional differential calculus on a differential manifold is studied in [2, 3, 4, 6, 13]. Even though fractional calculus is a handy and important topic, however, the research on geometric interpretation and applications are limited ,and not many in current literature. Thus, in this study, we focus on the Riemannian curvature tensor, Livi-Civita connection and Bianchi's identity on fractional differentiable manifolds and discuss some related properties. We also give some examples.

II. Fractional Differential Calculus on Manifolds

Assume that N be an m-dimensional differential manifold (V, x_i) a local coordinate system on N and $V_0 = \{x \in V : 0 \leq x_i \leq b_i, i = 1, 2, ..., m\}$ [5].

For a function $f: V_0 \to \mathbb{R}$, the fractional derivative with respect to x_i:

$\partial_i^\alpha f(x) =$

$$\frac{1}{\Gamma(n-\alpha)}\partial_{x_i}^n \int_0^{x_i} \frac{f(x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_m) - f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)}{(x_i - s)^{\alpha-n+1}} ds,$$

Ref

2. M. Axtell, M.E. Bise, Fractional Calculus Application in Control Systems. IEEE, Nat, Aerospace and Electronics Conf., New York, (1990) 563–566.

where $\partial_{x_i}^n = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_i} \circ ... \circ \frac{\partial}{\partial x_i}$ (n times, i is fixed, $\alpha \geq 0$).

For $\alpha \in (0,1), \gamma > -1$,

$$\partial_i^\alpha (x_i)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} \quad ; \partial_i^\alpha = \delta_i^j.$$

Notes

A fractional vector field $V \subset N$ is an object of the form $X^\alpha = X_i^\alpha \partial_i^\alpha$, where $X_i^\alpha \in \Im_V(N) \quad i = 1, ..., m$.

The fractional vector fields on V and χ_V^α is generated by the operators $\partial_i^\alpha, i = 1, 2, ..., m$ are denoted by χ_V^α . If c: $x = x(t), t \in I$ is a parameterized curve in U then the fractional tangent vector field of c is given by

$$x^\alpha(t) = \frac{1}{\Gamma(1+\alpha)} \partial_t^\alpha x_i(t) \partial_i^\alpha.$$

A fractional covariant derivative is given by

$$\nabla_{X^\alpha}^\alpha Y^\alpha = X_i^\alpha (\partial_i^\alpha Y_j^\alpha + \tilde{\Gamma}_{ik}^j Y_k^\alpha) \partial_j^\alpha$$

where $X^\alpha, Y^\alpha \in \chi_U^\alpha$ and $\tilde{\Gamma}_{ik}^j$ the functions defining the coefficients of a fractional linear connection on N. They are determined by the relations

$$\nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha = \tilde{\Gamma}_{ik}^j \partial_j^\alpha.$$

Since it is essential to study fractional vector fields on a differentiable manifold N. For $\mathbb{R}^n$, there is an obvious way to do this. Recall that $\chi^\alpha(\mathbb{R}^n)$ denotes the space of fractional differentiable vector fields defined on $\mathbb{R}$. Examples are the fractional vector fields $\frac{\partial^\alpha}{\partial u_1^\alpha}, ..., \frac{\partial^\alpha}{\partial u_n^\alpha}$ determined by the natural coordinate functions $u_1, ..., u_n$.

Definition 2.1. *Fractional Riemannian metric F on m-dimensional manifold N defines for every point $p \in N$, the scalar product of fractional tangent vectors in the fractional tangent space $T_p^\alpha N$ depending on the point p.*

Let $A^\alpha = A_i^\alpha \partial_i^\alpha$ and $B^\alpha = B_j^\alpha \partial_j^\alpha$ any two fractional vectors tangent to the manifold N at the point p with coordinates $x = (x_1, ..., x_m)$ $(A^\alpha, B^\alpha \in T_p^\alpha N)$ the scalar product is equal to

$$\begin{aligned} \langle A^\alpha, B^\alpha \rangle_F |_p &= A_i^\alpha(x) \tilde{g}_{ij}(x) B_j^\alpha(x) \\ &= (A_1^\alpha, ..., A_n^\alpha) \begin{pmatrix} \tilde{g}_{11} & ... & \tilde{g}_{1n} \\ . & ... & . \\ . & ... & . \\ . & ... & . \\ \tilde{g}_{n1} & ... & \tilde{g}_{nn} \end{pmatrix} \begin{pmatrix} B_1^\alpha \\ . \\ . \\ . \\ B_n^\alpha \end{pmatrix} \end{aligned}$$

where

1. $F(A^\alpha, B^\alpha) = F(B^\alpha, A^\alpha)$, i.e., $\tilde{g}_{ij} = \tilde{g}_{ji}$ (symmetricity condition).
2. $F(A^\alpha, A^\alpha) > 0$ if $A^\alpha \neq 0$, i.e. $\tilde{g}_{ij} u_i^\alpha u_j^\alpha \geq 0$, $\tilde{g}_{ij} u_i^\alpha u_j^\alpha = 0$ iff $u_1^\alpha = ... = u_n^\alpha = 0$ (positive definiteness).
3. $F(A^\alpha, B^\alpha) |_{p=x}$, i.e. $\tilde{g}_{ij}(x)$ are smooth function where $0 < \alpha < 1$.

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Components of tensor field F in coordinate system are matrix valued functions $\tilde{g}_{ij}(x)$

$$F = \tilde{g}_{ij}(x) d^{\alpha} x_i \otimes d^{\alpha} x_j.$$

III. Rule of Transformation for Entries of the Matrix $\tilde{g}_{ij}(x)$

Notes

$\tilde{g}_{ij}(x)$- entries of the matrix $\| \tilde{g}_{ij} \|$ are components of tensor field F in a given coordinate system.

How do these components transform under transformation of coordinates $\{x_i\} \to \{x_{i'}\}$?

$$\begin{aligned} F &= \tilde{g}_{ij} dx_i^{\alpha} \otimes dx_j^{\alpha} \\ &= \tilde{g}_{ij} \left(\frac{\partial x_i^{\alpha}}{\partial x_{i'}^{\alpha}} dx_{i'}^{\alpha} \right) \otimes \quad \frac{\partial x_j^{\alpha}}{\partial x_{j'}^{\alpha}} dx_{j'}^{\alpha} \Bigg) \\ &= \frac{\partial x_i^{\alpha}}{\partial x_{i'}^{\alpha}} \tilde{g}_{ij} \frac{\partial x_j^{\alpha}}{\partial x_{j'}^{\alpha}} dx_{i'}^{\alpha} \otimes dx_{j'}^{\alpha} \\ &= \tilde{g}_{i'j'} dx_{i'}^{\alpha} \otimes dx_{j'}^{\alpha}. \end{aligned}$$

Hence,

$$\tilde{g}_{i'j'} = \frac{\partial x_i^{\alpha}}{\partial x_{i'}^{\alpha}} \tilde{g}_{ij} \frac{\partial x_j^{\alpha}}{\partial x_{j'}^{\alpha}}.$$

Example 2.2. Consider $\mathbb{R}^2$ with fractional polar coordinates in the domain $y > 0$, $X = (r^{\alpha} cos^{\alpha}\varphi, r^{\alpha} sin^{\alpha}\varphi)$, then

$$\frac{\partial^{\alpha} X}{\partial r^{\alpha}} = (\alpha! \cos^{\alpha} \varphi, \alpha! \sin^{\alpha} \varphi).$$

$$\frac{\partial^{\alpha} X}{\partial \varphi^{\alpha}} = \left(\alpha! e^{i\alpha\pi} r^{\alpha} \sin^{\alpha} \varphi, \alpha! r^{\alpha} \cos^{\alpha} \varphi\right).$$

$$\tilde{g}_{ij} = \begin{pmatrix} (\alpha!)^2 \left[\cos^{2\alpha} \varphi + \sin^{2\alpha} \varphi\right] & (\alpha!)^2 r^{\alpha} \left[e^{2\alpha\pi} + 1\right] \sin^{\alpha} \varphi \cos^{\alpha} \varphi \\ (\alpha!)^2 r^{\alpha} \left[e^{\alpha\pi i} + 1\right] \sin^{\alpha} \varphi \cos^{\alpha} \varphi & (\alpha!)^2 r^{2\alpha} \left[e^{2\alpha\pi i} \sin^{2\alpha} \varphi + \cos^{2\alpha} \varphi\right] \end{pmatrix}$$

We have that

$$F = (\alpha!)^2 \left[\cos^{2\alpha} \varphi + \sin^{2\alpha} \varphi\right] (dr^{\alpha})^2$$

$$+2(\alpha!)^2 r^{\alpha} \left[\left[e^{i\alpha\pi} + 1\right] \sin^{\alpha} \varphi \cos^{\alpha} \varphi\right] dr^{\alpha} d\varphi^{\alpha}$$

$$+(\alpha!)^2 r^{2\alpha} \left[e^{2i\alpha\pi} \sin^{2\alpha} \varphi + \cos^{2\alpha} \varphi\right] (d\varphi^{\alpha})^2 \}.$$

Notice that , as expected, when $\alpha = 1$, one recovers the classical formula.

$$F = (dr)^2 + r^2 (d\varphi)^2.$$

Notes

Table 1: $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.1	0.2
$\tilde{g}_{rr}$	$0.905[C^{0.2}+S^{0.2}]$	$0.843[C^{0.4}+S^{0.4}]$
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.905r^{0.1}[i+1]S^{0.2}C^{0.2}$	0
$\tilde{g}\varphi\varphi$	$0.905r^{0.2}[-S^{0.2}-C^{0.2}]$	$0.843r^{0.4}[-S^{0.4}+C^{0.4}]$
F	$0.905[C^{0.2}+S^{0.2}]dr^{0.2}$ $+1.8101r^{0.1}[i+1]S^{0.2}C^{0.2}(dr)^{.1}(d\varphi)^{.1}$ $+0.905r^{0.2}[-S^{0.2}-C^{0.2}](d\varphi)^{.2}$	$0.843[C^{0.4}+S^{0.4}]dr^{0.4}$ $+$ $0.843r^{0.4}[-S^{0.4}+C^{0.4}](d\varphi)^{0.4}$

Table 2: $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.3	0.4
$\tilde{g}_{rr}$	$0.805[C^{0.6}+S^{0.6}]$	$0.787[C^{0.8}+S^{0.8}]$
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.805r^{0.3}[[i+1]S^{0.3}C^{0.3}]$	$1.574r^{0.4}S^{0.3}C^{0.3}$
$\tilde{g}\varphi\varphi$	$0.805r^{0.6}[-S^{0.6}+C^{0.6}]$	$0.787r^{0.8}[S^{0.8}+C^{0.8}]$
F	$0.805[C^{0.6}+S^{0.6}](dr)^{0.6}$ $+1.61r^{0.3}[[i+1]S^{0.3}C^{0.3}](dr)^{.3}(d\varphi)^{.3}$ $+0.805r^{0.6}[-S^{0.6}+C^{0.6}](d\varphi)^{.6}$	$0.787[C^{0.8}+S^{0.8}](dr)^{0.8}$ $+3.148r^{0.4}S^{0.4}C^{0.4}dr^{.4}d\varphi^{.4}$ $+0.787r^{0.8}[S^{0.8}+C^{0.8}](d\varphi)^{.8}$

Remark 2.1. Let N is an m-dimensional Riemannian manifold with fractional metric tensor $\tilde{g}$,then we shall denote the fractional derivatives of the elements of tensor $\tilde{g}$ as follows:

$$\tilde{g}_{ij,k} = \frac{\partial^\alpha}{\partial x_k^\alpha}\tilde{g}_{ij},$$

and

$$\begin{aligned}\tilde{g}_{ij,kl} &= \frac{\partial^\alpha}{\partial x_l^\alpha}\frac{\partial^\alpha}{\partial x_k^\alpha}\tilde{g}_{ij} \\ &= \frac{\partial^{2\alpha}}{\partial x_l^\alpha \partial x_k^\alpha}\tilde{g}_{ij}, i,j,k,l = 1,...,n.\end{aligned}$$

Definition 2.3. Asymmetric fractional connection is called Levi-Civita fractional connection if it is compatible with metric, i.e., if it preserves the scalar product:

$$\partial^\alpha_{X^\alpha}\langle Y^\alpha, Z^\alpha\rangle = \langle \nabla^\alpha_{X^\alpha}Y^\alpha, Z^\alpha\rangle + \langle Y^\alpha, \nabla^\alpha_{X^\alpha}Z^\alpha\rangle$$

for arbitrary fractional vector fields X^α, Y^α, and Z^α.

In local coordinates Christoffel symbols of Levi-Civita fractional connection are given by:

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\partial^\alpha_j\tilde{g}_{il} + \partial^\alpha_i\tilde{g}_{lj} - \partial^\alpha_l\tilde{g}_{ij}).$$

Table 4: $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.5	0.6
$\tilde{g}_{rr}$	$0.785[C+S]$	$0.798[C^{1.2}+S^{1.2}]$
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.785r^{0.5}[i+1]S^{0.5}C^{0.5}$	0
$\tilde{g}\varphi\varphi$	$0.785r[-S+C]$	$0.798r^{1.2}[S^{1.2}+C^{1.2}]$
F	$0.785[C+S]r(dr)$ $+1.57r^{0.5}[i+1]S^{0.5}C^{0.5}(dr)^{0.5}(d\varphi)^{0.5}$ $0.785r[-S+C]d\varphi$	$0.798[C^{1.2}+S^{1.2}](dr)^{1.2}$ $+$ $+0.798r^{1.2}[S^{1.2}+C^{1.2}](d\varphi)^{1.2}$

Table 5: $C = \cos\varphi, S = \sin\varphi$

$\alpha =$	0.7	0.8
$\tilde{g}_{rr}$	$0.826[C^{1.4}+S^{1.4}]$	$0.867[C^{1.6}+S^{1.6}]$
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.826r^{0.7}[i+1]S^{0.7}C^{0.7}$	$1.734r^{0.8}S^{0.8}C^{0.8}$
$\tilde{g}\varphi\varphi$	$0.826r^{1.4}[-S^{1.4}+C^{1.4}]$	$0.867r^{1.6}[S^{1.6}+C^{1.6}]$
F	$0.826[C^{1.4}+S^{1.4}](dr)^{1.4}$ $+1.652r^{0.7}[i+1]S^{0.7}C^{0.7}(dr)^{0.7}(d\varphi)^{0.7}$ $+0.826r^{1.4}[-S^{1.4}+C^{1.4}](d\varphi)^{1.4}$	$0.867[C^{1.6}+S^{1.6}](dr)^{1.6}$ $+3.468r^{0.8}S^{0.8}C^{0.8}dr^{0.8}d\varphi^{0.8}$ $+0.867r^{1.6}[S^{1.6}+C^{1.6}](d\varphi)^{1.6}$

$\alpha =$	0.9	1
$\tilde{g}_{rr}$	$0.925[C^{1.8}+S^{1.8}]$	1
$\tilde{g}_{r\varphi}=\tilde{g}_{\varphi r}$	$0.925r^{0.9}[i+1]S^{0.9}C^{0.9}$	0
$\tilde{g}\varphi\varphi$	$0.925r^{1.8}[-S^{1.8}+C^{1.8}]$	r^2
F	$0.925[C^{1.8}+S^{1.8}](dr)^{1.8}$ $+1.85r^{0.9}[i+1]S^{0.9}C^{0.9}(dr)^{0.9}(d\varphi)^{0.9}$ $+0.925r^{1.8}[-S^{1.8}+C^{1.8}](d\varphi)^{1.8}$	$(dr)^2$ $+$ $r^2\,(d\varphi)^2$

Proof. Since

$$\partial_j^\alpha e_i = \tilde{\Gamma}_{ij}^m e_m \tag{2.1}$$

$$\tilde{\Gamma}_{ij}^m e_m e_l = (\partial_j^\alpha e_i)e_l \tag{2.2}$$

$$\begin{aligned} \tilde{\Gamma}_{ij}^m g_{ml} &= \partial_j^\alpha (e_i.e_l) - e_i(\partial_j^\alpha e_l) \\ &= \partial_j^\alpha g_{il} - \tilde{\Gamma}_{lj}^m e_m e_i \\ &= \partial_j^\alpha g_{il} - \tilde{\Gamma}_{lj}^m g_{mi}, \end{aligned} \tag{2.3}$$

then

$$\tilde{\Gamma}_{ij}^m g_{ml} + \tilde{\Gamma}_{lj}^m g_{mi} = \partial_j^\alpha g_{il}, \tag{2.4}$$

which implies that

$$\tilde{\Gamma}_{ij}^m \tilde{g}_{ml} + \tilde{\Gamma}_{lj}^m \tilde{g}_{mi} = \partial_j^\alpha \tilde{g}_{il}. \tag{2.5}$$

In this equation, the index m is a dummy, so only the indices i,j ,and l are specified. We can cyclically permute these indices to generate two more equations:

Notes

$$\tilde{\Gamma}^m_{jl}\tilde{g}_{mi} + \tilde{\Gamma}^m_{il}\tilde{g}_{mj} = \partial^\alpha_l \tilde{g}_{ji} \tag{2.6}$$

$$\tilde{\Gamma}^m_{li}\tilde{g}_{mj} + \tilde{\Gamma}^m_{ji}\tilde{g}_{ml} = \partial^\alpha_i \tilde{g}_{lj} \tag{2.7}$$

since $\tilde{\Gamma}^m_{ij} = \tilde{\Gamma}^m_{ji}$, then

Notes

$$\tilde{\Gamma}^m_{lj}\tilde{g}_{mi} + \tilde{\Gamma}^m_{il}\tilde{g}_{mj} = \partial^\alpha_l \tilde{g}_{ij} \tag{2.8}$$

$$\tilde{\Gamma}^m_{il}\tilde{g}_{mj} + \tilde{\Gamma}^m_{ij}\tilde{g}_{ml} = \partial^\alpha_i \tilde{g}_{lj}. \tag{2.9}$$

We can now add (2.5)to (2.9) and subtract (2.8)to get

$$2\tilde{\Gamma}^m_{ij}\tilde{g}_{ml} = \partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij}$$

$$2\tilde{\Gamma}^m_{ij}\tilde{g}_{ml}\tilde{g}^{kl} = \tilde{g}^{kl}(\partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij})$$

since $\tilde{g}_{ml}\tilde{g}^{kl} = \delta^k_m$, then

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\partial^\alpha_j \tilde{g}_{il} + \partial^\alpha_i \tilde{g}_{lj} - \partial^\alpha_l \tilde{g}_{ij}),$$

we can write

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\tilde{g}_{il,j} + \tilde{g}_{lj,i} - \tilde{g}_{ij,l}).$$

Example 2.2. For 2-dimential polar coordinates $X = (r^\alpha cos^\alpha\varphi, r^\alpha sin^\alpha\varphi)$.

The metric tensor and its inverse here are:

$$\tilde{g}_{ij} = \begin{pmatrix} (\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right] & (\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ (\alpha!)^2 r^\alpha\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & (\alpha!)^2 r^{2\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] \end{pmatrix}$$

$$\tilde{g}^{ij} = \begin{pmatrix} A^{-1}(\alpha!)^2 r^{2\alpha}\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] & -A^{-1}(\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ -A^{-1}(\alpha!)^2 r^\alpha\left[e^{2\alpha\pi}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & A^{-1}(\alpha!)^2\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right] \end{pmatrix}.$$

where

$$A = (\alpha!)^4 r^{2\alpha}\left\{\left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi\right]\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] - \left[e^{\alpha\pi i}+1\right]\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi\right\}$$

Therefore,

$$\partial^\alpha_r \tilde{g}_{ij} = \begin{pmatrix} 0 & (\alpha!)^3\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \\ (\alpha!)^3\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi & (\alpha!)(2\alpha)! r^\alpha\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] \end{pmatrix}$$

$$\partial^\alpha_\varphi \tilde{g}_{ij} = \begin{pmatrix} (\alpha!)(2\alpha)!\left[e^{\alpha\pi i}+1\right] cos^\alpha\varphi sin^\alpha\varphi & (\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] \\ (\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right] & (\alpha!)(2\alpha)! r^{2\alpha}\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi \end{pmatrix}.$$

Then ,

$$\tilde{\Gamma}^r_{rr} = \frac{1}{2}\tilde{g}^{rl}(\partial^\alpha_r \tilde{g}_{rl} + \partial^\alpha_r \tilde{g}_{lr} - \partial^\alpha_l \tilde{g}_{rr})$$

$$
\begin{aligned}
&= -A^{-1}(\alpha!)^5 r^\alpha \left[e^{\alpha\pi i}+1\right]^2 \sin^{2\alpha}\varphi\cos^{2\alpha}\varphi,\\
\tilde{\Gamma}^r_{r\varphi} &= \frac{1}{2}\tilde{g}^{rl}(\partial^\alpha_\varphi \tilde{g}_{rl} + \partial^\alpha_r \tilde{g}_{l\varphi} - \partial^\alpha_l \tilde{g}_{r\varphi})\\
&= 0\\
&= \tilde{\Gamma}^r_{\varphi r},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^r_{\varphi\varphi} &= \frac{1}{2}\tilde{g}^{rl}(\partial^\alpha_\varphi \tilde{g}_{\varphi l} + \partial^\alpha_\varphi \tilde{g}_{l\varphi} - \partial^\alpha_l \tilde{g}_{\varphi\varphi})\\
&= (\alpha!)^2 r^\alpha (2A)^{-1}\left\{\left[e^{2\alpha\pi i} sin^{2\alpha}\varphi + \cos^{2\alpha}\varphi\right]\left[2(\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right.\right.\\
&\left.\left. -\ (\alpha!)(2\alpha)! r^\alpha\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\right] - (\alpha!)(2\alpha)! r^\alpha\left[e^{\alpha\pi i}+1\right]\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi\right\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^\varphi_{rr} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^\alpha_\varphi \tilde{g}_{rl} + \partial^\alpha_r \tilde{g}_{lr} - \partial^\alpha_l \tilde{g}_{rr})\\
&= -(\alpha!)^2(2A)^{-1}\left\{(\alpha!)(2\alpha)! r^\alpha\left[e^{\alpha\pi i}+1\right]^2\sin^{2\alpha}\varphi\cos^{2\alpha}\varphi + \left[\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\left[e^{\alpha\pi i}+1\right]\right.\\
&\left.\times\left[(\alpha!)^3 r^\alpha\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right] + (\alpha!)^3\sin^\alpha\varphi\cos^\alpha\varphi - (\alpha!)(2\alpha)!\cos^\alpha\varphi\sin^\alpha\varphi\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^\varphi_{r\varphi} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^\alpha_\varphi \tilde{g}_{rl} + \partial^\alpha_r \tilde{g}_{l\varphi} - \partial^\alpha_l \tilde{g}_{r\varphi})\\
&= (\alpha!)^2(2A)^{-1}\left\{-(\alpha!)(2\alpha)!^2 r^\alpha\left[e^{\alpha\pi i}+1\right]\cos^{2\alpha}\varphi\sin^{2\alpha}\varphi\right.\\
&\left. +(\alpha!)(2\alpha)! r^\alpha\left[\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi\right]\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right\}\\
&= \tilde{\Gamma}^\varphi_{\varphi r},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^\varphi_{\varphi\varphi} &= \frac{1}{2}\tilde{g}^{\varphi l}(\partial^\alpha_\varphi \tilde{g}_{\varphi l} + \partial^\alpha_\varphi \tilde{g}_{l\varphi} - \partial^\alpha_l \tilde{g}_{\varphi\varphi})\\
&= (\alpha!)^2 r^{2\alpha}(2A)^{-1}\left\{-\left[e^{\alpha\pi i}+1\right]\sin^\alpha\varphi\cos^\alpha\varphi\left[2(\alpha!)^3\left[e^{\alpha\pi i}+1\right]\left[e^{\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right.\right.\\
&\left.\left. -(\alpha!)(2\alpha)!\left[e^{2\alpha\pi i}\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\right] + (\alpha!)(2\alpha)!\left[e^{\alpha\pi i}+1\right]\left[\sin^{2\alpha}\varphi+\cos^{2\alpha}\varphi\right]\sin^\alpha\varphi\cos^\alpha\varphi\right\}.
\end{aligned}
$$

IV. Fractional Curvature

Definition 3.1. *The fractional curvature $\tilde{R}$ of order α of a Riemannian manifold N is a correspondence that associates to every pair $X^\alpha, Y^\alpha \in \chi^\alpha$ a mapping $\tilde{R}(X^\alpha, Y^\alpha)\colon \chi^\alpha(N)\times\chi^\alpha(N)\to\chi^\alpha(N)$ given by*

$$\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha = \nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}Z^\alpha - \nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}Z^\alpha - \nabla^\alpha_{[X^\alpha,Y^\alpha]}Z^\alpha,$$

where $Z^\alpha \in \chi^\alpha$ and ∇^α is the fractional Riemannian connection.

Notes

Notes

Remark 3.1.

$$\begin{aligned}\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha &= \nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}Z^\alpha - \nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}Z^\alpha - \nabla^\alpha_{[X^\alpha,Y^\alpha]}Z^\alpha \\ &= -(\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{X^\alpha}Z^\alpha - \nabla^\alpha_{X^\alpha}\nabla^\alpha_{Y^\alpha}Z^\alpha - \nabla^\alpha_{[Y^\alpha,X^\alpha]}Z^\alpha) \\ &= -\tilde{R}(Y^\alpha, X^\alpha)Z^\alpha.\end{aligned}$$

Proposition 3.2. The fractional curvature $\tilde{R}$ of a Riemannian manifold has the following properties:

1. *$\tilde{R}$ is bilinear in $\chi^\alpha(N)\times\chi^\alpha(N)$, that is,*

$$\tilde{R}(fX^\alpha + gY^\alpha, Z^\alpha)W^\alpha = f\tilde{R}(X^\alpha, Z^\alpha)W^\alpha + g\tilde{R}(Y^\alpha, Z^\alpha)W^\alpha,$$

$$\tilde{R}(X^\alpha, fY^\alpha + gZ^\alpha)W^\alpha = f\tilde{R}(X^\alpha, Y^\alpha)W^\alpha + g\tilde{R}(X^\alpha, Z^\alpha)W^\alpha,$$

where $f, g \in \Im(M), X^\alpha, Y^\alpha, Z^\alpha, W^\alpha \in \chi^\alpha(N)$

2. *For any $X^\alpha, Y^\alpha \in \chi^\alpha(N), \tilde{R}(X^\alpha, Y^\alpha)$ is linear*

$$\tilde{R}(X^\alpha, Y^\alpha)(Z^\alpha + W^\alpha) = \tilde{R}(X^\alpha, Y^\alpha)Z^\alpha + \tilde{R}(X^\alpha, Y^\alpha)W^\alpha,$$

$$\tilde{R}(X^\alpha, Y^\alpha)(fZ^\alpha) = f\tilde{R}(X^\alpha, Y^\alpha)Z^\alpha,$$

where $Z^\alpha, W^\alpha \in \chi^\alpha(N)$

Proof. 1.

$$\begin{aligned}&\tilde{R}(fX^\alpha + gY^\alpha, Z^\alpha)W^\alpha \\ &= \nabla^\alpha_{fX^\alpha+gY^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha - \nabla^\alpha_{Z^\alpha}\nabla^\alpha_{fX^\alpha+gY^\alpha}W^\alpha - \nabla^\alpha_{[fX^\alpha+gY^\alpha,Z^\alpha]}W^\alpha \\ &= (f\nabla^\alpha_{X^\alpha} + g\nabla^\alpha_{Y^\alpha})\nabla^\alpha_{Z^\alpha}W^\alpha - \nabla^\alpha_{Z^\alpha}(f\nabla^\alpha_{X^\alpha}W^\alpha + g\nabla^\alpha_{Y^\alpha}W^\alpha) \\ &\quad - \nabla^\alpha_{f[X^\alpha,Z^\alpha]+g[Y^\alpha,Z^\alpha]-(Z^\alpha f)X^\alpha-(Z^\alpha g)Y^\alpha}W^\alpha \\ &= f\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha + g\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha - (Z^\alpha f)\nabla^\alpha_{X^\alpha}W^\alpha \\ &\quad - f\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha - (Z^\alpha g)\nabla^\alpha_{Y^\alpha}W^\alpha - g\nabla^\alpha_{Z^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha - f\nabla^\alpha_{[X^\alpha,Z^\alpha]}W^\alpha \\ &\quad - g\nabla^\alpha_{[Y^\alpha,Z^\alpha]}W^\alpha + (Z^\alpha f)\nabla^\alpha_{X^\alpha}W^\alpha + (Z^\alpha g)\nabla^\alpha_{Y^\alpha}W^\alpha \\ &= f(\nabla^\alpha_{X^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha - \nabla^\alpha_{Z^\alpha}\nabla^\alpha_{X^\alpha}W^\alpha - \nabla^\alpha_{[X^\alpha,Z^\alpha]}W^\alpha) \\ &\quad + g(\nabla^\alpha_{Y^\alpha}\nabla^\alpha_{Z^\alpha}W^\alpha - \nabla^\alpha_{Z^\alpha}\nabla^\alpha_{Y^\alpha}W^\alpha - \nabla^\alpha_{[Y^\alpha,Z^\alpha]}W^\alpha) \\ &= f\tilde{R}(X^\alpha, Z^\alpha)W^\alpha + gf\tilde{R}(Y^\alpha, Z^\alpha)W^\alpha.\end{aligned}$$

Also,

$$\begin{aligned}\tilde{R}(X^\alpha, fY^\alpha + gZ^\alpha)W^\alpha &= -\tilde{R}(fY^\alpha + gZ^\alpha, X^\alpha) \\ &= -f\tilde{R}(Y^\alpha, X^\alpha)W^\alpha - g\tilde{R}(Z^\alpha, X^\alpha)W^\alpha \\ &= f\tilde{R}(X^\alpha, Y^\alpha)W^\alpha + g\tilde{R}(X^\alpha, Z^\alpha)W^\alpha.\end{aligned}$$

2.

$$\tilde{R}(X^\alpha, Y^\alpha)(Z^\alpha + W^\alpha)$$

$$= \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} (Z^\alpha + W^\alpha) - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} (Z^\alpha + W^\alpha) - \nabla^\alpha_{[X^\alpha, Y^\alpha]}(Z^\alpha + W^\alpha)$$

$$= \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} Z^\alpha + \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} W^\alpha - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} Z^\alpha$$

$$- \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} W^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} W^\alpha$$

$$= (\nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} Z^\alpha - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} Z^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha)$$

$$+(\nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} W^\alpha - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} W^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} W^\alpha)$$

$$= \tilde{R}(X^\alpha, Y^\alpha) Z^\alpha + \tilde{R}(X^\alpha, Y^\alpha) W^\alpha.$$

Also,

$$\tilde{R}(X^\alpha, Y^\alpha)(f Z^\alpha)$$

$$= \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} (f Z^\alpha) - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} (f Z^\alpha) - \nabla^\alpha_{[X^\alpha, Y^\alpha]}(f Z^\alpha)$$

$$= \nabla^\alpha_{X^\alpha}((Y^\alpha f) Z^\alpha + f \nabla^\alpha_{Y^\alpha} Z^\alpha) - \nabla^\alpha_{Y^\alpha}((X^\alpha f) Z^\alpha + f \nabla^\alpha_{X^\alpha} Z^\alpha)$$

$$-(([X^\alpha, Y^\alpha] f) Z^\alpha + f \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha)$$

$$= X^\alpha (Y^\alpha f) Z^\alpha + (Y^\alpha f) \nabla^\alpha_{X^\alpha} Z^\alpha + (X^\alpha f) \nabla^\alpha_{Y^\alpha} Z^\alpha + f \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} Z^\alpha$$

$$-Y^\alpha (X^\alpha f) Z^\alpha + (X^\alpha f) \nabla^\alpha_{Y^\alpha} Z^\alpha + (Y^\alpha f) \nabla^\alpha_{X^\alpha} Z^\alpha + f \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} Z^\alpha$$

$$-([X^\alpha, Y^\alpha] f) Z^\alpha - f \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha$$

$$= ([X^\alpha, Y^\alpha] f) Z^\alpha + f(\nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} Z^\alpha - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} Z^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha) - ([X^\alpha, Y^\alpha] f) Z^\alpha$$

$$= f \tilde{R}(X^\alpha, Y^\alpha) Z^\alpha.$$

Proposition 3.3 (Bianchi Fractional Identity)**.**

$$\tilde{R}(X^\alpha, Y^\alpha) Z^\alpha + \tilde{R}(Y^\alpha, Z^\alpha) X^\alpha + \tilde{R}(Z^\alpha, X^\alpha) Y^\alpha = 0.$$

Proof.

$$\tilde{R}(X^\alpha, Y^\alpha) Z^\alpha + \tilde{R}(Y^\alpha, Z^\alpha) X^\alpha + \tilde{R}(Z^\alpha, X^\alpha) Y^\alpha$$

$$= \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Y^\alpha} Z^\alpha - \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{X^\alpha} Z^\alpha - \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha$$

$$+ \nabla^\alpha_{Y^\alpha} \nabla^\alpha_{Z^\alpha} X^\alpha - \nabla^\alpha_{Z^\alpha} \nabla^\alpha_{Y^\alpha} X^\alpha - \nabla^\alpha_{[Y^\alpha, Z^\alpha]} X^\alpha$$

$$+ \nabla^\alpha_{Z^\alpha} \nabla^\alpha_{X^\alpha} Y^\alpha - \nabla^\alpha_{X^\alpha} \nabla^\alpha_{Z^\alpha} Y^\alpha - \nabla^\alpha_{[Z^\alpha, X^\alpha]} Y^\alpha$$

$$= \nabla^\alpha_{X^\alpha}[Y^\alpha, Z^\alpha] + \nabla^\alpha_{Y^\alpha}[Z^\alpha, X^\alpha] + \nabla^\alpha_{Z^\alpha}[X^\alpha, Y^\alpha]$$

$$- \nabla^\alpha_{[X^\alpha, Y^\alpha]} Z^\alpha - \nabla^\alpha_{[Y^\alpha, Z^\alpha]} X^\alpha - \nabla^\alpha_{[Z^\alpha, X^\alpha]} Y^\alpha$$

$$= [X^\alpha, [Y^\alpha, Z^\alpha]] + [Y^\alpha, [Z^\alpha, X^\alpha]] + [Z^\alpha, [X^\alpha, Y^\alpha]] = 0.$$

Notes

In local coordinates

$$\tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha = \tilde{R}_{ijk}^l \partial_l^\alpha,$$

and

$$\begin{aligned} \tilde{R}_{ijkm} &= \left\langle \tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha, \partial_m^\alpha \right\rangle \\ &= \left\langle \tilde{R}_{ijk}^l \partial_l^\alpha, \partial_m^\alpha \right\rangle \\ &= \tilde{R}_{ijk}^l \left\langle \partial_l^\alpha, \partial_m^\alpha \right\rangle \\ &= \tilde{R}_{ijk}^l \tilde{g}_{ml}. \end{aligned}$$

The fractional Riemannian curvature tensor acts on fractional vector fields as follows:

$$\tilde{R}(X^\alpha, Y^\alpha, Z^\alpha, W^\alpha) = \left\langle \tilde{R}(X^\alpha, Y^\alpha)Z^\alpha, W^\alpha \right\rangle.$$

Proposition 3.4. *1.* $\tilde{R}_{ijkl} + \tilde{R}_{jkil} + \tilde{R}_{kijl} = 0.$

2. $\tilde{R}_{ijkl} = -\tilde{R}_{jikl}.$

3. $\tilde{R}_{ijkl} = -\tilde{R}_{ijlk}.$

4. $\tilde{R}_{ijkl} = \tilde{R}_{klij}.$

Proof. 1. is just the Bianchi fractional identity again.

2.

$$\begin{aligned} \tilde{R}_{ijkl} &= \left\langle \tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= \left\langle -\tilde{R}(\partial_j^\alpha, \partial_i^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= -\left\langle \tilde{R}(\partial_j^\alpha, \partial_i^\alpha)\partial_k^\alpha, \partial_l^\alpha \right\rangle \\ &= -\tilde{R}_{jikl}. \end{aligned}$$

3. is equivalent to $R_{ijkk} = 0$, whose proof follows:

$$\begin{aligned} \tilde{R}_{ijkk} &= \left\langle \tilde{R}(\partial_i^\alpha, \partial_j^\alpha)\partial_k^\alpha, \partial_k^\alpha \right\rangle \\ &= \left\langle \nabla_{\partial_i^\alpha}^\alpha \nabla_{\partial_j^\alpha}^\alpha \partial_k^\alpha - \nabla_{\partial_j^\alpha}^\alpha \nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha - \nabla_{[\partial_i^\alpha, \partial_j^\alpha]}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle, \end{aligned}$$

but

$$\left\langle \nabla_{\partial_j^\alpha}^\alpha \nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle = \partial_j^\alpha \left\langle \nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle - \left\langle \nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha, \nabla_{\partial_j^\alpha}^\alpha \partial_k^\alpha \right\rangle,$$

and

$$\left\langle \nabla_{[\partial_i^\alpha, \partial_j^\alpha]}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle = \frac{1}{2} \left[\partial_i^\alpha, \partial_j^\alpha\right] \left\langle \partial_k^\alpha, \partial_k^\alpha \right\rangle,$$

then

$$\tilde{R}_{ijkk} = \partial_j^\alpha \left\langle \nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle - \partial_i^\alpha \left\langle \nabla_{\partial_j^\alpha}^\alpha \partial_k^\alpha, \partial_k^\alpha \right\rangle + \frac{1}{2} \left[\partial_i^\alpha, \partial_j^\alpha\right] \left\langle \partial_k^\alpha, \partial_k^\alpha \right\rangle$$

$$= \frac{1}{2}\partial_j^\alpha(\partial_i^\alpha\langle\partial_k^\alpha,\partial_k^\alpha\rangle) - \frac{1}{2}\partial_i^\alpha\left(\partial_j^\alpha\langle\partial_k^\alpha,\partial_k^\alpha\rangle\right) + \frac{1}{2}\left[\partial_i^\alpha,\partial_j^\alpha\right]\langle\partial_k^\alpha,\partial_k^\alpha\rangle$$

$$= -\frac{1}{2}\left[\partial_i^\alpha,\partial_j^\alpha\right]\langle\partial_k^\alpha,\partial_k^\alpha\rangle + \frac{1}{2}\left[\partial_i^\alpha,\partial_j^\alpha\right]\langle\partial_k^\alpha,\partial_k^\alpha\rangle = 0.$$

4. By Bianchi fractional identity we have

$$\tilde{R}_{ijkl} + \tilde{R}_{jkil} + \tilde{R}_{kijl} = 0$$

$$\tilde{R}_{jkli} + \tilde{R}_{klji} + \tilde{R}_{ljki} = 0$$

$$\tilde{R}_{klij} + \tilde{R}_{likj} + \tilde{R}_{iklj} = 0$$

$$\tilde{R}_{lijk} + \tilde{R}_{ijlk} + \tilde{R}_{jlik} = 0$$

summing the equations above, we obtain

$$2\tilde{R}_{kijl} + 2\tilde{R}_{ljki} = 0,$$

then

$$\tilde{R}_{kijl} = -\tilde{R}_{ljki} = \tilde{R}_{jlki}.$$

Proposition 3.5. *The following expression holds*

$$2\tilde{R}_{ijkm} = \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} - 2\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im}\tilde{g}_{rs} + 2\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm}\tilde{g}_{rs}.$$

Proof. From the definition of the Christoffel symbols, $\nabla^\alpha_{\partial_i^\alpha}\partial_j^\alpha = \tilde{\Gamma}^k_{ij}\partial_k^\alpha$,

$$\begin{aligned} 2\left\langle\nabla^\alpha_{\partial_i^\alpha}\partial_j^\alpha,\partial_m^\alpha\right\rangle &= 2\left\langle\tilde{\Gamma}^k_{ij}\partial_k^\alpha,\partial_m^\alpha\right\rangle \\ &= 2\tilde{\Gamma}^k_{ij}\tilde{g}_{mk} \\ &= \tilde{g}_{im,j} + \tilde{g}_{jm,i} - \tilde{g}_{ij,m}, \end{aligned}$$

an appropriate rearrangement of the indices yields the following expression:

$$\begin{aligned} 2\left\langle\nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\partial_m^\alpha\right\rangle &= 2\left\langle\tilde{\Gamma}^i_{jk}\partial_i^\alpha,\partial_m^\alpha\right\rangle \\ &= 2\tilde{\Gamma}^i_{jk}\tilde{g}_{im} \\ &= \tilde{g}_{jm,k} + \tilde{g}_{km,j} - \tilde{g}_{jk,m}. \qquad (3.1) \end{aligned}$$

$$\partial_i^\alpha < \nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\partial_m^\alpha > = < \nabla^\alpha_{\partial_i^\alpha}\nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\partial_m^\alpha > + < \nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\nabla^\alpha_{\partial_i^\alpha}\partial_m^\alpha >$$

whence, by (3.1)

$$\begin{aligned} 2\left\langle\nabla^\alpha_{\partial_i^\alpha}\nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\partial_m^\alpha\right\rangle + 2\left\langle\nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\nabla^\alpha_{\partial_i^\alpha}\partial_m^\alpha\right\rangle &= 2\partial_i^\alpha\left\langle\nabla^\alpha_{\partial_j^\alpha}\partial_k^\alpha,\partial_m^\alpha\right\rangle \\ &= \partial_i^\alpha\left(\tilde{g}_{im}g^{il}(\partial_k^\alpha g_{jl} + \partial_j^\alpha g_{kl} - \partial_l^\alpha g_{jk})\right) \\ &= \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi}. \qquad (3.2) \end{aligned}$$

By switching i and j we also have that

$$2\left\langle\nabla^\alpha_{\partial_j^\alpha}\nabla^\alpha_{\partial_i^\alpha}\partial_k^\alpha,\partial_m^\alpha\right\rangle + 2\left\langle\nabla^\alpha_{\partial_i^\alpha}\partial_k^\alpha,\nabla^\alpha_{\partial_j^\alpha}\partial_m^\alpha\right\rangle = \tilde{g}_{im,kj} + \tilde{g}_{km,ij} - \tilde{g}_{ik,mj}. \quad (3.3)$$

Notes

Notes

Combining (3.2) and (3.3) yields

$$2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{i}} \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k}, \partial^{\alpha}_{m} \right\rangle - 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{j}} \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}, \partial^{\alpha}_{m} \right\rangle =$$

$$\tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj}$$

$$-2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{m} \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{m} \right\rangle.$$

By definition

$$\tilde{R}\left(\partial^{\alpha}_{i}, \partial^{\alpha}_{j}\right) \partial^{\alpha}_{k} = \nabla^{\alpha}_{\partial^{\alpha}_{i}} \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k} - \nabla^{\alpha}_{\partial^{\alpha}_{j}} \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}$$

whence

$$\begin{aligned} 2\tilde{R}_{ijkm} &= 2\left\langle \tilde{R}(\partial^{\alpha}_{i}, \partial^{\alpha}_{j})\partial^{\alpha}_{k}, \partial^{\alpha}_{m} \right\rangle \\ &= 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{i}} \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k}, \partial^{\alpha}_{m} \right\rangle - 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{j}} \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}, \partial^{\alpha}_{m} \right\rangle, \end{aligned}$$

so we have proven that

$$2\tilde{R}_{ijkm} =$$

$$\tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj}$$

$$-2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{m} \right\rangle + 2\left\langle \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{m} \right\rangle.$$

By the definition of the Christoffels,

$$\begin{aligned} \left\langle \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{m} \right\rangle &= \left\langle \tilde{\Gamma}^{r}_{ik} \partial^{\alpha}_{r}, \tilde{\Gamma}^{s}_{jm} \partial^{\alpha}_{s} \right\rangle \\ &= \tilde{\Gamma}^{r}_{ik} \tilde{\Gamma}^{s}_{jm} \left\langle \partial^{\alpha}_{r}, \partial^{\alpha}_{s} \right\rangle \\ &= \tilde{\Gamma}^{r}_{ik} \tilde{\Gamma}^{s}_{jm} \tilde{g}_{rs}, \\ \left\langle \nabla^{\alpha}_{\partial^{\alpha}_{j}} \partial^{\alpha}_{k}, \nabla^{\alpha}_{\partial^{\alpha}_{i}} \partial^{\alpha}_{m} \right\rangle &= \left\langle \tilde{\Gamma}^{r}_{jk} \partial^{\alpha}_{r}, \tilde{\Gamma}^{s}_{im} \partial^{\alpha}_{s} \right\rangle \\ &= \tilde{\Gamma}^{r}_{jk} \tilde{\Gamma}^{s}_{im} \left\langle \partial^{\alpha}_{r}, \partial^{\alpha}_{s} \right\rangle \\ &= \tilde{\Gamma}^{r}_{jk} \tilde{\Gamma}^{s}_{im} \tilde{g}_{rs}, \end{aligned}$$

then

$$2\tilde{R}_{ijkm} = \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} - 2\tilde{\Gamma}^{r}_{jk} \tilde{\Gamma}^{s}_{im} \tilde{g}_{rs} + 2\tilde{\Gamma}^{r}_{ik} \tilde{\Gamma}^{s}_{jm} \tilde{g}_{rs}.$$

Remark 3.6. *If $\alpha = 1$, then*

$$2R_{ijkm} = g_{jm,ki} + g_{km,ji} - g_{jk,mi} - g_{im,kj} - g_{km,ij} + g_{ik,mj} - 2\Gamma^{r}_{jk} \Gamma^{s}_{im} g_{rs} + 2\Gamma^{r}_{ik} \Gamma^{s}_{jm} g_{rs}.$$

Since $g_{km,ji} = g_{km,ij}$, then

$$2R_{ijkm} = g_{jm,ki} - g_{jk,mi} - g_{im,kj} + g_{ik,mj} - 2\Gamma^{r}_{jk} \Gamma^{s}_{im} g_{rs} + 2\Gamma^{r}_{ik} \Gamma^{s}_{jm} g_{rs},$$

then

$$2R_{ijkm} = \tilde{g}_{jm,ki} + \tilde{g}_{km,ji} - \tilde{g}_{jk,mi} - \tilde{g}_{im,kj} - \tilde{g}_{km,ij} + \tilde{g}_{ik,mj} - 2\Gamma^r_{jk}\Gamma^s_{im}g_{rs} + 2\Gamma^r_{ik}\Gamma^s_{jm}g_{rs}.$$

Remark 3.7. *For any pair of fractional tangent vectors* $X^\alpha, Y^\alpha \in T^\alpha_p N$ *we shall denote with* $\tilde{\Gamma}(X^\alpha, Y^\alpha)$ *the following fractional vector in* $T^\alpha_p N$*:*

$$\tilde{\Gamma}(X^\alpha, Y^\alpha) = \tilde{\Gamma}^k_{ij} X^\alpha_i Y^\alpha_j \partial^\alpha_k.$$

Proposition 3.8. *The following expressions hold for any pair* $X^\alpha, Y^\alpha \in T^\alpha_p N$*:*

$$\begin{aligned} 2\tilde{R}\left(X^\alpha, Y^\alpha, Y^\alpha, X^\alpha\right) & \\ &= \partial^\alpha_i\left(\tilde{g}_{im}g^{il}\right)\left(\partial^\alpha_k g_{jl} + \partial^\alpha_j g_{kl} - \partial^\alpha_l g_{jk}\right) + \tilde{g}_{im}g^{il}\left(\partial^\alpha_i\partial^\alpha_k g_{jl} + \partial^\alpha_i\partial^\alpha_j g_{kl} - \partial^\alpha_i\partial^\alpha_l g_{jk}\right) \\ &\quad -\partial^\alpha_j\left(\tilde{g}_{jm}g^{jl}\right)\left(\partial^\alpha_k g_{il} + \partial^\alpha_i g_{kl} - \partial^\alpha_l g_{ik}\right) - \tilde{g}_{jm}g^{jl}\left(\partial^\alpha_j\partial^\alpha_k g_{il} + \partial^\alpha_j\partial^\alpha_i g_{kl} - \partial^\alpha_j\partial^\alpha_l g_{ik}\right) \\ &\quad +2 \parallel \tilde{\Gamma}\left(X^\alpha, Y^\alpha\right) \parallel^2 - 2\left\langle \tilde{\Gamma}\left(X^\alpha, X^\alpha\right), \tilde{\Gamma}\left(Y^\alpha, Y^\alpha\right)\right\rangle. \end{aligned}$$

Proof. Since

$$\begin{aligned} \tilde{g}_{rs}X^\alpha_i Y^\alpha_j Y^\alpha_k X^\alpha_m \tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jm} &= \left\langle X^\alpha_i Y^\alpha_k \tilde{\Gamma}^r_{ik}\partial^\alpha_r, Y^\alpha_j X^\alpha_m \tilde{\Gamma}^s_{jm}\partial^\alpha_s\right\rangle \\ &= \left\langle \tilde{\Gamma}(X^\alpha, Y^\alpha), \tilde{\Gamma}(X^\alpha, Y^\alpha)\right\rangle = \parallel \tilde{\Gamma}(X^\alpha, Y^\alpha) \parallel^2, \end{aligned}$$

and

$$\begin{aligned} \tilde{g}_{rs}X^\alpha_i Y^\alpha_j Y^\alpha_k X^\alpha_m \tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{im} &= \left\langle X^\alpha_i X^\alpha_m \tilde{\Gamma}^s_{im}\partial^\alpha_s, Y^\alpha_j Y^\alpha_k \tilde{\Gamma}^r_{jk}\partial^\alpha_r\right\rangle \\ &= \left\langle \tilde{\Gamma}(X^\alpha, X^\alpha), \tilde{\Gamma}(Y^\alpha, Y^\alpha)\right\rangle, \end{aligned}$$

This completes the proof.

Remark 3.9. *If* $\alpha = 1$*, then*

$$\begin{aligned} 2R(X, Y, Y, X) &= g_{jm,ki} - g_{jk,mi} - g_{im,kj} + g_{ik,mj} \\ &+2 \parallel \Gamma(X, Y) \parallel^2 - 2\left\langle \Gamma(X, X), \Gamma(Y, Y)\right\rangle. \end{aligned}$$

References Références Referencias

1. D. Albu, M. Neamtu, D. Opris, The geometry of fractional osculator bundle of higher order and applications. Anal. St. Univ."Al.I.Cuza" Iasi, Proceedings of the Conf. on Diff. Geometry: Lagrange and Hamilton spaces, Sept.3-8, Iasi, (2007) 21–31.
2. M. Axtell, M.E. Bise, Fractional Calculus Application in Control Systems. IEEE, Nat, Aerospace and Electronics Conf., New York, (1990) 563–566.

Notes

3. Ding, Yong, and Xinfeng Wu, Fractional integrals on product manifolds. Potential Analysis, 304 (2009), 371–383.
4. Guy Jumarie. Riemann-Christoffel Tensor in Differntial Geometry of Fractional Order Application to Fractal Space-Time. Fractals, 211 (2013).
5. Ion Doru Albu and Dumitru Opris. The geometry of fractional tangent bundle and applications. Balkan Society of Geometers (2009).
6. A. Kılı, cman, and W. Saleh. Note on the Fractional Mittag-Leffler Functions by Applying the Modified Riemann-Liouville Derivatives. Boletim da Sociedade Paranaense de Matematica, textbf40 2022,1-16
7. K. M. Kolwankar. Studies of Fractal structures and Processes Using Methods of Fractional Calculus; http://arxiv.org/abs/chaodyn/9811008.
8. Mehdi Delkhosh. Introduction of Derivatives and Integrals of Fractional order and Its Applications. Applied Mathematics and physics, 14 (2013), 103–119.
9. K. S. Miller and B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons, Inc.; New York (1993).
10. K. B. Oldham and J. Spanier. The Fractional Calculus. Academic Press; New York (1974).
11. I. Podlubny. Fractional Order System and Fractional Order Controllers. UEF- 03-94, Inst. Exp. Phys, Slovak Acad. Sci., Kosice (1994).
12. B. Ross. In Fractional Calculus and Its Applications: Lecture Notes in Mathematics; Springer- Verlag; New York; 457 (1975).
13. W. Saleh and A. Kılı cman. Some Remarks on Geometry of Fractional Calculus. Applied Mathematics & Information Sciences (AMIS), 9 3(2015), 1265–1275.
14. W. Saleh and A. Kılı, cman. Some Inequalities For Generalized s-Convex Functions, JP Journal of Geometry and Topology, 17 (2015), 63–82.
15. I. N. Sneddon. Special functions of mathematical physics and chemistry. Oliver & Boyd, (1956).
16. M. H. Tavassoli, A. Tavassoli and M. R. Ostad Rahimi. The geometric and physical interpretation of fractional order derivatives of polynomial functions. Differential Geometry-Dynamical Systems, 15(2013), 93–104.
17. A. Kılı, cman and B. Fisher. Some Results on the Composition of Singular Distribution. Applied Mathematics & Information Sciences (AMIS), 53 (2011), 597–608.
18. A. Kılı, cman and W. Saleh. Notions of Generalized s-Convex Functions on Fractal Sets. Journal of Inequalities and Applications, 20151(2015).

19. A. Kılı, cman and W. Saleh. Some Generalized Hermite-Hadamard Type Integral Inequalities for Generalized s-Convex Functions on Fractal Sets. Advances in Difference Equations, 20151(2015).
20. B. Fisher, B. Jolevska-Tuneska and A. Kılı, cman. On defining the incomplete gamma function. Integral Transforms Spec. Funct. 14 4(2003), 293–299.
21. B. Fisher, A. Kılı, cman and D. Nicholas. On the beta function and the neutrix product of distributions. Integral Transforms Spec. Funct., 7 1-2(1998), 35–42.
22. B. Fisher and A. Kılı, cman. Results on the Gamma function. Applied Mathematics & Information Sciences (AMIS), 62 (2012), 173–176.

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Certain Oberhettinger's Integrals Associated with Product of General Polynomials and Incomplete H- Function

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Abstract- In this paper, we developed Oberhettinger's integral formulas containing the product of general polynomials and incomplete H-functions. These integral formulas are very useful to obtain the Mellin transform of various simpler special functions. The Mellin transform of special functions find their applications in mathematical statistics, number theory and the theory of asymptotic expansions. The main findings of the present work are very useful in solving the problems arising in digital signals, image processing, finance and ship target recognition by sonar system and radar signals.

Keywords: *incomplete H-functions, gamma functions, improper integrals, general polynomials.*

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6. Brychkov, Y.A. Handbook of Special Functions, Derivatives, Integrals, Series and Other formulas, CRC Press, Boca Raton, FL, USA,2008.

Certain Oberhettinger's Integrals Associated with Product of General Polynomials and Incomplete H- Function

Harshita Garg

Abstract- In this paper, we developed Oberhettinger's integral formulas containing the product of general polynomials and incomplete H-functions. These integral formulas are very useful to obtain the Mellin transform of various simpler special functions. The Mellin transform of special functions find their applications in mathematical statistics, number theory and the theory of asymptotic expansions. The main findings of the present work are very useful in solving the problems arising in digital signals, image processing, finance and ship target recognition by sonar system and radar signals.

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I. Introduction

Several integral formulas containing generalized special functions have been explored by many authors [1-5]. These integrals play a focal role in solving scientific and engineering problems. Here we develop Oberhettinger's integral formulas containing the product of general polynomials and incomplete H- functions. Many authors established several unified integral formulas containing a various kind of special functions [6-8]. The findings of this work are general in nature and very useful in science, engineering and finance. Here we find some special cases by specializing the parameters of general polynomials and incomplete H-functions (for example, Fox's H-function, Incomplete Fox-Wright functions, Fox-Wright functions and incomplete generalized hypergeometric functions) and also listed few known results. The main results of this work are very useful in solving the problems arising in digital signals, image processing, finance and ship target recognition by sonar system and radar signals [9-12].

The incomplete Gamma functions $\Gamma(\xi, z)$ and $\Upsilon(\xi, z)$ are defined as following:

$$\Upsilon(\xi, z) = \int_0^z t^{\xi-1} e^{-t}\, dt \qquad (\Re(\xi) > 0\,;\, z \geq 0) \qquad \ldots (1)$$

And

$$\Gamma(\xi, z) = \int_z^\infty t^{\xi-1} e^{-t}\, dt \qquad (z \geq 0\,;\, \Re(\xi) > 0 \text{ when } z = 0) \qquad \ldots (2)$$

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Respectively, satisfy the decomposition formula given by:

$$\Gamma(\xi, z) + \Upsilon(\xi, z) = \Gamma(\xi) \qquad (\Re(\xi) > 0) \qquad \ldots (3)$$

The condition which we have used on the parameter z in and anywhere else in this paper is unrestricted of $\Re(x)$ (x∈ C).

The incomplete generalized hypergeometric functions ${}_e\Gamma_f$ and ${}_e\gamma_f$ are defined by Srivastava et al. [13] in terms of incomplete Gamma functions $\Gamma(s, z)$ and $\Upsilon(s, z)$ as following:

$${}_e\Gamma_f\left[\begin{matrix}(u_1, z), u_2, \ldots, u_e; \\ v_1, \ldots, v_f;\end{matrix} x\right] = \frac{\prod_{j=1}^{f}\Gamma(v_j)}{\prod_{j=1}^{e}\Gamma(u_j)} \sum_{\ell=0}^{\infty} \frac{\Gamma(u_1+\ell, z)\prod_{j=2}^{e}\Gamma(u_j+\ell)}{\prod_{j=1}^{f}\Gamma(v_j+\ell)} \frac{x^{\ell}}{\ell!}$$

$$= \frac{1}{2\pi\iota} \frac{\prod_{j=1}^{f}\Gamma(v_j)}{\prod_{j=1}^{e}\Gamma(u_j)} \int_{\pounds} \frac{\Gamma(u_1+s, z)\prod_{j=2}^{e}\Gamma(u_j+s)}{\prod_{j=1}^{f}\Gamma(v_j+s)} \Gamma(-s)(-x)^s\, ds \qquad \ldots (4)$$

(|arg (-x)|< π,

And

$${}_e\gamma_f\left[\begin{matrix}(u_1, z), u_2, \ldots, u_e; \\ v_1, \ldots, v_f;\end{matrix} x\right] = \frac{\prod_{j=1}^{f}\Gamma(v_j)}{\prod_{j=1}^{e}\Gamma(u_j)} \sum_{\ell=0}^{\infty} \frac{\gamma(u_1+\ell, z)\prod_{j=2}^{e}\Gamma(u_j+\ell)}{\prod_{j=1}^{f}\Gamma(v_j+\ell)} \frac{x^{\ell}}{\ell!}$$

$$= \frac{1}{2\pi\iota} \frac{\prod_{j=1}^{f}\Gamma(v_j)}{\prod_{j=1}^{e}\Gamma(u_j)} \int_{\pounds} \frac{\gamma(u_1+s, z)\prod_{j=2}^{e}\Gamma(u_j+s)}{\prod_{j=1}^{f}\Gamma(v_j+s)} \Gamma(-s)(-x)^s\, ds \qquad \ldots (5)$$

(|arg (-x)|< π,

Where $\mathscr{L}$ is the Mellin- Barnes type contour having ꞇ-í∞ as the starting point and ꞇ+í∞ (ꞇ ϵ R) as the end point with the usual indentations to separate a set of poles from another of the integrand in each and every case.

The incomplete H- function $\gamma_{e,f}^{c,d}(x)$ and $\Gamma_{e,f}^{c,d}(x)$ are introduced by Srivastava et. Al. [14] as following:

Notes

$$\Gamma_{e,f}^{c,d}(x)=\Gamma_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e}\\(v_j,V_j)_{1,f}\end{matrix}\right.\right]=\Gamma_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1,z),(u_2,U_2),...,(u_e,U_e)\\(v_1,V_1),(v_2,V_2),...,(v_f,V_f)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\iota}\int_{£} f(s,z)x^{-s}ds \quad \ldots (6)$$

Where $f(s,z)=\dfrac{\Gamma(1-u_1-U_1s,z)\prod_{j=1}^{c}\Gamma(v_j+V_js)\prod_{j=2}^{d}\Gamma(1-u_j-U_js)}{\prod_{j=c+1}^{f}\Gamma(1-v_j-V_js)\prod_{j=d+1}^{e}\Gamma(u_j+U_js)}$

And

$$\gamma_{e,f}^{c,d}(x)=\gamma_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e}\\(v_j,V_j)_{1,f}\end{matrix}\right.\right]=\gamma_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1,z),(u_2,U_2),...,(u_e,U_e)\\(v_1,V_1),(v_2,V_2),...,(v_f,V_f)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\iota}\int_{£} F(s,z)x^{-s}ds \quad \ldots (7)$$

Where

$$F(s,z)=\frac{\gamma(1-u_1-U_1s,z)\prod_{j=1}^{c}\Gamma(v_j+V_js)\prod_{j=2}^{d}\Gamma(1-u_j-U_js)}{\prod_{j=c+1}^{f}\Gamma(1-v_j-V_js)\prod_{j=d+1}^{e}\Gamma(u_j+U_js)}$$

The incomplete H- functions $\gamma_{e,f}^{c,d}(x)$ and $\Gamma_{e,f}^{c,d}(x)$ in (6) and (7) respectively exists for all $z\geq0$ under the same set of conditions and same set of contour stated in the articles presented by Kilbas et al. [15], Mathai and Saxena [16] and Mathai et al. [17].

Some special cases of incomplete H-function are as following:

(i)

If we take z=0 in (6), then the incomplete H- function $\Gamma_{e,f}^{c,d}(x)$ reduces to Fox's H-function [18].

$$\Gamma_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1,0),(u_2,U_2),...,(u_e,U_e)\\(v_1,V_1),(v_2,V_2),...,(v_f,V_f)\end{matrix}\right.\right]=H_{e,f}^{c,d}\left[x\left|\begin{matrix}(u_1,U_1),(u_2,U_2),...,(u_e,U_e)\\(v_1,V_1),(v_2,V_2),...,(v_f,V_f)\end{matrix}\right.\right] \quad \ldots (8)$$

(ii) If we take c=1, d=e and replace f by f+1 and take suitable parameter, then the function (6) and (7) reduces to incomplete Fox-Wright function ${}_e\Psi_f^{(\Gamma)}$ and ${}_e\Psi_f^{(\gamma)}$ (for details see [14]).

Ref

17. Mathai, A.M., Saxena, R.K., Haubold, H.J. The H Function: Theory and Applications; Springer; New York, USA, 2009.

$$\Gamma_{e,f+1}^{1,e}\left[-x\left|\begin{matrix}(1-u_1,U_1,z),(1-u_j,U_j)_{2,e}\\(0,1),(1-v_j,V_j)_{1,f}\end{matrix}\right.\right]={}_e\Psi_f^{(\Gamma)}\left[\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e};\\(v_j,V_J)_{1,f};\end{matrix}\;x\right] \quad \ldots (9)$$

And

$$\gamma_{e,f+1}^{1,e}\left[-x\left|\begin{matrix}(1-u_1,U_1,z),(1-u_j,U_j)_{2,e}\\(0,1),(1-v_j,V_j)_{1,f}\end{matrix}\right.\right]={}_e\Psi_f^{(\gamma)}\left[\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e};\\(v_j,V_J)_{1,f};\end{matrix}\;x\right] \quad \ldots (10)$$

(iii)

If we take z=0 in (9), then incomplete Fox-Wright function ${}_e\Psi_f^{(\Gamma)}$ reduces to well-known Fox-Wright function ${}_e\Psi_f$ (for details see [18]).

$${}_e\Psi_f^{(\Gamma)}\left[\begin{matrix}(u_1,U_1,0),(u_j,U_j)_{2,e};\\(v_j,V_j)_{1,f};\end{matrix}\;x\right]={}_e\Psi_f\left[\left|\begin{matrix}(u_1,U_1),(u_j,U_j)_{2,e};\\(v_j,V_j)_{1,f};\end{matrix}\;x\right.\right] \quad \ldots (11)$$

(iv) If we take $U_j = V_k = 1$ (j=1,…,e, k=1,…,f) in (9) and (10), then incomplete Fox-Wright function reduces to the incomplete generalized hypergeometric functions ${}_e\gamma_f$ and ${}_e\Gamma_f$ (see [13]).

$${}_e\Psi_f^{(\Gamma)}\left[\begin{matrix}(u_1,1,z),(u_j,1)_{2,e};\\(v_j,1)_{1,f};\end{matrix}\;x\right]={}_e\Gamma_f\left[\left|\begin{matrix}(u_1,z),u_2,\ldots,u_e;\\v_1,\ldots v_f;\end{matrix}\;x\right.\right] \quad \ldots (12)$$

And

$${}_e\Psi_f^{(\gamma)}\left[\begin{matrix}(u_1,1,z),(u_j,1)_{2,e};\\(v_j,1)_{1,f};\end{matrix}\;x\right]={}_e\gamma_f\left[\left|\begin{matrix}(u_1,z),u_2,\ldots,u_e;\\v_1,\ldots v_f;\end{matrix}\;x\right.\right] \quad \ldots (13)$$

The general polynomials are defined by Srivastava [19] as following:

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[t_1,\ldots,t_s]=\sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}.\ A[n_1,\alpha_1;\ldots;n_s,\alpha_s]t_1^{\alpha_1}\ldots t_s^{\alpha_s} \quad \ldots (14)$$

Where $n_i = 0, 1, 2\ldots\ \forall i=(1,\ldots,s)$; $m_1,\ldots,m_s$ are arbitrary positive integers and the coefficients $A[n_1,\alpha_1;\ldots;n_s,\alpha_s]$ are arbitrary constants, real or complex.

In this paper we use the following integral formula [20],

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}dy=2\tau k^{-\tau}\left(\frac{k}{2}\right)^{\delta}\frac{\Gamma(2\delta)\Gamma(\tau-\delta)}{\Gamma(1+\tau+\delta)} \quad \ldots (15)$$

Ref

18. Srivastava, H.M., Gupta, K.C., Goyal, S.P. The H- Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, India, 1982.

II. The Main Integral Formula

In this section we obtain Oberhettinger's integral formulas containing product of incomplete H-function and general polynomials. These integral formulas are very useful to obtain the Mellin transform of several simpler special functions.

Notes

Theorem 1: If τ, δ ϵ C with 0 ≤ R (δ) < R (τ) and y>0, then the following integral formula holds:

$$\int_0^\infty y^{\delta\text{-}1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau} .S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},\ldots,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]$$

$$.\Gamma_{e,f}^{c,d}\left[\frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e}\\(v_j,V_j)_{1,f}\end{matrix}\right.\right]dy=2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]}\ldots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-\mathrm{n}_1)_{\mathrm{m}_1\alpha_1}}{\alpha_1!}\ldots\frac{(-\mathrm{n}_s)_{\mathrm{m}_s\alpha_s}}{\alpha_s!}.A[\mathrm{n}_1,\alpha_1;\ldots;\mathrm{n}_s,\alpha_s]\mathrm{t}_1^{\alpha_1}\ldots\mathrm{t}_s^{\alpha_s}$$

$$.\Gamma_{e+2,f+2}^{c,d+2}\left[\frac{x}{k}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e},\left(-\tau-\sum_{i=1}^{s}\alpha_i,1\right),\left(1-\tau-\sum_{i=1}^{s}\alpha_i+\delta,1\right)\\(v_j,V_j)_{1,f},\left(1-\tau-\sum_{i=1}^{s}\alpha_i,1\right),\left(-\tau-\sum_{i=1}^{s}\alpha_i-\delta,1\right)\end{matrix}\right.\right] \quad \ldots (16)$$

All the conditions of incomplete H- function $\Gamma_{e,f}^{c,d}(x)$ in (6) are satisfied.

Proof: Let the left hand side of the assertion (16) is denoted by Δ and using (6) and (14) in the left hand side of (16), we get

$$\Delta=\int_0^\infty y^{\delta\text{-}1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}..\sum_{\alpha_1=0}^{[n_1/m_1]}\ldots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-\mathrm{n}_1)_{\mathrm{m}_1\alpha_1}}{\alpha_1!}\ldots\frac{(-\mathrm{n}_s)_{\mathrm{m}_s\alpha_s}}{\alpha_s!}A[\mathrm{n}_1,\alpha_1;\ldots;\mathrm{n}_s,\alpha_s]$$

$$.\left(\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{\alpha_1},\ldots,\left(\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{\alpha_s}\frac{1}{2\pi\iota}\int_L\left(\frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{-\xi}f(\xi,z)d\xi dy$$

Where f(ξ , z) is defined in (6).

Now changing the order of summation, integration and contour integral involved therein (which is permissible under the stated conditions), we get

$$\Delta = \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1 !} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s !} A[n_1,\alpha_1;\ldots;n_s,\alpha_s].t_1^{\alpha_1}\ldots.t_s^{\alpha_s}.\frac{1}{2\pi\iota}\int_L f(\xi,z)x^{-\xi}$$

$$.\int_0^\infty y^{\delta\text{-}1}\left(y+k+\sqrt{y^2+2ky}\right)^{-(\tau+\alpha_1+\ldots+\alpha_s-\xi)} dyd\xi$$

Now by using (15) in above integral and reinterpreting it in the form of incomplete H-function $\Gamma_{e,f}^{c,d}(x)$, we get the result (16).

Theorem- 2

If τ, δ Є C with 0 <R (δ) < R (τ) and y > 0, then the following integral formula holds:

$$\int_0^\infty y^{\delta\text{-}1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}.S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},\ldots,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right].$$

$$.\gamma_{e,f}^{c,d}\left[\frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e}\\(v_j,V_j)_{1,f}\end{matrix}\right.\right]dy = 2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1 !} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s !} .A[n_1,\alpha_1;\ldots;n_s,\alpha_s]t_1^{\alpha_1}\ldots t_s^{\alpha_s}$$

$$.\gamma_{e+2,f+2}^{c,d+2}\left[\frac{x}{k}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e},\left(-\tau-\sum_{i=1}^{s}\alpha_i,1\right),\left(1-\tau-\sum_{i=1}^{s}\alpha_i+\delta,1\right)\\(v_j,V_j)_{1,f},\left(1-\tau-\sum_{i=1}^{s}\alpha_i,1\right),\left(-\tau-\sum_{i=1}^{s}\alpha_i-\delta,1\right)\end{matrix}\right.\right] \quad \ldots (17)$$

All the conditions of incomplete H- function $\gamma_{e,f}^{c,d}(x)$ in (7) are satisfied.

Proof: Let the left hand side of the assertion (17) is denoted by Δ and using (7) and (14) in the left hand side of (17), we get

$$\Delta = \int_0^\infty y^{\delta\text{-}1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}..\sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1 !} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s !} A[n_1,\alpha_1;\ldots;n_s,\alpha_s]$$

$$.\left(\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{\alpha_1},\ldots,\left(\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{\alpha_s}\frac{1}{2\pi\iota}\int_L\left(\frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\right)^{-\xi}F(\xi,z)d\xi dy$$

Notes

Notes

Where F (ξ, z) is defined in (7)

Now changing the order of summation, integration and contour integral involved therein (which is permissible under the stated conditions), we get

$$\Delta = \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!} A[n_1,\alpha_1;\ldots;n_s,\alpha_s].t_1^{\alpha_1}\ldots t_s^{\alpha_s}.\frac{1}{2\pi\iota}\int_L F(\xi,z)x^{-\xi}$$

$$.\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-(\tau+\alpha_1+\ldots+\alpha_s-\xi)} dy d\xi$$

Now by using (15) in above integral and reinterpreting it in the form of incomplete H-function $\gamma_{e,f}^{c,d}(x)$, we get the result (17).

III. Special Cases

In this section, we obtain some interesting special cases of main results (16) and (17).

(i) If τ, δ Є C with 0 < R(δ) < R(τ) and y > 0 and incomplete H-function reduces into incomplete hypergeometric function with the help of (12) and (13), then the following integral formula occurs:

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau} .S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},\ldots,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]$$

$$._e\Gamma_f\left[\begin{matrix}(u_1,z),u_2,\ldots,u_e; \\ v_1,\ldots,v_f;\end{matrix} \frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]dy = 2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!} .A[n_1,\alpha_1;\ldots;n_s,\alpha_s]t_1^{\alpha_1}\ldots t_s^{\alpha_s}$$

$$._{e+2}\Gamma_{f+2}\left[\frac{x}{k}\left|\begin{matrix}(u_1,z),u_2,\ldots,u_e,-\tau-\sum_{i=1}^{s}\alpha_i,1-\tau-\sum_{i=1}^{s}\alpha_i+\delta \\ v_1,\ldots,v_f,1-\tau-\sum_{i=1}^{s}\alpha_i,-\tau-\sum_{i=1}^{s}\alpha_i-\delta\end{matrix}\right.\right] \quad \ldots (18)$$

And

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau} .S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},\ldots,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]$$

$$\cdot_e\gamma_f\left[\begin{matrix}(u_1,z),u_2,...,u_e;\\ v_1,...,v_f;\end{matrix}\ \frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]dy=2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}.A[n_1,\alpha_1;...;n_s,\alpha_s]t_1^{\alpha_1}...t_s^{\alpha_s}$$

$$\cdot_{e+2}\gamma_{f+2}\left[\frac{x}{k}\left|\begin{matrix}(u_1,z),u_2,...,u_e,-\tau-\sum_{i=1}^{s}\alpha_i,1-\tau-\sum_{i=1}^{s}\alpha_i+\delta\\ v_1,...,v_f,1-\tau-\sum_{i=1}^{s}\alpha_i,-\tau-\sum_{i=1}^{s}\alpha_i-\delta\end{matrix}\right.\right] \quad \ldots (19)$$

Given that both integrals exist.

Proof: Again if we take $U_j=V_j=1$ in (16) and (17), we get the required results.

(ii) If τ, δ Є C with $0<R(\delta)<R(\tau)$ and $y>0$ and incomplete H-function reduces into incomplete Wright functions, then the following integral formula occurs:

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}.S_{n_1,...,n_s}^{m_1,...,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},...,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]$$

$$\cdot_e\Psi_f^{(\Gamma)}\left[\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e};\\ (v_j,V_j)_{1,f};\end{matrix}\ \frac{x}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]dy=2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}.A[n_1,\alpha_1;...;n_s,\alpha_s]t_1^{\alpha_1}...t_s^{\alpha_s}$$

$$\cdot_{e+2}\Psi_{f+2}^{(\Gamma)}\left[\frac{x}{k}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e},(1+\tau+\sum_{i=1}^{s}\alpha_i,1),(\tau+\sum_{i=1}^{s}\alpha_i-\delta,1)\\ (v_j,V_j)_{1,f},(\tau+\sum_{i=1}^{s}\alpha_i,1),(1+\tau+\sum_{i=1}^{s}\alpha_i+\delta,1)\end{matrix}\right.\right] \quad \ldots (20)$$

And

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}.S_{n_1,...,n_s}^{m_1,...,m_s}\left[\frac{t_1}{\left(y+k+\sqrt{y^2+2ky}\right)},...,\frac{t_s}{\left(y+k+\sqrt{y^2+2ky}\right)}\right]$$

Notes

$$._e\Psi_f^{(\gamma)}\left[\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e}; & x \\ (v_j,V_j)_{1,f}; & \overline{\left(y+k+\sqrt{y^2+2ky}\right)}\end{matrix}\right]dy = 2^{1-\delta}k^{\delta-\tau-\sum_{i=1}^{s}\alpha_i}\Gamma(2\delta)$$

$$.\sum_{\alpha_1=0}^{[n_1/m_1]}\cdots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\cdots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}.A[n_1,\alpha_1;\ldots;n_s,\alpha_s]t_1^{\alpha_1}\ldots t_s^{\alpha_s}$$

$$._{e+2}\Psi_{f+2}^{(\gamma)}\left[\frac{x}{k}\left|\begin{matrix}(u_1,U_1,z),(u_j,U_j)_{2,e},(1+\tau+\sum_{i=1}^{s}\alpha_i,1),(\tau+\sum_{i=1}^{s}\alpha_i-\delta,1) \\ (v_j,V_j)_{1,f},(\tau+\sum_{i=1}^{s}\alpha_i,1),(1+\tau+\sum_{i=1}^{s}\alpha_i+\delta,1)\end{matrix}\right.\right] \quad \ldots (21)$$

Given that each member of assertion (20) and (21) are exist.

Proof: with the help of (9) and (10), we get the above results.

(iii) If τ, δ ∈ C with $0 < R(\delta) < R(\tau)$, $y > 0$ and incomplete H-function reduces into Fox-Wright generalized hypergeometric function and general polynomials reduces into unity, then the following integral formula holds:

$$\int_0^\infty y^{\delta-1}\left(y+k+\sqrt{y^2+2ky}\right)^{-\tau}\ ._e\Psi_f\left[\begin{matrix}(u_j,U_j)_{1,e}; & x \\ (v_j,V_j)_{1,f}; & \overline{\left(y+k+\sqrt{y^2+2ky}\right)}\end{matrix}\right]dy = 2^{1-\delta}k^{\delta-\tau}\Gamma(2\delta)$$

$$._{e+2}\Psi_{f+2}\left[\frac{x}{k}\left|\begin{matrix}(u_j,U_j)_{1,e},(1+\tau,1),(\tau-\delta,1) \\ (v_j,V_j)_{1,f},(\tau,1),(1+\tau+\delta,1)\end{matrix}\right.\right] \quad \ldots (22)$$

Given that each member of assertion (22) is exist.

Proof: Let general polynomials reduces into unity and with the help of (11), we get the above result.

IV. Conclusions

In this paper, we obtained some engrossing integrals containing the product of incomplete H-function and general polynomials, which are expressed in terms of incomplete H-functions. We have also given some special cases by specializing the parameters of general polynomials and incomplete H-functions (Incomplete Fox-Wright functions, incomplete hypergeometric functions, Fox - Wright generalized hypergeometric functions). These results are general in nature and very useful in science, engineering and finance.

Notes

References Références Referencias

1. Bansal, M.K., Choi, J. A Note on Pathway Fractional Integral Formulas Associated with the Incomplete H- Functions. Int. J. Appl. Comput. Math. 2019,5,133. [CrossRef]
2. Bansal, M.K., Jolly, N., Jain, R. Kumar, D. An integral operator involving generalized Mittag-Leffler function and associated fractional calculus results J.Anal.2019, 27,727-740. [Cross Ref]
3. Chaurasia, V.B.L., Kumar, D. The integration of certain product involving special functions. Sci. Ser. A Math. Sci. 2010, 19, 7-12.
4. Chaurasia, V.B.L., Singh, J. Certain integral properties pertaining to special functions. Sci. Ser. A Math. Sci. 2010, 19, 1-6.
5. Kumar, D., Ayant, F. A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function. Kragujevac J. Math. 2020, 44, 539-550.
6. Brychkov, Y.A. Handbook of Special Functions, Derivatives, Integrals, Series and Other formulas, CRC Press, Boca Raton, FL, USA,2008.
7. Garg, M., Mittal, S. On a new unified integral. Proc. Indian Acad. Sci. 2004, 114, 99-101. [Cross Ref]
8. Srivastava, H.M., Exton, H. A generalization of the Weber-Schafheitlin integral. J. Reine Angew. Math.1979, 309, 1-6.
9. Epstein, B. Some Applications of the Mellin Transform in statistics. Ann.Math.Stat. 1948, 19, 370-379.[CrossRef]
10. Yao K. Spherically Invariant Random Processes, Theory and Applications. In communications, information and Network security, The Springer International series in Engineering and computer science(Communications and Information Theory); Bhagava, V.K., Poor, H.V., Tarokh V., Yoon, S., Eds.; Springer: Berlin/Heidelberg, Germany, 2003; Volume 712, pp. 315-332.
11. Yao,K., Simon, M.K., Biglieri,E. A Unified Theory on Wireless Communications Fading Statistics based on SIRP. In Proceedings of the fifth IEEE Workshop on Signal Processing Advances in Wireless Communications, Lisbon, Portugal, 11-14, July 2004, pp. 134-139.
12. Chaurasia, V.B.L., Kumar, D. Applications of special functions, and SIRP in Wireless Communication Fading statistics. Glob. J. Sci. Front.Res.2010, 10, 14-19.
13. Srivastava, H.M., Chaudhary, M.A., Agarwal, R.P. The Incomplete Pochhamer symbols and Their Applications to Hypergeometric and Related Functions. Integral Transform. Spec. Funct. 2012, 23, 659-683.[CrossRef]
14. Srivastava, H.M., Saxena, R.K., Parmar, R.K. Some families of the Incomplete Hfunctions and the Incomplete H -functions and Associated Integral Transforms and Operators of Fractional Caalculus with Applications. Russ.J.Math.Phys. 2018, 25, 116-138.[CrossRef]
15. Kilbas, A.A., Srivastava, H.M. Trujillo, J.J. Theory and Applications of Fractional Differential Equations; (North-Holland Mathematical Studies); Elsevier(North Holland) Science Publishers: Amsterdam, The Netherland' London, UK, New York, NY, USA, 2006, P.204.
16. Mathai, A.M., Saxena, R.K. The H function with Applications in Statistics other Disciplines; Wiley Eastern, New Delhi, India; Wiley Halsted: New York, NY, USA, 1978.

Notes

Notes

17. Mathai, A.M., Saxena, R.K., Haubold, H.J. The H Function: Theory and Applications; Springer; New York, NY, USA, 2009.
18. Srivastava, H.M., Gupta, K.C., Goyal, S.P. The H- Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, India, 1982.
19. Srivastava, H.M., A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math. 1985, 117, 183-191. Oberhettinger, F. Tables of Mellin Transforms, Springer, New York, NY, USA, 1974.

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4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

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Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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