Non-Decaying Initial Data
The General Service Readiness

Highlights

Positive Integers as Entries
Pathway Fractional Integral Operator

Discovering Thoughts, Inventing Future

VOLUME 22    ISSUE 2    VERSION 1.0

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Yoga, Meditation and Mental Health well-being during Covid-19 Pandemic

By Priyanka Verma & Sheela Misra

University of North Carolina

Abstract- Yoga and meditation have been playing vital roles in our holistic wellbeing and attaining our spiritual goals since ancient time. During second wave of Covid-19 and lock-down its importance became more significant and visible around the globe. Many patients have easily recovered with the help of their boosted immunity by doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation etc) and could keep themselves stress-free. During and post pandemic maintaining mental health is a great challenge. It is very difficult to be relaxed, peaceful and healthy due to increased level of anxiety, stress and depression as a consequence of loss of health, lives, jobs, migration, inaccessibility to health education and other basic facilities, changes in life styles and so on. Yoga can help us to stay calm, manage our health and anxiety without any extra intervention if understood well and made a part of our daily routine. For the young adults it becomes even more important to do yoga and meditation because they are the future of India.

Keywords: yoga, meditation, stress, mental health, ordinal logistic regression analysis.


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Yoga, Meditation and Mental Health well-being during Covid-19 Pandemic

Priyanka Verma & Sheela Misra

Abstract- Yoga and meditation have been playing vital roles in our holistic wellbeing and attaining our spiritual goals since ancient time. During second wave of Covid-19 and lock-down its importance became more significant and visible around the globe. Many patients have easily recovered with the help of their boosted immunity by doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation etc) and could keep themselves stress-free. During and post pandemic maintaining mental health is a great challenge. It is very difficult to be relaxed, peaceful and healthy due to increased level of anxiety, stress and depression as a consequence of loss of health, lives, jobs, migration, inaccessibility to health education and other basic facilities, changes in life styles and so on. Yoga can help us to stay calm, manage our health and anxiety without any extra intervention if understood well and made a part of our daily routine. For the young adults it becomes even more important to do yoga and meditation because they are the future of India.

We have conducted a survey among the young adults age group (18 – 35 years) of Uttar Pradesh, India by a questionnaire tool with the help of Google form. Questionnaire includes questions related to socio demographic status, yoga, meditation, and mental health well-being during Covid-19 pandemic. This study is an exploratory study based on primary data.

The objective of this survey is to know the proportion of young adults who are doing yoga and meditation regularly, their satisfaction level with it, their changing behaviour, what the benefits of it that they experienced are, and its graphical representation. The ordinal logistic regression analysis carried out to know the satisfaction level of independent variables (Gender, Age, Education Qualification, and Native Area) and satisfaction level as dependent variable.

Keywords: yoga, meditation, stress, mental health, ordinal logistic regression analysis.

I. Introduction

Yoga and meditation has been playing a vital role to fit our physical and mental health since ancient time. During second wave of Covid-19 and lock-down its importance become more than compare to previous. Many patients have easily recovered with the help of habit of doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation). During this scenario good mental health is very important to survive because level of anxiety, stress and depression growing very fast. Yoga can help us to stay calm, manage our blood pressure and anxiety. For the young adults it becomes more essential to do yoga and meditation because they are the future of country.

The pandemic has been difficult for everyone. According to the Centers for Disease Control and Prevention, one in five Americans report struggling with mental or behavioral health issues associated with COVID-19, including anxiety, depression,
increased substance use, and suicidal thoughts. The highest burden of distress has been reported by the younger generation (ages 18-29) and minority communities. [1]

Keeping all these points in mind we have conducted online survey among young adults group with the help of Google form including questions related to yoga and meditation, to know the present scenario regarding habit of yoga, its benefits and satisfaction level of young adults towards mental health well-being.

II. Literature Review

Mental health issues increased during COVID-19 pandemic. Yoga and meditation can help in reducing mental stress and improving psychological wellbeing. The frequency of practice is positively associated with a higher level of mental wellbeing in case of both yoga and meditation, with daily practice having the highest wellbeing scores. [8]

Change in eating and sleeping pattern during the pandemic was significantly higher in people who did not practice yoga and meditation, and it was least in those practicing both. A large proportion of study subjects reported a change in relationship with family members during the COVID-19 pandemic. [8]

A significant effect of duration of practice was found on illness perception, and wellbeing related measures. Long term practitioners reported higher personal control and lower illness concern in contracting COVID-19 than the mid-term or beginner group. The improved physiological functions are believed to reduce stress, anxiety, depression, and enhance overall well-being. [9]

III. Objective

- To find out the proportion of young adults of Uttar Pradesh who are doing yoga and meditation regularly.
- To find out satisfaction level of young adult, how much yoga and meditation is helpful for their mental health well-being.
- To find out Behavior change in young adult during Covid-19 pandemic regarding yoga and meditation.
- To find out benefits of yoga and meditation for young adults (18-35 years).
- To find satisfaction level (dependent variable), ordinal logistic regression analysis used for independent variables (Gender, Age, Education Qualification, and Native Area) and dependent variable (satisfaction level).

IV. Methodology

- A survey was conducted among the young adults (18-35 years) of Uttar Pradesh, India with the help of Google form using Convenient Sampling Technique (Non-probability sampling technique).
- There were total 11 questions in the Questionnaire including sociodemographic variables (like age, education qualification, gender, district, native area), out of them 4 questions related to yoga and meditation was close-ended and 1 question based on likert-scale (1-5) related to satisfaction level and one question was open-ended (benefits of yoga and meditation).
- Total 203 responses were used for analysis.
- Analysis is based on Frequency, Cross-tab, Ordinal logistic Regression Analysis.
- For analysis SPSS version.21 and Excel software used.
This study is an exploratory study based on primary data.
Data come from the following Districts of Uttar Pradesh, India:
- Maximum data collected from Lucknow district.

V. Interpretation

Table 1: Demographic details of 203 respondent

<table>
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<tr>
<th>Variables</th>
<th>Frequency (%)</th>
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<tr>
<td><strong>Gender</strong></td>
<td></td>
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<tr>
<td>Female</td>
<td>54.7</td>
</tr>
<tr>
<td>Male</td>
<td>45.3</td>
</tr>
<tr>
<td>Others</td>
<td>0</td>
</tr>
<tr>
<td><strong>Age</strong></td>
<td></td>
</tr>
<tr>
<td>18 - 24 Years</td>
<td>61.1</td>
</tr>
<tr>
<td>24 - 30 Years</td>
<td>30.5</td>
</tr>
<tr>
<td>30 - 35 Years</td>
<td>8.4</td>
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<tr>
<td><strong>Education qualification</strong></td>
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</tr>
<tr>
<td>Higher than postgraduate</td>
<td>8.4</td>
</tr>
<tr>
<td>Intermediate</td>
<td>9.4</td>
</tr>
<tr>
<td>Other</td>
<td>2.5</td>
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<tr>
<td>Postgraduate</td>
<td>42.9</td>
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<tr>
<td>Undergraduate</td>
<td>36.9</td>
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<tr>
<td><strong>Districts</strong></td>
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<tr>
<td>Lucknow</td>
<td>65.5</td>
</tr>
<tr>
<td>Other than Lucknow</td>
<td>34.5</td>
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<tr>
<td><strong>Native area</strong></td>
<td></td>
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<tr>
<td>Rural</td>
<td>28.1</td>
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<tr>
<td>Urban</td>
<td>71.9</td>
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Fig. 1: Out of the 203 respondent 56 percent (114) of respondent said yes, they are doing yoga and meditation and 44 percent of respondent said No.

Fig. 2: Out of the 114 respondent there are 36 percent of respondents who have started yoga and meditation during Covid-19 pandemic and 64 percent of respondent started before Covid-19 pandemic.
Fig. 3: There are 89 percent of respondent who have doing yoga and meditation in the Morning, 11 percent of respondent are doing it in the Evening and only1 percent of respondent doing yoga and meditation in the Afternoon.

Fig. 4: There are 59 percent respondent who are doing yoga and meditation daily, 38 percent of respondent doing weekly and 4 percent respondent doing in monthly.
For model fitting we have taken-

Ho: there is no significant different between baseline model to final model

The significant value is 0.255 which is > 0.05 so we accept null hypothesis and conclude that there is no significant different between baseline model to final model

For goodness of fit we have taken-

Ho: the observed data is having goodness of fit with the fitted model

Significant value is 0.806 which is > 0.05 so we accept the null hypothesis and conclude that the observed data is having goodness of fit with the fitted model.

From Pseudo R-Square

The Nagel kerke value should be 0.7 but here the value is 0.099, means 0.099 variation proportion of variance the independent variable (Gender, Age, Education Qualification, and Native Area) is explaining on the dependent variable (satisfaction level).

It means more independent variable should be used.

VI. Ordinal Logistic Regression Analysis

We have apply Ordinal logistic Regression on the dependent variable (satisfaction level) which is ordinal variable and independent variable (Gender, Age, Education Qualification, and Native Area) which is categorical variable.

To know the satisfaction level of independent variables that yoga and meditation is helpful to their mental health

We find the following results

- Total respondents are 114, who are doing yoga and meditation.
- Out of them 57 percent of respondent who are very satisfied that yoga and meditation is helpful to their mental health well-being and 33 percent are moderately satisfied.
- 43 percent are Male respondent and 57 percent are Female respondent.

For model fitting we have taken-

Ho: there is no significant different between baseline model to final model

The significant value is 0.255 which is > 0.05 so we accept null hypothesis and conclude that there is no significant different between baseline model to final model

For goodness of fit we have taken-

Ho: the observed data is having goodness of fit with the fitted model

Significant value is 0.806 which is > 0.05 so we accept the null hypothesis and conclude that the observed data is having goodness of fit with the fitted model.

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The Nagel kerke value should be 0.7 but here the value is 0.099, means 0.099 variation proportion of variance the independent variable (Gender, Age, Education Qualification, and Native Area) is explaining on the dependent variable (satisfaction level).

It means more independent variable should be used.
We can see the satisfaction level of Gender, Age, Education Qualification, and Native Area

For gender null hypothesis was -

**Ho:** There is no significant difference between the gender towards satisfaction level.

The estimate for male is positive which indicate that male have more positive satisfaction level than female, the sig value for male is 0.132, > 0.05 we accept null hypothesis and conclude that male and female does not have any significant difference in level of satisfaction.

Similarly, we had taken null hypothesis for other independent variables also.

**Interpretation from Table 2**

- Estimate value of male is positive means male is more positive satisfied than female or we can say female have less satisfied than male. Compare to female male are not significant.
- Students of Undergraduate, Postgraduate, and Higher than Postgraduate having more positive satisfaction level than other education level but Students of Intermediate having less satisfaction than other education level. Compare to students of other education level, Students of Intermediate, Undergraduate, Postgraduate, and Higher than Postgraduate are not significant.
• Students of Urban area having more positive satisfaction level than rural area. Compare to rural area students, students of urban area are not significant.
• Students of (30-35) age group having more satisfaction than other age groups. Compare to (30-35) age group other age groups are not significant.

**Fig. 6:** Male is having 1.764 times more satisfaction than female

**Key-Points**

<table>
<thead>
<tr>
<th>Data in Percentage Total 144 (Doing yoga and meditation)</th>
<th>yoga started during Covid-19 (36%)114</th>
<th>Yoga in morning</th>
<th>Yoga on daily</th>
<th>Yoga on weekly</th>
<th>Yoga is helpful for Mental health well-being, Very satisfied</th>
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**Benefits of yoga and meditation experience by respondents**

- Mind stability, Positive thinking, Healthy lifestyle, Make calm, increase creativity and thinking level, Enhance focus and concentration power, Manage anxiety, stress, and anger, Make us happier, Boost our immune system, Feel energised and active
• Clarity in thoughts, powerful connection with almighty, Help in weight gain/loss, Help in breathing and good health, increase blood flow, flexibility, Good for physical and mental health, feel relax
• Increase confidence level, fitness of body, improve sleep quality, Healthy and fresh start of the day, Helps to stay away from disease (like fever, cough), Helps in time management, balance life, Improve body posture and stamina.
• Increase overall metabolism of the body, organs of body work properly, Improve decision-making ability, less frequent headache, Maintain discipline in morning routine, Increase body awareness
• It is helpful for recovering from diseases like brain cancer, kidney, stress, heavyweight gain, headache and many more diseases
• Good for heart health, respiratory, digestion system, Keep glowing skin, getting to know one own self, back pain relief.

VII. Conclusion

• Out of 203 young adult respondent more than half (56%) respondent are doing yoga and meditation. So we need to generate awareness among young adults about importance and benefits of doing yoga and meditation on regular basis, so that we can get much better data for the same.
• From analysis we find positive result that out of the 114 respondents there are 36 % respondents who have start doing yoga and meditation during Covid-19 pandemic out of them 44% are male and 56% are female, and 64 percent have started before the pandemic. This indicates the behavior change in young adults during pandemic.
• Young adults prefer to do yoga and meditation in the morning (89%) and 11 % prefer to do it in evening time. Means morning and evening are the good time to do yoga and meditation.
• 59 % respondent doing yoga and meditation on Daily basis. This is also a good result.
• 57 % respondent are very satisfied that yoga and meditation is helpful to their mental health well-being. This shows that yoga and meditation are actually helpful to our mental health well-being.

• From ordinal logistic regression analysis
  • Since the significance value of Model Fitting information is 0.255 which is >0.05, so we are unable to fit a good model.
  • Since the significance value of Goodness of fit is 0.806 which is >0.05, so we accept the null hypothesis and conclude that the observed data is having goodness of fit with the fitted model.
  • Further we can go with more independent variable for better result because Pseudo R-square should be >=0.7 and data results 0.099
  • Male and female does not have any significant difference in level of satisfaction, from Parameter Estimate.
  • Male have more positive satisfaction level than female.
  • With the help of exponential value chart we conclude that male is having 1.80398842 times more satisfaction than female.
Students of Undergraduate, Postgraduate, and Higher than Postgraduate having more positive satisfaction level than other education level but Students of Intermediate having less satisfaction than other education level.

Students of Urban area having more positive satisfaction level than rural area.

Students of (30-35) age group having more satisfaction than other age groups.

In the same way we can conduct survey for other states also.

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Life Span of Solutions for a Time Fractional Reaction-Diffusion Equation with Non-Decaying Initial Data

By Takefumi Igarashi
Nihon University

Abstract- We consider the Cauchy problem of time fractional reaction-diffusion equation

$$\partial_t^\alpha u = \Delta u + u^p \quad \text{in} \quad \mathbb{R}^n \quad (n \geq 1),$$

where $0 < \alpha < 1$, $p > 1$ and $\partial_t^\alpha$ denotes the Caputo time fractional derivative of order $\alpha$. The initial condition $u_0$ is assumed to be nonnegative and bounded continuous function. For the non-decaying initial data at space infinity, we show that the positive solution blows up in finite time and give the estimate of the life span of positive solutions. It is also given blow-up time of the solutions when the initial data attain its maximum at space infinity.

Keywords: life span, fractional diffusion equation, cauchy problem, non-decaying initial data, blow-up.

GJSFR-F Classification: MSC 2020: 35B44, 35K15, 35R11, 26A33, 35K57

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Life Span of Solutions for a Time Fractional Reaction-Diffusion Equation with Non-Decaying Initial Data

Takefumi Igarashi

Abstract: We consider the Cauchy problem of time fractional reaction-diffusion equation
\[
\partial_t^\alpha u = \Delta u + u^p \quad \text{in } \mathbb{R}^n \quad (n \geq 1),
\]
where \(0 < \alpha < 1, p > 1\) and \(\partial_t^\alpha\) denotes the Caputo time fractional derivative of order \(\alpha\). The initial condition \(u_0\) is assumed to be nonnegative and bounded continuous function. For the non-decaying initial data at space infinity, we show that the positive solution blows up in finite time and give the estimate of the life span of positive solutions. It is also given blow-up time of the solutions when the initial data attain its maximum at space infinity.

Keywords: life span, fractional diffusion equation, Cauchy problem, non-decaying initial data, blow-up.

I. Introduction

We study the Cauchy problem for a time fractional reaction-diffusion equation
\[
\begin{cases}
\partial_t^\alpha u = \Delta u + u^p, & \text{in } \mathbb{R}^n, \quad t > 0, \\
u(x, 0) = u_0(x) \geq 0, & \text{in } \mathbb{R}^n,
\end{cases}
\]
(1.1)

where \(n \geq 1, 0 < \alpha < 1, p > 1\), \(u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), and \(\partial_t^\alpha\) denotes the Caputo time fractional derivative of order \(\alpha\) defined by
\[
\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x, s) ds, \quad 0 < \alpha < 1.
\]
(1.2)

Here, \(\Gamma(\cdot)\) is the Gamma function. Moreover, the Caputo time fractional derivative (1.2) is related to the Riemann-Liouville derivative by
\[
\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (u(x, s) - u_0(x)) ds.
\]
(1.3)

In this paper, we show that every solution of (1.1) blows up in finite time with the non-decaying initial data at space infinity, and also present the estimate on the life span of the solutions for (1.1). Then, we define the life span (or blow-up time) \(T^*\) as
\[
T^* = \sup\{T > 0; \text{ there exists a mild solution } u \text{ of (1.1) in } C([0, T], C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))\},
\]

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where the definition of “mild solution” and the local “existence” of a mild solution are described in section 2. If $T^* = \infty$, the solution is global. On the other hand, if $T^* < \infty$, then the solution is not global in time in the sense that it blows up at $t = T^*$ such as

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = \infty.$$ 

A brief review on the blow-up and global existence results obtained for Cauchy problem (1.1) is given below:

(A) Kirane et al. [12] obtained that the following results.
- If $1 < p \leq 1 + 2\alpha/\{\alpha n + 2(1 - \alpha)\}$, then (1.1) admits no global weak nonnegative solutions other than the trivial one.
- Let $u$ be a local solution to (1.1). Then, there exists a constant $C > 0$ such that
  $$\liminf_{|x| \to \infty} |x|^\frac{\alpha}{p-1} u_0(x) \leq C T^{\frac{\alpha}{1-p}},$$
  
  where $0 < t \leq T < +\infty$.
- Suppose that (1.1) has a nontrivial global nonnegative weak solution. Then, there is a constant $K > 0$ such that
  $$\liminf_{|x| \to \infty} |x|^{\frac{\alpha}{p-1}} u_0(x) \leq K.$$

(B) When $u_0 \in C_0(\mathbb{R}^n) := \{u \in C(\mathbb{R}^n)| \lim_{|x| \to \infty} u(x) = 0\}$, the following results were proved by Zhang and Sun [28] and Zhang et al. [29]:
- If $1 < p < 1 + 2/n$, then any nontrivial positive solution of (1.1) blows up in finite time.
- If $p \geq 1 + 2/n$ and $\|u_0\|_{L^q_c}$ is sufficiently small, where $q_c = n(p-1)/2$, then (1.1) has a global solution.
- If $\int_{\mathbb{R}^n} u_0(x) \chi(x) \, dx > 1$, where
  $$\chi(x) = \left(\int_{\mathbb{R}^n} e^{-\sqrt{n^2 + |x|^2}} \, dx\right)^{-1} e^{-\sqrt{n^2 + |x|^2}},$$
  
  then the solutions of (1.1) blow up in finite time.

(C) The following results were also obtained in Ahmad et al. [1] when $u_0 \in C_0(\mathbb{R}^n)$:
- If $p \geq 1 + 2/n$ and $\|u_0\|_{L^1} + \|u_0\|_{L^\infty} \leq \epsilon_0$ with some $\epsilon_0 > 0$, there exists $s > p$ such that (1.1) admits a global solution with $u \in L^\infty([0, \infty), L^\infty(\mathbb{R}^n)) \cap L^\infty([0, \infty), L^s(\mathbb{R}^n))$. Furthermore, for all $\delta > 0$,
  $$\max \left\{1 - \frac{p-1}{\alpha p}, 2-p\right\} < \delta < \min \left\{1, \frac{n(p-1)}{2p}\right\},$$
  
  then
  $$\|u(t)\|_{L^s} \leq C(t + 1)^{-\frac{(1-\delta)\alpha}{p-1}}, \quad t \geq 0.$$
In addition, if $pn < 2s$, or $n > 2$ and $pn \geq 2s$

\[
\max \left\{ \frac{1}{p^2}, \frac{p-1}{p^2}, \frac{\alpha}{p}, \sqrt{\frac{\alpha}{p^2}} \right\} < \alpha < 1,
\]

then $u \in L^\infty([0, \infty), L^\infty(\mathbb{R}^n))$,

\[
\|u(t)\|_{L^\infty} \leq C(t + 1)^{-\sigma}, \quad t \geq 0,
\]

for some constant $\sigma > 0$.

- If $Z_0 := \int_{\mathbb{R}^n} u_0(x) \chi(x) dx > 2^{1/(p-1)}$, then the solutions of (1.1) blow up in finite time, and the estimate of the blow-up time is

\[
T^* \leq \left[ \frac{\log \left( 1 - 2^p Z_0^{1-p} \right)}{2(1-p)\Gamma(\alpha+1)} \right]^{1/\alpha}.
\]

Several studies have been made on the life span of solutions. The results are given below:

(A) Gui and Wang [6] and Mukai et al. [17] considered

\[
\left\{ \begin{array}{ll}
\partial_t v = \Delta v^m + v^{p_1}, & x \in \mathbb{R}^n, \quad t > 0, \\
v(x, 0) = v_0(x) \geq 0, & x \in \mathbb{R}^n,
\end{array} \right.
\]

for $m = 1$ and $m > 1$, respectively, and proved the following life span results when an initial datum takes the form $v_0(x) = \lambda \phi(x)$, where $\lambda > 0$ and $\phi(x)$ is a bounded continuous in $\mathbb{R}^n$:

- If $\| \phi \|_{L^\infty(\mathbb{R}^n)} = \phi(0) > 0$, then there exists $\lambda_1 \geq 0$ such that $T^* < \infty$ for any $\lambda > \lambda_1$, and

\[
\lim_{\lambda \to \infty} \lambda^{p_1-1}T^* = \frac{1}{p_1-1} \phi(0)^{-p_1-1}.
\]

- If $\| \phi \|_{L^\infty(\mathbb{R}^n)} = \lim_{|x| \to \infty} \phi(x) = \phi_\infty > 0$, then $T^* < \infty$ for any $\lambda > 0$, and

\[
\lim_{\lambda \to 0} \lambda^{p_1-1}T^* = \frac{1}{p_1-1} \phi_\infty^{-(p_1-1)}.
\]

(B) Giga and Umeda [4, 5], Seki [19] and Seki et al. [20] showed the solution of (1.4) blows up at minimal blow-up time (see Remark 1 below); that is,

\[
T^* = \frac{1}{p_1-1} \| v_0 \|_{L^\infty(\mathbb{R}^n)}^{1-p_1}.
\]
if and only if there exists a sequence \( \{x_j\} \subset \mathbb{R}^n \) such that

\[
\lim_{j \to \infty} |x_j| = \infty \quad \text{and} \quad \lim_{j \to \infty} v_0(x + x_j) = \|v_0\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \quad \text{a.e. in } \mathbb{R}^n.
\]

**Remark 1.** Applying the comparison principal to (1.4), it follows that

\[
T^* \geq \frac{1}{p_1 - 1} \|v_0\|_{\mathbb{L}^{1-p_1}(\mathbb{R}^n)}^{1-p_1}.
\]  

(1.6)

So, when (1.5) holds, we call the time \( T^* \) the “minimal blow-up time” and the solution \( v \) to (1.4) a “blow-up solution with the minimal blow-up time”.

(C) Maingé [15] considered (1.4) for \( \max(0, 1 - 2/n) < m < 1 \), and proved if the initial data satisfies

\[
v_0(x) \geq c_0 \max \{0, 1 - |x - x_0|^2 \phi_0\}^s,
\]

where \( x_0 \in \mathbb{R}^n, s > 2, \) and \( c_0^{p_1 - m} > C_b \phi_0 \) for some constant \( C_b > 0 \) and \( \phi_0 > 0 \), then the solution of (1.4) blows up in finite time, and

\[
\frac{1}{p_1 - 1} \|v_0\|_{\mathbb{L}^{1-p_1}(\mathbb{R}^n)} \leq T^* \leq \max \left\{ \frac{d_1}{c_0^{p_1 - m}}, \frac{d_2}{c_0^{p_1 - m} - C_b \phi_0} \right\},
\]

where \( d_1 > 0 \) and \( d_2 > 0 \).

(D) Yamauchi [18, 25, 26, 27] considered (1.4) for \( m = 1 \), the author [8, 9] for \( \max(0, 1-2/n) < m < 1 \) or \( 1 < m < p_1 \), and showed the following life span results:

(a) Let \( n \geq 2 \). For some \( \xi \in S^{n-1} \) and \( \delta > 0 \), we set the conic neighborhood \( D_\xi(\delta) \):

\[
D_\xi(\delta) = \left\{ \eta \in \mathbb{R}^n \setminus \{0\}; \left| \xi - \frac{\eta}{|\eta|} \right| < \delta \right\},
\]

(1.7)

and set \( S_\xi(\delta) = D_\xi(\delta) \cap S^{n-1} \). Define

\[
N_\infty := \sup_{\xi \in S^{n-1}, \delta > 0} \left\{ \text{ess.inf}_{\theta \in S_\xi(\delta)} \left( \liminf_{r \to +\infty} v_0(r\theta) \right) \right\},
\]

where \( r = |x|, \theta = x/r \).

- If \( N_\infty > 0 \), then the solution of (1.4) blows up in finite time, and

\[
\frac{1}{p_1 - 1} \|v_0\|_{\mathbb{L}^{1-p_1}(\mathbb{R}^n)} \leq T^* \leq \frac{1}{p_1 - 1} N_\infty^{1-p_1}.
\]

- If \( N_\infty = \|v_0\|_{\mathbb{L}^\infty(\mathbb{R}^n)} \), then the solution of (1.4) blows up at minimal blow-up time; that is,

\[
T^* = \frac{1}{p_1 - 1} \|v_0\|_{\mathbb{L}^{1-p_1}(\mathbb{R}^n)} = \frac{1}{p_1 - 1} N_\infty^{1-p_1}.
\]
(b) Let $n = 1$. Define

$$n_\infty := \max \left( \liminf_{x \to \infty} v_0(x), \liminf_{x \to -\infty} v_0(x) \right).$$

- If $n_\infty > 0$, then the solution of (1.4) blows up in finite time, and

$$\frac{1}{p_1 - 1} \|v_0\|^{1 - p_1}_{L^{\infty}(\mathbb{R})} \leq T^* \leq \frac{1}{p_1 - 1} n_\infty^{1 - p_1}.$$

- If $n_\infty = \|v_0\|_{L^{\infty}(\mathbb{R})}$, then the solution of (1.4) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_1 - 1} \|v_0\|^{1 - p_1}_{L^{\infty}(\mathbb{R})} = \frac{1}{p_1 - 1} n_\infty^{1 - p_1}.$$

(E) The author [10] also considered

$$\frac{\partial_t v = v^{p_2}(\Delta v + v^q), \quad x \in \mathbb{R}^n, \quad t > 0,}{v(x, 0) = v_0(x) > 0, \quad x \in \mathbb{R}^n},$$

(1.8)

for $p_2 \geq 1$ or $q \geq 1$, and showed the following life span results:

(a) Let $n \geq 2$.

- If $N_\infty > 0$, then the solution of (1.8) blows up in finite time, and

$$\frac{1}{p_2 + q - 1} \|v_0\|^{1 - p_2 - q}_{L^{\infty}(\mathbb{R}^n)} \leq T^* \leq \frac{1}{p_2 + q - 1} N_\infty^{1 - p_2 - q}.$$

- If $N_\infty = \|v_0\|_{L^{\infty}(\mathbb{R}^n)}$, then the solution of (1.8) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_2 + q - 1} \|v_0\|^{1 - p_2 - q}_{L^{\infty}(\mathbb{R}^n)} = \frac{1}{p_2 + q - 1} N_\infty^{1 - p_2 - q}.$$

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- If $n_\infty = \|v_0\|_{L^{\infty}(\mathbb{R})}$, then the solution of (1.8) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_2 + q - 1} \|v_0\|^{1 - p_2 - q}_{L^{\infty}(\mathbb{R})} = \frac{1}{p_2 + q - 1} n_\infty^{1 - p_2 - q}.$$
Several recent studies show that the minimal blow-up time is strongly associated with blow-up at space infinity. Related researchers are Giga and Umeda [4, 5], Mochizuki and Suzuki [16], Ozawa and Yamauchi [18], Seki [19], Seki et al. [20], Shimojō [22], Yamaguchi and Yamauchi [27], Yamauchi [25, 26] and the author [8, 9].

Here, we state the main results.

Theorem 1. Consider the Cauchy problem (1.1) for $0 < \alpha < 1$ and $p > 1$.

(a) Let $n \geq 2$. Suppose that there exist $\xi \in S^{n-1}$ and $\delta > 0$ such that

$$M_\infty := \sup_{\xi \in S^{n-1}, \delta > 0} \left\{ \text{ess inf}_{\theta \in S_\xi(\delta)} \liminf_{r \to +\infty} u_0(r\theta) \right\} > 0,$$

where $r = |x|$, $\theta = x/r$, $S_\xi(\delta) = D_\xi(\delta) \cap S^{n-1}$ and $D_\xi(\delta)$ is the conic neighborhood defined by (1.7). Then the solution of (1.1) blows up in finite time, and we have

$$\left( \frac{(p-1)p^{-1}\Gamma(\alpha+1)}{p^p} \|u_0\|_{L^\infty(\mathbb{R}^n)}^{1-p} \right)^{1/\alpha} \leq T^* \leq \left( \frac{\Gamma(\alpha+1)}{p-1} M_\infty^{1-p} \right)^{1/\alpha}.$$

(1.10)

In particular, assuming that

$$M_\infty = \|u_0\|_{L^\infty(\mathbb{R}^n)},$$

(1.11)

the solution of (1.1) blows up at

$$T^* = \left( \frac{\Gamma(\alpha+1)}{p-1} \|u_0\|_{L^\infty(\mathbb{R}^n)}^{1-p} \right)^{1/\alpha} = \left( \frac{\Gamma(\alpha+1)}{p-1} M_\infty^{1-p} \right)^{1/\alpha}.$$

(1.12)

(b) Let $n = 1$. Suppose that

$$m_\infty := \max \left( \liminf_{x \to +\infty} u_0(x), \liminf_{x \to -\infty} u_0(x) \right) > 0.$$

(1.13)

Then the solution of (1.1) blows up in finite time, and we have

$$\left( \frac{(p-1)p^{-1}\Gamma(\alpha+1)}{p^p} \|u_0\|_{L^\infty(\mathbb{R})}^{1-p} \right)^{1/\alpha} \leq T^* \leq \left( \frac{\Gamma(\alpha+1)}{p-1} m_\infty^{1-p} \right)^{1/\alpha}.$$

(1.14)

In particular, assuming that

$$m_\infty = \|u_0\|_{L^\infty(\mathbb{R})},$$

(1.15)

the solution of (1.1) blows up at

$$T^* = \left( \frac{\Gamma(\alpha+1)}{p-1} \|u_0\|_{L^\infty(\mathbb{R})}^{1-p} \right)^{1/\alpha} = \left( \frac{\Gamma(\alpha+1)}{p-1} m_\infty^{1-p} \right)^{1/\alpha}.$$

(1.16)
Theorem 1 allows us the information of the life span for the initial data of intermediate size and the non-decaying initial data at space infinity; (1.9) and (1.13).

**Remark 2.** We show some examples of the initial data $u_0$ which satisfy $M_\infty > 0$ in the space dimensions $n \geq 2$. For simplicity, we employ polar coordinates.

(i) $u_0(r, \alpha) = 1 - \exp(-r^2)$.

Since $\liminf_{r \to +\infty} u_0(r, \alpha) = 1$, we have $M_\infty = 1$.

(ii) $u_0(r, \alpha) = \{1 - \exp(-r^2)\}(2 - \cos r)$.

Since $\liminf_{r \to +\infty} u_0(r, \alpha) = 1$, we have $M_\infty = 1$.

(iii) $u_0(r, \alpha) = \{1 - \exp(-r^2)\}(1 + \cos \alpha)$.

Since $\liminf_{r \to +\infty} u_0(r, \alpha) = 1 + \cos \alpha$, we have $M_\infty = 2$.

(iv) $u_0(r, \alpha) = \{1 - \exp(-r^2)\}(1 + \cos \alpha)(2 - \cos r)$.

Since $\liminf_{r \to +\infty} u_0(r, \alpha) = 1 + \cos \alpha$, we have $M_\infty = 2$.

For the examples (i) and (iii), the initial data $u_0$ satisfies (1.11). However, for the examples (ii) and (iv), since $\|u_0\|_{L^\infty(R^n)} = 3$ and $\|u_0\|_{L^\infty(R^n)} = 6$, respectively, it follows that $M_\infty \neq \|u_0\|_{L^\infty(R^n)}$.

The outline of the rest of this paper is organized as follows. In section 2, we give the existence theorem of a local solution to (1.1). In section 3, we prove the main results by improving the method in the author [8, 9], Ozawa and Yamauchi [18] and Yamauchi [25, 26].

### II. Existence of a Local Mild Solution

In this section, we show the local existence and uniqueness theorem of a mild solution to problem (1.1).

**Definition.** Let $T^* > 0$. We say $u \in C([0, T^*], C(R^n))$ is a mild solution of (1.1) if $u$ satisfies the integral equation

\[
u(t) = E_{\alpha,1}(-t^\alpha A)u_0 + \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-s^\alpha A)f(u(t - s))ds,
\]

where $f(u(s)) = u^p(s)$, and $A$ is realization of $-\Delta$ and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function (see [11]):

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.
\]

**Theorem 2.** Suppose that $u_0 \in C(R^n) \cap L^\infty(R^n)$. Then there exists a unique local mild solution $u \in C([0, T^*], C(R^n))$ for the problem (1.1).

**Proof.** See [21, Theorem 1] noting that the nonlinear term $f(u(s)) = u^p(s)$ is a locally Lipschizian function. (See also [24, Theorem 2.2].)

**Remark 3.** If $u$ solves (1.1), then $u$ satisfies (2.1) by the method of the proof for [21, Lemma 1].
In this section, we shall estimate the life span $T^*$ both from below and from above. Here, we improve the method in Yamauchi [25, 26] and the author [8, 9, 10].

First, we shall show a lower estimate of $T^*$ in the space dimensions $n \geq 1$. This is obtained by comparing the solution $u$ of (1.1) with the solution $U$ of the ordinary differential equation

$$
\begin{cases}
    \partial_t^\alpha U(t) = U^p(t), & t > 0, \\
    U(0) = \|u_0\|_{L^\infty(\mathbb{R}^n)}. 
\end{cases}
$$

(3.1)

The solution $U$ of (3.1) satisfies the integral equation

$$
U(t) = E_{\alpha,1}(0)U(0) + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(0)U^p(t-s)ds
$$

$$
= U(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}U^p(s)ds,
$$

(3.2)

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function by defined in (2.2). Now, we take the same strategy as in [7, Theorem 3.2] and [13, Theorem 3.1]. Here, changing of variables

$$
U(t) = U(0)[V(t) + 1] \quad \text{and} \quad k(t) = \gamma t^{\alpha-1} \quad \text{with} \quad \gamma = \frac{[U(0)]^{p-1}}{\Gamma(\alpha)},
$$

(3.3)

the integral equation (3.2) can be expressed as

$$
V(t) = \int_0^t k(t-s)[V(s) + 1]^pds.
$$

(3.4)

Then, the solution $U$ blows up in finite time $T^*(U)$ such that

$$
\left(\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p}\|u_0\|_{L^\infty(\mathbb{R}^n)}^{1-p}\right)^{1/\alpha} \leq T^*(U) \leq \left(\frac{\Gamma(\alpha+1)}{p-1}\|u_0\|_{L^\infty(\mathbb{R}^n)}^{1-p}\right)^{1/\alpha}.
$$

(3.5)

By a comparison argument, we obtain

$$
T^* \geq T^*(U).
$$

(3.6)

Next, we shall prove a upper estimate of $T^*$ by two case of $n \geq 2$ and $n = 1$.

**a) Case (a): $n \geq 2$**

For $\xi \in S^{n-1}$ and $\delta > 0$ as in the theorem, we determine the sequences $\{a_j\} \subset \mathbb{R}^n$ and $\{R_j\} \subset (0, \infty)$. Let $\{a_j\} \subset \mathbb{R}^n$ be a sequence satisfying that $|a_j| \to \infty$ as $j \to \infty$, and that $a_j/|a_j| = \xi$ for any $j \in \mathbb{N}$.

Put $R_j = (\delta \sqrt{4 - \delta^2}/2)|a_j|$ for $\delta \in (0, \sqrt{2})$. For $R_j > 0$, let $\rho_{R_j}$ be the first eigenfunction of $-\Delta$ on

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with zero Dirichlet boundary condition under the normalization

\[ \int_{B_{R_j}(0)} \rho_{R_j}(x) dx = 1. \]

Moreover, let \( \mu_{R_j} \) be the corresponding first eigenvalue. For the solutions of (1.1), we define

\[ w_j(t) := \int_{B_{R_j}(0)} u(x + a_j, t) \rho_{R_j}(x) dx. \]  

(3.7)

Then we have the following propositions.

**Proposition 1.** We have

\[ \liminf_{j \to +\infty} w_j(0) \geq \text{ess.inf}_{\theta \in S(\delta)} \left( \liminf_{r \to \infty} u_0(r\theta) \right), \]

(3.8)

and

\[ \lim_{j \to +\infty} \frac{\log \left( 1 - \mu_{R_j} w_j^{1-p}(0) \right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1. \]  

(3.9)

**Proof.** See [25, Proposition 1].

**Proposition 2.** Let \( 0 < \alpha < 1 \) and \( p > 1 \). Suppose that

\[ w_j(0) > \frac{1}{\mu_{R_j}^{\frac{1}{p-1}}}. \]  

(3.10)

Then \( u \) blows up in finite time, and we have

\[ T^* \leq \left[ \frac{\log \left( 1 - \mu_{R_j} w_j^{1-p}(0) \right)}{(1-p)\mu_{R_j}} \right]^{1/\alpha} \Gamma(\alpha + 1) . \]  

(3.11)

**Proof.** We use the method in [1, Theorem 3.7] and [3, Theorem 2.2].

By (1.1) and (3.7), we have

\[ \partial_t^\alpha w_j(t) = \int_{B_{R_j}(0)} \partial_t^\alpha u(x + a_j, t) \rho_{R_j}(x) dx \]

\[ = \int_{B_{R_j}(0)} \{ \Delta u(x + a_j, t) + u^p(x + a_j, t) \} \rho_{R_j}(x) dx \]
\[ \geq -\mu_{R_j} \int_{B_{R_j}(0)} u(x + a_j, t) \rho_{R_j}(x) dx + \int_{B_{R_j}(0)} u^p(x + a_j, t) \rho_{R_j}(x) dx. \] (3.12)

Since \( p > 1 \) and

\[ \int_{B_{R_j}(0)} \rho_{R_j}(x) dx = 1, \]

by Jensen’s inequality, we have

\[ \int_{B_{R_j}(0)} u^p(x + a_j, t) \rho_{R_j}(x) dx \geq \left( \int_{B_{R_j}(0)} u(x + a_j, t) \rho_{R_j}(x) dx \right)^p. \] (3.13)

Thus, by (3.12)-(3.13), we obtain

\[ \partial_t^\alpha \xi_j(t) \geq -\mu_{R_j} \xi_j(t) + w_j^p(t). \] (3.14)

By (1.3), the inequality (3.14) implies

\[ \frac{d}{dt} (k \ast [w_j - w_j(0)])(t) \geq -\mu_{R_j} w_j(t) + w_j^p(t) \quad \text{with} \quad k(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}. \] (3.15)

We put \( F(\zeta) = -\mu_{R_j} \zeta + \zeta^p \). Then the function \( F \) is convex in \( \zeta \geq 0 \), and we get

\[ \frac{d}{dt} (k \ast [w_j - w_j(0)])(t) \geq F(w_j(t)). \] (3.16)

in (3.15). \( F \) is positive and increasing for all \( \zeta > \mu_{R_j}^{\frac{1}{p-1}} \). If \( w_j(0) \) satisfies (3.10), then (3.16) implies that \( w_j(t) > \mu_{R_j}^{\frac{1}{p-1}} \) for all \( t \in (0, T^*) \) (see [1, P.24–25]). Knowing that \( w_j(t) \geq w_j(0) > \mu_{R_j}^{\frac{1}{p-1}} \) for all \( t \in (0, T^*) \), it follows from (3.16) that

\[ \partial_t^\alpha w_j(t) = \frac{d}{dt} (k \ast [w_j - w_j(0)])(t) \geq F(w_j(t)) > 0, \quad \text{for all} \quad t \in (0, T^*). \] (3.17)

Therefore the function \( w_j(t) \) satisfying (3.17) is an upper solution of the problem

\[ \partial_t^\alpha \zeta = F(\zeta) = -\mu_{R_j} \zeta + \zeta^p, \quad \zeta(0) = w_j(0), \] (3.18)

we have \( w_j(t) \geq \zeta(t) \) by comparison principle (see [14, Theorem 2.3]).

On the other hand, since \( F(0) \geq 0, F(\zeta) > 0 \) and \( F'(\zeta) > 0 \) for all \( \zeta \geq w_j(0) > \mu_{R_j}^{\frac{1}{p-1}} \). Then, it follows from [1, Lemma 3.8] (see also [23, Lemma 3.10]) that \( v(t) = g \left( \frac{t^{\alpha}}{F'(\alpha + 1)} \right) \) is a lower solution for (3.18), where \( v(t) \) satisfies

\[ \partial_t^\alpha v \leq F(v) = -\mu_{R_j} v + v^p, \quad v(0) \leq w_j(0), \]
and \( g(t) \) solves the ordinary differential equation

\[
\frac{dg}{dt} = F(g) = -\mu R_j g + g^p, \quad g(0) = w_j(0).
\] (3.19)

By comparison principle (see [14, Theorem 2.3]), we obtain \( \zeta(t) \geq v(t) \). Solving the initial value problem (3.19), we have the solution

\[
g(t) = w_1^{1-p}(0) - \frac{1 - \exp\left\{ (1-p)\mu R_j t \right\}}{\mu R_j} \exp\left( -\frac{\mu R_j t}{\Gamma(\alpha+1)} \right),
\]

and obtain that \( g(t) \to \infty \) as \( t \to \frac{\log(1-\mu R_j w_1^{1-p}(0))}{(1-p)\mu R_j} \). By comparison principle (see [14, Theorem 2.3]), we conclude that

\[
w_j(t) \geq \zeta(t) \geq v(t) = g\left( \frac{t^\alpha}{\Gamma(\alpha+1)} \right) = w_1^{1-p}(0) - \frac{1 - \exp\left\{ (1-p)\mu R_j t^\alpha \right\}}{\mu R_j} \exp\left( -\frac{\mu R_j t^\alpha}{\Gamma(\alpha+1)} \right).
\] (3.20)

By (3.20), if \( w_j(0) \) satisfies (3.10), then we obtain that \( v(t) \to \infty \) as

\[
t \to \left[ \frac{\log(1-\mu R_j w_1^{1-p}(0))}{(1-p)\mu R_j} \Gamma(\alpha+1) \right]^{1/\alpha},
\] (3.21)

and that \( w_j(t) \) blows up in finite time. Therefore, the solution \( u \) blows up in finite time, and it follows that the estimate (3.11) holds, the proof of Proposition 2 is complete.

Now let us prove the Case (a).

By Propositions 1 and 2, we obtain that

\[
T^* \leq \limsup_{j \to \infty} \left[ \frac{\log\left( 1 - \mu R_j w_1^{1-p}(0) \right)}{(1-p)\mu R_j} \Gamma(\alpha+1) \right]^{1/\alpha}
\]

\[
= \limsup_{j \to \infty} \left[ \frac{\log\left( 1 - \mu R_j w_1^{1-p}(0) \right)}{-\mu R_j w_1^{1-p}(0)} \cdot \frac{w_1^{1-p}(0)}{p-1} \Gamma(\alpha+1) \right]^{1/\alpha}
\]

\[
= \left( \frac{\Gamma(\alpha+1)}{p-1} \right)^{1/\alpha} \lim_{j \to \infty} \left[ \frac{\log\left( 1 - \mu R_j w_1^{1-p}(0) \right)}{-\mu R_j w_1^{1-p}(0)} \right]^{1/\alpha} \cdot \left( \liminf_{j \to \infty} w_j(0) \right)^{\frac{1-p}{\alpha}}
\]

\[
\leq \left( \frac{\Gamma(\alpha+1)}{p-1} \right)^{1/\alpha} \left\{ \text{ess.inf} \left( \theta \in S_r(\delta) \right) \liminf_{r \to \infty} u_0(r\theta) \right\}^{\frac{1-p}{\alpha}}. \quad (3.22)
\]

From arbitrariness of \( \xi \in S^{n-1} \) and \( \delta > 0 \), by (3.22), we obtain
By (3.6) and (3.23), we have

\[
T^* \leq \left( \frac{\Gamma(\alpha + 1)}{p - 1} \right)^{1/\alpha} \left[ \sup_{\xi \in S^{n-1}, \delta > 0} \left\{ \text{ess.inf}_{\theta \in S_\xi(\delta)} \left( \liminf_{r \to \infty} u_0(r\theta) \right) \right\} \right]^{1-p/\alpha}
\]

\[
= \left[ \frac{\Gamma(\alpha + 1)}{p - 1} M_{\infty}^{1-p} \right]^{1/\alpha}.
\]

(3.23)

By (3.6) and (3.23), we have

\[
\left[ \frac{(p-1)^{p-1}\Gamma(\alpha + 1)}{p^p} \|u_0\|_{L^\infty(\mathbb{R}^n)} \right]^{1/\alpha} \leq T^* \leq \left[ \frac{\Gamma(\alpha + 1)}{p - 1} M_{\infty}^{1-p} \right]^{1/\alpha}.
\]

(3.24)

Therefore, we obtain (1.10). Moreover, by (1.10) and (1.11), we have (1.12). This completes the proof.

b) Case (b): \( n = 1 \)

Let \( a_j = j \) or \(-j\). Put \( R_j = j/2 \). For \( R_j > 0 \), let \( \rho_{R_j} \) be the first eigenfunction of \(-\partial_x^2\) on \((-R_j, R_j)\) with zero Dirichlet boundary condition under the normalization

\[
\int_{-R_j}^{R_j} \rho_{R_j}(x)dx = 1.
\]

Moreover, let \( \mu_{R_j} \) be the corresponding first eigenvalue. For the solutions of (1.1), we define

\[
w_j(t) := \int_{-R_j}^{R_j} u(x + a_j, t)\rho_{R_j}(x)dx.
\]

(3.25)

Then we have the following propositions.

Proposition 3. We have

\[
\liminf_{j \to +\infty} w_j(0) \geq \max \left( \liminf_{x \to +\infty} u_0(x), \liminf_{x \to -\infty} u_0(x) \right)
\]

and

\[
\lim_{j \to +\infty} \frac{\log \left( 1 - \mu_{R_j} w_j^{1-p}(0) \right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1.
\]

(3.26)

Proof. See [25, Proposition 2].

Proposition 4. Let \( 0 < \alpha < 1 \) and \( p > 1 \). Suppose that

\[
w_j(0) > \frac{1}{\mu_{R_j}^{p-1}}.
\]

(3.27)

Then \( u \) blows up in finite time, and we have
\[ T^* \leq \left[ \log \left( \frac{1 - \mu R_j w_j^{1-p}(0)}{(1-p)\mu R_j} \right) \frac{\Gamma(\alpha + 1)}{1-p} \right]^{1/\alpha} \quad (3.29) \]

**Proof.** It is shown in the same way as in Proposition 2.

Finally, let us prove the Case (b). The rest of the proof is the same as in that of the Case (a).

By Propositions 3 and 4, we see that

\[ T^* \leq \limsup_{j \to \infty} \left[ \log \left( \frac{1 - \mu R_j w_j^{1-p}(0)}{(1-p)\mu R_j} \right) \frac{\Gamma(\alpha + 1)}{1-p} \right]^{1/\alpha} \]

\[ = \limsup_{j \to \infty} \left[ \log \left( \frac{1 - \mu R_j w_j^{1-p}(0)}{-\mu R_j w_j^{1-p}(0)} \right) \cdot \frac{w_j^{1-p}(0)}{1-p} \frac{\Gamma(\alpha + 1)}{1-p} \right]^{1/\alpha} \]

\[ = \left( \frac{\Gamma(\alpha + 1)}{p-1} \right)^{1/\alpha} \lim_{j \to \infty} \left[ \log \left( \frac{1 - \mu R_j w_j^{1-p}(0)}{-\mu R_j w_j^{1-p}(0)} \right) \right]^{1/\alpha} \cdot \left( \liminf_{j \to \infty} w_j^{1-p}(0) \right)^{1-\alpha/p} \]

\[ \leq \left( \frac{\Gamma(\alpha + 1)}{p-1} \right)^{1/\alpha} \left\{ \max \left( \liminf_{x \to +\infty} u_0(x), \liminf_{x \to -\infty} u_0(x) \right) \right\}^{1-\alpha/p}. \quad (3.30) \]

From (3.6) and (3.30), we have

\[ \left[ \frac{(p-1)^p \Gamma(\alpha + 1)}{p^p} \|u_0\|^{1-p}_{L^\infty(\mathbb{R}^n)} \right]^{1/\alpha} \leq T^* \leq \left[ \frac{\Gamma(\alpha + 1)}{p-1} m^{1-p}_{L^\infty} \right]^{1/\alpha}. \quad (3.31) \]

Therefore, we obtain (1.14). Moreover, by (1.14) and (1.15), we have (1.16). This completes the proof.

**References Références Referencias**

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Product of Special Function and Polynomial Associated Via Pathway Fractional Integral Operator

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Abstract- In present paper we introduce four theorems using pathway fractional integral operator involving product of Srivastava polynomial and generalized Struve function. Our results are quite general in nature. We obtain our results in term of hypergeometric function. Certain special cases of the main results are also obtained here. Our results will help to extend some classical statistical distribution to wider classes of distribution, these are useful in practical applications.

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GJSFR-F Classification: MSC 2010: 33C60, 26A33

Strictly as per the compliance and regulations of:
Product of Special Function and Polynomial Associated Via Pathway Fractional Integral Operator

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Abstract - In present paper we introduce four theorems using pathway fractional integral operator involving product of Srivastava polynomial and generalized Struve function. Our results are quite general in nature. We obtain our results in term of hypergeometric function. Certain special cases of the main results are also obtained here. Our results will help to extend some classical statistical distribution to wider classes of distribution, these are useful in practical applications.

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I. Introduction

Let $f(x) \in L(a,b), \alpha \in \mathbb{C}, R(\alpha) > 0$, then left sided Reimann – Liouville fractional integral operator is defined as [9]

$$(I_0^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \text{where } R(\alpha) > 0. \quad \text{(1.1)}$$

Let $f(x) \in L(a,b), \eta \in \mathbb{C}, R(\eta) > 0, a > 0$ and a "Pathway parameter" $\alpha < 1$. Then the pathway fractional integration operator is defined by [6], also see [10]

$$(p_0^{(\eta,\alpha)} f) = x^\eta \int_0^x a^{1-a} (1 - a(a-1)t)^{\eta-1} f(t)dt \quad \text{(1.2)}$$

when $\alpha = 0$, $a = 1$ and $\eta$ is replaced by $\eta - 1$ in (1.2) it yields

$$(I_0^\eta f)(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt \quad \text{(1.3)}$$

Fractional integration operators play an important role in the solution of several problems of diversified fields of science and engineering. Many fractional integral operators like Riemann – Liouville, Weyl, Kober, Erdely – Kober and Saigo operator are studied by various workers due to their applications in the solutions of integral equation arising in several problems of many areas of physical, engineering and technological science. A detailed description of these operators can be found in the survey paper by Srivastava and Saxena [17].
In this paper, we consider following functions defined as follows:

The Struve function of order \( p \) is given by

\[
H_p(z) = \left(\frac{z}{2}\right)^{p+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)} \frac{Z^{2k}}{\Gamma\left(k+\frac{1}{2}\right)} \frac{2k}{k+1} \quad \text{(1.4)}
\]

The Struve function and its more generalization are found in many papers [1,2,4,12,7,13,14,15]. The generalized Struve function studied by [13] as follows:

\[
H_{\lambda,\varepsilon}^{\lambda,\varepsilon}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(p+\frac{1}{2}\right)} \frac{Z^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)} \quad \lambda > 0, \varepsilon > 0 \quad \text{(1.5)}
\]

The generalized Struve function of the first kind \( H_{p,b,c}(z) \) [see 7] defined for complex \( z \in \mathbb{C} \) and \( b, c, p \in \mathbb{C}, (\text{Re}(p) > -1) \) by:

\[
H_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{\Gamma\left(p+1+\frac{b}{2}\right)} \frac{Z^{2k+p+1}}{\Gamma\left(k+\frac{3}{2}\right)} \quad \text{(1.6)}
\]

Where \( \Gamma \) is the classical gamma function whose Euler's integral is given by Srivastava and Choi (see [11])

\[
\Gamma(y) = \int_0^\infty e^{-t} t^{y-1} \, dt, \text{ Re}(y) > 0 \quad \text{(1.7)}
\]

The special cases of Struve function are as follows [7]:

\[
H_{-\frac{b}{2},b,c^2} = \frac{1}{c\sqrt{\pi}} \left[\frac{2}{z}\right]^{b/2} (1 - \cos(cz)) b \in \mathbb{R}, c \neq 0 \quad \text{(1.8)}
\]

\[
H_{-\frac{b}{2},b,-c^2} = \frac{1}{c\sqrt{\pi}} \left[\frac{2}{z}\right]^{b/2} (\cosh(cz) - 1) b \in \mathbb{R}, c \neq 0 \quad \text{(1.9)}
\]

\[
H_{-\frac{b}{2},-b,c^2} = \frac{1}{c\sqrt{\pi}} \left[\frac{2}{z}\right]^{b/2} \sin(cz) \quad b \in \mathbb{R}, c \neq 0 \quad \text{(1.10)}
\]

\[
H_{-\frac{b}{2},-b,-c^2} = \frac{1}{c\sqrt{\pi}} \left[\frac{2}{z}\right]^{b/2} \sinh(cz) \quad b \in \mathbb{R}, c \neq 0 \quad \text{(1.11)}
\]

The generalized Wright hypergeometric function \( r\Psi_s(z) \) defined for \( a_i, b_j \in \mathbb{C} \), and real \( a_i, b_j \in \mathbb{R} \) (\( a_i, b_j \neq 0; i = 1,2, \ldots, r; j = 1,2, \ldots s \)) is given by the series:

\[
r\Psi_s(z) = r\Psi_s \left( \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i+\alpha \cdot 1) z^k}{\prod_{k=1}^{s} \Gamma(b_j+\beta \cdot 1) k!} \right) \quad \text{(1.12)}
\]

where \( \Gamma(z) \) is the Euler gamma function and the asymptotic behavior of this function for large values of argument of \( z \in \mathbb{C} \) were studied in [3] and under the condition:
\[ \sum_{i=1}^{r} \beta_i - \sum_{i=1}^{s} \alpha_i > -1 \quad \text{..................... (1.13)} \]

For detailed Study of various properties, generalization and applications of Wright function and generalized Wright function, we refer to paper (for instance see [19] [20] [21]). The generalized hypergeometric function represented as follows [8]:

\[ r \Psi \left[ \begin{array}{c} \alpha_r; \\ z \end{array} \right] = \sum_{R=0}^{\infty} \frac{\prod_{i=1}^{R} (\alpha_i)_{n} z^n}{\prod_{j=1}^{R} (\beta_j)_{n} n!} \quad \text{..................... (1.14)} \]

Provided \( r \leq s; r = s + 1 \) and \(|z| < 1\) where \((\lambda)_n\) is well known pochhammer symbol defined for \( \lambda \in \mathbb{C} \) [3][8]

\[ (\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} = 1, 2, 3, \ldots) \end{cases} \quad \text{........ (1.15)} \]

\[ = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} (\lambda \in \mathbb{C}/Z_0^-) \]

where \( Z_0^- \) is the set of non – positive integers. If we put \( \alpha_1 = \ldots = \alpha_r = \beta_1 = \ldots = \beta_s \) in equation (1.8), then (1.10) is a special case of the generalized Wright function [16]

\[ r \Psi_s = r \Psi \left[ \begin{array}{c} \alpha_1, 1; \\ (\beta_1, 1), \ldots, (\beta_s, 1); \\ z \end{array} \right] = \frac{\prod_{i=1}^{s} (\alpha_i)_{n} z^n}{\prod_{j=1}^{s} (\beta_j)_{n} n!} r \Psi \left[ \begin{array}{c} (\alpha_1), \ldots, (\alpha_r); \\ (\beta_1), \ldots, (\beta_s); \\ z \end{array} \right] \quad \text{..................... (1.16)} \]

The Srivastava polynomial defined by Srivastava [18](pp. 1, eq.1), [5](pp. 11, eq. 7) in the following manner

\[ S_w^u [x] = \sum_{s=0}^{\infty} \frac{(-w)_{u+s}}{s!} A_{w,s} x^s, \quad w = 0, 1, 2, \ldots \quad \text{..................... (1.17)} \]

where \( w \) is an arbitrary positive integer and the coefficient \( A_{w,s} (w, s) > 0 \) are the arbitrary constant real or complex. This polynomial provide a large number spectrum of well-known polynomial as one of its particular cases on appropriately specializing the coefficient \( A_{w,s} \) particularly by setting \( u = 1, A_{w,s} \) for \( s = k \) and \( A_{w,s} = 0 \) for \( s \neq k \) the above polynomial leads to a power function.

\[ S_w^u [x] = x^k \quad (k \in \mathbb{Z}^+ \text{ with } k \leq w) \quad \text{..................... (1.18)} \]

II. Main Results

Theorem 1: Let \( \eta, \rho, \beta, \gamma, \mu, \delta \in \mathbb{C}, \mathcal{R} \left(1 + \frac{\eta}{(1-\alpha)}\right) > 0 \min\{\text{Re}(\rho), \text{Re}(\beta), \text{Re}(\gamma), \text{Re}(\mu), \text{Re}(\delta), \text{Re}(\eta)\} > 0 \) and \( p_i, q_i > 0, \alpha < 1 \). Then the pathaway fractional integral operator \( (p_{0+}^{(n,\alpha)}) \) defined by (1.2) then the following formula holds:
Proof: Making the use of (1.2), (1.5) and (1.17) in LHS of the theorem first and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.1).

Theorem 2: Let $\eta, \mu, p, b, c \in \mathbb{C}$ and $\alpha < 1$ such that $\{\text{Re}(\eta), \text{Re}(\mu), \text{Re}(\mu + p)\} > 0$, $\text{Re}(p + 1 + \frac{b}{2}) > -1$ and $\text{Re}(\frac{\eta}{1-\alpha}) > -1$ then the following formula hold:

\[
p_{0}^{(\eta,\alpha)}\{t^{\mu-1}H_{p,b,c}(t)S_{w}^{u}[\sigma t^{p}]\}(x) =
\]

\[
x^{\eta+p+1} \Gamma\left(1 + \frac{\eta}{1-\alpha}\right) \left(\frac{1}{2}\right)^{p+1} \sum_{s=0}^{(w/u)} (-W)u, s \frac{s!}{A_{w,s}} \left[\sigma \left(\frac{x}{[a(1-\alpha)]}\right)^{p}\right]^{s}
\]

\[
\times 1^{\psi} 3 \left[\left(\frac{p+\mu + \rho s + 1,2}{(\frac{b}{2} + 1.1)}, \left(\frac{\eta}{(1-\alpha)} + \mu + \rho s + 2,2\right)\right) - \frac{(cx)^{2}}{4[a(1-\alpha)]^{2}}\right] \quad \text{..........(2.2)}
\]

Proof: Making the use of (1.2), (1.6) and (1.18) in LHS of the theorem 2 and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.2).

Theorem 3: Let $\eta, \mu, p, b, c \in \mathbb{C}$ and $\alpha < 1$ such that $\{\text{Re}(\eta), \text{Re}(\mu), \text{Re}(\mu + p)\} > 0$, and $\text{Re}(\frac{\eta}{1-\alpha}) > -1$ then the following formula hold:

\[(i) \quad p_{0}^{(\eta,\alpha)}\{t^{\mu-1}\sin(ct)S_{w}^{u}[\sigma t^{p}]\}(x) =
\]

\[
\frac{1}{4} c \sqrt{\pi} x^{\eta+1} \Gamma\left(1 + \frac{\eta}{1-\alpha}\right) \left(\frac{1}{2}\right)^{\mu+1} \sum_{s=0}^{(w/u)} (-W)u, s \frac{s!}{A_{w,s}} \left[\sigma \left(\frac{x}{[a(1-\alpha)]}\right)^{\mu}\right]^{s}
\]

\[
\times 1^{\psi} 3 \left[\left(\frac{\mu+\rho s + 1,2}{(\frac{\eta}{(1-\alpha)} + \mu + \rho s + 2,2)\right)} - \frac{(cx)^{2}}{4[a(1-\alpha)]^{2}}\right] \quad \text{..........(2.3)}
\]

Proof: Making the use of (1.2), (1.10) and (1.18) in LHS of the theorem 3 (part I) and then interchange the order of integration and summation, we evaluate the inner integral
by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.3)

(ii) \( P_{0+}^{(\eta, \alpha)} \{ t^{\mu-1} \sinh(ct) S_w^u [\sigma t^\rho] \} (x) = \)

\[
\frac{1}{4} c^2 \sqrt{\pi} \left[ \frac{x^{\eta+\mu+1}}{[a(1-\alpha)]^{\mu+1}} \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right) \sum_{s=0}^{(w/u)} \frac{(-W)_u s}{s!} \frac{\sigma \left( \frac{x}{[a(1-\alpha)]} \right)^s}{A_{w,s}} \right] 
\]

\[
\times 1^\Psi 3 \left[ \frac{(\mu + \rho + 2, 2)}{([2, 1], (1, 1))} ; - \frac{(c x)^2}{4[a(1-\alpha)]^2} \right] \quad \text{…………………. (2.4)}
\]

**Proof:** Making the use of (1.2), (1.11) and (1.18) in LHS of the theorem 3 (part II) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.4).

**Theorem 4:** Let \( \eta, \mu, p, b, c \in \mathbb{C} \) and \( \alpha < 1 \) such that \( \text{Re}(\alpha) > 0 \), and \( \text{Re}(\beta - \eta) > 2 \) then the following formula hold:

(i) \( P_{0+}^{(\eta, \alpha)} \{ t^{\mu-1} (1 - \cos(ct)) S_w^u [\sigma t^\rho] \} (x) = \)

\[
\frac{1}{4} c^2 \sqrt{\pi} \left[ \frac{x^{\eta+\mu+2}}{[a(1-\alpha)]^{\mu+2}} \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right) \sum_{s=0}^{(w/u)} \frac{(-W)_u s}{s!} \frac{\sigma \left( \frac{x}{[a(1-\alpha)]} \right)^s}{A_{w,s}} \right] 
\]

\[
\times 1^\Psi 3 \left[ \frac{(\mu + \rho + 2, 2)}{([3, 1], (2, 1), (\eta, (1-\alpha)^\mu + \rho + 2, 2))} ; - \frac{(c x)^2}{4[a(1-\alpha)]^2} \right] \quad \text{…………………. (2.5)}
\]

**Proof:** Making the use of (1.2), (1.8) and (1.18) in LHS of the theorem 4 (part I) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and we arrive at the desired result RHS of (2.5)

(ii) \( P_{0+}^{(\eta, \alpha)} \{ t^{\mu-1} (\cosh(ct) - 1) S_w^u [\sigma t^\rho] \} (x) = \)

\[
\frac{1}{4} c^2 \sqrt{\pi} \left[ \frac{x^{\eta+\mu+2}}{[a(1-\alpha)]^{\mu+2}} \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right) \sum_{s=0}^{(w/u)} \frac{(-W)_u s}{s!} \frac{\sigma \left( \frac{x}{[a(1-\alpha)]} \right)^s}{A_{w,s}} \right] 
\]

\[
\times 1^\Psi 3 \left[ \frac{(\mu + \rho + 2, 2)}{([3, 1], (2, 1), (\eta, (1-\alpha)^\mu + \rho + 3, 2))} ; - \frac{(c x)^2}{4[a(1-\alpha)]^2} \right] \quad \text{…………………. (2.6)}
\]

**Proof:** Making the use of (1.2), (1.9) and (1.18) in LHS of the theorem 4 (part II) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.6)
III. Special Cases

1. If we take $\alpha = 0, a = 1$ and $\eta$ is replaced by $\eta - 1$ in (2.1), then pathway fractional integral operator will reduce Riemann – Liouville fractional integral defined in (1.1). Then we get the following result:

$$I_0^a + \{t^{\mu-1}H_{\lambda,\epsilon}^\lambda(t)S_w^\mu[\sigma t^\rho]\}(x) = x^{\eta+\mu+1} \Gamma(\eta) \left(\frac{1}{2}\right)^{1+1} \sum_{s=0}^{(w/u)} \frac{(-W)u, s}{s!} A_{w,s} (\sigma x^\rho)^s$$

$$\times 1^\psi 3 \left[ \begin{array}{c} \frac{l}{3} + \frac{3}{2}, \lambda , \frac{3}{2}, 1, (l + \eta + \mu + \rho s + 1, 2); \\ (-x^2) \end{array} \right]$$

2. If we take $\alpha = 0, a = 1$, $\eta$ is replaced by $\eta - 1$ and also on setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.1), then pathway fractional integral operator will reduce Riemann – Liouville fractional integral defined in (1.1) and general class of polynomial will reduce to 1 defined in (1.13). then we get the following result:

$$I_0^a + \{t^{\mu-1}H_{\lambda,\epsilon}^\lambda(t)\}(x)$$

$$= x^{\eta+\mu+1} \Gamma(\eta) \left(\frac{1}{2}\right)^{1+1}$$

$$\times 1^\psi 3 \left[ \begin{array}{c} \frac{l}{3} + \frac{3}{2}, \lambda , \frac{3}{2}, 1, (l + \eta + \mu + 1, 2); \\ (-x^2) \end{array} \right]$$

3. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.2), we arrive at the known result given by Nisar K .S [10, pp. 66, eq. 13]

4. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.3), we arrive at the known result given by Nisar K .S [10, pp. 67, theorem (3.1)(part I)].

5. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.4), we arrive at the known result given by Nisar K .S [10, pp. 67, theorem (3.1)(part II)].

6. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.5), we arrive at the known result given by Nisar K .S [10, pp. 68 theorem (3.2), (part I)].

7. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^\mu [x] \rightarrow 1$ in (2.6), we arrive at the known result given by Nisar K .S [10, pp. 68 theorem (3.2), (part II)].

IV. Conclusion

In this paper, we have presented Struve function, generalized Struve function and Srivastava polynomial via pathway fractional integral operator. As in this operator $\alpha$ establishes a path of going from one distribution to another and to different classes of distribution. we conclude this investigation by remarking that the result obtained here are general in character and useful in deriving various integral formulas in the theory of the pathway fractional integration operator and also our result will help to extend some
classical statistical distribution to wider classes of distribution, useful in practical application.

**References Références Referencias**


On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

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Abstract- We construct sequences of triples of circulant matrices with positive integers as entries which are solutions of the equation $x^2 + y^2 = z^2$. We introduce Mouanda’s choice function for matrices which allows us to construct galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of galaxies of circulant matrices. The characterization of the matrix solutions of the equation $x^2 + y^2 = z^2$ allows us to show that the equation $x^{2n} + y^{2n} = z^{2n}$ ($n \geq 2$) has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation $x^n + y^n = z^n$ ($n \geq 3$) has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we prove Fermat's Last Theorem for complex polynomials over $\mathbb{D}$ associated to circulant matrices.

Keywords: Fermat's equation, polynomials, model theory, circulant matrices, Mouanda’s choice function, galaxy, Toeplitz matrices.

GJSFR-F Classification: MSC 2010: 11D41, 11C08, 03C95.
On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

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Abstract: We construct sequences of triples of circulant matrices with positive integers as entries which are solutions of the equation \( x^n + y^n = z^n \). We introduce Mouanda's choice function for matrices which allows us to construct galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of galaxies of circulant matrices. The characterization of the matrix solutions of the equation \( x^n + y^n = z^n \) allows us to show that the equation \( x^{2n} + y^{2n} = z^{2n} \) \((n \geq 2)\) has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation \( x^n + y^n = z^n \) \((n \geq 3)\) has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we prove Fermat's Last Theorem for complex polynomials over \( \mathbb{D} \) associated to circulant matrices.

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I. Introduction and Main Result

It is well known that there are many solutions in integers to the equation \( x^2 + y^2 = z^2 \), for instance \((3,4,5); (5,12,13)\). Around 1500 B.C, the Babylonians were aware of the solution \((4961, 6480, 8161)\) and the Egyptians knew the solutions \((148, 2736, 2740)\) and \((514, 66048, 66050)\). Also Greek mathematicians were attracted to the solutions of this equation. We notice that this equation has sequences of complex number solutions

\[
(1 + 2i \times a^k, 2i \times a^k - 2 \times a^{2k}, 1 + 2i \times a^k - 2 \times a^{2k}), a \in \mathbb{C}, k \in \mathbb{N}
\]

and matrix solutions

\[
(1 + 2i \times a^k, 2i \times a^k - 2 \times a^{2k}, 1 + 2i \times a^k - 2 \times a^{2k}), a \in \mathbb{C}, k \in \mathbb{N}
\]
In 1637, Pierre de Fermat wrote a note in the margin of his copy of Diophantus Arithmetica [1] stating that the equation

$$x^n + y^n = z^n, n \in \mathbb{N}(n > 2), xyz \neq 0$$  \hspace{1cm} (1.1)$$

has no integer solutions. This is the Fermat Last Theorem. He claimed that he had found the proof of this Theorem. The only case Fermat actually wrote down a proof is the case $n = 4$. In his proof, Fermat introduced the idea of infinite descent which is still one the main tools in the study of Diophantine equations. He proved that the equation $x^4 + y^4 = z^2$ has no solutions in relatively prime integers with $xyz \neq 0$. Solutions to this equation correspond to rational points on the elliptic curve $v^2 = u^3 - 4u$. The proof of the case $n = 3$ was given first by Karl Gauss. In 1753, Leonhard Euler gave a different prove of Fermat’s Last Theorem for $n = 3$ [2, 3]. In 1823, Sophie Germain proved that if $l$ is a prime and $2l + 1$ is also prime, the equation $x^l + y^l = z^l$ has no solutions $(x,y,z)$ with $xyz \neq 0 (mod l)$. The case $n = 5$ was proved simultaneously by Adrien Marie Legendre in 1825 [4, 5] and Peter Lejeune Dirichlet [6] in 1832. In 1839, Gabriel Lame proved the case $n = 7$ [7, 8, 9, 10]. Between 1840 - 1843, V. A. Lebesque worked on Fermat’s Last Theorem [11, 12]. Between 1847 and 1853, Ernst Eduard Kummer published some masterful papers about this Theorem. Fermat’s Last Theorem attracted the attention of many researchers and many studies have been developed around this Theorem. For example the work of Arthur Wieferich (1909), Andre Weil (1940), John Tate (1950), Gerhard Frey (1986), who was the first to suggest that the existence of a solution of the Fermat equation might contradict the modality conjeture of Taniyama, Shimura and Weil [29]; Jean Pierre Serre (1985 - 1986) [14, 15, 16], who gave an interested formulation and (with J. F. Mestre) tested numerically a precise conjecture about modular forms and Galois representations mod p and proved how a small piece of this conjecture the so called epsilon conjecture together Modularity Conjecture would imply Fermat’s Last Theorem; Kennedy Ribet (1986) [17], who proved Serre’s epsilon conjecture, thus reducing the proof of Fermat’s Last Theorem; Barry Mazur (1986), who introduced a significant piece of work on the deformation of Galois representations [18, 19]. However, no final proof was given to this Theorem. This Theorem was unsolved for nearly 350 years. In 1995, using Mazur’s deformation theory of Galois representations, recent results on Serre’s conjecture on the modularity of Galois representations, and deep arithmetical properties of Hecke algebras, Andrew Wiles with Richard Taylor succeeded in proving that all semi-stable elliptic curves defined over the rational numbers are modular. This result is less than the full Shimura-Taniyama conjecture. This result does imply that the elliptic curve given above is modular. Therefore, proving Fermat’s Last Theorem [20, 21]. Many mathematicians are still heavenly involved on studying Fermat’s Last Theorem [22, 23, 24]. In 2021, Nag introduced an elementary proof of Fermat’s
Last Theorem for epsilons[25]. In 2022, Mouanda constructed the galaxies of sequences of triples of positive integers solutions of the equation \( x^2 + y^2 = z^2 \). The unique characterization of the solutions of this equation allowed him to provide an elementary analytic proof of Fermat’s Last Theorem [26]. The Fermat Last Theorem for positive integers has been extended over some number fields. In 1966, Domiaty proved that the equation \( X^4 + Y^4 = Z^4 \) is solvable in \( M_2(\mathbb{Z}) \) [27]. Let \( GL_n(\mathbb{Z}) \) be the group of units of ring \( M_n(\mathbb{Z}) \). Denote by

\[
SL_n(\mathbb{Z}) = \{ A \in M_n(\mathbb{Z}) : \det A = 1 \}.
\] (1.2)

In 1989, Vaserstein investigated the question of the solvability of the equation

\[
X^n + Y^n = Z^n, \quad n \geq 2, \quad n \in \mathbb{N},
\] (1.3)

for matrices of the group \( GL_2(\mathbb{Z}) \) [28]. In 1993, Frejman studied the solvability of the equation (1.3) in the set of positive integer powers of a matrix \( A \) with elements \( a_{11} = 0, a_{12} = a_{21} = a_{22} = 1 \) [29]. In 1995, the same case was studied by Grytczuk [30]. The same year, Khazanov proved that in \( GL_3(\mathbb{Z}) \) solutions of the equation (1.3) do not exist if \( n \) is a multiple of either 21 or 96, and in \( SL_3(\mathbb{Z}) \) solutions do not exist if \( n \) is a multiple of 48 [31]. In 1996, Qin gave another proof of Khazanov’s result on the solvability of the equation (1.3) in \( SL_2(\mathbb{Z}) \) [32]. In 2002, Patay and Szakacs described the periodic elements in \( GL_2(\mathbb{Z}) \) and gave the answer to some problems concerning the equation (1.3) in matrix groups and in irreducible elements of matrix rings [33]. In 2021, Mao-Ting and Jie proved that Fermat’s matrix equation has many solutions in a set of 2-by-2 positive semi-definite integral matrices, and has no nontrivial solutions in some classes including 2-by-2 symmetric rational and stochastic quadratic field matrices [34]. Fermat’s Last Theorem has been extended to the field of complex polynomials of one variable [35].

This Theorem has many applications in Cryptography.

In this paper, we are mainly concerned with Fermat’s Last Theorem for circulant matrices with positive integers as entries. Firstly, we focus our attention on the construction of the galaxies of sequences of triples of circulant matrices with positive integers as entries solutions of the equation \( X^2 + Y^2 = Z^2 \). In particular, Mouanda’s matrix choice function allows us to construct practical examples of such galaxies. The elementary characterization of these matrix solutions allows us to prove Fermat’s Last Theorem for circulant matrices with positive integers as entries.

**Theorem 1.1.** The equation

\[
X^n + Y^n = Z^n, \quad XY \neq 0, \quad n \in \mathbb{N}(n \geq 3)
\]

has no circulant matrix with positive integers as entries solutions.

We construct a galaxy of sequences of eigenvalues of circulant matrices and we prove Fermat’s Last Theorem for eigenvalues of circulant matrices. Also, we construct a galaxy of sequences of complex polynomials over the unit disk \( \mathbb{D} \) associated to circulant matrices and we prove Fermat’s Last Theorem for complex polynomials over \( \mathbb{D} \).
II. Preliminaries

Definition 2.1. Let $\mathbb{A}$ be a unital Banach algebra. We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = 1$. In this case $b$ is unique and written $a^{-1}$. The set

$$Inv(\mathbb{A}) = \{ a \in \mathbb{A} : \exists b \in \mathbb{A}, ab = ba = 1 \}$$

is a group under multiplication. If $a$ is an element of $\mathbb{A}$, the spectrum of $a$ is defined as

$$\sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \notin Inv(\mathbb{A}) \},$$

and its spectral radius is defined to be

$$r(a) = \sup \{ |\lambda| : \lambda \in \sigma(a) \}.$$

Let $V = \{ a_0, a_1, \ldots, a_{m-1} \} \subset \mathbb{C}$ be a subset of the set of complex numbers, denote by $C_V$ the following Toeplitz matrix:

$$C_V = \begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_{m-1} & a_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & \cdots & a_{m-1} & a_0 \end{pmatrix}.$$

This matrix is called a $m \times m$-complex circulant matrix or a complex circulant matrix of order $m$. Denote by $C_m(\mathbb{C})$ the commutative algebra of $m \times m$-complex circulant matrices. Let $\epsilon = e^{\frac{2\pi i}{m}}$ be a primitive $m$-th root of unity. Let us denote by $U$ the following matrix:

$$U = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \epsilon & \cdots & \cdots & \epsilon^{(m-3)} & \epsilon^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \\ 1 & \epsilon^{m-3} & \cdots & \cdots & \epsilon^{(m-3)(m-2)} & \epsilon^{(m-3)(m-1)} \\ 1 & \epsilon^{m-2} & \cdots & \cdots & \epsilon^{(m-2)(m-3)} & \epsilon^{(m-2)(m-1)} \\ 1 & \epsilon^{m-1} & \cdots & \cdots & \epsilon^{(m-1)(m-3)} & \epsilon^{(m-1)(m-2)} \end{pmatrix}.$$ 

This matrix is called Vandermonde matrix. It is well known that this matrix has the following properties:

$$det(U) = \frac{1}{m^m} \prod_{i,j=0}^{m-1} (\epsilon^j - \epsilon^i) \neq 0,$$
U is non-singular, unitary, $U^{-1} = U^T$, $U^T = U$ and $U^{-1} = U^*$. It is well known that all the elements of $C_m(\mathbb{C})$ are simultaneously diagonalized by the same unitary matrix U, that is, for A in $C_m(\mathbb{C})$, one has

$$U^*AU = D_A$$

with $D_A$ is a diagonal matrix with diagonal entries given by the ordered eigenvalues of A: $\lambda_1^A, \lambda_2^A, \ldots, \lambda_m^A$. The factorization $U^*AU = D_A$ is called the spectral factorization of A [36, 37, 38, 39]. It is possible to write the matrix $C_V$ as one variable complex polynomial. Indeed, let P be the cyclic permutation $m \times m$-matrix given by

$$P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}.$$  

It is simple to see that

$$C_V = \sum_{k=0}^{m-1} a_k P^k.$$ 

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk. The complex polynomial

$$f_V(z) = \sum_{k=0}^{m-1} a_k z^k$$

over $\mathbb{D}$ is called the associated complex polynomial of the matrix $C_V = f_V(P)$. It follows that if

$$X = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}$$

is a $m \times m$-complex matrix, then
\[ f_V(X) = \sum_{k=0}^{m-1} a_k X^k = \begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{pmatrix} \]

is a \( m \times m \)-upper complex triangular Toeplitz matrix. The complex polynomial

\[ f_V(z) = \sum_{k=0}^{m-1} a_k z^k \]

is also called the associated complex polynomial of the matrix \( f_V(X) \).

### III. The Universe of an Algebra

**Definition 3.1.** Let \( x, y, z \in \mathbb{C} \) be complex numbers. Denote by

\[(x, y, z)^n = (x^n, y^n, z^n), n = \frac{p}{q}, p, q \in \mathbb{N}, q \neq 0.\]

The triple \((x^n, y^n, z^n)\) is called the triple \((x, y, z)\) to the power \(n\).

**Definition 3.2.** Let \( x, y, z \in \mathbb{C} \) be complex numbers. Denote by

\[\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z), (x, y, z) + (a, b, c) = (x + a, y + b, z + c).\]

**Definition 3.3.** A universe of degree \(\frac{p}{q}\) of the algebra \(B\) is the set \(F_{\frac{p}{q}}(B)\) of triples \((x, y, z)\) of elements of \(B\) which satisfy the law of stability

\[x^p + y^p = z^p, xyz \neq 0, p, q \in \mathbb{N}, q \neq 0.\]

The element \((x, y, z)\) is called a star (or a planet) of the universe \(F_{\frac{p}{q}}(B)\).

Every sequence \((x_k, y_k, z_k)_{n \geq 0}\) of elements of the universe \(F_{\frac{p}{q}}(B)\) is called a planet system of elements of \(B\).

The set

\[F_{\frac{p}{q}}(C_m(\mathbb{C})) = \{(X, Y, Z) \in \mathbb{C}^3 : X^p + Y^p = Z^p, XYZ \neq 0\}, p, q \in \mathbb{N}, q \neq 0,\]

is called the complex ciculant universe of degree \(\frac{p}{q}\). In particular, the set

\[F_n(C_m(\mathbb{N})) = \{(X, Y, Z) \in C_m(\mathbb{N})^3 : X^n + Y^n = Z^n, XYZ \neq 0\}, n \in \mathbb{N}, n \geq 2,\]
is called the natural circulant universe of degree $n$. We are going to show that

$$\mathbb{F}_n(C_m(\mathbb{N})) = \emptyset, n \geq 3.$$  

In other words, there are matrix complex universes which don’t have triples of matrices of positive integers as entries elements.

IV. Mouanda’s Choice Function for Matrices

Denote by $C_*(C_m(\mathbb{C})) = \{h/h : C_m(\mathbb{C}) \rightarrow C_m(\mathbb{C})\}$, the set of complex functions over $\mathbb{C}$. Let

$$\Omega(\mathbb{F}_2(C_m(\mathbb{C}))) = \{P : P \subseteq \mathbb{F}_2(C_m(\mathbb{C}))\}$$

be the set of all subsets of $\mathbb{F}_2(C_m(\mathbb{C}))$. Theorem 2.5 of [26] allows us to claim that the appropriate choice of the values of $m_0(k)$ and $n_0(k)$ such that

$$2(m_0(k) - n_0(k)) \pm \sqrt{8m_0(k)(m_0(k) - n_0(k))} \in C_m(\mathbb{C})$$

leads to the construction of sequences of triples of circulant matrices with positive (or negative) integers as entries which satisfy the equation

$$X^2 + Y^2 = Z^2.$$  

Let $f_M : C_*(C_m(\mathbb{C})) \times C_*(C_m(\mathbb{C})) \rightarrow \Omega(\mathbb{F}_2(C_m(\mathbb{C})))$ be the function defined by

$$f_M(m_0(k), n_0(k)) = \begin{bmatrix}
    m_0(k) = a^{\beta(k)}, k, a, \beta(k) \in C_m(\mathbb{C}), \beta \in C_*(C_m(\mathbb{C})) \\
    m_0(k) - n_0(k) \in C_m(\mathbb{C}) \\
    \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} \in C_m(\mathbb{C}) \\
    \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + n_0(k) \\
    \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + m_0(k)
\end{bmatrix},$$

This type of function is called Mouanda’s choice function for matrices. Mouanda’s choice function for matrices is a galaxy valued function. This function allows us to construct galaxies of sequences of matrices.

V. A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries

All the galaxies defined in this section have been deduced from the galaxies already introduced in [26].
Definition 5.1. A multi-galaxy is a galaxy which contains other galaxies. The order of a galaxy is the number of variables of the galaxy.

Let

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

be a \(m \times m\)–matrix. Denote by \(T_k = P^k, k = 1, 2, \ldots, m - 1\). Let

\[
U_m = \{I_m, T_1, T_2, \ldots, T_{m-1}\}
\]

be a finite set of unitary circulant matrices. The elements of the set \(U_m\) satisfy the following:

\[
T_i T_j = T_{i+j}, T_i T_{m-i} = I_m, T_{m+i} = T_i, T_k = P^k.
\]

Denote by

\[
P^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.
\]

Let

\[
f(z) = \sum_{k=0}^{m-1} a_k z^k
\]

be a complex polynomial over \(\mathbb{D}\). The Toeplitz matrix

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\
 a_{m-1} & a_0 & a_1 & \cdots & a_{m-2} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
a_2 & \cdots & \cdots & a_{m-1} & a_0 & a_1 \\
a_1 & a_2 & \cdots & \cdots & a_{m-1} & a_0
\end{pmatrix} = f(P)
\]

is called the circulant matrix with complex numbers as entries. The polynomial \(f(z)\) is called associated polynomial of the matrix \(f(P)\). Recall that the
set $C_m(\mathbb{C})$ is the commutative algebra of $m \times m$-complex circulant matrices. In other words,

$$C_m(\mathbb{C}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2 & \cdots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{C} \right\}.$$ 

It follows that

$$C_m(\mathbb{C}) = \{ f(P) : f \in \mathcal{P}(m)(\mathbb{C}) \}$$

and

$$C_m(\mathbb{N}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2 & \cdots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{N} \right\}.$$ 

It is quiet clear that

$$C_m(\mathbb{N}) \subset C_m(\mathbb{Z}) \subset C_m(\mathbb{Q}) \subset C_m(\mathbb{R}) \subset C_m(\mathbb{C}).$$

Mouanda’s matrix choice function $f_M$ allows us to construct galaxies of circulant matrices. For instance, if we choose $m_0 = A^{2k} \times 2, m_0 - n_0 = \alpha I_m, A \in C_m(\mathbb{N})$ the model

$$Gala(\mathbb{N}I_m, C_m(\mathbb{N})) = \left\{ \begin{array}{c} X_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k \\
Y_k(\alpha I_m, A) = 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\
Z_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\
\alpha = r^2, k, r \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array} \right\}$$

is called the galaxy of sequences of circulant matrices with positive integers as entries of order 2. For $\alpha_0$ and $A$ fixed, the triple

$$(X_0(\alpha_0 I_m, A), Y_0(\alpha_0 I_m, A), Z_0(\alpha_0 I_m, A))$$
is called the origin of the galaxy $\text{Gala}(\alpha_0 I_m, C_m(\mathbb{N}))$. The elements of $\text{Gala}(\alpha_0 I_m, C_m(\mathbb{N}))$ satisfy

$$X_k^2(\alpha_0 I_m, A) + Y_k^2(\alpha_0 I_m, A) = Z_k^2(\alpha_0 I_m, A), k \in \mathbb{N}, A \in C_m(\mathbb{N})$$

and

$$(X_k(\alpha_0 I_m, A), Y_k(\alpha_0 I_m, A), Z_k(\alpha_0 I_m, A)) \neq \left(D_{\alpha}^{k}, B_{\alpha}^{k}, C_{\alpha}^{k}\right), A, B, C, D \in C_m(\mathbb{N})$$

with $k, p, q \in \mathbb{N}, q \neq 0$.

**Example 5.2. A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries**

- The model

$$\text{Gala}(4I_m, U_m) = \begin{bmatrix}
X_k(4I_m, A) = 4I_m + 4 \times A^k \\
Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\
Z_k(4I_m, A) = 4 + 4 \times A^k + 2 \times A^{2k} \\
(X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m)
\end{bmatrix}, k \in \mathbb{N}, A \in U_m$$

is called the finite galaxy of sequences of circulant matrices with positive integers as entries of order 1. The triple $\left(X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)\right)$ is called the origin of the galaxy $\text{Gala}(4I_m, U_m)$. The triple

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A))$$

satisfies

$$X_k^2(4I_m, A) + Y_k^2(4I_m, A) = Z_k^2(4I_m, A), k \in \mathbb{N},$$

$$X_0(4I_m, A) + Y_0(4I_m, A) + Z_0(4I_m, A) = 24I_m$$

and

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A)) \neq \left(D_{\alpha}^{k}, B_{\alpha}^{k}, C_{\alpha}^{k}\right), k, p, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The finite galaxy $\text{Gala}(4I_m, U_m)$ allows the construction of the infinite galaxy

$$\text{Gala}(4I_m, C_m(\mathbb{N})) = \begin{bmatrix}
X_k(4I_m, A) = 4I_m + 4 \times A^k \\
Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\
Z_k(4I_m, A) = 4 + 4 \times A^k + 2 \times A^{2k} \\
(X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m)
\end{bmatrix}, k \in \mathbb{N}, A \in C_m(\mathbb{N})$$

which has the same origin and stability law than $\text{Gala}(4I_m, U_m)$. Therefore, we can say that the galaxy $\text{Gala}(\mathbb{N}I_m, C_m(\mathbb{N}))$ is a multi-galaxy.
• Assume that \( m = 5, U_5 = \{I_5, T_1, T_2, T_3, T_4\} \) and
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}^2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} = T_2 \in U_5.
\]

The triples \((X_k(4I_5, A), Y_k(4I_5, A), Z_k(4I_5, A))\) of the galaxy
\[
\begin{align*}
X_k(4I_5, A) &= \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}^k + 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^{2k}, \\
Y_k(4I_5, A) &= 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^k + 2 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^{2k}, \\
Z_k(4I_5, A) &= 4I_5 + 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^k + 2 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^{2k},
\end{align*}
\]

satisfy the equation
\[
X_k^2(4I_5, A) + Y_k^2(4I_5, A) = Z_k^2(4I_5, A), \quad k \in \mathbb{N}
\]
and
\[
(X_k(4I_5, A), Y_k(4I_5, A), Z_k(4I_5, A)) \neq \left( D^x, B^y, C^z \right), D, B, C \in U_m, k, q, p \in \mathbb{N}, q \neq 0.
\]

• The elements of the galaxy
\[
Gala(9I_m, U_m) = \begin{cases}
X_k(9I_m, A) = 9I_m + 6 \times A^k, \\
Y_k(9I_m, A) = 6 \times A^k + 2 \times A^{2k}, \\
Z_k(9I_m, A) = 9I_m + 6 \times A^k + 2 \times A^{2k}, \quad & k \in \mathbb{N}, A \in U_m \\
(X_0(9I_m, A), Y_0(9I_m, A), Z_0(9I_m, A)) = (15I_m, 8I_m, 17I_m)
\end{cases}
\]
satisfy
\[ X_k^2(9I_m, A) + Y_k^2(9I_m, A) = Z_k^2(9I_m, A), \ k \in \mathbb{N} \]

and
\[
(X_k(9I_m, A), Y_k(9I_m, A), Z_k(9I_m, A)) \neq \left(D^p, B^q, C^r\right), \ p, q, k \in \mathbb{N}, q \neq 0,
\]

\(D, B, C \in U_m\). Again, from the galaxy \(Gala(9I_m, U_m)\), we can construct a galaxy which has an infinite number of elements. Indeed, the galaxy

\[ Gala(9I_m, C_m(\mathbb{N})) = \begin{cases} 
X_k(9I_m, A) = 9I_m + 6 \times A^k, \\
Y_k(9I_m, A) = 6 \times A^k + 2 \times A^{2k}, \\
Z_k(9I_m, A) = 9I_m + 6 \times A^k + 2 \times A^{2k} 
\end{cases}, \ \ k \in \mathbb{N}, A \in C_m(\mathbb{N})
\]

has the same origin and stability law than the galaxy \(Gala(9I_m, U_m)\). This galaxy has an infinite number of elements.

- The elements of the galaxy

\[ Gala(16I_m, U_m) = \begin{cases} 
X_k(16I_m, A) = 16I_m + 8 \times A^k, \\
Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k}, \\
Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} 
\end{cases}, \ \ k \in \mathbb{N}, A \in U_m
\]

satisfy
\[ X_k^2(16I_m, A) + Y_k^2(16I_m, A) = Z_k^2(16I_m, A), \ k \in \mathbb{N} \]

and
\[
(X_k(16I_m, A), Y_k(16I_m, A), Z_k(16I_m, A)) \neq \left(D^p, B^q, C^r\right), \ p, q, k \in \mathbb{N}, q \neq 0,
\]

\(D, B, C \in U_m\). The galaxy \(Gala(16, U_m)\) has a finite number of elements (or planets). However, the galaxy

\[ Gala(16I_m, C_m(\mathbb{N})) = \begin{cases} 
X_k(16I_m, A) = 16I_m + 8 \times A^k, \\
Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k}, \\
Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} 
\end{cases}, \ \ k \in \mathbb{N}, A \in C_m(\mathbb{N})
\]

has an infinite number of elements (or planets).
Example 5.3. Assume that \( m_0 - n_0 = 2I_m, m_0 = A^{2k}, A \in C_m(\mathbb{N}) \). We can define the galaxy

\[
\Delta(2I_m, C_m(\mathbb{N})) = \begin{bmatrix}
X_k(2I_m, A) = 2I_m + 2 \times A^k \\
Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\
Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k}
\end{bmatrix},
\]

in which the triples \((X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A))\), \(k \in \mathbb{N}\), satisfy

\[
X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), \quad k \in \mathbb{N},
\]

\[
X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m
\]

and

\[
(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left( D^\frac{p}{2}, B^\frac{q}{2}, C^\frac{q}{2} \right),
\]

\(p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in C_m(\mathbb{N})\).

Example 5.4. A Finite Galaxy

The triples \((X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A))\), \(k \in \mathbb{N}\), of the galaxy

\[
\Delta(2I_m, U_m) = \begin{bmatrix}
X_k(2I_m, A) = 2I_m + 2 \times A^k \\
Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\
Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k}
\end{bmatrix},
\]

satisfy

\[
X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), \quad k \in \mathbb{N},
\]

\[
X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m
\]

and

\[
(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left( D^\frac{p}{2}, B^\frac{q}{2}, C^\frac{q}{2} \right),
\]

\(p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in U_m\).

Example 5.5. Assume that \( m = 10, U_{10} = \{ I_{10}, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9 \} \) with

\[
T_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
The triples \( (X_k(2I_{10}, T_1), Y_k(2I_{10}, T_1), Z_k(2I_{10}, T_1)) \) of the galaxy \( \Delta(2I_{10}, T_1) \)

\[
\begin{bmatrix}
X_k(2I_{10}, T_1) = 2I_{10} + 2 \times T_1^k \\
Y_k(2I_{10}, T_1) = 2 \times T_1^k + T_1^{2k} \\
Z_k(2I_{10}, T_1) = 2I_{10} + 2 \times T_1^k + T_1^{2k}
\end{bmatrix}
\]

\( k \in \mathbb{N} \)

satisfy

\( X_k^2(2I_{10}, T_1) + Y_k^2(2I_{10}, T_1) = Z_k^2(2I_{10}, T_1), k \in \mathbb{N} \)

and

\( (X_k(2I_{10}, T_1), Y_k(2I_{10}, T_1), Z_k(2I_{10}, T_1)) \neq \left( D^k, B^k, C^k \right), p, q, k \in \mathbb{N}, q \neq 0, D, B, C \in U_{10} \).

VI. \( \Sigma \)-Model

The triples \( (X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \) of the galaxy \( \Sigma(2I_m, U_m) \)

\[
\begin{bmatrix}
X_k(2I_m, A) = 2I_m + 2 \times A^{4k} \\
Y_k(2I_m, A) = 2 \times A^{4k} + A^{8k} \\
Z_k(2I_m, A) = 2I_m + 2 \times A^{4k} + A^{8k}
\end{bmatrix}
\]

\( k \in \mathbb{N}, A \in U_m \)

satisfy

\( X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N}, \)

\( X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m \)

and

\( (X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left( D^k, B^k, C^k \right), p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in U_m \). The multi-galaxy \( \Sigma(2I_m, C_m(\mathbb{N})) \) has an infinite number of planets.

Example 6.1. The triples \( (X_k(2I_m, T_4), Y_k(2I_m, T_4), Z_k(2I_m, T_4)) \) of the galaxy \( \Sigma(2I_m, T_4) \)

\[
\begin{bmatrix}
X_k(2I_m, T_4) = 2I_m + 2 \times T_4^{4k} \\
Y_k(2I_m, T_4) = 2 \times T_4^{4k} + T_4^{8k} \\
Z_k(2I_m, T_4) = 2I_m + 2 \times T_4^{4k} + T_4^{8k}
\end{bmatrix}
\]

\( k \in \mathbb{N} \).
Notes

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\[ X_k^2(2I_m, T_4) + Y_k^2(2I_m, T_4) = Z_k^2(2I_m, T_4), k \in \mathbb{N}, \]
\[ X_0(2I_m, T_4) + Y_0(2I_m, T_4) + Z_0(2I_m, T_4) = 12I_m \]

and

\[ (X_k(2I_m, T_4), Y_k(2I_m, T_4), Z_k(2I_m, T_4)) \neq \left( D^\frac{p}{q}, B^\frac{p}{q}, C^\frac{p}{q} \right), p, q, k \in \mathbb{N}, q \neq 0, \]
\[ D, B, C, T_4 \in U_m. \]

Example 6.2. The triples \((X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3))\) of the sequence

\[ \Sigma(2I_m, T_3) = \begin{bmatrix}
X_k(2I_m, T_3) &=& 2I_m + 2 \times T_3^{4k} \\
Y_k(2I_m, T_3) &=& 2 \times T_3^{3k} + T_3^{8k} \\
Z_k(2I_m, T_3) &=& 2I_m + 2 \times T_3^{4k} + T_3^{8k}
\end{bmatrix} \quad k \in \mathbb{N} \]
\[ (X_0(2I_m, T_3), Y_0(2I_m, T_3), Z_0(2I_m, T_3)) = (4I_m, 3I_m, 5I_m) \]

satisfy

\[ X_k^2(2I_m, T_3) + Y_k^2(2I_m, T_3) = Z_k^2(2I_m, T_3), k \in \mathbb{N}, \]
\[ X_0(2I_m, T_3) + Y_0(2I_m, T_3) + Z_0(2I_m, T_3) = 12I_m \]

and

\[ (X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3)) \neq \left( D^\frac{p}{q}, B^\frac{p}{q}, C^\frac{p}{q} \right), p, q, k \in \mathbb{N}, q \neq 0, \]
\[ D, B, C, T_3 \in U_m, \Sigma(2I_m, T_3) \subset \Sigma(2I_m, U_m) \subset \Sigma(2I_m, C_m(\mathbb{N})). \]

VII. Power Models of Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries of Order 3

A model of a galaxy is a power model if the power of the lead of the model is a power. For example, if we choose \(m_0 = A^{2\lambda k}\) and \(m_0 - n_0 = 2 \times \alpha^2 \times \lambda^2 I_m\), the model

\[ URS(NI_m, U_m, NI_m) = \begin{bmatrix}
X_k(\alpha I_m, A, \lambda I_m) &=& 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda k} \\
Y_k(\alpha I_m, A, \lambda I_m) &=& 2\alpha \times \lambda \times A^{\lambda k} + A^{2\lambda k} \\
Z_k(\alpha I_m, A, \lambda I_m) &=& 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda k} + A^{2\lambda k}
\end{bmatrix} \quad k, \alpha, a, \lambda \in \mathbb{N}, \alpha \neq 0, a \neq 0, \lambda \neq 0, A \in U_m \]
is a power model. The elements of the model \( URS(U_m, U_m, U_m) \) satisfy
\[
X_k^2(\alpha I_m, A, \lambda I_m) + Y_k^2(\alpha I_m, A, \lambda I_m) = Z_k^2(\alpha I_m, A, \lambda I_m), \ k \in \mathbb{N}
\]
and
\[
(X_k(\alpha I_m, A, \lambda I_m), Y_k(\alpha I_m, A, \lambda I_m), Z_k(\alpha I_m, A, \lambda I_m)) \neq \left( D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}} \right), \ k, p, q \in \mathbb{N},
\]
\( p, q, k \in \mathbb{N}, p \neq 0, D, B, C \in U_m. \)

**Example 7.1.** The elements of the galaxy
\[
URS(2I_m, U_m, 2I_m) = \begin{bmatrix}
X_k(2I_m, A, 2I_m) &= 32I_m + 8 \times A^{2k} \\
Y_k(2I_m, A, 2I_m) &= 8 \times A^{2k} + A^{2k+1} \\
Z_k(2I_m, A, 2I_m) &= 32I_m + 8 \times A^{2k} + A^{2k+1}
\end{bmatrix}
\]
satisfy
\[
X_k^2(2I_m, A, 2I_m) + Y_k^2(2I_m, A, 2I_m) = Z_k^2(2I_m, A, 2I_m), \ k \in \mathbb{N}
\]
and
\[
(X_k(2I_m, A, 2I_m), Y_k(2I_m, A, 2I_m), Z_k(2I_m, A, 2I_m)) \neq \left( D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}} \right),
\]
\( p, q, k \in \mathbb{N}, q \neq 0, D, B, C \in U_m. \)

**Example 7.2.** If we choose \( m_0 = A^{2k+1} \) and \( m_0 - n_0 = 2 \times \alpha^2 I_m \), we could construct the galaxy
\[
\Omega(\alpha I_m, U_m) = \begin{bmatrix}
X_k(\alpha I_m, A) &= \alpha^2 I_m + 2 \times \alpha \times A^{2k} \\
Y_k(\alpha I_m, A) &= 2 \times \alpha \times A^{2k} + A^{2k+1} \\
Z_k(\alpha I_m, A) &= \alpha^2 I_m + 2 \times \alpha \times A^{2k} + A^{2k+1}
\end{bmatrix}, \ \alpha \in \mathbb{N}.
\]
The elements of the galaxy \( \Omega(\alpha I_m, U_m) \) satisfy
\[
X_k^2(\alpha I_m, A) + Y_k^2(\alpha I_m, A) = Z_k^2(\alpha I_m, A), \ k \in \mathbb{N}
\]
and
\[
(X_k(\alpha I_m, A), Y_k(\alpha I_m, A), Z_k(\alpha I_m, A)) \neq \left( D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}} \right), \ p, k, q \in \mathbb{N}, q \neq 0,
\]
\( D, B, C \in U_m. \) The galaxy
The characterization of the elements of the set \( F_2(C_m(N)) \) is completely the same as the characterization of the elements of the set \( F_2(N) \) [26].

Let \((X,Y,Z)\) and \((X_1,Y_1,Z_1)\) be two elements of \( F_2(C_m(N)) \). Then

\[
\begin{align*}
(X,Y,Z) \neq (A^{p_k}, B^{p_k}, C^{p_k}),
(X_1,Y_1,Z_1) \neq (A^q_1, B^q_1, C^q_1),
(X,Y,Z) \neq (X_1,Y_1,Z_1),
\end{align*}
\]

\( A, B, C, A_1, B_1, C_1 \in F_2(C_m(N)) \).

Remark 7.3. Let \((X,Y,Z)\) and \((X_1,Y_1,Z_1)\) be two elements of \( F_2(C_m(N)) \).

Then

\[
\begin{align*}
(X,Y,Z) \neq (A^{p_k}, B^{p_k}, C^{p_k}),
(X_1,Y_1,Z_1) \neq (A^q_1, B^q_1, C^q_1),
(X,Y,Z) \neq (X_1,Y_1,Z_1),
\end{align*}
\]

\( A, B, C, A_1, B_1, C_1 \in F_2(C_m(N)) \).

Let us observe that the characterization of one element of the set \( F_2(C_m(N)) \) allows us to deduce the characterization of the elements of the set \( F_2(C_m(N)) \). In other words, the set \( F_2(C_m(N)) \) has no power elements. Remark 7.3 allows us to prove the following result:

**Theorem 7.4.** The equation

\[
X^{2n} + Y^{2n} = Z^{2n}, \quad n \in \mathbb{N}(n \geq 2)
\]

has no circulant matrix with positive integers as entries solutions.

**Proof.** Assume that there exist \( X, Y, Z \in C_m(N) \) such that

\[
X^{2n} + Y^{2n} = Z^{2n}, \quad n \geq 2, \quad n \in \mathbb{N}.
\]

This means that

\[
(X^n)^2 + (Y^n)^2 = (Z^n)^2.
\]

Therefore,

\[
(X^n, Y^n, Z^n) \in F_2(C_m(N)) = \{(A, B, C) \in C_m(N)^3 : A^2 + B^2 = C^2\}.
\]

Remark 7.3 allows us to claim that we have a contradiction because the universe \( F_2(C_m(N)) \) has no power elements. Finally, there exist no circulant matrices with positive integers as entries \( X, Y, Z \in C_m(N) \) such that

\[
X^{2n} + Y^{2n} = Z^{2n}, \quad n \in \mathbb{N}, \quad n \geq 2.
\]

This result allows to claim that the equation

\[
(X^2)^n + (Y^2)^n = (Z^2)^n, \quad n \geq 2,
\]

has no solution in \( C_m(N) \). We can now prove our main result.
Proof of Theorem 1.1

We just need to show that if \((X, Y, Z) \in F_n(C_m(\mathbb{C})), n \in \mathbb{N}, n \geq 3\), then \((X, Y, Z) \notin C_m(\mathbb{N})\). Let \((X, Y, Z)\) be an element of the universe \(F_n(C_m(\mathbb{C})), n \geq 3\). Then

\[X^n + Y^n = Z^n.\]

This implies that

\[(\sqrt{X})^{2n} + (\sqrt{Y})^{2n} = (\sqrt{Z})^{2n} \iff (X^{2n})^{\frac{1}{2}} + (Y^{2n})^{\frac{1}{2}} = (Z^{2n})^{\frac{1}{2}}.\]

and

\[(X^{2n})^{\frac{1}{2}} + (Y^{2n})^{\frac{1}{2}} = (Z^{2n})^{\frac{1}{2}} \iff (X^{\frac{n}{2}})^2 + (Y^{\frac{n}{2}})^2 = (Z^{\frac{n}{2}})^2.\]

Theorem 7.4 and Remark 7.3 allow us to claim that

\[(\sqrt{X}, \sqrt{Y}, \sqrt{Z}) \notin F_2(C_m(\mathbb{N})), (X^{2n}, Y^{2n}, Z^{2n}) \notin F_2(C_m(\mathbb{N}))\]

and

\[(X^{\frac{n}{2}}, Y^{\frac{n}{2}}, Z^{\frac{n}{2}}) \notin F_2(C_m(\mathbb{N})), n \geq 3,\]

since \(F_2(C_m(\mathbb{N}))\) has no power elements. In other words,

\[(\sqrt{X}, \sqrt{Y}, \sqrt{Z}) \notin C_m(\mathbb{N}), (X^{2n}, Y^{2n}, Z^{2n}) \notin C_m(\mathbb{N})\]

and

\[(X^{\frac{n}{2}}, Y^{\frac{n}{2}}, Z^{\frac{n}{2}}) \notin C_m(\mathbb{N}), n \geq 3.\]

The fact that

\[(X^{2n}, Y^{2n}, Z^{2n}) \notin C_m(\mathbb{N}), n \geq 3\]

implies that

\[(X, Y, Z) \notin C_m(\mathbb{N}).\]

VIII. Eigenvalues of Circulant Matrices

It is well known that if \(A = C(\Omega)\), where \(\Omega\) is a compact Hausdorff space, then \(\sigma(f) = f(\Omega)\) for all \(f \in A\). Let

\[\varphi(z) = \sum_{k=0}^{m-1} a_k z^k\]

be a complex polynomial over \(\mathbb{D}\). Then \(\sigma(\varphi) = \varphi(\mathbb{D})\).
The Spectral Mapping Theorem 1. [40]. Let $T \in B(H)$ be a normal bounded linear operator on the Hilbert space $H$ and let $f : \sigma(T) \rightarrow \mathbb{C}$ be a continuous function on $\sigma(T)$. Then $\sigma(f(T)) = f(\sigma(T))$.

Let us introduce the well known spectrum of circulant matrices associated to complex polynomials over $\mathbb{D}$. Let

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

be an $m \times m$- matrix. The matrix $P$ is normal. Indeed, $PP^* = P^*P = I_m$. Assume that

$$A_0 = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & \cdots & a_{m-2} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & \cdots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{m-1} & a_0 \end{pmatrix}$$

A simple calculation shows that

$$A_0 = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.$$ 

The matrix $A_0$ is considered as a polynomial of one variable. Let us compute
the spectrum of the normal matrix $P$. Let

$$f(\lambda) = \det(P - \lambda I_m) = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\lambda \\ 1 & 0 & \cdots & 0 & 0 & -\lambda \end{vmatrix} = 1 - \lambda^m.$$ 

be the characteristic polynomial of $P$. Thus, $\lambda$ is a primitive $m$-th root of unity. Therefore,

$$\sigma(P) = \left\{ \lambda_k^P = e^{2\pi ki/m} : k = 0, 1, 2, \ldots, m-1 \right\}.$$ 

In other words,

$$\sigma(P) = \left\{ 1, e^{2\pi i/m}, e^{4\pi i/m}, e^{6\pi i/m}, e^{8\pi i/m}, e^{10\pi i/m}, e^{12\pi i/m}, e^{14\pi i/m}, \ldots, e^{2(m-1)\pi i/m} \right\}.$$ 

Finally,

$$\sigma(P) = \{ \lambda_0^P, \lambda_1^P, \ldots, \lambda_{m-1}^P \} \subset \mathbb{D}.$$ 

The spectral mapping Theorem allows us to claim that

$$\varphi(\sigma(P)) = \sigma(\varphi(P)) = \sigma(A_0).$$

Therefore,

$$\sigma(A_0) = \{ \varphi(\lambda_0^P), \varphi(\lambda_1^P), \ldots, \varphi(\lambda_{m-1}^P) \}.$$ 

IX. GALAXY OF SEQUENCES OF EIGENVALUES OF CIRCULANT MATRICES

In this section, we construct galaxies of sequences of eigenvalues of circulant matrices.

**Theorem 9.1.** Let $X, Y, Z \in C_m(\mathbb{C})$ be three circulant matrices with complex numbers as entries such that

$$X^n + Y^n = Z^n.$$ 

Then

$$(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, \ldots, m-1.$$
In other words, the triples \((\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m - 1\). That is, the planet system

\[
M(X, Y, Z) = \left[ \begin{array}{ccc}
\lambda_k^X & 0 & 0 \\
0 & \lambda_k^X & 0 \\
\vdots & \ddots & \ddots \\
0 & 0 & \lambda_k^{m-2} \\
0 & 0 & \lambda_k^{m-1}
\end{array} \right] 
\subset \mathbb{F}_n(\mathbb{C}).
\]

Proof: Let \((X, Y, Z) \in C_m(\mathbb{C}), n \in \mathbb{N}, n \geq 3\), be an element of the universe \(\mathbb{F}_n(C_m(\mathbb{C}))\). The spectral factorization of the matrices \(X, Y, Z\) [36, 37, 38] allows us to claim that there exists a unitary matrix \(U\) such that

\[
X = U \left( \begin{array}{cccccc}
\lambda_0^X & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_1^X & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & \lambda_{m-2}^X & 0 \\
0 & \cdots & 0 & \lambda_{m-1}^X & 0
\end{array} \right) U^* = UD_X U^*,
\]

\[
Y = U \left( \begin{array}{cccccc}
\lambda_0^Y & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_1^Y & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & \lambda_{m-2}^Y & 0 \\
0 & \cdots & 0 & \lambda_{m-1}^Y & 0
\end{array} \right) U^* = UD_Y U^*,
\]

and

\[
Z = U \left( \begin{array}{cccccc}
\lambda_0^Z & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_1^Z & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & \lambda_{m-2}^Z & 0 \\
0 & \cdots & 0 & \lambda_{m-1}^Z & 0
\end{array} \right) U^* = UD_Z U^*.
\]

The equation \(X^n + Y^n = Z^n\) implies

\[
UD_X^n U^* + UD_Y^n U^* = UD_Z^n U^*.
\]
It follows that

\[ U \left[ D_X \alpha + D_Y \alpha \right] U^* = U D_Z \alpha U^*. \]

We can claim that

\[ D_X \alpha + D_Y \alpha = D_Z \alpha. \]

Finally,

\[(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, ..., m-1.\]

In other words, the triples \((\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m - 1\). That is, the planet system

\[ M(X, Y, Z) = \begin{bmatrix} \lambda_k^X \\ \lambda_k^Y \\ \lambda_k^Z \\ k = 0, 1, 2, ..., m - 1 \end{bmatrix} \subset \mathbb{F}_n(\mathbb{C}). \]

Every triple \((X, Y, Z)\) of the universe \(\mathbb{F}_n(C_m(\mathbb{C}))\) generates a planet system \(M(X, Y, Z)\) which has exactly \(m\) elements of the universe \(\mathbb{F}_n(\mathbb{C})\). We can say that the galaxies of sequences of circulant matrices are linked to the galaxies of sequences of eigenvalues of circulant matrices. Let us consider the galaxies

\[ \Omega(\alpha I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2k} + A^{2k+1} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2k} + A^{2k+1} \end{bmatrix}, \alpha \in \mathbb{N}, \]

of circulant matrices. We can construct the galaxies of sequences of eigenvalues of the triples of circulant matrices of the galaxies \(\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}\). For example, the galaxies

\[ \Omega(\alpha, \sigma(C_m(\mathbb{N}))) = \begin{bmatrix} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2k} + \lambda^{2k+1} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2k} + \lambda^{2k+1} \end{bmatrix} \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N}, \]

are galaxies of sequences of eigenvalues of triples of circulant matrices of the galaxies \(\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}\). As we can see that the galaxies

\[ \Omega(\alpha, \sigma(A)) = \begin{bmatrix} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2k} + \lambda^{2k+1} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2k} + \lambda^{2k+1} \end{bmatrix} \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N}, A \in C_m(\mathbb{N}), \]
have each a finite number of planet systems. In our case, each galaxy has 
$m$ planet systems. Every galaxy of the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ generates a new
galaxy of eigenvalues of elements of $C_m(\mathbb{N})$. Let us consider the galaxy

$$
\Sigma(2I_m, C_m(\mathbb{N})) = \begin{bmatrix}
X_k(2I_m, A) = 2I_m + 2 \times A^{4k} \\
Y_k(2I_m, A) = 2 \times A^{4k} + A^{8k} \\
Z_k(2I_m, A) = 2I_m + 2 \times A^{4k} + A^{8k}
\end{bmatrix}.
$$

We know that the triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A))$ of the galaxy
$\Sigma(2I_m, C_m(\mathbb{N}))$ satisfy

$$
X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N}.
$$

Define the galaxy

$$
\Sigma(2, \sigma(C_m(\mathbb{N}))) = \begin{bmatrix}
X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\
Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\
Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k}
\end{bmatrix}. 
$$

The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$ of the galaxy $\Sigma(2, \sigma(C_m(\mathbb{N})))$ satisfy

$$
X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.
$$

We can deduce the galaxies

$$
\Sigma(2, \sigma(A)) = \begin{bmatrix}
X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\
Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\
Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k}
\end{bmatrix}, A \in C_m(\mathbb{N}),
$$

which have a finite number of planet systems. The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$
of the galaxy $\Sigma(2, \sigma(A))$ also satisfy

$$
X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.
$$

The first eigenvalue of every matrix of $C_m(\mathbb{N})$ is a positive integer.

**Theorem 9.2.** Let $A \in C_m(\mathbb{N})$ be a circulant matrix with positive integers
as entries. Then the first eigenvalue $\lambda_0^A$ of $A$ is a positive integer. In other
words, $\lambda_0^A \in \mathbb{N}$.
Proof. Let
\[
A = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\
a_{m-1} & a_0 & a_1 & \cdots & a_{m-2} & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
a_2 & \cdots & a_{m-1} & a_0 & a_1 & \cdots \\
a_1 & a_2 & \cdots & a_{m-1} & a_0 & \cdots
\end{pmatrix} \in C_m(\mathbb{N})
\]
be a circulant matrix with positive integers as entries. Then there exists a polynomial

\[
\varphi(z) = \sum_{k=0}^{m-1} a_k z^k, a_k \in \mathbb{N}, z \in \mathbb{D},
\]
such that

\[
A = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.
\]

We know that

\[
\sigma(A) = \{ \varphi(\lambda_0^P), \varphi(\lambda_1^P), \ldots, \varphi(\lambda_{m-1}^P) \}
\]

with

\[
\{ \lambda_0^P, \lambda_1^P, \ldots, \lambda_{m-1}^P \} = \left\{ 1, e^{\frac{2\pi i}{m}}, e^{\frac{4\pi i}{m}}, e^{\frac{6\pi i}{m}}, e^{\frac{8\pi i}{m}}, e^{\frac{10\pi i}{m}}, e^{\frac{12\pi i}{m}}, \ldots, e^{\frac{2(m-1)\pi i}{m}} \right\}.
\]

Therefore,

\[
\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in \mathbb{N}.
\]

Remark 9.3. Let \(A\) be an algebra and let \(A \in C_m(A)\). Then

\[
\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in A.
\]

Theorem 9.1 and Theorem 9.2 allow us to provide another proof of our main result.
Second Proof of Theorem 1.1

Assume that there exist $X, Y, Z \in C_m(\mathbb{N}), n \in \mathbb{N}, n \geq 3$, three circulant matrices with positive integers as entries such that

$$X^n + Y^n = Z^n.$$

Theorem 9.1 and Theorem 9.2 allow us to claim that

$$(\lambda_0^X)^n + (\lambda_0^Y)^n = (\lambda_0^Z)^n, n \geq 3.$$  

This implies that the equation $x^n + y^n = z^n, n \geq 3$ has positive integer solutions. We have a contradiction. Therefore, the equation

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no circulant matrix with positive integers as entries solutions.

Let $A$ be an algebra and let

$$\mathcal{P}^{(m)}(A) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in A, z \in \mathbb{D} \right\}$$

be the algebra of polynomials over $\mathbb{D}$. Complex polynomials of the algebra $\mathcal{P}^{(m)}(\mathbb{N})$ allow us to provide Fermat’s Last Theorem for eigenvalues of circulant matrices.

Theorem 9.4. The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no positive integer eigenvalues of circulant matrices solutions.

Proof. Assume that there exists a triple $(\lambda, \eta, \mu)$ of positive integer eigenvalues of circulant matrices $X, Y$ and $Z$ of $C_m(\mathbb{N})$ such that

$$\lambda^n + \eta^n = \mu^n, \lambda \eta \mu \neq 0, n \in \mathbb{N}, n \geq 3, \lambda \in \sigma(X), \eta \in \sigma(Y), \mu \in \sigma(Z).$$

Therefore, there exist three $f, g, h$ complex polynomials of $\mathcal{P}^{(m)}(\mathbb{N})$ such that

$$f(z)^n + g(z)^n = h(z)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}.$$  

In particular,

$$f(P)^n + g(P)^n = h(P)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}$$
with \(P\) the cyclic permutation \(m \times m\)-matrix given by

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

In other words,

\[X^n + Y^n = Z^n, \quad XYZ \neq 0, \quad n \in \mathbb{N}, n \geq 3.\]

We have a contradiction. Finally, The equation

\[x^n + y^n = z^n, \quad xyz \neq 0, \quad n \in \mathbb{N}(n \geq 3)\]

has no positive integer eigenvalues of circulant matrices solutions.

**X. Fermat's Last Theorem for Complex Polynomials associated to Circulant Matrices**

We can now construct models of galaxies of complex polynomials associated to circulant matrices. Recall that

\[\mathcal{P}^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.\]

The galaxies of the universe \(\mathbb{F}_n(C_m(\mathbb{C}))\) generate the galaxies of the universe \(\mathbb{F}_n(\mathcal{P}^{(m)}(\mathbb{C}))\). For example, from the galaxy \(\Sigma(2I_m, C_m(\mathbb{C}))\), we can construct the galaxy

\[
\Sigma(2, \mathcal{P}^{(m)}(\mathbb{C})) = \left[\begin{array}{c}
X_k(f) = 2 + 2 \times f^{4k} \\
Y_k(f) = 2 \times f^{4k} + f^{8k} \\
Z_k(f) = 2 + 2 \times f^{4k} + f^{8k} \\
k \in \mathbb{N}, f \in \mathcal{P}^{(m)}(\mathbb{C})
\end{array}\right] \subset \mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C})).
\]

We can continue doing the same identification process with the remaining galaxies of \(\mathbb{F}_2(C_m(\mathbb{C}))\). This process will lead to the construction of the universe \(\mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C}))\). Now, we are able to provide Fermat's Last Theorem for complex polynomials over the unit disk \(\mathbb{D}\) associated to circulant matrices of the set \(C_m(\mathbb{N})\).
The equation

\[ x^n + y^n = z^n, \quad xyz \neq 0, \quad n \in \mathbb{N}(n \geq 3) \]

has no solutions in \( \mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0 \).

**Proof.** Assume that there exists a triple \((f, g, h)\) of complex polynomials of the set \( \mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0 \), such that

\[ f(z)^n + g(z)^n = h(z)^n, \quad n \in \mathbb{N}, n \geq 3, \quad z \in \mathbb{D}. \]

This implies that

\[ f(P)^n + g(P)^n = h(P)^n, \quad n \in \mathbb{N}, n \geq 3. \]

with \( P \) the cyclic permutation \( m \times m \)-matrix given by

\[
 P = \begin{pmatrix}
 0 & 1 & 0 & \ldots & 0 & 0 \\
 0 & 0 & 1 & \ldots & 0 & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \ldots & 0 & 0 & 1 \\
 1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

In other words, there exist \( X, Y, Z \in C_m(\mathbb{N}) \) such that

\[ X^n + Y^n = Z^n, \quad XYZ \neq 0, \quad n \in \mathbb{N}, n \geq 3. \]

We have a contradiction. Finally, The equation

\[ x^n + y^n = z^n, \quad xyz \neq 0, \quad n \in \mathbb{N}(n \geq 3) \]

has no solutions in \( \mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0 \).

Theorem 1.1, Theorem 9.4 and Theorem 10.1 are equivalent.

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Derived Subgroup and Direct Product of Groups Embedded into Wreath Product

By Enoch Suleiman & Muhammed Salihu Audu

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Abstract- In this paper, we showed that derived subgroup and direct product groups can be embedded into wreath products of groups with examples.

Keywords: derived subgroup, direct product, wreath product, homomorphisms, embedding.

GJSFR-F Classification: DDC Code: 658 LCC Code: HD31

Strictly as per the compliance and regulations of:
Derived Subgroup and Direct Product of Groups Embedded into Wreath Product

Enoch Suleiman α & Muhammed Salihu Audu σ

Abstract: In this paper, we showed that derived subgroup and direct product groups can be embedded into wreath products of groups with examples.

Keywords: derived subgroup, direct product, wreath product, homomorphisms, embedding.

I. Introduction

Over the years, many people have worked on wreath products and embedment of groups into wreath products as seen in [3,4,6,7,8], in this work we considered the case where derived subgroup and direct product are embedded into wreath products.

II. Basic Definition

If \( G \) and \( H \) are groups, then \( G \times H \) is a group called the Direct Product of \( G \) and \( H \) where \( G \times H = \{(g,h) | g \in G, h \in H\} \) and multiplication is defined by

\[
(g_1,h_1)(g_2,h_2) = (g_1g_2,h_1h_2)
\]  

(1)

If \( 1_G \) is the identity for \( G \), and \( 1_H \) is the identity for \( H \), then \((1_G,1_H)\) is the identity for \( G \times H \) and \((g,h)^{-1} = (g^{-1},h^{-1})\).

If \( \Gamma \) and \( \Delta \) are nonempty sets, then we call \( \Gamma^\Delta \) to denote the set of all functions from \( \Delta \) to \( \Gamma \). In the case that \( \mathcal{C} \) is a group, we turn \( \mathcal{C}^\Delta \) into a group by defining product “pointwise”

\[
f \cdot g (\gamma) := f(\gamma)g(\gamma)
\]  

(2)

for all \( f, g \in \mathcal{C}^\Delta \) and \( \gamma \in \Delta \) where the product in the right is in \( \mathcal{C} \).

Let \( \mathcal{C} \) and \( D \) be groups and suppose \( D \) acts on the nonempty set \( \Delta \). Then the wreath product of \( \mathcal{C} \) by \( D \) is defined with respect to this action is defined to be the semi direct product \( \mathcal{C}^\Delta \rtimes D = CwrD \) where \( D \) acts on the group \( \mathcal{C}^\Delta \) via

\[
f^d(\gamma) := f(\gamma^d^{-1})
\]  

(3)

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for all \( f \in C^\Delta, \gamma \in \Delta \) and \( d \in D \) and multiplication for all \((f_1, d_1), (f_2, d_2) \in C \wr D \) is given by

\[
(f_1, d_1)(f_2, d_2) = \left(f_1 f_2^{d_1^{-1}}, d_1 d_2\right) \quad \text{..........................(4)}
\]

Clearly,

\[
|C \wr D| = |C|^{|\Delta|}|D| \quad \text{..........................(5)}
\]

A homomorphism \( \phi: G \to H \) that is one-to-one(injective) is called an embedding: the group \( G \) “embeds” into \( H \) as a subgroup. If \( \phi \) is not one-to-one, then it is a quotient. Note that if \( \phi: G \to H \) is an embedding, then \( \ker(\phi) = \{e_G\} \) and from the First Isomorphism Theorem, \( \text{Im}(\phi) \cong G/\{e_G\} \cong G \). Now \( \text{Im}(\phi) \leq H \) as \( \phi: G \to H \) is a homomorphism, and so we conclude that in an embedding, \( G \) is isomorphic to a subgroup of \( H \).

### III. Main Results

**Proposition 1:** If \( A \) is an abelian group, then the derived subgroup of the wreath product \( A \wr C_2 \) is embedded into the wreath product.

**Proof:** Since \( A \) is an abelian group, then the derived subgroup \( \{(a, a^{-1}): a \in A\} \) of the base group \( A^2 \), of the wreath product which is clearly isomorphic to \( A \) (See [5]). Thus embedded in \( A \wr C_2 \), as \( A \) is isomorphic to a subgroup of \( A \wr C_2 \).

**Example:** Let \( A := \{(12), (34)\} = \{(1), (12), (34), (12)(34)\} \) which is abelian and \( C_2 := \{(12)\} = \{(1), (12)\} \) then the Wreath Product

\[
A \wr C_2 = \{(12), (34), (56), (78), (15)(26)(37)(48)\}
\]

\[
= \{(1), (78), (56), (56)(78), (34), (34)(78), (34)(56), (34)(56)(78), (12), (12)(78),
\]

\[
(12)(56), (12)(56)(78), (12)(34), (12)(34)(78), (12)(34)(56), (12)(34)(56)(78),
\]

\[
\]

\[
\]

\[
\]

which is a group of order 32. Then the derived subgroup is

\[
((12)(34)(56)(78), (12)(56)) = \{(1), (34)(78), (12)(56), (12)(34)(56)(78)\} \cong A.
\]

**Proposition 2:** Let \( A \) be a direct product of \( p - 1 \) cyclic groups of order \( p^n \), then \( A \) is embedded into the wreath product \( W = C_{p^n} \wr C_p \).

**Proof:** Since \( A \) is a direct product of \( p - 1 \) cyclic groups and \( W = C_{p^n} \wr C_p \), then \( W' \cong A \) (See [2]). Now since \( W' \subseteq W \), then \( A \) is embedded in \( W = C_{p^n} \wr C_p \).

**Example:** Let \( p = 3 \) and \( n = 2 \). Then we have: \( C_3 = \{(123)\} = \{(1), (123), (132)\} \) and \( C_{3^2} = C_9 = \{(123456789)\} = \{(1), (123456789), (135792468), (147)(258)(369),
\]

\[
(159483726), (162738495), (174)(285)(396), (186429753), (198765432)\}. \) Then the Wreath Product
\[ W = C_9 \text{ wr } C_3 \]
\[ = \langle (123456789), (10 11 12 13 14 15 16 17 18), (19 20 21 22 23 24 25 26 27), (1 10 19)(2 11 20)(3 12 21)(4 13 22)(5 14 23)(6 15 24)(7 16 25)(8 17 26)(9 18 27) \rangle \]

which is a group of order 2187 and the derived subgroup
\[ W' = \langle (123456789)(10 18 17 16 15 14 13 12 11), (10 11 12 13 14 15 16 17 18)(19 27 26 25 24 23 22 21 20) \rangle \]

Which is a group of order 81 and it isomorphic to
\[ C_9 \times C_9 = \langle (123456789), (10 11 12 13 14 15 16 17 18) \rangle \]
which is also a group of order 81.

IV. Conclusion

We proved with examples how derived subgroup and direct product of groups were embedded into wreath products.

Acknowledgment

The authors are grateful to Prof. B. Sury of the Indian Statistical Institute, Bengalore, India for suggestions and helpful insight.

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The General Service Readiness in Health Facilities: Evidence based on Bangladesh Health Facility Survey, 2017 Data

By Tazia Hossain
Notre Dame University

Abstract- Although health outcomes have improved in the past few decades, still Bangladesh is working towards the target of sustainable goal 3 (SGD 3). In the recent age, anew insight is at hand that having access to the health care is not that having the quality care. Increasing people expectations, health need and determined new health goals are raising the agenda for health systems to insure better health outcomes and bigger social value. The objective of the study is to give center attention to the percentage attended in basic amenities as well as the average percentage of general service readiness to ensure high quality health systems for Bangladesh in the SDG era. In this study different domains of general service readiness of high quality care are explored, found different resource constrains and suggestions are given to improve quality care. This study found that the average general service readiness score of health facilities in Bangladesh is 47.202 % (basic amenities = 47.1667 %, basic equipment = 78.55%, diagnostic capacity = 10.7833%, for standard precaution is 53.712% and essential medicine = 45.8%). Main concern should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities.

Keywords: general service readiness, basic amenities, bangladesh health facility survey, 2018.

GJSFR-F Classification: LCC Code: KF27
The General Service Readiness in Health Facilities: Evidence based on Bangladesh Health Facility Survey, 2017 Data

Tazia Hossain

Abstract: Although health outcomes have improved in the past few decades, still Bangladesh is working towards the target of sustainable goal 3 (SGD 3). In the recent age, a new insight is at hand that having access to the health care is not that having the quality care. Increasing people expectations, health need and determined new health goals are raising the agenda for health systems to insure better health outcomes and bigger social value. The objective of the study is to give center attention to the percentage attended in basic amenities as well as the average percentage of general service readiness to ensure high quality health systems for Bangladesh in the SDG era. In this study different domains of general service readiness of high quality care are explored, found different resource constrains and suggestions are given to improve quality care. This study found that the average general service readiness score of health facilities in Bangladesh is 47.202% (basic amenities = 47.1667%, basic equipment = 78.55%, diagnostic capacity = 10.7833%, for standard precaution is 53.712% and essential medicine = 45.8%). Main concern should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities. Policymakers should give insights to quality care by improving the facilities up to 100% in health services.

Keywords: general service readiness, basic amenities, Bangladesh health facility survey, 2018.

I. Introduction

Last 20 years is known as the golden age for the global health care system. The major achievement acquired in health determinants (eg, clean water, and sanitation) and health services (eg, vaccination and antenatal care).9-11. High quality care involves diagnosis, appropriate and treatment. High quality health systems require strong financing, trained service providers, service delivery, and also community involvement. Poor-quality care can causes adverse health outcomes, health-related suffering even it build lack of trust and confidence in health systems.

Bangladesh Government is giving continuously effort and working hard to achieve SDG 3. The Government has been taken many initiatives to improve the health condition of our people. Government has established many community clinics from corner to corner the country to ensure better health and give free access to health care to the people. These community clinics provide free health care and free medicines to the people.

Service Provision Assessment (SPA) or health facility surveys have been conducted throughout East and South Asia as well as sub-Saharan Africa to determine primary health care readiness. SPA survey is carried out In Bangladesh named as Bangladesh Health Facility Survey (BHFS).4. To addition the Bangladesh Demographic and Health Survey (BDHS) data by providing useful descriptive...
information on health system focus on reproductive, maternal, child health services, non-communicable diseases etc to measure the quality care and health services at the national level.6

II. Objectives of the Study

The main objectives of the study are to:

• Find out the average percentage of basic amenities, basic equipments, standard precautions for infection prevention, diagnostic capacity and essential medicines given by health facility service.
• Determine the average percentage of general service readiness in health facility care
• Give recommendations to improve or develop the existing strategies to ensure 100% general service readiness in a quality health care.

III. Literature Review

The Lancet Global Health Commission is working for the High Quality Health Systems in this SDG Era. The word Quality of care is a insightful theme. This Commission is focus on improving the quality of care of people.

According to The Lancet Global Health Commission almost 9 million lives are lost in every year for lack of good quality care and that a shocking 60% of those deaths were among them who actually had got access to care. That is why having access to the health care is not that much enough?

According to The Lancent Global Health Commission, high-quality health systems could save more than 8 million lives each year and 2.5 million deaths due to cardiovascular disease, 1 million newborn deaths and 50% of all maternal deaths in each year in the low and middle countries. 1

In low and middle income countries, over 8 million people per year are dying from situation that could be treated by the health system. These deaths causes in US$6 trillion in economic losses in year 2015. Poor health quality care becomes the barrier for reducing mortality. 1

A global report on quality of health care is published earlier in 2018 by WHO, the World Bank, and the Organization for Economic Co-operation and Development (OECD). 12

According to WHO the overall capability of health facilities to provide general health services is defined as the general service readiness. People-centred health systems is a system where all people should have the same access to quality health services they actually needed in their life. 13

The availability of components essential to provide services is known as Readiness, for example, basic amenities, basic equipment, standard precautions for infection prevention, diagnostic capacity and essential medicines. In this study, the general service readiness is defined by five general service readiness domains provided by WHO tracer. 14

IV. Data and Methodology

a) Data

The data is extracted from Bangladesh Health Facility Survey (2017 BHFS) which is nationally representative health facility survey. The information is collected on general facility readiness. A stratified random sample of 1600 observations (health facilities) are selected from 8 division of Bangladesh which includes district hospitals.
(DH), mother and child welfare centers (MCWCs), upazila health complexes (UHCCs), union health and family welfare centers (UHFWCs), union subcenters or rural dispensaries (US or RD), and community clinics (CCs). The survey was conducted under the management of the National Institute of Population Research and Training (NIPORT) of the Ministry of Health and Family Welfare (MOHFW) funded by the Government of Bangladesh and the U.S. Agency for International Development (USAID).

b) Methodology

This study data has been extracted from the 2017 Bangladesh Health Facility Survey (BHFS). The 2017 BHFS is successfully collected a stratified random sample of 1,524 health facilities from all formal-sector health facilities in Bangladesh. It is not possible to measure general service readiness directly. The WHO identifies specific tracers or items to measure the readiness indices. By WHO tracer the domains of general service readiness are basic amenities, basic equipments, standard precautions for infection prevention, diagnostic capacity and essential medicines. A percentage is a number or ratio which is calculated as a fraction of 100. A percentage of facilities in each domain are observed. Then the average of each domain is calculated by adding the percentage of indicators dividing by the number of indicators. Using the average of five domains, the average general service readiness is calculated. The domains are:

**Table 1:** The domain with corresponding indicators for measuring general service readiness

<table>
<thead>
<tr>
<th>Serial No.</th>
<th>Domain</th>
<th>Indicators</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>Basic amenities</td>
<td>improved water supply, emergency transport, computer with internet, regular electricity availability, separate latrine or toilet for female clients, room with privacy, communication equipment</td>
</tr>
<tr>
<td>b.</td>
<td>Basic equipment</td>
<td>thermometer, stethoscope, blood pressure apparatus, adult scale, child scale, light source availability</td>
</tr>
<tr>
<td>c.</td>
<td>Standard precautions for infection prevention</td>
<td>appropriate storage of sharps waste and safe ultimate disposal of sharps, appropriate storage of infectious waste and safe ultimate disposal of infectious wastes, disinfectant and guidelines for standard precautions, single-use disposable/auto-disable syringes, soap and running water or alcohol-based hand rub, latex gloves</td>
</tr>
<tr>
<td>d.</td>
<td>Diagnostic capacity</td>
<td>haemoglobin, blood glucose, urine dipstick for protein, urine dipstick for glucose, syphilis rapid diagnostic test, urine pregnancy test, amitriptyline tablet, amoxicillin (tablets capsules), Atenolol tablets, Captopril tablets, Ceftriaxone injectable, Ciprofloxacin tablets, Cotrimoxazole oral suspension, Diazepam tablets, Diclofenac tablets, Glibenclamide</td>
</tr>
<tr>
<td>e.</td>
<td>Essential medicines</td>
<td></td>
</tr>
</tbody>
</table>
The 2017 BHFS collected several types of health facilities. The seven amenities have been observed in the study. The amenities are: separate latrine or toilet for female clients, emergency transport, computer with internet, regular electricity availability, improved water supply, room with privacy, communication equipment.

Table 2: Frequency distribution of basic amenities and equipment for client services

<table>
<thead>
<tr>
<th>Basic amenities</th>
<th>Percentage (N= 1524)</th>
<th>Basic equipments</th>
<th>Percentage (N= 1524)</th>
</tr>
</thead>
<tbody>
<tr>
<td>improved water supply</td>
<td>90</td>
<td>thermometer</td>
<td>86.3</td>
</tr>
<tr>
<td>emergency transport</td>
<td>5</td>
<td>stethoscope</td>
<td>94.2</td>
</tr>
<tr>
<td>computer with internet</td>
<td>58</td>
<td>blood pressure apparatus,</td>
<td>85.4</td>
</tr>
<tr>
<td>regular electricity availability</td>
<td>43</td>
<td>adult scale</td>
<td>85.9</td>
</tr>
<tr>
<td>room with privacy</td>
<td>70</td>
<td>child scale</td>
<td>61.6</td>
</tr>
<tr>
<td>separate latrine or toilet for female clients</td>
<td>17</td>
<td>light source availability</td>
<td>51.9</td>
</tr>
<tr>
<td>communication equipment</td>
<td>11</td>
<td>All basic equipment</td>
<td>28</td>
</tr>
</tbody>
</table>

The availability of at least 5 basic amenities in health facilities is only 19% (BDHS 2017) which is improved from 11% (BDHF 2014).

The communication equipments (land line or mobile phone) is highest in private hospital (95%), upazila and district level (82%) but very low in community clinics (3%) and overall it is 11%. Available emergency transport is high in district hospital (97%), upazila district level (79%), private hospital (62%) Including all community clinics and NGO facility clinics this percentage is very low (5%). Still the facilities of emergency transport, communication equipment have to be improved in huge scale. The percentage has to be improved to regular electricity availability, computer internet, and separate female latrine for the client’s satisfaction with health services.

The facilities are most likely to have WHO and USAID proposed basic equipments such as thermometer, stethoscope, blood pressure apparatus, adult scale child scale light source availability.
The availability of all six basic equipments in health facilities is 28% (BDHS 2017)) which is improved from 26% (BDHF 2014). This percentage is high in private hospitals and NGO facilities which are 80% or more. Community clinics (CCs) and union level public facilities are meager indicating 23%.

The average general service readiness score of health facilities in Bangladesh for basic amenities is 47.1667% and for basic equipments is 78.55%.

Table 3: Frequency distribution of standard precautions for infection prevention and diagnostic capacity for client services

<table>
<thead>
<tr>
<th>Standard precautions</th>
<th>Percentage (N=1524)</th>
<th>diagnostic capacity</th>
<th>Percentage (N=1524)</th>
</tr>
</thead>
<tbody>
<tr>
<td>safe ultimate disposal of sharps and appropriate storage of sharps waste</td>
<td>72.5</td>
<td>hemoglobin</td>
<td>17</td>
</tr>
<tr>
<td>safe ultimate disposal of infectious wastes</td>
<td>64.1</td>
<td>blood glucose</td>
<td>19.7</td>
</tr>
<tr>
<td>appropriate storage of infectious waste and disinfectant and guidelines for standard precautions,</td>
<td>66.3</td>
<td>urine pregnancy test</td>
<td>12.4</td>
</tr>
<tr>
<td>single-use disposable/auto-disable syringes</td>
<td>32.9</td>
<td>urine dipstick for protein</td>
<td>10.8</td>
</tr>
<tr>
<td>soap and running water or alcohol-based hand rub</td>
<td>17.3</td>
<td>urine dipstick for glucose</td>
<td>10</td>
</tr>
<tr>
<td>latex gloves</td>
<td>76.9</td>
<td>Syphilis rapid diagnostic test</td>
<td>4.8</td>
</tr>
</tbody>
</table>

The average general service readiness score of health facilities in Bangladesh for diagnostic capacity is 10.7833% and for standard precaution is 53.712%.
### Table 4: Frequency distribution of essential medicines for client services

<table>
<thead>
<tr>
<th>Facility type</th>
<th>District and upazila public facilities</th>
<th>Community clinic (CC)</th>
<th>Private hospital</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amitriptyline tablet</td>
<td>7.7</td>
<td>0</td>
<td>57.6</td>
<td>2.4</td>
</tr>
<tr>
<td>Aamoxicillin (tablets capsules)</td>
<td>85</td>
<td>85.6</td>
<td>61.5</td>
<td>83.6</td>
</tr>
<tr>
<td>Atenolol tablets</td>
<td>30</td>
<td>0</td>
<td>61.6</td>
<td>3.9</td>
</tr>
<tr>
<td>Captopril tablets</td>
<td>53.9</td>
<td>0</td>
<td>74.2</td>
<td>6.4</td>
</tr>
<tr>
<td>Ceftriaxone injectable</td>
<td>66</td>
<td>2.8</td>
<td>77.8</td>
<td>9.4</td>
</tr>
<tr>
<td>Ciprofloxacin tablets</td>
<td>96</td>
<td>95.6</td>
<td>78.2</td>
<td>94.8</td>
</tr>
<tr>
<td>Cotrimoxazole oral suspension</td>
<td>73.8</td>
<td>86.7</td>
<td>73.8</td>
<td>82.7</td>
</tr>
<tr>
<td>Diazepam tablets</td>
<td>77.4</td>
<td>6.4</td>
<td>75.1</td>
<td>22.8</td>
</tr>
<tr>
<td>Diclofenac tablets</td>
<td>90.6</td>
<td>60.8</td>
<td>80.4</td>
<td>63.8</td>
</tr>
<tr>
<td>Glibenclamide tablets</td>
<td>41.1</td>
<td>0</td>
<td>69.6</td>
<td>5</td>
</tr>
<tr>
<td>Omeprazole or cimetidine tablets</td>
<td>95.5</td>
<td>93.5</td>
<td>80.7</td>
<td>92.7</td>
</tr>
<tr>
<td>Paracetamol oral suspension</td>
<td>75.5</td>
<td>88.3</td>
<td>79.6</td>
<td>83.7</td>
</tr>
<tr>
<td>Salbutamol inhaler</td>
<td>95.1</td>
<td>89.9</td>
<td>78.5</td>
<td>86.6</td>
</tr>
<tr>
<td>Simvastatin or atorvastatin tablets</td>
<td>24.2</td>
<td>0.2</td>
<td>64.3</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Essential medicines are comparatively more available in private sector than district and upazila level and lower in community clinics. The average score of health facilities for essential medicines is 45.8%.

### VI. Findings

The average of general service readiness is calculated by adding the average percentage of 5 domains, dividing by the number 5 which is 47.202 % (average basic amenities = 47.1667 %, average basic equipment = 78.55%, average diagnostic capacity = 10.7833%, average standard precaution is 53.712% and average essential medicine = 45.8%) less than 50%. By constantly improving existing facilities to
changing population needs a high-quality health system can enhanced in general health care.

**Recommendation**

Bangladesh government has incredible improved in health sector which is valued and trusted by the all over the country. To give 100% service facility Government should:

- Develop our community clinics by improving basic amenities, diagnostic capacity, standard precautions and essential medicines.
- Give special attention to Union level public facilities as union health and family welfare centers (UHFWCs), union subcenters or rural dispensaries (US or RD), and community clinics (CCs) so that these facilities can follow IMCI guidelines properly.
- Ensure the adequacy of basic amenities, standard precautions, especially for improving the diagnostic capacity the authorities of the facilities should take necessary measures.
- Stringently supervise the performances of health providers.

**Further study**

There is enormous scope to work with this study by advanced analysis such as ordinal logistic regression model assuming proportional odds assumption i.e. proportional odds model could have been applied in the study.

**VII. Conclusion**

The study found the significant lacking of the general service readiness in basic amenities: emergency transport, communication equipment. Special attention should be given in different diagnostic capacity improving such as hemoglobin, blood glucose, urine pregnancy test, urine dipstick for protein, urine dipstick for glucose, Syphilis rapid diagnostic test. All the facilities should maintain the disinfectant and guidelines for standard precautions for quality care. Only half of the essential medicines are available in community clinics. Higher general service readiness in private hospitals is observed. This study finds that priority should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities.

**References Références Referencias**

4. NIPORT. (2016). Bangladesh health facility survey 2014 Dhaka, Bangladesh: National Institute of population research and training (NIPORT), associates for community and population research (ACPR), and ICF international.
6. Bangladesh health facility survey 2017 Dhaka, Bangladesh: National Institute of population research and training (NIPORT), associates for community and population research (ACPR), and ICF international.


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22. **Report concluded results:** Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. **Upon conclusion:** Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

**Informal Guidelines of Research Paper Writing**

**Key points to remember:**
- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

**Final points:**

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

**The introduction:** This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

**The discussion section:**

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

**General style:**

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

**To make a paper clear:** Adhere to recommended page limits.
Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don’t address the reviewer directly. Don’t use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

**Abstract:** This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

**Reason for writing the article**—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

**Approach:**

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

**Introduction:**

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.
The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.
Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

**Approach:**

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

### THE ADMINISTRATION RULES

**Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.**

*Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.*

**Segment draft and final research paper:** You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

**Written material:** You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.
## Criterion for Grading a Research Paper (Compilation)

**By Global Journals**

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

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