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Mathematics and Decision Science

Highlights

Non-Decaying Initial Data

The General Service Readiness

Positive Integers as Entries

Pathway Fractional Integral Operator

Discovering Thoughts, Inventing Future

VOLUME 22 ISSUE 2 VERSION 1.0

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Mathematics & Decision Sciences

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Yoga, Meditation and Mental Health well-being during Covid-19 Pandemic

By Priyanka Verma & Sheela Misra
University of North Carolina

Abstract- Yoga and meditation have been playing vital roles in our holistic wellbeing and attaining our spiritual goals since ancient time. During second wave of Covid-19 and lock-down its importance became more significant and visible around the globe. Many patients have easily recovered with the help of their boosted immunity by doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation etc) and could keep themselves stress-free. During and post pandemic maintaining mental health is a great challenge. It is very difficult to be relaxed, peaceful and healthy due to increased level of anxiety, stress and depression as a consequence of loss of health, lives, jobs, migration, inaccessibility to health education and other basic facilities, changes in life styles and so on. Yoga can help us to stay calm, manage our health and anxiety without any extra intervention if understood well and made a part of our daily routine. For the young adults it becomes even more important to do yoga and meditation because they are the future of India.

Keywords: *yoga, meditation, stress, mental health, ordinal logistic regression analysis.*

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Yoga, Meditation and Mental Health well-being during Covid-19 Pandemic

Priyanka Verma [α] & Sheela Misra [σ]

Abstract- Yoga and meditation have been playing vital roles in our holistic wellbeing and attaining our spiritual goals since ancient time. During second wave of Covid-19 and lock-down its importance became more significant and visible around the globe. Many patients have easily recovered with the help of their boosted immunity by doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation etc) and could keep themselves stress-free. During and post pandemic maintaining mental health is a great challenge. It is very difficult to be relaxed, peaceful and healthy due to increased level of anxiety, stress and depression as a consequence of loss of health, lives, jobs, migration, inaccessibility to health education and other basic facilities, changes in life styles and so on. Yoga can help us to stay calm, manage our health and anxiety without any extra intervention if understood well and made a part of our daily routine. For the young adults it becomes even more important to do yoga and meditation because they are the future of India.

We have conducted a survey among the young adults age group (18 – 35 years) of Uttar Pradesh, India by a questionnaire tool with the help of Google form. Questionnaire includes questions related to socio demographic status, yoga, meditation, and mental health well-being during Covid-19 pandemic. This study is an exploratory study based on primary data.

The objective of this survey is to know the proportion of young adults who are doing yoga and meditation regularly, their satisfaction level with it, their changing behaviour, what the benefits of it that they experienced are, and its graphical representation. The ordinal logistic regression analysis carried out to know the satisfaction level of independent variables (Gender, Age, Education Qualification, and Native Area) and satisfaction level as dependent variable.

Keywords: yoga, meditation, stress, mental health, ordinal logistic regression analysis.

I. Introduction

Yoga and meditation has been playing a vital role to fit our physical and mental health since ancient time. During second wave of Covid-19 and lock-down its importance become more than compare to previous. Many patients have easily recovered with the help of habit of doing yoga and meditation (like breathing exercise, Bhramari Pranayama and meditation). During this scenario good mental health is very important to survive because level of anxiety, stress and depression growing very fast. Yoga can help us to stay calm, manage our blood pressure and anxiety. For the young adults it becomes more essential to do yoga and meditation because they are the future of country.

The pandemic has been difficult for everyone. According to the Centers for Disease Control and Prevention, one in five Americans report struggling with mental or behavioral health issues associated with COVID-19, including anxiety, depression,

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increased substance use, and suicidal thoughts. The highest burden of distress has been reported by the younger generation (ages 18-29) and minority communities. [1]

Keeping all these point in mind we have conducted online survey among young adults group with the help of Google form including questions related to yoga and meditation, to know the present scenario regarding habit of yoga, its benefits and satisfaction level of young adults towards mental health well-being.

II. Literature Review

Mental health issues increased during COVID-19 pandemic. Yoga and meditation can help in reducing mental stress and improving psychological wellbeing. The frequency of practice is positively associated with a higher level of mental wellbeing in case of both yoga and meditation, with daily practice having the highest wellbeing scores. [8]

Change in eating and sleeping pattern during the pandemic was significantly higher in people who did not practice yoga and meditation, and it was least in those practicing both. A large proportion of study subjects reported a change in relationship with family members during the COVID-19 pandemic. [8]

A significant effect of duration of practice was found on illness perception, and wellbeing related measures. Long term practitioners reported higher personal control and lower illness concern in contracting COVID-19 than the mid-term or beginner group. The improved physiological functions are believed to reduce stress, anxiety, depression, and enhance overall well-being. [9]

III. Objective

- To find out the proportion of young adults of Uttar Pradesh who are doing yoga and meditation regularly.
- To find out satisfaction level of young adult, how much yoga and meditation is helpful for their mental health well-being.
- To find out Behavior change in young adult during Covid-19 pandemic regarding yoga and meditation.
- To find out benefits of yoga and meditation for young adults (18-35 years).
- To find satisfaction level (dependent variable), ordinal logistic regression analysis used for independent variables (Gender, Age, Education Qualification, and Native Area) and dependent variable (satisfaction level).

IV. Methodology

- A survey was conducted among the young adults (18- 35 years) of Uttar Pradesh, India with the help of Google form using Convenient Sampling Technique (Non-probability sampling technique).
- There were total 11 questions in the Questionnaire including sociodemographic variables (like age, education qualification, gender, district, native area), out of them 4 questions related to yoga and meditation was close-ended and 1 question based on likert- scale (1-5) related to satisfaction level and one question was open-ended (benefits of yoga and meditation).
- Total 203 responses were used for analysis.
- Analysis is based on Frequency, Cross-tab, Ordinal logistic Regression Analysis.
- For analysis SPSS version.21 and Excel software used.

Ref

8. Priyanka, S. K., Rasania.(2021). A cross-sectional study of mental wellbeing with practice of yoga and meditation during COVID-19 pandemic. *Journal of Family Medicine and Primary Care*, IP: 117.197.19.2

Notes

- This study is an exploratory study based on primary data.
- Data come from the following Districts 0f Uttar Pradesh, India
- Agra, Allahabad, Ambedkar Nagar, Amethi, Ballia, Banda. Barabanki, Basti, Bijnor, Bulandshahr, Deoria, Ghazipur, Gonda, Gorakhpur, Hamirpur, Hardoi, Jhansi, Kanpur, Kushinagar, Lakhimpur-Kheri, Lucknow, Maharajganj, Mathura, Muzaffarnagar, Pratapgarh, RaeBareli, Shahjahanpur, Siddharth Nagar, Sitapur, Unnao, Varanasi.
- Maximum data collected from Lucknow district.

V. Interpretation

Table 1: Demographic details of 203 respondent

Variables	**Frequency(%)**
Gender	
Female	54.7
Male	45.3
Others	0
Age	
18 - 24 Years	61.1
24 - 30 Years	30.5
30 - 35 Years	8.4
Education qualification	
Higher than postgraduate	8.4
Intermediate	9.4
Other	2.5
Postgraduate	42.9
Undergraduate	36.9
Districts	
Lucknow	65.5
Other than Lucknow	34.5
Native area	
Rural	28.1
Urban	71.9

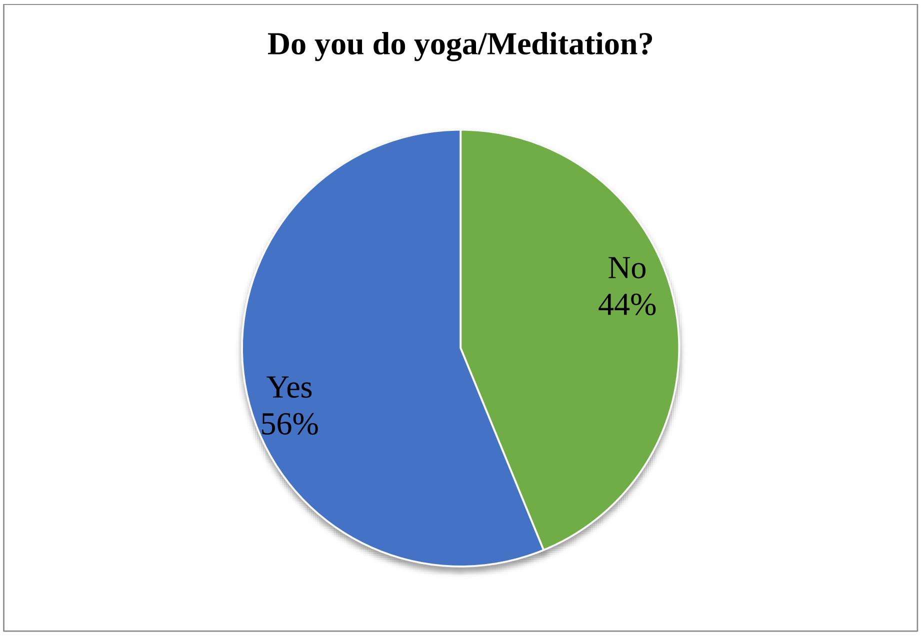

Fig. 1: Out of the 203 respondent 56 percent (114) of respondent said yes, they are doing yoga and meditation and 44 percent of respondent said No

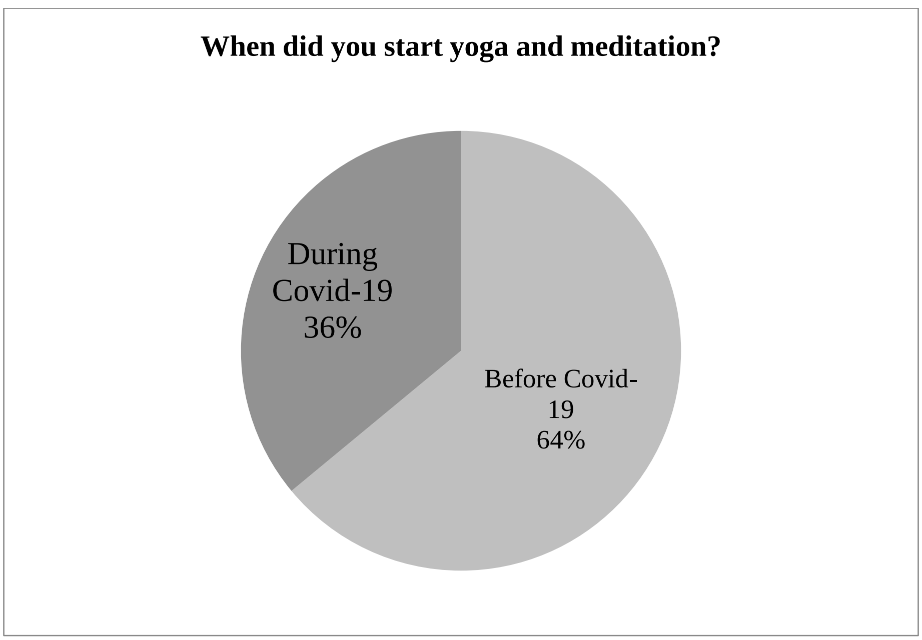

Fig. 2: Out of the 114 respondent there are 36 percent of respondents who have started yoga and meditation during Covid-19 pandemic and 64 percent of respondent started before Covid-19 pandemic

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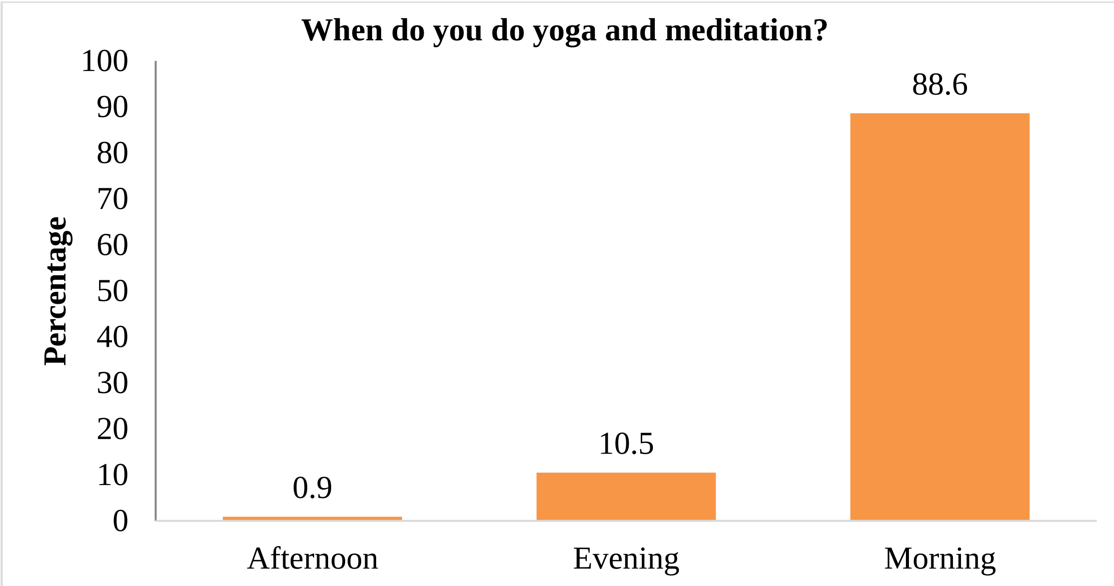

Fig. 3: There are 89 percent of respondent who have doing yoga and meditation in the Morning, 11 percent of respondent are doing it in the Evening and only1 percent of respondent doing yoga and meditation in the Afternoon

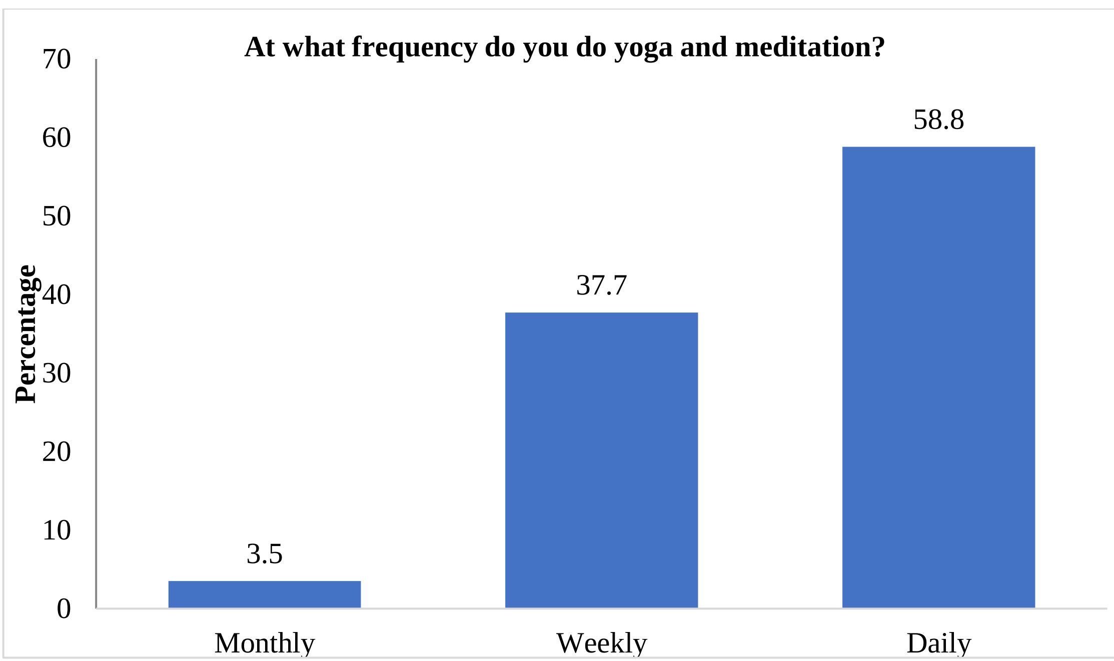

Fig. 4: There are 59 percent respondent who are doing yoga and meditation daily, 38 percent of respondent doing weekly and 4 percent respondent doing in monthly

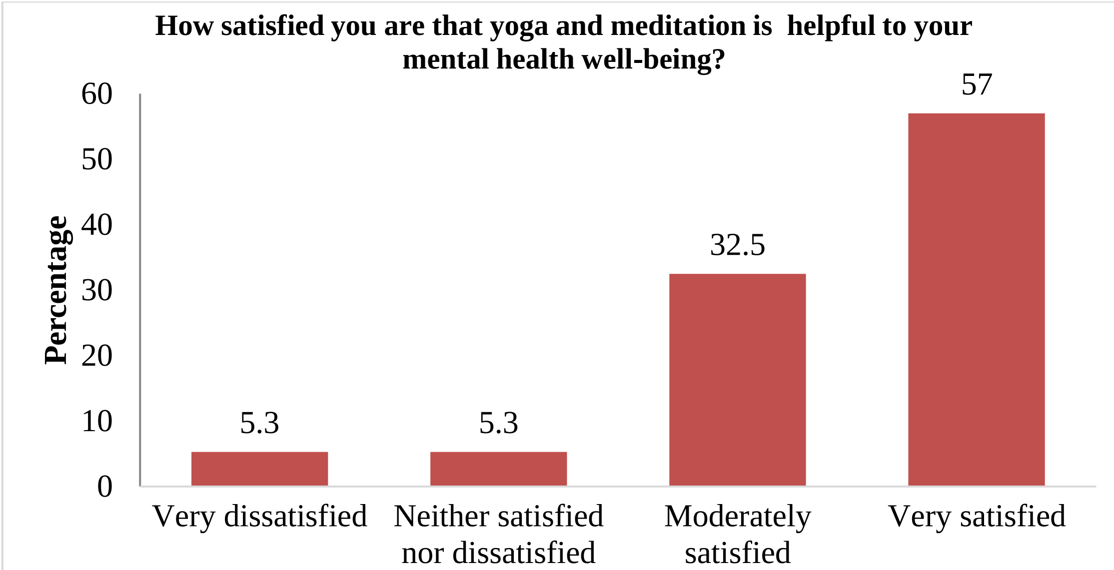

Fig.5: There are 57 percent respondent who are very satisfied that yoga and meditation is helpful to their mental health well-being, 33 percent respondent are moderately satisfied and only 5 percent of respondent are very dissatisfied

VI. Ordinal Logistic Regression Analysis

We have apply Ordinal logistic Regression on the dependent variable (satisfaction level) which is ordinal variable and independent variable (Gender, Age, Education Qualification, and Native Area) which is categorical variable.

To know the satisfaction level of independent variables that yoga and meditation is helpful to their mental health

We find the following results

- Total respondents are 114, who are doing yoga and meditation.
- Out of them 57 percent of respondent who are very satisfied that yoga and meditation is helpful to their mental health well-being and 33 percent are moderately satisfied.
- 43 percent are Male respondent and 57 percent are Female respondent.

For model fitting we have taken-

Ho: there is no significant different between baseline model to final model

The significant value is 0.255 which is > 0.05 so we accept null hypothesis and conclude that there is no significant different between baseline model to final model

For goodness of fit we have taken-

Ho: the observed data is having goodness of fit with the fitted model

Significant value is 0.806 which is > 0.05 so we accept the null hypothesis and conclude that the observed data is having goodness of fit with the fitted model.

From Pseudo R-Square

The Nagel kerke value should be 0.7 but here the value is 0.099, means 0.099 variation proportion of variance the independent variable (Gender, Age, Education Qualification, and Native Area) is explaining on the dependent variable (satisfaction level).

It means more independent variable should be used.

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Notes

Table 2

Parameter Estimates								
		Estimate	Std. Error	Wald	df	Sig.	95% Confidence Interval	
							Lower Bound	Upper
Threshold	[Howsatisfiedyouarethatyogaandmeditationishelpfultoyourmentalheal = 1 (Very dissatisfied)	-3.736	1.538	5.902	1	0.015	-6.751	-0.722
	[Howsatisfiedyouarethatyogaandmeditationishelpfultoyourmentalheal = 3](Neither satisfied nor dissatisfied)	-2.968	1.511	3.862	1	0.049	-5.929	-0.008
	[Howsatisfiedyouarethatyogaandmeditationishelpfultoyourmentalheal = 4](Moderately satisfied)	-0.999	1.49	0.45	1	0.502	-3.92	1.921
Location	[Male=1]	0.59	0.392	2.265	1	0.132	-0.178	1.358
	[Female=2]	0^a	.	.	0	.	.	.
	18 - 24 Years	-1.613	1.349	1.429	1	0.232	-4.258	1.032
	24 - 30 Years	-2.17	1.289	2.835	1	0.092	-4.695	0.356
	30 - 35 Years	0^a	.	.	0	.	.	.
	Intermediate	-0.272	1.633	0.028	1	0.868	-3.472	2.928
	Undergraduate	0.519	1.543	0.113	1	0.737	-2.505	3.543
	Postgraduate	0.846	1.492	0.321	1	0.571	-2.079	3.77
	Higher than postgraduate	0.702	1.538	0.208	1	0.648	-2.314	3.717
	Other	0^a	.	.	0	.	.	.
	[Nativearea=Urban]	0.278	0.409	0.462	1	0.497	-0.524	1.08
	[Nativearea=Rural]	0^a	.	.	0	.	.	.
Link function: Logit.								
a. This parameter is set to zero because it is redundant.								

We can see the satisfaction level of Gender, Age, Education Qualification, and Native Area

For gender null hypothesis was -

Ho: There is no significant difference between the gender towards satisfaction level.

The estimate for male is positive which indicate that male have more positive satisfaction level than female, the sig value for male is 0.132, > 0.05 we accept null hypothesis and conclude that male and female does not have any significant difference in level of satisfaction.

Similarly, we had taken null hypothesis for other independent variables also.

Interpretation from Table 2

- Estimate value of male is positive means male is more positive satisfied than female or we can say female have less satisfied than male. Compare to female male are not significant.
- Students of Undergraduate, Postgraduate, and Higher than Postgraduate having more positive satisfaction level than other education level but Students of Intermediate having less satisfaction than other education level. Compare to students of other education level, Students of Intermediate, Undergraduate, Postgraduate, and Higher than Postgraduate are not significant.

- Students of Urban area having more positive satisfaction level than rural area. Compare to rural area students, students of urban area are not significant.
- Students of (30-35) age group having more satisfaction than other age groups. Compare to (30-35) age group other age groups are not significant.

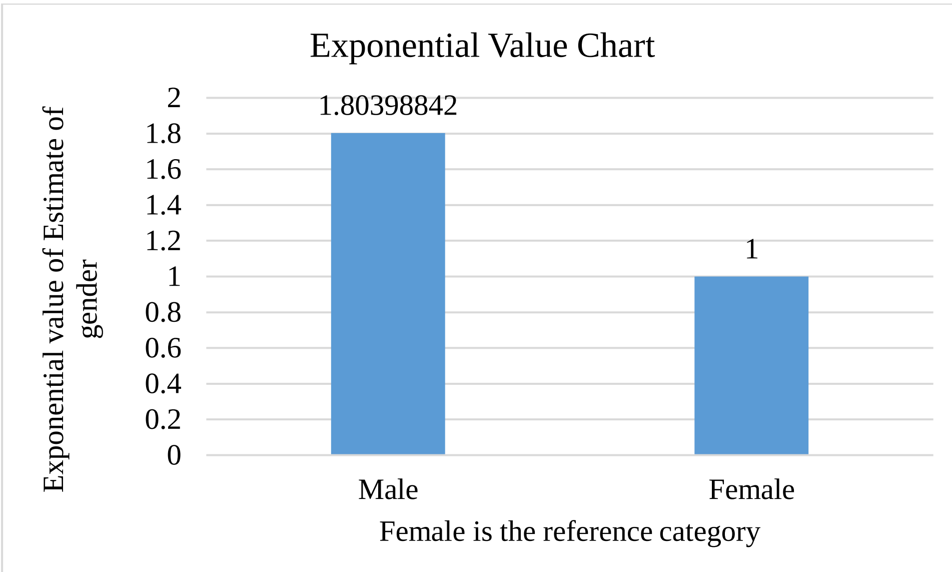

Fig. 6: Male is having 1.764 times more satisfaction than female

Key-Points

Data in Percentage Total 144 (Doing yoga and meditation)	yoga started during Covid-19 (36%)114	Yoga in morning	Yoga on daily	Yoga on weekly	Yoga is helpful for Mental health well-being, Very satisfied
Male	44	56	52	65	51
Female	56	44	48	35	49
18-24 years	63	55	64	42	57
24-30 years	29	39	28	51	32
Rural	29	35	34	28	29
Urban	71	65	66	72	71
Postgraduate	42	44	37	51	45
Undergraduate	27	38	46	21	35
Intermediate	17	7	7	9	6
Higher education	10	10	9	12	11

Benefits of yoga and meditation experience by respondents

- Mind stability, Positive thinking, Healthy lifestyle, Make calm, increase creativity and thinking level, Enhance focus and concentration power, Manage anxiety, stress, and anger, Make us happier, Boost our immune system, Feel energised and active

Notes

- Clarity in thoughts, powerful connection with almighty, Help in weight gain/loss, Help in breathing and good health, increase blood flow, flexibility, Good for physical and mental health, feel relax
- Increase confidence level, fitness of body, improve sleep quality, Healthy and fresh start of the day, Helps to stay away from disease (like fever, cough), Helps in time management, balance life, Improve body posture and stamina.
- Increase overall metabolism of the body, organs of body work properly, Improve decision-making ability, less frequent headache, Maintain discipline in morning routine, Increase body awareness
- It is helpful for recovering from diseases like brain cancer, kidney, stress, heavyweight gain, headache and many more diseases
- Good for heart health, respiratory, digestion system, Keep glowing skin, getting to know one own self, back pain relief.

VII. Conclusion

- Out of 203 young adult respondent more than half (56%) respondent are doing yoga and meditation. So we need to generate awareness among young adults about importance and benefits of doing yoga and meditation on regular basis, so that we can get much better data for the same.
- From analysis we find positive result that out of the 114 respondents
  - there are 36 % respondents who have start doing yoga and meditation during Covid-19 pandemic out of them 44% are male and 56% are female, and 64 percent have started before the pandemic. This indicates the behavior change in young adults during pandemic.
  - Young adults prefer to do yoga and meditation in the morning (89%) and 11 % prefer to do it in evening time. Means morning and evening are the good time to do yoga and meditation.
  - 59 % respondent doing yoga and meditation on Daily basis. This is also a good result.
  - 57 % respondent are very satisfied that yoga and meditation is helpful to their mental health well-being. This shows that yoga and meditation are actually helpful to our mental health well-being.
- From ordinal logistic regression analysis
  - Since the significance value of Model Fitting information is 0.255 which is ¿0.05, so we are unable to fit a good model.
  - Since the significance value of Goodness of fit is 0.806 which is ¿0.05, so we accept the null hypothesis and conclude that the observed data is having goodness of fit with the fitted model.
  - Further we can go with more independent variable for better result because Pseudo R-square should be >=0.7 and data results 0.099
  - Male and female does not have any significant difference in level of satisfaction, from Parameter Estimate.
  - Male have more positive satisfaction level than female.
  - With the help of exponential value chart we conclude that male is having 1.80398842 times more satisfaction than female.

- Students of Undergraduate, Postgraduate, and Higher than Postgraduate having more positive satisfaction level than other education level but Students of Intermediate having less satisfaction than other education level.
- Students of Urban area having more positive satisfaction level than rural area.
- Students of (30-35) age group having more satisfaction than other age groups.

- In the same way we can conduct survey for other states also.
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Life Span of Solutions for a Time Fractional Reaction-Diffusion Equation with Non-Decaying Initial Data

By Takefumi Igarashi
Nihon University

Abstract- We consider the Cauchy problem of time fractional reaction-diffusion equation

$$\partial_t^\alpha u = \Delta u + u^p \quad \text{in } \mathbf{R}^n \ (n \geq 1),$$

where $0 < \alpha < 1$, $p > 1$ and ∂_t^α denotes the Caputo time fractional derivative of order α. The initial condition u_0 is assumed to be nonnegative and bounded continuous function. For the non-decaying initial data at space infinity, we show that the positive solution blows up in finite time and give the estimate of the life span of positive solutions. It is also given blow-up time of the solutions when the initial data attain its maximum at space infinity.

Keywords: life span, fractional diffusion equation, Cauchy problem, non-decaying initial data, blow-up.

GJSFR-F Classification: *MSC 2020: 35B44, 35K15, 35R11, 26A33, 35K57*

LIFESPANOFSOLUTIONSFORATIMEFRACTIONALREACTIONDIFFUSIONEQUATIONWITHNONDECAYINGINITIALDATA

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Life Span of Solutions for a Time Fractional Reaction-Diffusion Equation with Non-Decaying Initial Data

Takefumi Igarashi

Abstract- We consider the Cauchy problem of time fractional reaction-diffusion equation

$$\partial_t^\alpha u = \Delta u + u^p \quad \text{in} \quad \mathbf{R}^n \ (n \geq 1),$$

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Keywords: *life span, fractional diffusion equation, Cauchy problem, non-decaying initial data, blow-up.*

I. Introduction

We study the Cauchy problem for a time fractional reaction-diffusion equation

$$\begin{cases} \partial_t^\alpha u = \Delta u + u^p, & x \in \mathbf{R}^n, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 1$, $0 < \alpha < 1$, $p > 1$, $u_0 \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, and ∂_t^α denotes the Caputo time fractional derivative of order α defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x,s)ds, \quad 0 < \alpha < 1. \tag{1.2}$$

Here, $\Gamma(\cdot)$ is the Gamma function. Moreover, the Caputo time fractional derivative (1.2) is related to the Riemann-Liouville derivative by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}(t^{-\alpha} * [u - u_0])(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}(u(x,s) - u_0(x))ds. \tag{1.3}$$

In this paper, we show that every solution of (1.1) blows up in finite time with the non-decaying initial data at space infinity, and also present the estimate on the life span of the solutions for (1.1). Then, we define the life span (or blow-up time) T^* as

$$T^* = \sup\{T > 0; \ \text{there exists a mild solution } u \text{ of (1.1) in } C([0,T], C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))\},$$

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where the definition of "mild solution" and the local "existence" of a mild solution are described in section 2. If $T^* = \infty$, the solution is global. On the other hand, if $T^* < \infty$, then the solution is not global in time in the sense that it blows up at $t = T^*$ such as

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

A brief review on the blow-up and global existence results obtained for Cauchy problem (1.1) is given below:

(A) Kirane et al. [12] obtained that the following results.

- If $1 < p \leq 1 + 2\alpha/\{\alpha n + 2(1-\alpha)\}$, then (1.1) admits no global weak nonnegative solutions other than the trivial one.
- Let u be a local solution to (1.1). Then, there exists a constant $C > 0$ such that

$$\liminf_{|x| \to \infty} u_0(x) \leq C T^{\frac{\alpha}{1-p}},$$

where $0 < t \leq T < +\infty$.

- Suppose that (1.1) has a nontrivial global nonnegative weak solution. Then, there is a constant $K > 0$ such that

$$\liminf_{|x| \to \infty} |x|^{\frac{\alpha}{p-1}} u_0(x) \leq K.$$

(B) When $u_0 \in C_0(\mathbf{R}^n) := \left\{ u \in C(\mathbf{R}^n) | \lim_{|x| \to \infty} u(x) = 0 \right\}$, the following results were proved by Zhang and Sun [28] and Zhang et al. [29]:

- If $1 < p < 1 + 2/n$, then any nontrivial positive solution of (1.1) blows up in finite time.
- If $p \geq 1 + 2/n$ and $\|u_0\|_{L^{q_c}}$ is sufficiently small, where $q_c = n(p-1)/2$, then (1.1) has a global solution.
- If $\int_{\mathbf{R}^n} u_0(x)\chi(x)dx > 1$, where

$$\chi(x) = \left(\int_{\mathbf{R}^n} e^{-\sqrt{n^2 + |x|^2}} dx \right)^{-1} e^{-\sqrt{n^2+|x|^2}},$$

then the solutions of (1.1) blow up in finite time.

(C) The following results were also obtained in Ahmad et.al. [1] when $u_0 \in C_0(\mathbf{R}^n)$:

- If $p \geq 1 + 2/n$ and $\|u_0\|_{L^1} + \|u_0\|_{L^\infty} \leq \epsilon_0$ with some $\epsilon_0 > 0$, there exists $s > p$ such that (1.1) admits a global solution with $u \in L^\infty([0,\infty), L^\infty(\mathbf{R}^n)) \cap L^\infty([0,\infty), L^s(\mathbf{R}^n))$. Furthermore, for all $\delta > 0$,

$$\max\left\{1 - \frac{p-1}{\alpha p}, 2 - p\right\} < \delta < \min\left\{1, \frac{n(p-1)}{2p}\right\},$$

$$\|u(t)\|_{L^s} \leq C(t+1)^{-\frac{(1-\delta)\alpha}{p-1}}, \quad t \geq 0.$$

Ref

28. Q. G. Zhang, H. R. Sun, *The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation*, Topol. Methods Nonlinear Anal. 46 (1) (2015), 69–92.

In addition, if $pn < 2s$, or

$$n > 2 \text{ and } pn \geq 2s$$

with

$$\max\left\{\frac{1}{p^2}, \frac{p-1}{p^2}, \frac{\alpha}{p}, \sqrt{\frac{\alpha}{p^2}}\right\} < \alpha < 1,$$

then $u \in L^\infty([0,\infty), L^\infty(\mathbf{R}^n))$,

$$\|u(t)\|_{L^\infty} \leq C(t+1)^{-\sigma}, \quad t \geq 0,$$

for some constant $\sigma > 0$.

- If $Z_0 := \int_{\mathbf{R}^n} u_0(x)\chi(x)dx > 2^{1/(p-1)}$, then the solutions of (1.1) blow up in finite time, and the estimate of the blow-up time is

$$T^* \leq \left[\frac{\log\left(1 - 2^p Z_0^{1-p}\right)}{2(1-p)}\Gamma(\alpha+1)\right]^{1/\alpha}.$$

Several studies have been made on the life span of solutions. The results are given below:

(A) Gui and Wang [6] and Mukai et al. [17] considered

$$\begin{cases} \partial_t v = \Delta v^m + v^{p_1}, & x \in \mathbf{R}^n,\ t > 0, \\ v(x,0) = v_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.4}$$

for $m = 1$ and $m > 1$, respectively, and proved the following life span results when an initial datum takes the form $v_0(x) = \lambda\phi(x)$, where $\lambda > 0$ and $\phi(x)$ is a bounded continuous in $\mathbf{R}^n$:

- If $\|\phi\|_{L^\infty(\mathbf{R}^n)} = \phi(0) > 0$, then there exists $\lambda_1 \geq 0$ such that $T^* < \infty$ for any $\lambda > \lambda_1$, and

$$\lim_{\lambda\to\infty} \lambda^{p_1-1}T^* = \frac{1}{p_1-1}\phi(0)^{-(p_1-1)}.$$

- If $\|\phi\|_{L^\infty(\mathbf{R}^n)} = \lim_{|x|\to\infty} \phi(x) = \phi_\infty > 0$, then $T^* < \infty$ for any $\lambda > 0$, and

$$\lim_{\lambda\to 0} \lambda^{p_1-1}T^* = \frac{1}{p_1-1}\phi_\infty^{-(p_1-1)}.$$

(B) Giga and Umeda [4, 5], Seki [19] and Seki et al. [20] showed the solution of (1.4) blows up at minimal blow-up time (see Remark 1 below); that is,

$$T^* = \frac{1}{p_1-1}\|v_0\|_{L^\infty(\mathbf{R}^n)}^{1-p_1} \tag{1.5}$$

Ref

6. C. Gui, X. Wang, *Life span of solutions on the Cauchy problem for a semilinear heat equation*, J. Differential Equations 115 (1995) 166–172.

if and only if there exists a sequence $\{x_j\} \subset \mathbf{R}^n$ such that

$$\lim_{j\to\infty} |x_j| = \infty \quad \text{and} \quad \lim_{j\to\infty} v_0(x+x_j) = \|v_0\|_{L^\infty(\mathbf{R}^n)} \quad \text{a.e. in } \mathbf{R}^n.$$

Remark 1. Applying the comparison principal to (1.4), *it follows that*

$$T^* \geq \frac{1}{p_1-1}\|v_0\|^{1-p_1}_{L^\infty(\mathbf{R}^n)}. \tag{1.6}$$

So, when (1.5) *holds, we call the time T^* the "minimal blow-up time" and the solution v to* (1.4) *a "blow-up solution with the minimal blow-up time".*

(C) Maingé [15] considered (1.4) for $\max(0, 1-2/n) < m < 1$, and proved if the initial data satisfies

$$v_0(x) \geq c_0 \max\left\{0, 1-|x-x_0|^2\phi_0\right\}^s,$$

where $x_0 \in \mathbf{R}^n$, $s > 2$, and $c_0^{p_1-m} > C_b\phi_0$ for some constant $C_b > 0$ and $\phi_0 > 0$, then the solution of (1.4) blows up in finite time, and

$$\frac{1}{p_1-1}\|v_0\|^{1-p_1}_{L^\infty(\mathbf{R}^n)} \leq T^* \leq \max\left\{\frac{d_1}{c_0^{p_1-1}}, \frac{d_2c_0^{1-m}}{c_0^{p_1-m}-C_b\phi_0}\right\},$$

where $d_1 > 0$ and $d_2 > 0$.

(D) Yamauchi [18, 25, 26, 27] considered (1.4) for $m = 1$, the author [8, 9] for $\max(0, 1-2/n) < m < 1$ or $1 < m < p_1$, and showed the following life span results:

(a) Let $n \geq 2$. For some $\xi \in S^{n-1}$ and $\delta > 0$, we set the conic neighborhood $D_\xi(\delta)$:

$$D_\xi(\delta) = \left\{\eta \in \mathbf{R}^n\backslash\{0\}; \left|\xi - \frac{\eta}{|\eta|}\right| < \delta\right\}, \tag{1.7}$$

and set $S_\xi(\delta) = D_\xi(\delta) \cap S^{n-1}$. Define

$$N_\infty := \sup_{\xi\in S^{n-1},\ \delta>0} \left\{\operatorname{ess.inf}_{\theta\in S_\xi(\delta)}\left(\liminf_{r\to+\infty} v_0(r\theta)\right)\right\},$$

where $r = |x|$, $\theta = x/r$.

- If $N_\infty > 0$, then the solution of (1.4) blows up in finite time, and

$$\frac{1}{p_1-1}\|v_0\|^{1-p_1}_{L^\infty(\mathbf{R}^n)} \leq T^* \leq \frac{1}{p_1-1}N_\infty^{1-p_1}.$$

- If $N_\infty = \|v_0\|_{L^\infty(\mathbf{R}^n)}$, then the solution of (1.4) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_1-1}\|v_0\|^{1-p_1}_{L^\infty(\mathbf{R}^n)} = \frac{1}{p_1-1}N_\infty^{1-p_1}.$$

Ref

15. P. Maingé, *Blow-up and propagation of disturbances for fast diffusion equations,* Nonlinear Anal. 68 (2008) 3913–3922.

(b) Let $n = 1$. Define

$$n_\infty := \max\left(\liminf_{x\to\infty} v_0(x), \liminf_{x\to-\infty} v_0(x)\right).$$

- If $n_\infty > 0$, then the solution of (1.4) blows up in finite time, and

$$\frac{1}{p_1 - 1}\|v_0\|_{L^\infty(\mathbf{R})}^{1-p_1} \le T^* \le \frac{1}{p_1 - 1} n_\infty^{1-p_1}.$$

- If $n_\infty = \|v_0\|_{L^\infty(\mathbf{R})}$, then the solution of (1.4) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_1 - 1}\|v_0\|_{L^\infty(\mathbf{R})}^{1-p_1} = \frac{1}{p_1 - 1} n_\infty^{1-p_1}.$$

(E) The author [10] also considered

$$\begin{cases} \partial_t v = v^{p_2}(\Delta v + v^q), & x \in \mathbf{R}^n,\ t > 0, \\ v(x,0) = v_0(x) > 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.8}$$

for $p_2 \ge 1$ or $q \ge 1$, and showed the following life span results:

(a) Let $n \ge 2$.

- If $N_\infty > 0$, then the solution of (1.8) blows up in finite time, and

$$\frac{1}{p_2 + q - 1}\|v_0\|_{L^\infty(\mathbf{R}^n)}^{1-p_2-q} \le T^* \le \frac{1}{p_2 + q - 1} N_\infty^{1-p_2-q}.$$

- If $N_\infty = \|v_0\|_{L^\infty(\mathbf{R}^n)}$, then the solution of (1.8) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_2 + q - 1}\|v_0\|_{L^\infty(\mathbf{R}^n)}^{1-p_2-q} = \frac{1}{p_2 + q - 1} N_\infty^{1-p_2-q}.$$

(b) Let $n = 1$.

- If $n_\infty > 0$, then the solution of (1.8) blows up in finite time, and

$$\frac{1}{p_2 + q - 1}\|v_0\|_{L^\infty(\mathbf{R})}^{1-p_2-q} \le T^* \le \frac{1}{p_2 + q - 1} n_\infty^{1-p_2-q}.$$

- If $n_\infty = \|v_0\|_{L^\infty(\mathbf{R})}$, then the solution of (1.8) blows up at minimal blow-up time; that is,

$$T^* = \frac{1}{p_2 + q - 1}\|v_0\|_{L^\infty(\mathbf{R})}^{1-p_2-q} = \frac{1}{p_2 + q - 1} n_\infty^{1-p_2-q}.$$

Ref

10. T. Igarashi, *Life span of solutions for a degenerate parabolic equation with non-decaying initial data*, Far East Journal of Mathematical Sciences (FJMS) 121 (2019) 39–61.

Several recent studies show that the minimal blow-up time is strongly associated with blow-up at space infinity. Related researchers are Giga and Umeda [4, 5], Mochizuki and Suzuki [16], Ozawa and Yamauchi [18], Seki [19], Seki et al. [20], Shimojō [22], Yamaguchi and Yamauchi [27], Yamauchi [25, 26] and the author [8, 9].

Here, we state the main results.

Theorem 1. Consider the Cauchy problem (1.1) *for* $0<\alpha<1$ *and* $p>1$.
(a) *Let* $n\geq 2$. *Suppose that there exist* $\xi\in S^{n-1}$ *and* $\delta>0$ *such that*

$$M_\infty := \sup_{\xi\in S^{n-1},\ \delta>0}\left\{\operatorname*{ess.inf}_{\theta\in S_\xi(\delta)}\left(\liminf_{r\to+\infty} u_0(r\theta)\right)\right\}>0, \tag{1.9}$$

where $r=|x|$, $\theta=x/r$, $S_\xi(\delta)=D_\xi(\delta)\cap S^{n-1}$ *and* $D_\xi(\delta)$ *is the conic neighborhood defined by* (1.7). *Then the solution of* (1.1) *blows up in finite time, and we have*

$$\left[\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p}\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha}\leq T^*\leq\left[\frac{\Gamma(\alpha+1)}{p-1}M_\infty^{1-p}\right]^{1/\alpha}. \tag{1.10}$$

In particular, assuming that

$$M_\infty=\|u_0\|_{L^\infty(\mathbf{R}^n)}, \tag{1.11}$$

the solution of (1.1) *blows up at*

$$T^*=\left[\frac{\Gamma(\alpha+1)}{p-1}\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha}=\left[\frac{\Gamma(\alpha+1)}{p-1}M_\infty^{1-p}\right]^{1/\alpha}. \tag{1.12}$$

(b) *Let* $n=1$. *Suppose that*

$$m_\infty := \max\left(\liminf_{x\to\infty} u_0(x),\liminf_{x\to-\infty} u_0(x)\right)>0. \tag{1.13}$$

Then the solution of (1.1) *blows up in finite time, and we have*

$$\left[\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p}\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha}\leq T^*\leq\left[\frac{\Gamma(\alpha+1)}{p-1}m_\infty^{1-p}\right]^{1/\alpha}. \tag{1.14}$$

In particular, assuming that

$$m_\infty=\|u_0\|_{L^\infty(\mathbf{R})}, \tag{1.15}$$

the solution of (1.1) *blows up at*

$$T^*=\left[\frac{\Gamma(\alpha+1)}{p-1}\|u_0\|_{L^\infty(\mathbf{R})}^{1-p}\right]^{1/\alpha}=\left[\frac{\Gamma(\alpha+1)}{p-1}m_\infty^{1-p}\right]^{1/\alpha}. \tag{1.16}$$

Ref

22. M. Shimojō, *The global profile of blow-up at space infinity in semilinear heat equations*, J. Math. Kyoto Univ. 48 (2008), 339–361.

Theorem 1 allows us the information of the life span for the initial data of intermediate size and the non-decaying initial data at space infinity; (1.9) and (1.13).

Remark 2. We show some examples of the initial data u_0 which satisfy $M_\infty > 0$ in the space dimensions $n \geq 2$. For simplicity, we employ polar coordinates.

(i) $u_0(r,\alpha) = 1 - \exp(-r^2)$.
Since $\liminf_{r\to+\infty} u_0(r,\alpha) = 1$, *we have* $M_\infty = 1$.

(ii) $u_0(r,\alpha) = \{1 - \exp(-r^2)\}(2 - \cos r)$.
Since $\liminf_{r\to+\infty} u_0(r,\alpha) = 1$, *we have* $M_\infty = 1$.

(iii) $u_0(r,\alpha) = \{1 - \exp(-r^2)\}(1 + \cos\alpha)$.
Since $\liminf_{r\to+\infty} u_0(r,\alpha) = 1 + \cos\alpha$, *we have* $M_\infty = 2$.

(iv) $u_0(r,\alpha) = \{1 - \exp(-r^2)\}(1 + \cos\alpha)(2 - \cos r)$.
Since $\liminf_{r\to+\infty} u_0(r,\alpha) = 1 + \cos\alpha$, *we have* $M_\infty = 2$.

For the examples (i) *and* (iii), *the initial data u_0 satisfies* (1.11). *However, for the examples* (ii) *and* (iv), *since* $\|u_0\|_{L^\infty(\mathbf{R}^n)} = 3$ *and* $\|u_0\|_{L^\infty(\mathbf{R}^n)} = 6$, *respectively, it follows that* $M_\infty \neq \|u_0\|_{L^\infty(\mathbf{R}^n)}$.

The outline of the rest of this paper is organized as follows. In section 2, we give the existence theorem of a local solution to (1.1). In section 3, we prove the main results by improving the method in the author [8, 9], Ozawa and Yamauchi [18] and Yamauchi [25, 26].

II. Existence of a Local Mild Solution

In this section, we show the local existence and uniqueness theorem of a mild solution to problem (1.1). Here, we state the definition of a mild solution of (1.1).

Definition. Let $T^* > 0$. We say $u \in C([0,T^*], C(\mathbf{R}^n))$ is a mild solution of (1.1) *if u satisfies the integral equation*

$$u(t) = E_{\alpha,1}(-t^\alpha A)u_0 + \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-s^\alpha A) f(u(t-s))ds, \tag{2.1}$$

where $f(u(s)) = u^p(s)$, and A is realization of $-\Delta$ and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function (see [11]):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{2.2}$$

Theorem 2. Suppose that $u_0 \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Then there exists a unique local mild solution $u \in C([0,T^*], C(\mathbf{R}^n))$ for the problem (1.1).

Proof. See [21, Theorem 1] noting that the nonlinear term $f(u(s)) = u^p(s)$ is a locally Lipschizian function. (See also [24, Theorem 2.2].)

Remark 3. If u solves (1.1), *then u satisfies* (2.1) *by the method of the proof for [21, Lemma 1].*

Ref

18. T. Ozawa, Y. Yamauchi, *Life span of positive solutions for a semilinear heat equation with general non-decaying initial data*, J. Math. Anal. Appl. 379 (2011) 518–523.

Remark 4. If $u_0(x) \geq 0$, then (1.1) admits a solution which satisfies $u(x,t) \geq 0$ by the maximum principle [2] (see also [3, Theorem 3.1] and [21, Theorem 3]).

III. Proof of Theorem 1

In this section, we shall estimate the life span T^* both from below and from above. Here, we improve the method in Yamauchi [25, 26] and the author [8, 9, 10].

First, we shall show a lower estimate of T^* in the space dimensions $n \geq 1$. This is obtained by comparing the solution u of (1.1) with the solution U of the ordinary differential equation

$$\begin{cases} \partial_t^\alpha U(t) = U^p(t), \quad t > 0, \\ U(0) = \|u_0\|_{L^\infty(\mathbf{R}^n)}. \end{cases} \tag{3.1}$$

The solution U of (3.1) satisfies the integral equation

$$\begin{aligned} U(t) &= E_{\alpha,1}(0)U(0) + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(0)U^p(t-s)ds \\ &= U(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U^p(s)ds, \end{aligned} \tag{3.2}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function by defined in (2.2). Now, we take the same strategy as in [7, Theorem 3.2] and [13, Theorem 3.1]. Here, changing of variables

$$U(t) = U(0)[V(t)+1] \quad \text{and} \quad k(t) = \gamma t^{\alpha-1} \text{ with } \gamma = \frac{[U(0)]^{p-1}}{\Gamma(\alpha)}, \tag{3.3}$$

the integral equation (3.2) can be expressed as

$$V(t) = \int_0^t k(t-s)[V(s)+1]^p ds. \tag{3.4}$$

Then, the solution U blows up in finite time $T^*(U)$ such that

$$\left[\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p}\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha} \leq T^*(U) \leq \left[\frac{\Gamma(\alpha+1)}{p-1}\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha}. \tag{3.5}$$

By a comparison argument, we obtain

$$T^* \geq T^*(U). \tag{3.6}$$

Next, we shall prove a upper estimate of T^* by two case of $n \geq 2$ and $n = 1$.

a) *Case (a): n ≥ 2*

For $\xi \in S^{n-1}$ and $\delta > 0$ as in the theorem, we determine the sequences $\{a_j\} \subset \mathbf{R}^n$ and $\{R_j\} \subset (0,\infty)$. Let $\{a_j\} \subset \mathbf{R}^n$ be a sequence satisfying that $|a_j| \to \infty$ as $j \to \infty$, and that $a_j/|a_j| = \xi$ for any $j \in \mathbf{N}$. Put $R_j = (\delta\sqrt{4-\delta^2}/2)|a_j|$ for $\delta \in (0,\sqrt{2})$. For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\Delta$ on

Ref

2. A. Alsaedi, B. Ahmad and M. Kirane, *Maximum principle for certain generalized time and space fractional diffusion equations*, Q. Appl. Math. 73 (2015) 163—175.

$$B_{R_j}(0) = \{x \in \mathbf{R}^n; |x| < R_j\}$$

with zero Dirichlet boundary condition under the normalization

$$\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1.$$

Notes

Moreover, let μ_{R_j} be the corresponding first eigenvalue. For the solutions of (1.1), we define

$$w_j(t) := \int_{B_{R_j}(0)} u(x+a_j,t)\rho_{R_j}(x)dx. \tag{3.7}$$

Then we have the following propositions.

Proposition 1. We have

$$\liminf_{j\to+\infty} w_j(0) \geq \operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} \left(\liminf_{r\to\infty} u_0(r\theta)\right), \tag{3.8}$$

and

$$\lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1. \tag{3.9}$$

Proof. See [25, Proposition 1].

Proposition 2. Let $0<\alpha<1$ and $p>1$. Suppose that

$$w_j(0) > \mu_{R_j}^{\frac{1}{p-1}}. \tag{3.10}$$

Then u blows up in finite time, and we have

$$T^* \leq \left[\frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}}\Gamma(\alpha+1)\right]^{1/\alpha}. \tag{3.11}$$

Proof. We use the method in [1, Theorem 3.7] and [3, Thorem 2.2].

By (1.1) and (3.7), we have

$$\partial_t^\alpha w_j(t) = \int_{B_{R_j}(0)} \partial_t^\alpha u(x+a_j,t)\rho_{R_j}(x)dx$$

$$= \int_{B_{R_j}(0)} \left\{\Delta u(x+a_j,t) + u^p(x+a_j,t)\right\}\rho_{R_j}(x)dx$$

$$\geq -\mu_{R_j} \int_{B_{R_j}(0)} u(x+a_j,t)\rho_{R_j}(x)dx + \int_{B_{R_j}(0)} u^p(x+a_j,t)\rho_{R_j}(x)dx. \tag{3.12}$$

Since $p > 1$ and

$$\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1,$$

by Jensen's inequality, we have

$$\int_{B_{R_j}(0)} u^p(x+a_j,t)\rho_{R_j}(x)dx \geq \left(\int_{B_{R_j}(0)} u(x+a_j,t)\rho_{R_j}(x)dx\right)^p. \tag{3.13}$$

Thus, by (3.12)-(3.13), we obtain

$$\partial_t^\alpha w_j(t) \geq -\mu_{R_j} w_j(t) + w_j^p(t). \tag{3.14}$$

By (1.3), the inequality (3.14) implies

$$\frac{d}{dt}(k * [w_j - w_j(0)])(t) \geq -\mu_{R_j} w_j(t) + w_j^p(t) \quad \text{with } k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{3.15}$$

We put $F(\zeta) = -\mu_{R_j}\zeta + \zeta^p$. Then the funtion F is convex in $\zeta \geq 0$, and we get

$$\frac{d}{dt}(k * [w_j - w_j(0)])(t) \geq F(w_j(t)). \tag{3.16}$$

in (3.15). F is positive and increasing for all $\zeta > \mu_{R_j}^{\frac{1}{p-1}}$. If $w_j(0)$ satisfies (3.10), then (3.16) implies that $w_j(t) > \mu_{R_j}^{\frac{1}{p-1}}$ for all $t \in (0, T^*)$ (see [1, P.24–25]). Knowing that $w_j(t) \geq w_j(0) > \mu_{R_j}^{\frac{1}{p-1}}$ for all $t \in (0, T^*)$, it follows from (3.16) that

$$\partial_t^\alpha w_j(t) = \frac{d}{dt}(k * [w_j - w_j(0)])(t) \geq F(w_j(t)) > 0, \quad \text{for all } t \in (0, T^*). \tag{3.17}$$

Therefore the function $w_j(t)$ satisfying (3.17) is an upper solution of the problem

$$\partial_t^\alpha \zeta = F(\zeta) = -\mu_{R_j}\zeta + \zeta^p, \quad \zeta(0) = w_j(0), \tag{3.18}$$

we have $w_j(t) \geq \zeta(t)$ by comparison principle (see [14, Theorem 2.3]).

On the other hand, since $F(0) \geq 0$, $F(\zeta) > 0$ and $F'(\zeta) > 0$ for all $\zeta \geq w_j(0) > \mu_{R_j}^{\frac{1}{p-1}}$. Then, it follows from [1, Lemma 3.8] (see also [23, Lemma 3.10]) that $v(t) = g\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)$ is a lower solution for (3.18), where $v(t)$ satisfies

$$\partial_t^\alpha v \leq F(v) = -\mu_{R_j} v + v^p, \quad v(0) \leq w_j(0),$$

Notes

and $g(t)$ solves the ordinary differential equation

$$\frac{dg}{dt} = F(g) = -\mu_{R_j} g + g^p, \quad g(0) = w_j(0). \tag{3.19}$$

By comparison principle (see [14, Theorem 2.3]), we obtain $\zeta(t) \geq v(t)$. Solving the initial value problem (3.19), we have the solution

$$g(t) = \left[w_j^{1-p}(0) - \frac{1 - \exp\{(1-p)\mu_{R_j} t\}}{\mu_{R_j}} \right]^{\frac{1}{1-p}} \exp\left(-\mu_{R_j} t\right),$$

and obtain that $g(t) \to \infty$ as $t \to \frac{\log\left(1-\mu_{R_j} w^{1-p}(0)\right)}{(1-p)\mu_{R_j}}$. By comparison principle (see [14, Theorem 2.3]), we conclude that

$$w_j(t) \geq \zeta(t) \geq v(t) = g\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) = \left[w_j^{1-p}(0) - \frac{1 - \exp\left\{\frac{(1-p)\mu_{R_j} t^\alpha}{\Gamma(\alpha+1)}\right\}}{\mu_{R_j}} \right]^{\frac{1}{1-p}} \exp\left(-\frac{\mu_{R_j} t^\alpha}{\Gamma(\alpha+1)}\right). \tag{3.20}$$

By (3.20), if $w_j(0)$ satisfies (3.10), then we obtain that $v(t) \to \infty$ as

$$t \to \left[\frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}} \Gamma(\alpha+1) \right]^{1/\alpha}, \tag{3.21}$$

and that $w_j(t)$ blows up in finite time. Therefore, the solution u blows up in finite time, and it follows that the estimate (3.11) holds, the proof of Proposition 2 is complete.

Now let us prove the Case (a).

By Propositions 1 and 2, we obtain that

$$\begin{aligned}
T^* &\leq \limsup_{j\to\infty} \left[\frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}} \Gamma(\alpha+1) \right]^{1/\alpha} \\
&= \limsup_{j\to\infty} \left[\frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \frac{w_j^{1-p}(0)}{p-1} \Gamma(\alpha+1) \right]^{1/\alpha} \\
&= \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{1/\alpha} \lim_{j\to\infty} \left[\frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \right]^{1/\alpha} \cdot \left(\liminf_{j\to\infty} w_j(0)\right)^{\frac{1-p}{\alpha}} \\
&\leq \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{1/\alpha} \left\{ \operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} \left(\liminf_{r\to\infty} u_0(r\theta)\right) \right\}^{\frac{1-p}{\alpha}}.
\end{aligned} \tag{3.22}$$

From arbitrariness of $\xi \in S^{n-1}$ and $\delta > 0$, by (3.22), we obtain

$$T^* \leq \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{1/\alpha} \left[\sup_{\xi\in S^{n-1},\delta>0} \left\{\operatorname{ess.inf}_{\theta\in S_\xi(\delta)} \left(\liminf_{r\to\infty} u_0(r\theta)\right)\right\}\right]^{\frac{1-p}{\alpha}}$$

$$= \left[\frac{\Gamma(\alpha+1)}{p-1} M_\infty^{1-p}\right]^{1/\alpha}. \tag{3.23}$$

By (3.6) and (3.23), we have

$$\left[\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p} \|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}\right]^{1/\alpha} \leq T^* \leq \left[\frac{\Gamma(\alpha+1)}{p-1} M_\infty^{1-p}\right]^{1/\alpha}. \tag{3.24}$$

Therefore, we obtain (1.10). Moreover, by (1.10) and (1.11), we have (1.12). This completes the proof.

b) Case (b): $n = 1$

Let $a_j = j$ or $-j$. Put $R_j = j/2$. For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\frac{\partial^2}{\partial x^2}$ on $(-R_j, R_j)$ with zero Dirichlet boundary condition under the normalization

$$\int_{-R_j}^{R_j} \rho_{R_j}(x)dx = 1.$$

Moreover, let μ_{R_j} be the corresponding first eigenvalue. For the solutions of (1.1), we define

$$w_j(t) := \int_{-R_j}^{R_j} u(x+a_j,t)\rho_{R_j}(x)dx. \tag{3.25}$$

Then we have the following propositions.

Proposition 3. We have

$$\liminf_{j\to+\infty} w_j(0) \geq \max\left(\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x)\right) \tag{3.26}$$

and

$$\lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1. \tag{3.27}$$

Proof. See [25, Proposition 2].

Proposition 4. Let $0<\alpha<1$ and $p>1$. Suppose that

$$w_j(0) > \mu_{R_j}^{\frac{1}{p-1}}. \tag{3.28}$$

Then u blows up in finite time, and we have

Notes

$$T^* \leq \left[\frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}} \Gamma(\alpha+1) \right]^{1/\alpha}. \tag{3.29}$$

Proof. It is shown in the same way as in Proposition 2.

Notes

Finally, let us prove the Case (b). The rest of the proof is the same as in that of the Case (a).

By Propositions 3 and 4, we see that

$$\begin{aligned}
T^* &\leq \limsup_{j\to\infty} \left[\frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}} \Gamma(\alpha+1) \right]^{1/\alpha} \\
&= \limsup_{j\to\infty} \left[\frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \frac{w_j^{1-p}(0)}{p-1} \Gamma(\alpha+1) \right]^{1/\alpha} \\
&= \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{1/\alpha} \lim_{j\to\infty} \left[\frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \right]^{1/\alpha} \cdot \left(\liminf_{j\to\infty} w_j(0)\right)^{\frac{1-p}{\alpha}} \\
&\leq \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{1/\alpha} \left\{ \max\left(\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x) \right) \right\}^{\frac{1-p}{\alpha}}.
\end{aligned} \tag{3.30}$$

From (3.6) and (3.30), we have

$$\left[\frac{(p-1)^{p-1}\Gamma(\alpha+1)}{p^p} \|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p} \right]^{1/\alpha} \leq T^* \leq \left[\frac{\Gamma(\alpha+1)}{p-1} m_\infty^{1-p} \right]^{1/\alpha}. \tag{3.31}$$

Therefore, we obtain (1.14). Moreover, by (1.14) and (1.15), we have (1.16). This completes the proof.

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Product of Special Function and Polynomial Associated Via Pathway Fractional Integral Operator

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GJSFR-F Classification: *MSC 2010: 33C60, 26A33*

PRODUCTOFSPECIALFUNCTIONANDPOLYNOMIALASSOCIATEDVIAPATHWAYFRACTIONALINTEGRALOPERATOR

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Product of Special Function and Polynomial Associated Via Pathway Fractional Integral Operator

Hemlata Saxena [α] & Danishwar Farooq [σ]

Abstract- In present paper we introduce four theorems using pathway fractional integral operator involving product of Srivastava polynomial and generalized Struve function. Our results are quite general in nature. We obtain our results in term of hypergeometric function. Certain special cases of the main results are also obtained here. Our results will help to extend some classical statistical distribution to wider classes of distribution, these are useful in practical applications.

Keywords: Pathway fractional integral operator/Srivastava polynomial/Struve function/generalized Struve function.

I. Introduction

Let $f(x)\,\epsilon\, L(a,b), \alpha\,\epsilon\,\mathbb{C}, R(\alpha) > 0,$ then left sided Reimann – Liouville fractional integral operator is defined as [9]

$$(\mathrm{I}_{0^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \text{where } R(\alpha) > 0. \qquad \ldots\ldots\ldots(1.1)$$

Let $f(x)\,\epsilon\, L(a,b), \eta\,\epsilon\,\mathbb{C}, R(\eta) > 0, a > 0$ and a ''Pathway parameter'' $\alpha < 1$. Then the pathway fractional integration operator is defined by [6], also see [10]

$$(\mathrm{p}_{0^+}^{(\eta,\alpha)} f\,) = x^{\eta} \int_0^{\frac{x}{a(1-\alpha)}} \left(1-\frac{a(1-\alpha)t}{x}\right)^{\frac{\eta}{1-\alpha}} f(t)dt \qquad \ldots\ldots\ldots\ldots(1.2)$$

when $\alpha = 0$, $a = 1$ and η is replaced by $\eta - 1$ in (1.2) it yields

$$(\mathrm{I}_{0^+}^{\eta} f)(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt \qquad \ldots(1.3)$$

Fractional integration operators play an important role in the solution of several problems of diversified fields of science and engineering. Many fractional integral operators like Riemann – Liouville, Weyl, Kober, Erdely – Koberand Saigo operator are studied by various workers due to their applications in the solutions of integral equation arising in several problems of many areas of physical, engineering and technological science. A detailed description of these operators can be found in the survey paper by Srivastava and Saxena [17].

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In this paper, we consider following functions defined as follows:
The Struve function of order p is given by

$$H_p(z) = \left(\frac{z}{2}\right)^{p+1} \sum_{K=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+p+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2k} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

The Struve function and its more generalization are found in many papers [1,2,4,12,7,13,14,15].The generalized Struve function studied by [13] as follows:

$$H^{\lambda}_{l,\varepsilon}(z) = \sum_{K=0}^{\infty} \frac{(-1)^k}{\Gamma\left(\lambda k+\frac{1}{\varepsilon}+\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k+l+1}, \lambda > 0, \varepsilon > 0 \ldots\ \ldots\ldots\ldots\ldots\ (1.5)$$

The generalized Struve function of the first kind $H_{p,b,c}(z)$ [see 7] defined for complex $z \in \mathbb{C}$ and $b, c, p \in \mathbb{C}$, $(Re(p) > -1$ by:

$$H_{p,b,c}(z) = \sum_{K=0}^{\infty} \frac{(-1)^k c^k}{\Gamma\left(p+1+\frac{b}{2}+k\right)\Gamma\left(k+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k+p+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1.6)$$

Where Γ is the classical gamma function whose Euler's integral is given by Srivastava and Choi(see [11])

$$\Gamma(y) = \int_0^{\infty} e^{-t}\, t^{y-1}\, dt\ Re(y) > 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.7)$$

The special cases of Struve function are as follows[7]:

$$H_{1-\frac{b}{2},b,c^2} = \frac{1}{c^2\sqrt{\pi}}\left[\frac{2}{z}\right]^{b/2}(1-\cos(cz))\, b \in \mathbb{R}, \&\ \ c \neq 0 \quad \ldots \qquad \ldots\ldots\ldots\ (1.8)$$

$$H_{1-\frac{b}{2},b,-c^2} = \frac{1}{c^2\sqrt{\pi}}\left[\frac{2}{z}\right]^{b/2}(\cosh(cz)-1)\quad b \in \mathbb{R}, \&\ \ c \neq 0 \qquad (1.9)$$

$$H_{-\frac{b}{2},b,c^2} = \frac{1}{c\sqrt{\pi}}\left[\frac{2}{z}\right]^{b/2}\sin(cz) \qquad b \in \mathbb{R}, \&\ \ c \neq 0 \qquad \ldots\ldots\ (1.10)$$

$$H_{-\frac{b}{2},b,-c^2} = \frac{1}{c\sqrt{\pi}}\left[\frac{2}{z}\right]^{b/2}\sinh(cz) \qquad b \in \mathbb{R}, \&\ \ c \neq 0 \qquad \ldots\ldots\ldots\ (1.11)$$

The generalized Wright hypergeometric function $r\Psi s\,(z)$ defined for a_i , $b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0$;$i = 1,2, \ldots, r$; $j = 1,2, \ldots s$) is given by the series :

$$r\Psi s\,(z) = r\Psi s\left[\begin{matrix}(a_i+\alpha\cdot 1)_{1,r}\\ ((a_i+\alpha\cdot 1)_{1,s})\end{matrix}\, z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(a_i+\alpha_i k)z^k}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j k)\,k!} \qquad \ldots\ldots\ldots\ (1.12)$$

where Γz is the Euler gamma function and the asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [3] and under the condition:

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13. Singh, R.P; Some integral representation of generalized Struve function, *Math Ed(Siwan)*, 22,(1988), 91-94.

$$\sum_{j=1}^{r} \beta_j - \sum_{i=1}^{s} \alpha_i > -1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.13)$$

For detailed Study of various properties, generalization and applications of wright function and generalized wright function, we refer to paper (for instance see [19] [20] [21]). The generalized hypergeometric function represented as fallows [8]:

$$r\Psi s \left[\begin{matrix} (\alpha_r)\,; \\ & z \\ (\beta_s)\,; \end{matrix} \right] = \sum_{R=0}^{\infty} \frac{\prod_{i=1}^{r} (\alpha_i)_n z^n}{\prod_{j=1}^{s} (\beta_i)_n \, n!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.14)$$

Provided $r \le s\,; r = s + 1$ and $|z| < 1$ where $(\lambda)_n$ is well known pochhammer symbol defined for$\lambda \in \mathbb{C}$ [3][8]

$$(\lambda)_n = \begin{Bmatrix} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in N = 1,2,3,\ldots) \end{Bmatrix} \quad \ldots\ldots (1.15)$$

$$= \frac{\Gamma\,(\lambda+n)}{\Gamma\,(\lambda)} (\lambda \in \mathbb{C}/Z_0^-)$$

where $Z_0^-$is the set of non – positive integers. If we put $\alpha_1 = \ldots\ldots = \alpha_r \ \ldots = \beta_1 = \ldots = \beta_s$ in equation (1.8), then (1.10) is a special case of the generalized wright function [16]

$$r\Psi s = r\Psi s \left[\begin{matrix} (\alpha_1, 1), \ldots, (\alpha_r, 1); \\ & z \\ (\beta_1, 1), \ldots, ; (\beta_s, 1); \end{matrix} \right] = \frac{\prod_{i=1}^{r} (\alpha_i)_n z^n}{\prod_{j=1}^{s} (\beta_i)_n \, n!} rFs \left[\begin{matrix} (\alpha_1), \ldots, (\alpha_r)\,; \\ & z \\ (\beta_1), \ldots, (\beta_s); \end{matrix} \right] \quad (1.16)$$

The Srivastava polynomial defined by Srivastava [18](pp. 1, eq.1), [5](pp. 11, eq. 7) in the following manner

$$S_w^u[x] = \sum_{s=0}^{(w/u)} \frac{(-W)u\,,s}{s!} A_{w,s}\, x^s, \qquad w = 0,1,2, \ldots\ldots (1.17)$$

where w is an arbitrary positive intenger and the coefficient $A_{w,s}\,(w\,,s\,) > 0$ are the arbitrary constant real or complex. This polynomial provide a large number spectrum of well-known polynomial as one of its particular cases on appropriately specializing the coefficient $A_{w,s}$ particularly by setting$u = 1\,, A_{w,s}$ for $s = k$ and $A_{w,s} = 0$ for $s \ne k$ the above polynomial leads to a power function.

$$S_w^u[x] = x^k \qquad (\, k \in Z^+ \, with \, k \le w) \ldots\ldots\ldots\ldots\ldots (1.18)$$

II. Main Results

Theorem 1: Let $\eta, \rho, \beta, \gamma, \mu, \delta \in \mathbb{C}$, $R\left(1 + \frac{\eta}{(1-\alpha)}\right) > 0$ $\min\{Re(\rho), Re(\beta), Re(\gamma), Re(\mu), Re(\delta), Re(\eta)\} > 0$ and $p_i, q_i > 0$, $\alpha < 1$. Then the pathaway fractional integral operator$(p_{0^+}^{(n,\alpha)})$ defined by (1.2) then the following formula holds:

Ref

8. Rainville, E. D; Special function, Macmillan, New York, 1960; Reprinted by Chelsea publishing Company, New York(1971).

$$P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}H_{l,\varepsilon}^{\lambda}(t)S_w^u[\sigma t^{\rho}]\}(x) =$$

$$\frac{x^{\eta+l+\mu+1}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{l+\mu+1}}\left(\frac{1}{2}\right)^{l+1}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^{s}$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(l+\mu+\rho s+1,2)\\ \left(\frac{l}{3}+\frac{3}{2},\lambda\right),\left(\frac{3}{2},1\right),\left(l+\frac{\eta}{(1-\alpha)}+\mu+\rho s+2,2\right)\end{matrix};-\frac{x^2}{4[a(1-\alpha)]^2}\right] \quad\ldots\ldots\ldots\ldots\ldots(2.1)$$

Proof: Making the use of (1.2), (1.5) and (1.17) in LHS of the theorem first and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.1).

Theorem 2: Let $\eta,\ \mu, p, b, c\ \epsilon\ \mathbb{C}$ and $\alpha\ < 1$ such that $\{Re(\eta),\ Re(\mu),\ Re(\mu+p)\} > 0$, $Re(\ p+1+\frac{b}{2}) > -1$ and $Re(\frac{\eta}{1-\alpha}) > -1$ then the following formula hold:

$$P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}H_{p,b,c}(t)S_w^u[\sigma t^{\rho}]\}(x) =$$

$$\frac{x^{\eta+p+\mu+1}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{p+\mu+1}}\left(\frac{1}{2}\right)^{p+1}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^{s}$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(p+\mu+\rho s+1,2)\\ \left(p+\frac{b}{2}+1,1\right),\left(\frac{3}{2},1\right),\left(p+\frac{\eta}{(1-\alpha)}+\mu+\rho s+2,2\right)\end{matrix}\ \Big|\ -\frac{cx^2}{4[a(1-\alpha)]^2}\right]\ldots\ldots\ldots\ldots\ldots(2.2)$$

Proof: Making the use of (1.2), (1.6) and (1.18) in LHS of the theorem 2 and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.2).

Theorem 3: Let $\eta, \mu, p, b, c\ \epsilon\ \mathbb{C}$ and $\alpha\ < 1$ such that $\{Re(\eta),\ Re(\mu),\ Re(\mu+p)\} > 0$, and $Re(\frac{\eta}{1-\alpha}) > -1$ then the following formula hold:

(i) $P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}\sin(ct)S_w^u[\sigma t^{\rho}]\}(x) =$

$$\frac{1}{4}c\sqrt{\pi}\frac{x^{\eta+\mu+1}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\mu+1}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^{s}$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(\mu+\rho s+1,2)\\ \left(\frac{3}{2},1\right),(1,1),\left(\frac{\eta}{(1-\alpha)}+\mu+\rho s+2,2\right)\end{matrix};-\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

Proof: Making the use of (1.2), (1.10) and (1.18) in LHS of the theorem 3 (part I) and then interchange the order of integration and summation, we evaluate the inner integral

Notes

Notes

by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.3)

(ii) $P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}\sinh(ct)S_w^u[\sigma t^\rho]\}(x) =$

$$\frac{1}{4}c\sqrt{\pi}\frac{x^{\eta+\mu+1}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\mu+1}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^s$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(\mu+\rho s+1,2)\\ \left(\frac{3}{2},1\right),(1,1),\left(\frac{\eta}{(1-\alpha)}+\mu+\rho s+2,2\right)\end{matrix};-\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]\quad\ldots\ldots\ldots\ldots\ \ldots.\ (2.4)$$

Proof: Making the use of (1.2), (1.11) and (1.18) in LHS of the theorem 3 (part II) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of(2.4).

Theorem 4: Let $\eta, \mu, p, b, c \in \mathbb{C}$ and $\alpha < 1$ such that $Re(\alpha) > 0$, and $\text{Re}(\beta - \eta) > 2$ then the following formula hold:

(i) $P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}(1-\cos(ct))S_w^u[\sigma t^\rho]\}(x) =$

$$\frac{1}{4}c^2\sqrt{\pi}\frac{x^{\eta+\mu+2}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\mu+2}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^s$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(\mu+\rho s+2,2)\,;\\ \left(\frac{3}{2},1\right),(2,1),\left(\frac{\eta}{(1-\alpha)}+\mu+\rho s+3,2\right);\end{matrix};-\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.5)$$

Proof: Making the use of (1.2), (1.8) and (1.18) in LHS of the theorem 4 (part I) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and we arrive at the desired result RHS of (2.5)

(ii) $P_{0^+}^{(\eta,\alpha)}\{t^{\mu-1}(\cosh(ct)-1)S_w^u[\sigma t^\rho]\}(x) =$

$$\frac{1}{4}c^2\sqrt{\pi}\frac{x^{\eta+\mu+2}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\mu+2}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,[\sigma\left(\frac{x}{[a(1-\alpha)]}\right)^{\rho}]^s$$

$$\times\, {}_1\Psi_3\left[\begin{matrix}(\mu+\rho s+2,2)\,;\\ \left(\frac{3}{2},1\right),(2,1),\left(\frac{\eta}{(1-\alpha)}+\mu+\rho s+3,2\right);\end{matrix};\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.6)$$

Proof: Making the use of (1.2), (1.9) and (1.18) in LHS of the theorem 4 (part II) and then interchange the order of integration and summation, we evaluate the inner integral by making use of beta function and using (1.12) we arrive at the desired result RHS of (2.6)

III. Special Cases

1. If we take $\alpha = 0, a = 1$ and η is replaced by $\eta - 1$ in (2.1), then pathway fractional integral operator will reduce Riemann – Liouville fractional integral defined in (1.1). Then we get the following result:

$$I_{0^+}^{\alpha}\{t^{\mu-1}H_{l,\varepsilon}^{\lambda}(t)S_w^u[\sigma t^{\rho}]\}(x) = x^{\eta+\mu+1}\,\Gamma(\eta)\left(\frac{1}{2}\right)^{l+1}\sum_{s=0}^{(w/u)}\frac{(-W)u\,,s}{s!}A_{w,s}\,(\sigma x^{\rho})^s$$

$$\times\ {}_1\Psi_3\left[\begin{matrix}(l+\mu+\rho s+1,\ 2)\,;\\ \left(\frac{l}{3}+\frac{3}{2},\lambda\right),\left(\frac{3}{2},1\right),\ (l+\eta+\mu+\rho s+1,2)\,;\end{matrix}(-x^2)\right]$$

2. If we take $\alpha = 0, a = 1$, η is replaced by $\eta - 1$ and also on setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.1), then pathway fractional integral operator will reduce Riemann – Liouville fractional integral defined in (1.1) and general class of polynomial will reduce to 1 defined in (1.13). then we get the following result:

$$I_{0^+}^{\alpha}\{t^{\mu-1}H_{l,\varepsilon}^{\lambda}(t)\}(x)$$

$$= x^{\eta+\mu+1}\,\Gamma(\eta)\left(\frac{1}{2}\right)^{l+1}$$

$$\times\ {}_1\Psi_3\left[\begin{matrix}(l+\mu+1,\ 2)\,;\\ \left(\frac{l}{3}+\frac{3}{2},\lambda\right),\left(\frac{3}{2},1\right),\ (l+\eta+\mu+1,2)\,;\end{matrix}(-x^2)\right]$$

3. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.2), we arrive at the known result given by Nisar K .S [10, pp. 66, eq. 13]
4. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.3), we arrive at the known result given by Nisar K .S [10, pp. 67, theorem (3.1)(part I)].
5. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.4), we arrive at the known result given by Nisar K .S [10, pp. 67, theorem (3.1)(part II)].
6. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.5), we arrive at the known result given by Nisar K .S [10, pp. 68 theorem (3.2), (part I)].
7. On setting $w = 0$, $A_{0,0} = 1$, then $S_0^u\,[x] \to 1$ in (2.6), we arrive at the known result given by Nisar K .S [10, pp. 68 theorem (3.2), (part II)].

IV. Conclusion

In this paper, we have presented Struve function, generalized Struve function and Srivastava polynomial via pathway fractional integral operator. As in this operator α establishes a path of going from one distribution to another and to different classes of distribution. we conclude this investigation by remarking that the result obtained here are general in character and useful in deriving various integral formulas in the theory of the pathway fractional integration operator and also our result will help to extend some

Notes

classical statistical distribution to wider classes of distribution, useful in practical application.

Notes

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On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

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Abstract- We introduce Mouanda's choice function for matrices which allows us to construct the galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of the galaxies of circulant matrices with positive integers as entries. The characterization of the matrix solutions of the equation $X^2+Y^2=Z^2$ allows us to show that the equation $X^{2n}+X^{2n}=Z^{2n}(n\geq 2)$ has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation $X^n+Y^n=Z^n(n\geq 3)$ has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we show Fermat's Last Theorem for complex polynomials over $\mathbb{D}$ associated to circulant matrices.

Keywords: *Fermat's equation, polynomials, model theory, circulant matrices, Mouanda's choise function, galaxy, Toeplitz matrices.*

GJSFR-F Classification: *MSC 2010: 11D41, 11C08, 03C95.*

ONFERMATSLASTTHEOREMMATRIXVERSIONANDGALAXIESOFSEQUENCESOFCIRCULANTMATRICESWITHPOSITIVEINTEGERSASENTRIES

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On Fermat's Last Theorem Matrix Version and Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries

Joachim Moussounda Mouanda [α], Jean Raoul Tsiba [σ] & Kinvi Kangni [ρ]

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatum in duos ejusdem nominis fas est dividere: cujes rei demonstrationem mirabilem sane detexi. Hane marginis exiguitas non caperet.

-Pierre de Fermat (1637).

Abstract- We introduce Mouanda's choice function for matrices which allows us to construct the galaxies of sequences of triples of circulant matrices with positive integers as entries. We give many examples of the galaxies of circulant matrices with positive integers as entries. The characterization of the matrix solutions of the equation $X^2 + Y^2 = Z^2$ allows us to show that the equation $X^{2n}+X^{2n}=Z^{2n}(n\geq 2)$ has no circulant matrix with positive integers as entries solutions. This allows us to prove that, in general, the equation $X^n+Y^n= Z^n(n\geq 3)$ has no circulant matrix with positive integers as entries solutions. We prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we show Fermat's Last Theorem for complex polynomials over $\mathbb{D}$ associated to circulant matrices.

Keywords: *Fermat's equation, polynomials, model theory, circulant matrices, Mouanda's choise function, galaxy, Toeplitz matrices.*

I. Introduction and Main Result

It is well known that there are many solutions in integers to the equation $x^2 + y^2 = z^2$, for instance (3,4,5); (5,12,13). Around 1500 B.C, the Babylonians were aware of the solution (4961, 6480, 8161) and the Egyptians knew the solutions (148, 2736, 2740) and (514, 66048, 66050). Also Greek mathematicians were attracted to the solutions of this equation. We notice that this equation has sequences of complex number solutions

$$(1 + 2i \times a^k, 2i \times a^k - 2 \times a^{2k}, 1 + 2i \times a^k - 2 \times a^{2k}), a \in \mathbb{C}, k \in \mathbb{N}$$

and matrix solutions

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$$\left(\begin{pmatrix} -2 & 2i & 0 \\ 0 & -2 & 2i \\ 2i & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 2i & 1 \\ 1 & 0 & 2i \\ 2i & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 2i & 1 \\ 1 & -2 & 2i \\ 2i & 1 & -2 \end{pmatrix}\right).$$

In 1637, Pierre de Fermat wrote a note in the margin of his copy of Diophantus Arithmetica [1] stating that the equation

$$x^n + y^n = z^n, n \in \mathbb{N}(n > 2), xyz \neq 0 \tag{1.1}$$

has no integer solutions. This is the Fermat Last Theorem. He claimed that he had found the proof of this Theorem. The only case Fermat actually wrote down a proof is the case $n = 4$. In his proof, Fermat introduced the idea of infinite descent which is still one the main tools in the study of Diophantine equations. He proved that the equation $x^4 + y^4 = z^2$ has no solutions in relatively prime integers with $xyz \neq 0$. Solutions to this equation correspond to rational points on the elliptic curve $v^2 = u^3 - 4u$. The proof of the case $n = 3$ was given first by Karl Gauss. In 1753, Leonhard Euler gave a different prove of Fermat's Last Theorem for $n = 3$[2, 3]. In 1823, Sophie Germain proved that if l is a prime and $2l + 1$ is also prime, the equation $x^l + y^l = z^l$ has no solutions (x,y,z) with $xyz \neq 0(modl)$. The case $n = 5$ was proved simultaneously by Adrien Marie Legendre in 1825 [4, 5] and Peter Lejeune Dirichlet [6] in 1832. In 1839, Gabriel Lame proved the case $n = 7$ [7, 8, 9, 10]. Between 1840 -1843, V. A. Lebesque worked on Fermat's Last Theorem [11, 12]. Between 1847 and 1853, Ernst Eduard Kummer published some masterful papers about this Theorem. Fermat's Last Theorem attracted the attention of many researchers and many studies have been developed around this Theorem. For example the work of Arthur Wieferich (1909), Andre Weil (1940), John Tate (1950), Gerhard Frey (1986), who was the first to suggest that the existence of a solution of the Fermat equation might contradict the modality conjecture of Taniyama, Shimura and Weil [29]; Jean Pierre Serre (1985 - 1986) [14, 15, 16], who gave an interested formulation and (with J. F. Mestre) tested numerically a precise conjecture about modular forms and Galois representations mod p and proved how a small piece of this conjecture the so called epsilon conjecture together Modularity Conjecture would imply Fermat's Last Theorem; Kennedy Ribet (1986) [17], who proved Serre's epsilon conjecture, thus reducing the proof of Fermat's Last Theorem; Barry Mazur (1986), who introduced a significant piece of work on the deformation of Galois representations [18, 19]. However, no final proof was given to this Theorem. This Theorem was unsolved for nearly 350 years. In 1995, using Mazur's deformation theory of Galois representations, recent results on Serre's conjecture on the modularity of Galois representations, and deep arithmetical properties of Hecke algebras, Andrew Wiles with Richard Taylor succeeded in proving that all semi-stable elliptic curves defined over the rational numbers are modular. This result is less than the full Shimura-Taniyama conjecture. This result does imply that the elliptic curve given above is modular. Therefore, proving Fermat's Last Theorem [20, 21]. Many mathematicians are still heavenly involved on studying Fermat's Last Theorem [22, 23, 24]. In 2021, Nag introduced an elementary proof of Fermat's

Ref

1. Diophantus of Alexandria, Arithmetica of Diophantus, 1637.

Last Theorem for epsilons[25]. In 2022, Mouanda constructed the galaxies of sequences of triples of positive integers solutions of the equation $x^2+y^2=z^2$. The unique characterization of the solutions of this equation allowed him to provide an elementary analytic proof of Fermat's Last Theorem [26]. The Fermat Last Theorem for positive integers has been extended over some number fields. In 1966, Domiaty proved that the equation $X^4+Y^4=Z^4$ is solvable in $M_2(\mathbb{Z})$ [27]. Let $GL_n(\mathbb{Z})$ be the group of units of ring $M_n(\mathbb{Z})$. Denote by

$$SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : detA = 1\}. \tag{1.2}$$

In 1989, Vaserstein investigated the question of the solvability of the equation

$$X^n + Y^n = Z^n, n \geq 2, n \in \mathbb{N}, \tag{1.3}$$

for matrices of the group $GL_2(\mathbb{Z})$ [28]. In 1993, Frejman studied the solvability of the equation (1.3) in the set of positive integer powers of a matrix A with elements $a_{11}=0, a_{12}=a_{21}=a_{22}=1$ [29]. In 1995, the same case was studied by Grytczuk [30]. The same year, Khazanov proved that in $GL_3(\mathbb{Z})$ solutions of the equation (1.3 do not exist if n is a multiple of either 21 or 96, and in $SL_3(\mathbb{Z})$ solutions do not exist if n is a multiple of 48 [31]. In 1996, Qin gave another proof of Khazanov's result on the solvability of the equation (1.3) in $SL_2(\mathbb{Z})$ [32]. In 2002, Patay and Szakacs described the periodic elements in $GL_2(\mathbb{Z})$ and gave the answer to some problems concerning the equation (1.3) in matrix groups and in irreducible elements of matrix rings [33]. In 2021, Mao-Ting and Jie proved that Fermat's matrix equation has many solutions in a set of 2-by-2 positive semi-definite integral matrices, and has no nontrivial solutions in some classes including 2-by-2 symmetric rational and stochastic quadratic field matrices [34]. Fermat's Last Theorem has been extended to the field of complex polynomials of one variable [35].

This Theorem has many applications in Cryptography.

In this paper, we are mainly concerned with Fermat's Last Theorem for circulant matrices with positive integers as entries. Firstly, we focus our attention on the construction of the galaxies of sequences of triples of circulant matrices with positive integers as entries solutions of the equation $X^2+Y^2=Z^2$. In particular, Mouanda's matrix choice function allows us to construct practical examples of such galaxies. The elementary characterization of these matrix solutions allows us to prove Fermat's Last Theorem for circulant matrices with positive integers as entries.

Theorem 1.1. The equation

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no circulant matrix with positive integers as entries solutions.

We construct a galaxy of sequences of eigenvalues of circulant matrices and we prove Fermat's Last Theorem for eigenvalues of circulant matrices. Also, we construct a galaxy of sequences of complex polynomials over the unit disk $\mathbb{D}$ associated to circulant matrices and we prove Fermat's Last Theorem for complex polynomials over $\mathbb{D}$.

Ref

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II. Preliminaries

Definition 2.1. *Let $\mathbb{A}$ be a unital Banach algebra. We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set*

$$Inv(\mathbb{A}) = \{a \in \mathbb{A} : \exists b \in \mathbb{A}, ab = ba = 1\}$$

is a group under multiplication. If a is an element of $\mathbb{A}$, the spectrum of a is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda.1 \notin Inv(\mathbb{A})\},$$

and its spectral radius is defined to be

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Let $V = \{a_0, a_1, \ldots, a_{m-1}\} \subset \mathbb{C}$ be a subset of the set of complex numbers, denote by C_V the following Toeplitz matrix:

$$C_V = \begin{pmatrix} a_0 & a_1 & \ddots & a_{m-1} \\ a_{m-1} & a_0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & \ldots & a_{m-1} & a_0 \end{pmatrix}.$$

This matrix is called a $m \times m$-complex circulant matrix or a complex circulant matrix of order m. Denote by $C_m(\mathbb{C})$ the commutative algebra of $m \times m$-complex circulant matrices. Let $\epsilon = e^{\frac{2\pi i}{m}}$ be a primitive m-th root of unity. Let us denote by U the following matrix:

$$U = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \ldots & \ldots & \epsilon^{(m-3)} & \epsilon^{m-2} & \epsilon^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \epsilon^{m-3} & \ldots & \ldots & \epsilon^{(m-3)^2} & \epsilon^{(m-3)(m-2)} & \epsilon^{(m-1)(m-3)} \\ 1 & \epsilon^{m-2} & \ldots & \ldots & \epsilon^{(m-2)(m-3)} & \epsilon^{(m-2)^2} & \epsilon^{(m-1)(m-2)} \\ 1 & \epsilon^{m-1} & \ldots & \ldots & \epsilon^{(m-1)(m-3)} & \epsilon^{(m-1)(m-2)} & \epsilon^{(m-1)^2} \end{pmatrix}.$$

This matrix is called Vandermonde matrix. It is well known that this matrix has the following properties:

$$det(U) = \frac{1}{m^{\frac{m}{2}}} \prod_{i,j=0}^{m-1} (\epsilon^j - \epsilon^i) \neq 0,$$

Notes

Notes

U is non-singular, unitary, $U^{-1} = \overline{U}^T, U^T = U$ and $U^{-1} = \overline{U} = U^*$. It is well known that all the elements of $C_m(\mathbb{C})$ are simultaneously diagonalized by the same unitary matrix U, that is, for A in $C_m(\mathbb{C})$, one has

$$U^* A U = D_A$$

with D_A is a diagonal matrix with diagonal entries given by the ordered eigenvalues of A: $\lambda_1^A, \lambda_2^A, \ldots, \lambda_m^A$. The factorization $U^*AU = D_A$ is called the spectral factorization of A [36, 37, 38, 39]. It is possible to write the matrix C_V as one variable complex polynomial. Indeed, let P be the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

It is simple to see that

$$C_V = \sum_{k=0}^{m-1} a_k P^k.$$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk. The complex polynomial

$$f_V(z) = \sum_{k=0}^{m-1} a_k z^k$$

over $\mathbb{D}$ is called the associated complex polynomial of the matrix $C_V = f_V(P)$. It follows that if

$$X = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

is a $m \times m$-complex matrix, then

$$f_V(X) = \sum_{k=0}^{m-1} a_k X^k = \begin{pmatrix} a_0 & a_1 & \ddots & a_{m-1} \\ 0 & a_0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & a_0 \end{pmatrix}$$

is a $m \times m$- upper complex triangular Toeplitz matrix. The complex polynomial

$$f_V(z) = \sum_{k=0}^{m-1} a_k z^k$$

is also called the associated complex polynomial of the matrix $f_V(X)$.

III. The Universe of an Algebra

Definition 3.1. Let $x, y, z \in \mathbb{C}$ be complex numbers. Denote by

$$(x, y, z)^n = (x^n, y^n, z^n), n = \frac{p}{q}, p, q \in \mathbb{N}, q \neq 0.$$

The triple (x^n, y^n, z^n) is called the triple (x, y, z) to the power n.

Definition 3.2. Let $x, y, z \in \mathbb{C}$ be complex numbers. Denote by

$$\alpha (x, y, z) = (\alpha x, \alpha y, \alpha z), (x, y, z) + (a, b, c) = (x + a, y + b, z + c).$$

Definition 3.3. A universe of degree $\frac{p}{q}$ of the algebra $\mathcal{B}$ is the set $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$ of triples (x, y, z) of elements of $\mathcal{B}$ which satisfy the law of stability

$$x^{\frac{p}{q}} + y^{\frac{p}{q}} = z^{\frac{p}{q}}, xyz \neq 0, p, q \in \mathbb{N}, q \neq 0.$$

The element (x, y, z) is called a star (or a planet) of the universe $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$. Every sequence $(x_k, y_k, z_k)_{n\geq 0}$ of elements of the universe $\mathbb{F}_{\frac{p}{q}}(\mathcal{B})$ is called a planet system of elements of $\mathcal{B}$.

The set

$$\mathbb{F}_{\frac{p}{q}}(C_m(\mathbb{C})) = \left\{ (X, Y, Z) \in \mathbb{C}^3 : X^{\frac{p}{q}} + Y^{\frac{p}{q}} = Z^{\frac{p}{q}}, XYZ \neq 0 \right\}, p, q \in \mathbb{N}, q \neq 0,$$

is called the complex ciculant universe of degree $\frac{p}{q}$. In particular, the set

$$\mathbb{F}_n(C_m(\mathbb{N})) = \left\{ (X, Y, Z) \in C_m(\mathbb{N})^3 : X^n + Y^n = Z^n, XYZ \neq 0 \right\}, n \in \mathbb{N}, n \geq 2,$$

Notes

is called the natural circulant universe of degree n. We are going to show that the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ is not empty. Fermat's Last Theorem for circulant matrices is equivalent to say that

$$\mathbb{F}_n(C_m(\mathbb{N})) = \{\} = \phi, n \geq 3.$$

In other words, there are matrix complex universes which don't have triples of matrices of positive integers as entries elements.

IV. Mouanda's Choice Function for Matrices

Denote by $C_*(C_m(\mathbb{C})) = \{h/h : C_m(\mathbb{C}) \longrightarrow C_m(\mathbb{C})\}$, the set of complex functions over $\mathbb{C}$. Let

$$\Omega(\mathbb{F}_2(C_m(\mathbb{C}))) = \{P : P \subseteq \mathbb{F}_2(C_m(\mathbb{C}))\}$$

be the set of all subsets of $\mathbb{F}_2(C_m(\mathbb{C}))$. Theorem 2.5 of [26] allows us to claim that the appropriate choice of the values of $m_0(k)$ and $n_0(k)$ such that

$$\frac{2(m_0(k) - n_0(k)) \pm \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} \in C_m(\mathbb{C})$$

leads to the construction of sequences of triples of circulant matrices with positive (or negative) integers as entries which satisfy the equation

$$X^2 + Y^2 = Z^2.$$

Let $f_{\mathbb{M}} : C_*(C_m(\mathbb{C})) \times C_*(C_m(\mathbb{C})) \longrightarrow \Omega(\mathbb{F}_2(C_m(\mathbb{C})))$ be the function defined by

$$f_{\mathbb{M}}(m_0(k), n_0(k)) = \left[\begin{array}{c} m_0(k) = a^{\beta(k)}, k, a, \beta(k) \in C_m(\mathbb{C}), \beta \in C_*(C_m(\mathbb{C})) \\ m_0(k) - n_0(k) \in C_m(\mathbb{C}) \\ X_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} \in C_m(\mathbb{C}) \\ Y_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + n_0(k) \\ Z_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + m_0(k) \end{array}\right],$$

This type of function is called Mouanda's choice function for matrices. Mouanda's choice function for matrices is a galaxy valued function. This function allows us to construct galaxies of sequences of matrices.

V. A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries

All the galaxies defined in this section have been deduced from the galaxies already introduced in [26].

Ref

26. J. Moussounda Mouanda, On Fermat's Last Theorem and Galaxies of sequences of positive integers, American Journal of Computational Mathematics, 12(2022), 162-189.

Definition 5.1. *A multi-galaxy is a galaxy which contains other galaxies. The order of a galaxy is the number of variables of the galaxy.*

Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

be a $m \times m-$matrix. Denote by $T_k = P^k, k = 1, 2, ..., m-1$. Let

$$U_m = \{I_m, T_1, T_2, ..., T_{m-1}\}$$

be a finite set of unitary circulant matrices. The elements of the set U_m satisfy the following:

$$T_i T_j = T_{i+j}, T_i T_{m-i} = I_m, T_{m+i} = T_i, T_k = P^k.$$

Denote by

$$\mathcal{P}^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.$$

Let

$$f(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. The Toeplitz matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} = f(P)$$

is called the circulant matrix with complex numbers as entries. The polynomial $f(z)$ is called associated polynomial of the matrix $f(P)$. Recall that the

Notes

set $C_m(\mathbb{C})$ is the commutative algebra of $m \times m$-complex circulant matrices. In other words,

$$C_m(\mathbb{C}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{C} \right\}.$$

It follows that

$$C_m(\mathbb{C}) = \left\{ f(P) : f \in \mathcal{P}^{(m)}(\mathbb{C}) \right\}$$

and

$$C_m(\mathbb{N}) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} : a_k \in \mathbb{N} \right\}.$$

It is quiet clear that

$$C_m(\mathbb{N}) \subset C_m(\mathbb{Z}) \subset C_m(\mathbb{Q}) \subset C_m(\mathbb{R}) \subset C_m(\mathbb{C}).$$

Mouanda's matrix choice function $f_{\mathbb{M}}$ allows us to construct galaxies of circulant matrices. For instance, if we choose $m_0 = A^{2k} \times 2, m_0 - n_0 = \alpha I_m, A \in C_m(\mathbb{N})$ the model

$$Gala(\mathbb{N}I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k \\ Y_k(\alpha I_m, A) = 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ Z_k(\alpha I_m, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ \alpha = r^2, k, r \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{bmatrix}$$

is called the galaxy of sequences of circulant matrices with positive integers as entries of order 2. For α_0 and A fixed, the triple

$$(X_0(\alpha_0 I_m, A), Y_0(\alpha_0 I_m, A), Z_0(\alpha_0 I_m, A))$$

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is called the origin of the galaxy $Gala(\alpha_0 I_m, C_m(\mathbb{N}))$. The elements of $Gala(\alpha_0 I_m, C_m(\mathbb{N}))$ satisfy

$$X_k^2(\alpha_0 I_m, A) + Y_k^2(\alpha_0 I_m, A) = Z_k^2(\alpha_0 I_m, A), k \in \mathbb{N}, A \in C_m(\mathbb{N})$$

and

$$(X_k(\alpha_0 I_m, A), Y_k(\alpha_0 I_m, A), Z_k(\alpha_0 I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), A, B, C, D \in C_m(\mathbb{N})$$

with $k, p, q \in \mathbb{N}, q \neq 0$.

Notes

Example 5.2. **A Finite Galaxy of Sequences of Circulant Matrices with Positive Integers as Entries**

- The model

$$Gala(4I_m, U_m) = \begin{bmatrix} X_k(4I_m, A) = 4I_m + 4 \times A^k \\ Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\ Z_k(4I_m, A) = 4I_m + 4 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in U_m \\ (X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m) \end{bmatrix}$$

is called the finite galaxy of sequences of circulant matrices with positive integers as entries of order 1. The triple $(X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A))$ is called the origin of the galaxy $Gala(4I_m, U_m)$. The triple

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A))$$

satisfies

$$X_k^2(4I_m, A) + Y_k^2(4I_m, A) = Z_k^2(4I_m, A), k \in \mathbb{N},$$

$$X_0(4I_m, A) + Y_0(4I_m, A) + Z_0(4I_m, A) = 24I_m$$

and

$$(X_k(4I_m, A), Y_k(4I_m, A), Z_k(4I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), k, p, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The finite galaxy $Gala(4I_m, U_m)$ allows the construction of the infinite galaxy

$$Gala(4I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(4I_m, A) = 4I_m + 4 \times A^k \\ Y_k(4I_m, A) = 4 \times A^k + 2 \times A^{2k} \\ Z_k(4I_m, A) = 4I_m + 4 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(4I_m, A), Y_0(4I_m, A), Z_0(4I_m, A)) = (8I_m, 6I_m, 10I_m) \end{bmatrix}$$

which has the same origin and stability law than $Gala(4I_m, U_m)$. Therefore, we can say that the galaxy $Gala(\mathbb{N}I_m, C_m(\mathbb{N}))$is a multi-galaxy.

- Assume that $m=5, U_5=\{I_5,T_1,T_2,T_3,T_4\}$ and

$$A=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\end{pmatrix}^2=\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}=T_2\in U_5.$$

Notes

The triples $(X_k(4I_5,A),Y_k(4I_5,A),Z_k(4I_5,A))$ of the galaxy

$$\left[\begin{array}{c}
X_k(4I_5,A)=\begin{pmatrix}4&0&0&0&0\\0&4&0&0&0\\0&0&4&0&0\\0&0&0&4&0\\0&0&0&0&4\end{pmatrix}+4\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}^k\\
Y_k(4I_5,A)=4\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}^k+2\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}^{2k}\\
Z_k(4I_5,A)=4I_5+4\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}^k+2\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}^{2k}\\
k\in\mathbb{N}\\
(X_0(4I_5,A),Y_0(4I_5,A),Z_0(4I_5,A))=(8I_5,6I_5,10I_5)\\
X_1(4I_5,A)=4I_5+4T_2\\
Y_1(4I_5,A)=4T_2+2T_4\\
Z_1(4I_5,A)=4I_5+4T_2+2T_4
\end{array}\right]$$

satisfy the equation

$$X_k^2(4I_5,A)+Y_k^2(4I_5,A)=Z_k^2(4I_5,A),k\in\mathbb{N}$$

and

$$(X_k(4I_5,A),Y_k(4I_5,A),Z_k(4I_5,A))\neq\left(D^{\frac{p}{q}},B^{\frac{p}{q}},C^{\frac{p}{q}}\right),D,B,C\in U_n,k,q,p\in\mathbb{N},q\neq 0.$$

- The elements of the galaxy

$$Gala(9I_m,U_m)=\left[\begin{array}{c}
X_k(9I_m,A)=9I_m+6\times A^k\\
Y_k(9I_m,A)=6\times A^k+2\times A^{2k}\\
Z_k(9I_m,A)=9I_m+6\times A^k+2\times A^{2k}\\
k\in\mathbb{N},A\in U_m\\
(X_0(9I_m,A),Y_0(9I_m,A),Z_0(9I_m,A))=(15I_m,8I_m,17I_m)
\end{array}\right]$$

satisfy

$$X_k^2(9I_m, A) + Y_k^2(9I_m, A) = Z_k^2(9I_m, A), k \in \mathbb{N}$$

and

$$(X_k(9I_m, A), Y_k(9I_m, A), Z_k(9I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. Again, from the galaxy $Gala(9I_m, U_m)$, we can construct a galaxy which has an infinite number of elements. Indeed, the galaxy

$$Gala(9I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(9I_m, A) = 9I_m + 6 \times A^k \\ Y_k(9I_m, A) = 6 \times A^k + 2 \times A^{2k} \\ Z_k(9I_m, A) = 9I_m + 6 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(9I_m, A), Y_0(9I_m, A), Z_0(9I_m, A)) = (15I_m, 8I_m, 17I_m) \end{array}\right]$$

has the same origin and stability law than the galaxy $Gala(9I_m, U_m)$. This galaxy has an infinite number of elements.

- The elements of the galaxy

$$Gala(16I_m, U_m) = \left[\begin{array}{c} X_k(16I_m, A) = 16I_m + 8 \times A^k \\ Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k} \\ Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in U_m \\ (X_0(16I_m, A), Y_0(16I_m, A), Z_0(16I_m, A)) = (24I_m, 10I_m, 26I_m) \end{array}\right]$$

satisfy

$$X_k^2(16I_m, A) + Y_k^2(16I_m, A) = Z_k^2(16I_m, A), k \in \mathbb{N}$$

and

$$(X_k(16I_m, A), Y_k(16I_m, A), Z_k(16I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, k, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The galaxy $Gala(16, U_m)$ has a finite number of elements (or planets). However, the galaxy

$$Gala(16I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(16I_m, A) = 16I_m + 8 \times A^k \\ Y_k(16I_m, A) = 8 \times A^k + 2 \times A^{2k} \\ Z_k(16I_m, A) = 16I_m + 8 \times A^k + 2 \times A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \\ (X_0(16I_m, A), Y_0(16I_m, A), Z_0(16I_m, A)) = (24I_m, 10I_m, 26I_m) \end{array}\right]$$

has an infinite number of elements (or planets).

Notes

Notes

Example 5.3. *Assume that* $m_0 - n_0 = 2I_m, m_0 = A^{2k}, A \in C_m(\mathbb{N})$. *We can define the galaxy*

$$\Delta(2I_m, C_m(\mathbb{N})) = \begin{bmatrix} X_k(2I_m, A) = 2I_m + 2 \times A^k \\ Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{bmatrix}$$

in which the triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)), k \in \mathbb{N}$, *satisfy*

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N},$$

$$X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m$$

and

$$(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in C_m(\mathbb{N})$.

Example 5.4. ***A Finite Galaxy***

The triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)), k \in \mathbb{N}$, *of the galaxy*

$$\Delta(2I_m, U_m) = \begin{bmatrix} X_k(2I_m, A) = 2I_m + 2 \times A^k \\ Y_k(2I_m, A) = 2 \times A^k + A^{2k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^k + A^{2k} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}$$

satisfy

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N},$$

$$X_0(2I_m, A) + Y_0(2I_m, A) + Z_0(2I_m, A) = 12I_m$$

and

$$(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, C, D, B \in U_m$.

Example 5.5. *Assume that* $m = 10, U_{10} = \{I_{10}, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9\}$ *with*

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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The triples $(X_k(2I_{10},T_1), Y_k(2I_{10},T_1), Z_k(2I_{10},T_1))$ *of the galaxy*

$$\Delta(2I_{10},T_1) = \left[\begin{array}{c} X_k(2I_{10},T_1) = 2I_{10} + 2\times T_1^k \\ Y_k(2I_{10},T_1) = 2\times T_1^k + T_1^{2k} \\ Z_k(2I_{10},T_1) = 2I_{10} + 2\times T_1^k + T_1^{2k} \\ k\in\mathbb{N} \end{array} \right]$$

satisfy

$$X_k^2(2I_{10},T_1) + Y_k^2(2I_{10},T_1) = Z_k^2(2I_{10},T_1), k\in\mathbb{N}$$

and

$$(X_k(2I_{10},T_1), Y_k(2I_{10},T_1), Z_k(2I_{10},T_1)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p,q,k\in\mathbb{N}, q\neq 0,$$

$D,B,C\in U_{10}$.

VI. Σ-Model

The triples $(X_k(2I_m,A), Y_k(2I_m,A), Z_k(2I_m,A))$ of the galaxy

$$\Sigma(2I_m,U_m) = \left[\begin{array}{c} X_k(2I_m,A) = 2I_m + 2\times A^{4k} \\ Y_k(2I_m,A) = 2\times A^{4k} + A^{8k} \\ Z_k(2I_m,A) = 2I_m + 2\times A^{4k} + A^{8k} \\ k\in\mathbb{N}, A\in U_m \end{array} \right]$$

satisfy

$$X_k^2(2I_m,A) + Y_k^2(2I_m,A) = Z_k^2(2I_m,A), k\in\mathbb{N},$$

$$X_0(2I_m,A) + Y_0(2I_m,A) + Z_0(2I_m,A) = 12I_m$$

and

$$(X_k(2I_m,A), Y_k(2I_m,A), Z_k(2I_m,A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p,q,k\in\mathbb{N}, q\neq 0,$$

$C,D,B\in U_m$. The multi-galaxy $\Sigma(2I_m, C_m(\mathbb{N}))$ has an infinite number of planets.

Example 6.1. The triples $(X_k(2I_m,T_4), Y_k(2I_m,T_4), Z_k(2I_m,T_4))$ *of the galaxy*

$$\Sigma(2I_m,T_4) = \left[\begin{array}{c} X_k(2I_m,T_4) = 2I_m + 2\times T_4^{4k} \\ Y_k(2I_m,T_4) = 2\times T_4^{4k} + T_4^{8k} \\ Z_k(2I_m,T_4) = 2I_m + 2\times T_4^{4k} + T_4^{8k} \\ k\in\mathbb{N}, \end{array} \right]$$

Notes

Notes

satisfy

$$X_k^2(2I_m, T_4) + Y_k^2(2I_m, T_4) = Z_k^2(2I_m, T_4), k \in \mathbb{N},$$

$$X_0(2I_m, T_4) + Y_0(2I_m, T_4) + Z_0(2I_m, T_4) = 12I_m$$

and

$$(X_k(2I_m, T_4), Y_k(2I_m, T_4), Z_k(2I_m, T_4)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C, T_4 \in U_m$.

Example 6.2. *The triples* $(X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3))$ *of the sequence*

$$\Sigma(2I_m, T_3) = \left[\begin{array}{c} X_k(2I_m, T_3) = 2I_m + 2 \times T_3^{4k} \\ Y_k(2I_m, T_3) = 2 \times T_3^{4k} + T_3^{8k} \\ Z_k(2I_m, T_3) = 2I_m + 2 \times T_3^{4k} + T_3^{8k} \\ k \in \mathbb{N} \\ (X_0(2I_m, T_3), Y_0(2I_m, T_3), Z_0(2I_m, T_3)) = (4I_m, 3I_m, 5I_m) \end{array}\right]$$

satisfy

$$X_k^2(2I_m, T_3) + Y_k^2(2I_m, T_3) = Z_k^2(2I_m, T_3), k \in \mathbb{N},$$

$$X_0(2I_m, T_3) + Y_0(2I_m, T_3) + Z_0(2I_m, T_3) = 12I_m$$

and

$$(X_k(2I_m, T_3), Y_k(2I_m, T_3), Z_k(2I_m, T_3)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, q, k \in \mathbb{N}, q \neq 0,$$

$D, B, C, T_3 \in U_m, \Sigma(2I_m, T_3) \subset \Sigma(2I_m, U_m) \subset \Sigma(2I_m, C_m(\mathbb{N}))$.

VII. Power Models of Galaxies of Sequences of Circulant Matrices with Positive Integers as Entries of Order 3

A model of a galaxy is a power model if the power of the lead of the model is a power. For example, if we choose $m_0 = A^{2\lambda^k}$ and $m_0 - n_0 = 2 \times \alpha^2 \times \lambda^2 I_m$, the model

$$URS(\mathbb{N}I_m, U_m, \mathbb{N}I_m) = \left[\begin{array}{c} X_k(\alpha I_m, A, \lambda I_m) = 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda^k} \\ Y_k(\alpha I_m, A, \lambda I_m) = 2\alpha \times \lambda \times A^{\lambda^k} + A^{2\lambda^k} \\ Z_k(\alpha I_m, A, \lambda I_m) = 2 \times \alpha^2 \times \lambda^2 I_m + 2\alpha \times \lambda \times A^{\lambda^k} + A^{2\lambda^k} \\ k, \alpha, a, \lambda \in \mathbb{N}, \alpha \neq 0, a \neq 0, \lambda \neq 0, A \in U_m \end{array}\right]$$

is a power model. The elements of the model $URS(U_m, U_m, U_m)$ satisfy

$$X_k^2(\alpha I_m, A, \lambda I_m) + Y_k^2(\alpha I_m, A, \lambda I_m) = Z_k^2(\alpha I_m, A, \lambda I_m), k \in \mathbb{N}$$

and

$$(X_k(\alpha I_m, A, \lambda I_m), Y_k(\alpha I_m, A, \lambda I_m), Z_k(\alpha I_m, A, \lambda I_m)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), k, p, q \in \mathbb{N},$$

$, q \neq 0, D, B, C \in U_m, URS(U_m, U_m, U_m) \subset URS(C_m(\mathbb{N}), C_m(\mathbb{N}), C_m(\mathbb{N}))$.

Example 7.1. The elements of the galaxy

$$URS(2I_m, U_m, 2I_m) = \begin{bmatrix} X_k(2I_m, A, 2I_m) = 32I_m + 8 \times A^{2^k} \\ Y_k(2I_m, A, 2I_m) = 8 \times A^{2^k} + A^{2^{k+1}} \\ Z_k(2I_m, A, 2I_m) = 32I_m + 8 \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}$$

satisfy

$$X_k^2(2I_m, A, 2I_m) + Y_k^2(2I_m, A, 2I_m) = Z_k^2(2I_m, A, 2I_m), k \in \mathbb{N}$$

and

$$(X_k(2I_m, A, 2I_m), Y_k(2I_m, A, 2I_m), Z_k(2I_m, A, 2I_m)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right),$$

$p, q, k \in \mathbb{N}, q \neq 0, D, B, C \in U_m$.

Example 7.2. If we choose $m_0 = A^{2^{k+1}}$ and $m_0 - n_0 = 2 \times \alpha^2 I_m$, we could construct the galaxy

$$\Omega(\alpha I_m, U_m) = \begin{bmatrix} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in U_m \end{bmatrix}, \alpha \in \mathbb{N}.$$

The elements of the galaxy $\Omega(\alpha I_m, U_m)$ satisfy

$$X_k^2(\alpha I_m, A) + Y_k^2(\alpha I_m, A) = Z_k^2(\alpha I_m, A), k \in \mathbb{N}$$

and

$$(X_k(\alpha I_m, A), Y_k(\alpha I_m, A), Z_k(\alpha I_m, A)) \neq \left(D^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), p, k, q \in \mathbb{N}, q \neq 0,$$

$D, B, C \in U_m$. The galaxy

Notes

$$\Omega(\alpha I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array}\right], \alpha \in \mathbb{N},$$

has an infinite number of planet systems.

Notes

The characterization of the elements of the set $\mathbb{F}_2(C_m(\mathbb{N}))$ is completely the same as the characterization of the elements of the set $\mathbb{F}_2(\mathbb{N})$ [26].

Remark 7.3. Let (X, Y, Z) and (X_1, Y_1, Z_1) be two elements of $\mathbb{F}_2(C_m(\mathbb{N}))$. Then

$$(X, Y, Z) \neq \left(A^{\frac{p}{q}}, B^{\frac{p}{q}}, C^{\frac{p}{q}}\right), (X_1, Y_1, Z_1) \neq \left(A_1^{\frac{p}{q}}, B_1^{\frac{p}{q}}, C_1^{\frac{p}{q}}\right), (X, Y, Z) \neq (X_1, Y_1, Z_1),$$

$A, B, C, A_1, B_1, C_1 \in \mathbb{F}_2(C_m(\mathbb{N}))$.

Let us observe that the characterization of one element of the set $\mathbb{F}_2(C_m(\mathbb{N}))$ allows us to deduce the characterization of the elements of the set $\mathbb{F}_2(C_m(\mathbb{N}))$. In other words, the set $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. Remark 7.3 allows us to prove the following result:

Theorem 7.4. The equation

$$X^{2n} + Y^{2n} = Z^{2n}, XYZ \neq 0, n \in \mathbb{N}(n \geq 2)$$

has no circulant matrix with positive integers as entries solutions.

Proof. Assume that there exist $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^{2n} + Y^{2n} = Z^{2n}, n \geq 2, n \in \mathbb{N}.$$

This means that

$$(X^n)^2 + (Y^n)^2 = (Z^n)^2.$$

Therefore,

$$(X^n, Y^n, Z^n) \in \mathbb{F}_2(C_m(\mathbb{N})) = \left\{(A, B, C) \in C_m(\mathbb{N})^3 : A^2 + B^2 = C^2\right\}.$$

Remark 7.3 allows us to claim that we have a contradiction because the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. Finally, there exist no circulant matrices with positive integers as entries $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^{2n} + Y^{2n} = Z^{2n}, n \in \mathbb{N}, n \geq 2.$$

This result allows to claim that the equation

$$\left(X^2\right)^n + \left(Y^2\right)^n = \left(Z^2\right)^n, n \geq 2,$$

has no solution in $C_m(\mathbb{N})$. We can now prove our main result.

Proof of Theorem 1.1

We just need to show that if $(X,Y,Z) \in \mathbb{F}_n(C_m(\mathbb{C})), n \in \mathbb{N}, n \geq 3$, then $(X,Y,Z) \notin C_m(\mathbb{N})$. Let (X,Y,Z) be an element of the universe $\mathbb{F}_n(C_m(\mathbb{C})), n \geq 3$. Then

$$X^n + Y^n = Z^n.$$

This implies that

$$(\sqrt{X})^{2n} + (\sqrt{Y})^{2n} = (\sqrt{Z})^{2n} \Longleftrightarrow (X^{2n})^{\frac{1}{2}} + (Y^{2n})^{\frac{1}{2}} = (Z^{2n})^{\frac{1}{2}}.$$

and

$$(X^{2n})^{\frac{1}{2}} + (Y^{2n})^{\frac{1}{2}} = (Z^{2n})^{\frac{1}{2}} \Longleftrightarrow (X^{\frac{n}{2}})^2 + (Y^{\frac{n}{2}})^2 = (Z^{\frac{n}{2}})^2.$$

Theorem 7.4 and Remark 7.3 allow us to claim that

$$(\sqrt{X}, \sqrt{Y}, \sqrt{Z}) \notin \mathbb{F}_2(C_m(\mathbb{N})), (X^{2n}, Y^{2n}, Z^{2n}) \notin \mathbb{F}_2(C_m(\mathbb{N}))$$

and

$$(X^{\frac{n}{2}}, Y^{\frac{n}{2}}, Z^{\frac{n}{2}}) \notin \mathbb{F}_2(C_m(\mathbb{N})), n \geq 3,$$

since $\mathbb{F}_2(C_m(\mathbb{N}))$ has no power elements. In other words,

$$(\sqrt{X}, \sqrt{Y}, \sqrt{Z}) \notin C_m(\mathbb{N}), (X^{2n}, Y^{2n}, Z^{2n}) \notin C_m(\mathbb{N})$$

and

$$(X^{\frac{n}{2}}, Y^{\frac{n}{2}}, Z^{\frac{n}{2}}) \notin C_m(\mathbb{N}), n \geq 3.$$

The fact that

$$(X^{2n}, Y^{2n}, Z^{2n}) \notin C_m(\mathbb{N}), n \geq 3$$

implies that

$$(X,Y,Z) \notin C_m(\mathbb{N}).$$

VIII. Eigenvalues of Circulant Matrices

It is well known that if $A = C(\Omega)$, where Ω is a compact Hausdorff space, then $\sigma(f) = f(\Omega)$ for all $f \in A$. Let

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. Then $\sigma(\varphi) = \varphi(\mathbb{D})$.

Notes

Notes

The Spectral Mapping Theorem 1. [40]. Let $T \in B(H)$ be a normal bounded linear operator on the Hilbert space H and let $f : \sigma(T) \longrightarrow \mathbb{C}$ be a continuous function on $\sigma(T)$. Then $\sigma(f(T)) = f(\sigma(T))$.

Let us introduce the well known spectrum of circulant matrices associated to complex polynomials over $\mathbb{D}$. Let

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k$$

be a complex polynomial over $\mathbb{D}$. Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

be an $m \times m$- matrix. The matrix P is normal. Indeed, $PP^* = P^*P = I_m$. Assume that

$$A_0 = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix}.$$

A simple calculation shows that

$$A_0 = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.$$

The matrix A_0 is considered as a polynomial of one variable. Let us compute the spectrum of the normal matrix P. Let

$$f(\lambda) = det(P - \lambda I_m) = \begin{vmatrix} -\lambda & 1 & 0 & \ddots & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -\lambda & 1 \\ 1 & 0 & \dots & 0 & 0 & -\lambda \end{vmatrix} = 1 - \lambda^m.$$

be the characteristic polynomial of P. Thus, λ is a primitive m-th root of unity. Therefore,

$$\sigma(P) = \left\{ \lambda_k^P = e^{\frac{2\pi k i}{m}} : k = 0, 1, 2, ..., m-1 \right\}.$$

In other words,

$$\sigma(P) = \left\{ 1, e^{\frac{2\pi i}{m}}, e^{\frac{4\pi i}{m}}, e^{\frac{6\pi i}{m}}, e^{\frac{8\pi i}{m}}, e^{\frac{10\pi i}{m}}, e^{\frac{12\pi i}{m}}, e^{\frac{14\pi i}{m}}, ..., e^{\frac{2(m-1)\pi i}{m}} \right\}.$$

Finally,

$$\sigma(P) = \left\{ \lambda_0^P, \lambda_1^P, ..., \lambda_{m-1}^P \right\} \subset \mathbb{D}.$$

The spectral mapping Theorem allows us to claim that

$$\varphi(\sigma(P)) = \sigma(\varphi(P)) = \sigma(A_0).$$

Therefore,

$$\sigma(A_0) = \left\{ \varphi(\lambda_0^P), \varphi(\lambda_1^P), ..., \varphi(\lambda_{m-1}^P) \right\}.$$

IX. Galaxy of Sequences of Eigenvalues of Circulant Matrices

In this section, we construct galaxies of sequences of eigenvalues of circulant matrices.

Theorem 9.1. *Let $X, Y, Z \in C_m(\mathbb{C})$ be three circulant matrices with complex numbers as entries such that*

$$X^n + Y^n = Z^n.$$

Then

$$(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, ..., m-1.$$

Notes

In other words, the triples $(\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m-1$. *Thatis, the planet system*

$$M(X,Y,Z) = \begin{bmatrix} \lambda_k^X \\ \lambda_k^Y \\ \lambda_k^Z \\ k = 0, 1, 2, ..., m-1 \end{bmatrix} \subset \mathbb{F}_n(\mathbb{C}).$$

Notes

Proof: Let $(X, Y, Z) \in C_m(\mathbb{C}), n \in \mathbb{N}, n \geq 3$, be an element of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$. The spectral factorization of the matrices X, Y, Z [36, 37, 38] allows us to claim that there exists a unitary matrix U such that

$$X = U \begin{pmatrix} \lambda_0^X & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^X & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^X & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^X \end{pmatrix} U^* = UD_XU^*,$$

$$Y = U \begin{pmatrix} \lambda_0^Y & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^Y & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^Y & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^Y \end{pmatrix} U^* = UD_YU^*$$

and

$$Z = U \begin{pmatrix} \lambda_0^Z & 0 & 0 & \ddots & 0 & 0 \\ 0 & \lambda_1^Z & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \lambda_{m-2}^Z & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \lambda_{m-1}^Z \end{pmatrix} U^* = UD_ZU^*.$$

The equation $X^n + Y^n = Z^n$ implies

$$UD_X^nU^* + UD_Y^nU^* = UD_Z^nU^*.$$

It follows that

$$U\left[D_X^n + D_Y^n\right]U^* = UD_Z^nU^*.$$

We can claim that

$$D_X^n + D_Y^n = D_Z^n.$$

Finally,

$$(\lambda_k^X)^n + (\lambda_k^Y)^n = (\lambda_k^Z)^n, \lambda_k^X \in \sigma(X), \lambda_k^Y \in \sigma(Y), \lambda_k^Z \in \sigma(Z), k = 0, 1, 2, ..., m-1.$$

In other words, the triples $(\lambda_k^X, \lambda_k^Y, \lambda_k^Z) \in \mathbb{F}_n(\mathbb{C}), k = 0, 1, 2, ..., m-1$. That is, the planet system

$$M(X, Y, Z) = \left[\begin{array}{c} \lambda_k^X \\ \lambda_k^Y \\ \lambda_k^Z \\ k = 0, 1, 2, ..., m-1 \end{array}\right] \subset \mathbb{F}_n(\mathbb{C}).$$

Every triple (X,Y,Z) of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$ generates a planet system M(X,Y,Z) which has exactly m elements of the universe $\mathbb{F}_n(\mathbb{C})$. We can say that the galaxies of sequences of circulant matrices are linked to the galaxies of sequences of eigenvalues of circulant matrices. Let us consider the galaxies

$$\Omega(\alpha I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha I_m, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha I_m, A) = \alpha^2 I_m + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array}\right], \alpha \in \mathbb{N},$$

of circulant matrices. We can construct the galaxies of sequences of eigenvalues of the triples of circulant matrices of the galaxies $\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}$. For example, the galaxies

$$\Omega(\alpha, \sigma(C_m(\mathbb{N}))) = \left[\begin{array}{c} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in C_m(\mathbb{N}) \end{array}\right] \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N},$$

are galaxies of sequences of eigenvalues of triples of circulant matrices of the galaxies $\Omega(\alpha I_m, C_m(\mathbb{N})), \alpha \in \mathbb{N}$. As we can see that the galaxies

$$\Omega(\alpha, \sigma(A)) = \left[\begin{array}{c} X_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} \\ Y_k(\alpha, \lambda) = 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ Z_k(\alpha, \lambda) = \alpha^2 + 2 \times \alpha \times \lambda^{2^k} + \lambda^{2^{k+1}} \\ k \in \mathbb{N}, \lambda \in \sigma(A) \end{array}\right] \subset \mathbb{F}_2(\mathbb{N}), \alpha \in \mathbb{N}, A \in C_m(\mathbb{N}),$$

Notes

have each a finite number of planet systems. In our case, each galaxy has m planet systems. Every galaxy of the universe $\mathbb{F}_2(C_m(\mathbb{N}))$ generates a new galaxy of eigenvalues of elements of $C_m(\mathbb{N})$. Let us consider the galaxy

$$\Sigma(2I_m, C_m(\mathbb{N})) = \left[\begin{array}{c} X_k(2I_m, A) = 2I_m + 2 \times A^{4k} \\ Y_k(2I_m, A) = 2 \times A^{4k} + A^{8k} \\ Z_k(2I_m, A) = 2I_m + 2 \times A^{4k} + A^{8k} \\ k \in \mathbb{N}, A \in C_m(\mathbb{N}) \end{array}\right].$$

We know that the triples $(X_k(2I_m, A), Y_k(2I_m, A), Z_k(2I_m, A))$ of the galaxy $\Sigma(2I_m, C_m(\mathbb{N}))$ satisfy

$$X_k^2(2I_m, A) + Y_k^2(2I_m, A) = Z_k^2(2I_m, A), k \in \mathbb{N}.$$

Define the galaxy

$$\Sigma(2, \sigma(C_m(\mathbb{N}))) = \left[\begin{array}{c} X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\ Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\ Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in C_m(\mathbb{N}) \end{array}\right].$$

The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$ of the galaxy $\Sigma(2, \sigma(C_m(\mathbb{N})))$ satisfy

$$X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.$$

We can deduce the galaxies

$$\Sigma(2, \sigma(A)) = \left[\begin{array}{c} X_k(\lambda) = 2 + 2 \times \lambda^{4k} \\ Y_k(\lambda) = 2 \times \lambda^{4k} + \lambda^{8k} \\ Z_k(\lambda) = 2 + 2 \times \lambda^{4k} + \lambda^{8k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), \end{array}\right], A \in C_m(\mathbb{N}),$$

which have a finite number of planet systems. The triples $(X_k(\lambda), Y_k(\lambda), Z_k(\lambda))$ of the galaxy $\Sigma(2, \sigma(A))$ also satisfy

$$X_k^2(\lambda) + Y_k^2(\lambda) = Z_k^2(\lambda), k \in \mathbb{N}.$$

The first eigenvalue of every matrix of $C_m(\mathbb{N})$ is a positive integer.

Theorem 9.2. Let $A \in C_m(\mathbb{N})$ *be a circulant matrix with positive integers as entries. Then the first eigenvalue* λ_0^A *of* A *is a positive integer. In other words,* $\lambda_0^A \in \mathbb{N}$.

Proof. Let

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots & a_{m-2} & a_{m-1} \\ a_{m-1} & a_0 & a_1 & a_2 & \ddots & a_{m-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & a_{m-1} & a_0 & a_1 \\ a_1 & a_2 & \ddots & \ddots & a_{m-1} & a_0 \end{pmatrix} \in C_m(\mathbb{N})$$

be a circulant matrix with positive integers as entries. Then there exists a polynomial

$$\varphi(z) = \sum_{k=0}^{m-1} a_k z^k, a_k \in \mathbb{N}, z \in \mathbb{D},$$

such that

$$A = \varphi(P) = \sum_{k=0}^{m-1} a_k P^k.$$

We know that

$$\sigma(A) = \left\{\varphi(\lambda_0^P), \varphi(\lambda_1^P), ..., \varphi(\lambda_{m-1}^P)\right\}$$

with

$$\left\{\lambda_0^P, \lambda_1^P, ..., \lambda_{m-1}^P\right\} = \left\{1, e^{\frac{2\pi i}{m}}, e^{\frac{4\pi i}{m}}, e^{\frac{6\pi i}{m}}, e^{\frac{8\pi i}{m}}, e^{\frac{10\pi i}{m}}, e^{\frac{12\pi i}{m}}, ..., e^{\frac{2(m-1)\pi i}{m}}\right\}.$$

Therefore,

$$\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in \mathbb{N}.$$

Remark 9.3. Let $\mathcal{A}$ be an algebra and let $A \in C_m(\mathcal{A})$. Then

$$\lambda_0^A = \varphi(1) = \sum_{k=0}^{m-1} a_k \in \mathcal{A}.$$

Theorem 9.1 and Theorem 9.2 allow us to provide another proof of our main result.

Notes

Notes

Second Proof of Theorem 1.1

Assume that there exist $X, Y, Z \in C_m(\mathbb{N}), n \in \mathbb{N}, n \geq 3$, three circulant matrices with positive integers as entries such that

$$X^n + Y^n = Z^n.$$

Theorem 9.1 and Theorem 9.2 allow us to claim that

$$(\lambda_0^X)^n + (\lambda_0^Y)^n = (\lambda_0^Z)^n, n \geq 3.$$

This implies that the equation $x^n + y^n = z^n, n \geq 3$ has positive integer solutions. We have a contradiction. Therefore, the equation

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no circulant matrix with positive integers as entries solutions.

Let $\mathcal{A}$ be an algebra and let

$$\mathcal{P}^{(m)}(\mathcal{A}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathcal{A}, z \in \mathbb{D} \right\}$$

be the algebra of polynomials over $\mathbb{D}$. Complex polynomials of the algebra $\mathcal{P}^{(m)}(\mathbb{N})$ allow us to provide Fermat's Last Theorem for eigenvalues of circulant matrices.

Theorem 9.4. The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no positive integer eigenvalues of circulant matrices solutions.

Proof. Assume that there exists a triple (λ, η, μ) of positive integer eigenvalues of circulant matrices X,Y and Z of $C_m(\mathbb{N})$ such that

$$\lambda^n + \eta^n = \mu^n, \lambda\eta\mu \neq 0, n \in \mathbb{N}, n \geq 3, \lambda \in \sigma(X), \eta \in \sigma(Y), \mu \in \sigma(Z).$$

Therefore, there exist three f, g, h complex polynoimals of $\mathcal{P}^{(m)}(\mathbb{N})$ such that

$$f(z)^n + g(z)^n = h(z)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}.$$

In particular,

$$f(P)^n + g(P)^n = h(P)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}$$

with P the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

In other words,

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}, n \geq 3.$$

We have a contradiction. Finally, The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no positive integer eigenvalues of circulant matrices solutions.

X. Fermat's Last Theorem for Complex Polynomials associated to Circulant Matrices

We can now construct models of galaxies of complex polynomials associated to circulant matrices. Recall that

$$\mathcal{P}^{(m)}(\mathbb{C}) = \left\{ f(z) = \sum_{k=0}^{m-1} a_k z^k : a_k \in \mathbb{C}, z \in \mathbb{D} \right\}.$$

The galaxies of the universe $\mathbb{F}_n(C_m(\mathbb{C}))$ generate the galaxies of the universe $\mathbb{F}_n(\mathcal{P}^{(m)}(\mathbb{C}))$. For example, from the galaxy $\Sigma(2I_m, C_m(\mathbb{C}))$, we can construct the galaxy

$$\Sigma(2, \mathcal{P}^{(m)}(\mathbb{C})) = \left[\begin{array}{c} X_k(f) = 2 + 2 \times f^{4k} \\ Y_k(f) = 2 \times f^{4k} + f^{8k} \\ Z_k(f) = 2 + 2 \times f^{4k} + f^{8k} \\ k \in \mathbb{N}, f \in \mathcal{P}^{(m)}(\mathbb{C}) \end{array} \right] \subset \mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C})).$$

We can continue doing the same identification process with the remaining galaxies of $\mathbb{F}_2(C_m(\mathbb{C}))$. This process will lead to the construction of the universe $\mathbb{F}_2(\mathcal{P}^{(m)}(\mathbb{C}))$. Now, we are able to provide Fermat's Last Theorem for complex polynomials over the unit disk $\mathbb{D}$ associated to circulant matrices of the set $C_m(\mathbb{N})$.

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Theorem 10.1. The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no solutions in $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0.$

Proof. Assume that there exists a triple (f, g, h) of complex polynomials of the set $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0$, such that

$$f(z)^n + g(z)^n = h(z)^n, n \in \mathbb{N}, n \geq 3, z \in \mathbb{D}.$$

This implies that

$$f(P)^n + g(P)^n = h(P)^n, n \in \mathbb{N}, n \geq 3.$$

with P the cyclic permutation $m \times m-$matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

In other words, there exist $X, Y, Z \in C_m(\mathbb{N})$ such that

$$X^n + Y^n = Z^n, XYZ \neq 0, n \in \mathbb{N}, n \geq 3.$$

We have a contradiction. Finally, The equation

$$x^n + y^n = z^n, xyz \neq 0, n \in \mathbb{N}(n \geq 3)$$

has no solutions in $\mathcal{P}^{(m)}(\mathbb{N}), m \in \mathbb{N}, m \neq 0.$

Theorem 1.1, Theorem 9.4 and Theorem 10.1 are equivalent.

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The General Service Readiness in Health Facilities: Evidence based on Bangladesh Health Facility Survey, 2017 Data

By Tazia Hossain
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Abstract- Although health outcomes have improved in the past few decades, still Bangladesh is working towards the target of sustainable goal 3 (SGD 3). In the recent age, anew insight is at hand thathaving access to the health care is not that having the quality care. Increasing people expectations, health need and determined new health goals are raising the agenda for health systems to insure better health outcomes and bigger social value. The objective of the study is to give center attention to the percentage attended in basic amenities as well as the average percentage of general service readiness to ensure high quality health systems for Bangladesh in the SDG era. IN this study different domains of general service readiness of high quality care are explored, found different resource constrains and suggestions are given to improve quality care. This study found that the average general service readiness score of health facilities in Bangladesh is 47.202 %(basic amenities= 47.1667 %, basic equipment =78.55%, diagnostic capacity = 10.7833%,for standard precaution is 53.712% and essential medicine = 45.8%).Main concern should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities.

Keywords: general service readiness, basic amenities, bangladesh health facility survey, 2018.

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THEGENERALSERVICEREADINESSINHEALTHFACILITIESEVIDENCEBASEDONBANGLADESHHEALTHFACILITYSURVEY2017DATA

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The General Service Readiness in Health Facilities: Evidence based on Bangladesh Health Facility Survey, 2017 Data

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Abstract- Although health outcomes have improved in the past few decades, still Bangladesh is working towards the target of sustainable goal 3 (SGD 3). In the recent age, anew insight is at hand thathaving access to the health care is not that having the quality care. Increasing people expectations, health need and determined new health goals are raising the agenda for health systems to insure better health outcomes and bigger social value. The objective of the study is to give center attention to the percentage attended in basic amenities as well as the average percentage of general service readiness to ensure high quality health systems for Bangladesh in the SDG era. IN this study different domains of general service readiness of high quality care are explored, found different resource constrains and suggestions are given to improve quality care. This study found that the average general service readiness score of health facilities in Bangladesh is 47.202 %(basic amenities= 47.1667 %, basic equipment =78.55%, diagnostic capacity = 10.7833%,for standard precaution is 53.712% and essential medicine = 45.8%).Main concern should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities. Policymakers should give insights to quality care by improving the facilities up to 100% in health services.

Keywords: general service readiness, basic amenities, bangladesh health facility survey, 2018.

I. Introduction

Last 20 years is known as the golden age for the global health care system. The major achievement acquired in health determinants (eg, clean water, and sanitation) and health services (eg, vaccination and antenatal care).9-11. High quality care involves diagnosis, appropriate and treatment. High quality health systems require strong financing, trained service providers, service delivery, and also community involvement. Poor-quality care can causes adverse health outcomes, health-related suffering even it build lack of trust and confidence in health systems.

Bangladesh Government is giving continuously effort and working hard to achieve SDG 3. The Government has been taken many initiatives to improve the health condition of our people. Government has established many community clinics from corner to corner the country to ensure better health and give free access to health care to the people. These community clinics provide free health care and free medicines to the people.

Service Provision Assessment (SPA) or health facility surveys have been conducted throughout East and South Asia as well as sub-Saharan Africa to determine primary health care readiness. SPA survey is carried out In Bangladesh named as Bangladesh Health Facility Survey (BHFS) .4. To addition the Bangladesh Demographic and Health Survey (BDHS) data by providing useful descriptive

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information on health system focus on reproductive, maternal, child health services, non-communicable diseases etc to measure the quality care and health services at the national level.6

II. Objectives of the Study

The main objectives of the study are to:

- Find out the average percentage of basic amenities, basic equipments, standard precautions for infection prevention, diagnostic capacity and essential medicines given by health facility service.
- Determine the average percentage of general service readiness in health facility care
- Give recommendations to improve or develop the existing strategies to ensure 100% general service readiness in a quality health care.

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III. Literature Review

The Lancet Global Health Commission is working for theHigh Quality Health Systems in this SDG Era. The word Quality of care is a insightful theme. This Commission is focus on improving the quality of care of people.

According to The Lancet Global Health Commission almost 9 million lives are lost in every year for lack of good quality care and that a shocking 60% of those deaths were among them who actually had got access to care. That is why having access to the health care is not that much enough?

According to The Lancent Global Health Commission, high-quality health systems could save more than 8 million lives each year and 2.5 million deaths due to cardiovascular disease, 1 million newborn deaths and 50% of all maternal deaths in each year in the low and middle countries. 1

In low and middle income countries, over 8 million people per year are dying from situation that could be treated by the health system. These deaths causes in US$6 trillion in economic losses in year 2015. Poor health quality care becomes the barrier for reducing mortality. 1

A global report on quality of health care is published earlier in 2018 by WHO, the World Bank, and the Organization for Economic Co-operation and Development (OECD). 12

According to WHO the overall capability of health facilities to provide general health services is defined as the general service readiness. People-centred health systems is a system where all people should have the same access to quality health services they actually needed in their life.13

The availability of components essential to provide services is known as Readiness, for example, basic amenities, basic equipment, standard precautions for infection prevention, diagnostic capacity and essential medicines. In this study, the general service readiness is defined by five general service readiness domains provided by WHO tracer. 14

IV. Data and Methodology

a) Data

The data is extracted from Bangladesh Health Facility Survey (2017 BHFS) which is nationally representative health facility survey. The information is collected on general facility readiness. A stratified random sample of 1600 observations (health facilities) are selected from 8 division of Bangladesh which includes district hospitals

(DH), mother and child welfare center(MCWCs), upazila health complexes (UHCc), union health and family welfare centers (UHFWCs), union subcenters or rural dispensaries (US or RD), and community clinics (CCs). The survey was conducted under the management of the National Institute of Population Research and Training (NIPORT) of the Ministry of Health and Family Welfare (MOHFW) funded by the Government of Bangladesh and the U.S. Agency for International Development (USAID).

Notes

b) Methodology

This study data has been extracted data from the 2017 Bangladesh Health Facility Survey (BHFS).The 2017 BHFS is successfully collected a stratified random sample of 1,524health facilities from all formal-sector health facilities in Bangladesh. It is not possible to measure general service readiness directly .The WHO identifies specific tracers or items to measure the readiness indices. By WHO tracer the domains of general service readiness are basic amenities, basic equipments, standard precautions for infection prevention, diagnostic capacity and essential medicines. A percentage is a number or ratio which is calculated as a fraction of 100. A percentage of facilities in each domain are observed. Then the average of each domain is calculated by adding the percentage of indicators dividing by the number of indicators. Using the average of five domains, the average general service readiness is calculated. The domains are:

Table 1: The domain with corresponding indicators for measuring general service readiness

Serial No.	Domain	Indicators
a.	Basic amenities	improved water supply, emergency transport , computer with internet, regular electricity availability , separate latrine or toilet for female clients, room with privacy, communication equipment
b.	Basic equipment	thermometer, stethoscope, blood pressure apparatus, adult scale, child scale, light source availability
c.	Standard precautions for infection prevention	appropriate storage of sharps waste and safe ultimate disposal of sharps, appropriate storage of infectious waste and safe ultimate disposal of infectious wastes, disinfectant and guidelines for standard precautions, single-use disposable/auto-disable syringes, soap and running water or alcohol-based hand rub, latex gloves
d.	Diagnostic capacity	haemoglobin, blood glucose , urine dipstick for protein, urine dipstick for glucose, syphilis rapid diagnostic test, urine pregnancy test
e.	Essential medicines	amitriptyline tablet, amoxicillin (tablets capsules), Atenolol tables, Captopril tablets, Ceftriaxone injectable, Ciprofloxacin tablets, Cotrimoxazole oral suspension, Diazepam tablets, Diclofenac tablets, Glibenclamide

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	tablets, Omeprazole or cimetidine tables, Paracetamol oral suspension, Salbutamol inhaler, Simvastatin or atorvastatin tablets.
Average General Service Readiness	(a+b+c+d+e) / 5

V. Analysis and Findings

The 2017 BHFS collected several types of health facilities. The seven amenities have been observed in the study. The amenities are: separate latrine or toilet for female clients, emergency transport, computer with internet, regular electricity availability, improved water supply, room with privacy, communication equipment.

Table 2: Frequency distribution of basic amenities and equipment for client services

Basic amenities	Percentage (N= 1524)	Basic equipments	Percentage (N= 1524)
improved water supply	90	thermometer	86.3
emergency transport	5	stethoscope	94.2
computer with internet	58	blood pressure apparatus,	85.4
regular electricity availability	43	adult scale	85.9
room with privacy	70	child scale	61.6
separate latrine or toilet for female clients	17	light source availability	51.9
communication equipment	11	All basic equipment	28

The availability of at least 5 basic amenities in health facilities is only 19% (BDHS 2017) which is improved from 11% (BDHF 2014).

The communication equipments (land line or mobile phone) is highest in private hospital (95%), upazila and district level (82%) but very low in community clinics (3%) and overall it is 11%. Available emergency transport is high in district hospital (97%), upazila district level (79%), private hospital (62%) Including all community clinics and NGO facility clinics this percentage is very low (5%). Still the facilities of emergency transport, communication equipment have to be improved in huge scale. The percentage has to be improved to regular electricity availability, computer internet, and separate female latrine for the client's satisfaction with health services.

The facilities are most likely to have WHO and USAID proposed basic equipments such as thermometer, stethoscope, blood pressure apparatus, adult scale child scale light source availability.

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The availability of all six basic equipments in health facilities is 28% (BDHS 2017)) which is improved from 26% (BDHF 2014). This percentage is high in private hospitals and NGO facilities which are 80% or more. Community clinics (CCs) and union level public facilities are meager indicating 23%.

The average general service readiness score of health facilities in Bangladesh for basic amenities is 47.1667% and for basic equipments is 78.55 %.

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Table 3: Frequency distribution of standard precautions for infection prevention and diagnostic capacity for client services

Standard precautions	Percentage (N= 1524)	diagnostic capacity	Percentage (N= 1524)
safe ultimate disposal of sharps	72.5	hemoglobin	17
and appropriate storage of sharps waste	64.1	blood glucose	19.7
safe ultimate disposal of infectious wastes	66.3	urine pregnancy test	12.4
appropriate storage of infectious waste and	32.9	urine dipstick for protein	10.8
disinfectant and guidelines for standard precautions,	17.3	urine dipstick for glucose	10
single-use disposable/auto-disable syringes	76.9	Syphilis rapid diagnostic test	4.8
soap and running water or alcohol-based hand rub	21.5		
latex gloves	78.2		

The average general service readiness score of health facilities in Bangladesh for diagnostic capacity is 10.7833% and for standard precaution is 53.712%.

Table 4: Frequency distribution of essential medicines for client services

Facility type	Percentage (N= 1524)			
	District and upazila public facilities	Community clinic (CC)	Private hospital	Total
Amitriptyline tablet	7.7	0	57.6	2.4
Aamoxicillin (tablets capsules)	85	85.6	61.5	83.6
Atenolol tables	30	0	61.6	3.9
Captopril tablets	53.9	0	74.2	6.4
Ceftriaxone injectable	66	2.8	77.8	9.4
Ciprofloxacin tablets	96	95.6	78.2	94.8
Cotrimoxazole oral suspension	73.8	86.7	73.8	82.7
Diazepam tablets	77.4	6.4	75.1	22.8
Diclofenac tablets	90.6	60.8	80.4	63.8
Glibenclamide tablets	41.1	0	69.6	5
Omeprazole or cimetidine tables	95.5	93.5	80.7	92.7
Paracetamol oral suspension	75.5	88.3	79.6	83.7
Salbutamol inhaler	95.1	89.9	78.5	86.6
Simvastatin or atorvastatin tablets	24.2	0.2	64.3	3.4

Essential medicines are comparatively more available in private sector than district and upazila level and lower in community clinics. The average score of health facilities for essential medicines is 45.8%.

VI. Findings

The average of general service readiness is calculated by adding the average percentage of 5 domains, dividing by the number 5 which is 47.202 %(average basic amenities= 47.1667 %, average basic equipment =78.55%, average diagnostic capacity = 10.7833%, average standard precaution is 53.712% and average essential medicine = 45.8%) less than 50 %. By constantly improving existing facilities to

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changing population needs a high-quality health system can enhanced in general health care.

Recommendation

Bangladesh government has incredible improved in health sector which is valued and trusted by the all over the country. T give 100% service facility Government should

- Develop our community clinics by improving basic amenities, diagnostic capacity, standard precautions and essential medicines.
- Give special attention to Union level public facilities as union health and family welfare centers (UHFWCs), union subcenters or rural dispensaries (US or RD), and community clinics (CCs)so that these facilities can follow IMCI guidelines properly.
- Ensure the adequacy of basic amenities, standard precautions, especially for improving the diagnostic capacity the authorities of the facilities should take necessary measures.
- Stringently supervise the performances of health providers.

Further study

There is enormous scope to work with this study by advanced analysis such as ordinal logistic regression model assuming proportional odds assumption i.e. proportional odds model could have been applied in the study.

VII. Conclusion

The study found the significant lacking of the general service readiness in basic amenities: emergency transport, communication equipment. Special attention should be given in different diagnostic capacity improving such as hemoglobin, blood glucose, urine pregnancy test, urine dipstick for protein, urine dipstick for glucose, Syphilis rapid diagnostic test. All the facilities should maintain the disinfectant and guidelines for standard precautions for quality care. Only half of the essential medicines are available in community clinics. Higher general service readiness in private hospitals is observed. This study finds that priority should be given for the healthcare in the primary level especially for the community clinics and the rural public facilities.

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The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Fellows are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Fellow Members.

Benefit

To the Institution

Get letter of appreciation

Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

Exclusive Network

Get access to a closed network

A FSFRC member gets access to a closed network of Tier 1 researchers and scientists with direct communication channel through our website. Fellows can reach out to other members or researchers directly. They should also be open to reaching out by other.

Career | Credibility | Exclusive | Reputation

Certificate

Receive a print ed copy of a certificate

Fellows receive a printed copy of a certificate signed by our Chief Author that may be used for academic purposes and a personal recommendation letter to the dean of member's university.

Career | Credibility | Exclusive | Reputation

Designation

Get honored title of membership

Fellows can use the honored title of membership. The “FSFRC” is an honored title which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., FSFRC or William Walldroff, M.S., FSFRC.

Career | Credibility | Exclusive | Reputation

Recognition on the Platform

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All the Fellow members of FSFRC get a badge of "Leading Member of Global Journals" on the Research Community that distinguishes them from others. Additionally, the profile is also partially maintained by our team for better visibility and citation. All fellows get a dedicated page on the website with their biography.

Career | Credibility | Reputation

Future Work

Get discounts on the future publications

Fellows receive discounts on future publications with Global Journals up to 60%. Through our recommendation programs, members also receive discounts on publications made with OARS affiliated organizations.

Career | Financial

GJ Internal Account

Unlimited forward of Emails

Fellows get secure and fast GJ work emails with unlimited forward of emails that they may use them as their primary email. For example,
john [AT] globaljournals [DOT] org.

Career | Credibility | Reputation

Premium Tools

Access to all the premium tools

To take future researches to the zenith, fellows and associates receive access to all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

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Organize seminar/conference

Fellows are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career | Credibility | Financial

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Early invitations to all the symposiums, seminars, conferences

All fellows receive the early invitations to all the symposiums, seminars, conferences and webinars hosted by Global Journals in their subject.

Exclusive

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Get a remuneration of 15% of author fees

Fellow members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

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Fellows may join as a member of the Editorial Board of Global Journals Incorporation (USA) after successful completion of three years as Fellow and as Peer Reviewer. Additionally, Fellows get a chance to nominate other members for Editorial Board.

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All members get access to 5 selected scientific museums and observatories across the globe. All researches published with Global Journals will be kept under deep archival facilities across regions for future protections and disaster recovery. They get 10 GB free secure cloud access for storing research files.

ASFRC

ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL

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> The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Associate membership can later be promoted to Fellow Membership. Associates are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Associate Members.

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Career | Credibility | Exclusive | Reputation

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Exclusive

Publishing Articles & Books

Earn 30-40% of sales proceeds

Associates can publish articles (limited) without any fees. Also, they can earn up to 30-40% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.

Exclusive Financial

Reviewers

Get a remuneration of 15% of author fees

Associate members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

And Much More

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Associate	Fellow	Research Group	Basic
$4800 **lifetime designation**	$6800 **lifetime designation**	$12500.00 **organizational**	APC **per article**
Certificate, LoR and Momento 2 discounted publishing/year **Gradation** of Research 10 research contacts/day 1 **GB** Cloud Storage **GJ** Community Access	**Certificate,** LoR and Momento **Unlimited** discounted publishing/year **Gradation** of Research **Unlimited** research contacts/day 5 **GB** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**Certificates,** LoRs and Momentos **Unlimited** free publishing/year **Gradation** of Research **Unlimited** research contacts/day **Unlimited** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**GJ** Community Access

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
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- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

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1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

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If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

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Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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	A-B	**C-D**	**E-F**
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Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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