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Obtaining the Sinusoid for Working with Membrane Vibration from the Bessel Differential Equation

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Notes

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Jose Mujica EE [α] & Ramon A. Mata-Toledo [σ]

Abstract- To study membranes' acoustic behavior there are several equations concerning their vibration and resonance which help us to understand better their physical properties. In the present paper we want to show the results of the Bessel sinusoid expressions when we change the values of their variables. Because the procedures to solve the Bessel equation are usually not shown in their entirety in books or the Internet this paper explains in detail each step to go from its differential expression to the sinusoid one.

I. Introduction

It is well known that the expression J_0 of Bessel equations is the graphic most widely published in Differential Equation textbooks. Every time we talk about the vibration of a circular membrane the graphics of Figure 1is always referred to. Mathematical software packages such as Maple™ always includes this curve in their libraries. If you hit once a circular membrane, it is easy to assume that the first undulation will have a higher amplitude than the following undulations which will decrease gradually as shown also in Figure 1.

$plot([\text{BesselJ}(0, x), \text{BesselJ}(1, x)], x = 0..30, y = -2..2, color = [red, blue]);$

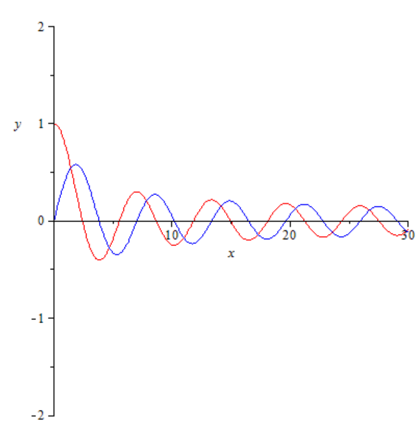

Figure 1: Plot of J0 and J1 of Bessel

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In this work we proceed to find the Jv(x) functions of the first kind and order v to satisfy the second order differential equation of Bessel.

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}+(x^2-v^2)y(x)=0$$

To study acoustical radiation and vibration analysis one usually solves problems in cylindrical coordinates that are associated with Besselfunctions of integer order.We can solve also spherical problems with half-integer order Bessel Function.

II. Bessel Functions

a) Differential Equation of Oder v

Wefocus this work on Besselfunctions of the first kind of positive order and real arguments[1]for

$$x^2y"+xy'+(x^2-v^2)y=0$$

Assuming

$$y=\sum_{0}^{\infty}c_n\,x^{n+r}$$

which leads to

Differentiating and substituting

$$y'=c_{n(n+r)X^{n+r-1}}$$

$$y"=c_{n(n+r)(n+r-1)X^{n+r-2}}$$

$$x^2y"+xy'+(x^2-v^2)y=0$$

$$x^2c_n(n+r)(n+r-1)x^{n+r}x^{-2}$$

$$c_n(n+r)(n+r-1)x^{n+r}$$

$$x^2y"+xy'+(x^2-v^2)y=0$$

$$xc_n(n+r)x^{n+r}x^{-1}$$

$$x^2y"+xy'+(x^2-v^2)y=0$$

$$c_n(x^2-v^2)x^{n+r}$$

Writing out the differential equation give us

$$\sum_{n=0}^{\infty}c_n(n+r)(n+r-1)\,x^{n+r}+\sum_{n=0}^{\infty}c_n(n+r)x^{n+r}+\sum_{n=0}^{\infty}c_nx^{n+r+2}-v^2\sum_{n=0}^{\infty}c_nx^{n+r}$$

Ref

1. Dennis Zill, A First course in Differential Equations with Applications, 2nd. Edition, PWS Publishers, Wadsworth, Inc. Belmont, California, 1982.

For n=0

$$c_0(0+r)(0+r-1)x^{0+r}+c_0(0+r)x^{0+r}+c_0x^{0+r+2}-c_0v^2x^{0+r}$$

$$c_0(r)(r-1)x^r+c_0(r^2-r)x^r+c_0rx^r+c_0x^{r+2}-c_0v^2x^r$$

$$c_{0(x^r r^2 - x^r r + r x^r + x^r x^2 - v^2 x^r)}$$

$$c_0(r^2-r+r+x^2-v^2)x^r$$

Substituting this back to the differential equation gives

$$c_0(r^2-r+r-v^2)x^r+x^r\sum_{n=1}^{\infty}c_n[(n+r)(n+r-1)+(n+r)-v^2]x^n+x^r\sum_{n=0}^{\infty}c_nx^{n+2}$$

$$(n+r)(n+r-1)+(n+r)$$

$$(n+r)(n+r+1)=n^2+2nr+r^2-n-r$$

$$n^2+2nr+r^2-n-r+(n+r)$$

$$n^2+2nr+r^2$$

$$(n+r)^2$$

Leading to

$$c_0(r^2-v^2)x^r+x^2\sum_{n=1}^{\infty}c_n[(n+r)^2-v^2]x^n+x^r\sum_{n=1}^{\infty}c_nx^{n+2}=0$$

If r_1 =v

$$[(n+v)^2-v^2]=n^2+2nv+v^2-v^2$$

$$n(n+2v)$$

giving

$$x^v\sum_{n=1}^{\infty}c_nn(n+2v)x^n+x^v\sum_{n=0}^{\infty}c_nx^{n+2}$$

or

$$x^v\left[c_1(1+2v)+\sum_{n=2}^{\infty}c_nn(n+2v)x^n+\sum_{n=0}^{\infty}c_nx^{n+2}\right]$$

Notes

Using k =n-2 in

$$\sum_{n=2}^{\infty} c_n n(n+2v)x^n$$

And k =n in

$$\sum_{n=0}^{\infty} c_n x^{n+2}$$

leads to

$$\sum_{k=0}^{\infty}[(k+2)(k+2+2v)c_{k+2}+c_k]x^{k+2}=o$$

Writing this out

$$x^v\left[c_1(1+2v)+\sum_{k=0}^{\infty}[(k+2)(k+2+2v)c_{k+2}+c_k]x^{k+2}\right]=0$$

So that

$$(1+2\mathrm{v})=0$$

$$(k+2)(k+2+2v)c_{k+2}+c_k=0$$

Or

$$c_{k+2}=\frac{-c_k}{(k+2)(k+2+2v)} \qquad \text{k=0,1,2..} \qquad (1)$$

c_1=0 in (1) leads to$c_3=c_5=c_7=\ldots=0$ so, for k=0,2,4. after makingk+2=2n, n=1, 2, 3..we will have that

$$c_{2n}=\frac{-c_{2n-2}}{2n(2n+2v)}$$

$$\frac{-c_{2n-2}}{2^2 n(n+v)}$$

The coefficients with even index are determined by the following formula:

$$c_2=-\frac{c_0}{2^2*1*(1+v)}$$

$$c_4=-\frac{c_2}{2^2*2*(2+v)}=\frac{c_0}{2^4*1*2(1+v)(2+v)}$$

$$c_6=-\frac{c_4}{2^2*3*(3+v)}=\frac{c_0}{2^6*1*2*3(1+v)(2+v)(3+v)}$$

.
.
.

$$c_{2n}=\frac{(-1^n)c_0}{2^n n!\,(1+v)(2+v)..(n+v)}$$

Notes

i. *The Gamma Function*

Since we can select a_0, we take it to be [2]

$$c_0 = \frac{1}{2^v \Gamma(1+\mathrm{v})}$$

The Gamma function Γ is defined as

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-t} dt$$

then

$$\Gamma(n+1) = n!$$

Where

$$\Gamma(1+\mathrm{v}+1) = (1+\mathrm{v})\, \Gamma(1+v)$$

$$\Gamma(1+\mathrm{v}+2) = (2+\mathrm{v})\, \Gamma(2+v)$$

$$= (2+\mathrm{v})(1+\mathrm{v})\, \Gamma(1+v)$$

Then we can write

$$c_{2n} = \frac{(-1^n)c_0}{2^{n+v} n!\,(1+v)(2+v)..(n+v)\Gamma(1+v)}$$

$$c_{2n} = \frac{(-1^n)c_0}{2^{n+v} n!\,\Gamma(1+v+n)}$$

The solution for this c_n is

$$y = \sum_{n=0}^{\infty} c_{2n} x^{2n+v}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

$$J_v(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

b) *Half Integer Order [3]*

$$j_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

Ref

2. Dennis Zill, Advance Engineering Mathematics, 7th Edition, Jones & Barlett Learning, Burlington, MA, 2022.

$$j_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\frac{3}{2}+n)} \left(\frac{x}{2}\right)^{2n} \left(\frac{x}{2}\right)^{\frac{1}{2}}$$

$$j_{\frac{1}{2}}(x) = \sqrt{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\frac{3}{2}+n)} \left(\frac{x}{2}\right)^{2n}$$

$$j_{\frac{1}{2}}(X) = \sqrt{\frac{1}{2}} \left[\frac{(1)}{\Gamma\left(\frac{3}{2}\right)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\left(\frac{5}{2}\right)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma\left(\frac{7}{2}\right)2!} - \cdots \right]$$

$$\Gamma_{\frac{1}{2}} = \sqrt{\pi}$$

$$\Gamma_{\frac{3}{2}} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma_{\frac{5}{2}} = \frac{15\sqrt{\pi}}{8}$$

$$j_{\frac{1}{2}}(x) = \sqrt{\frac{1}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\frac{8}{15\sqrt{\pi*2}} - \cdots \right]$$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots \right]$$

$$= \sqrt{\frac{x}{2\pi}} * \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$j_{1/2(x)} = \sqrt{\frac{2}{\pi x}} * \sin(x) \quad (2)$$

c) Spherical Bessel [4]

If we solve the Helmholtz's equation we obtain the following differential

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left[1 + \frac{l(l+1)}{x^2}\right] y = 0 \quad (3)$$

One of the solutions of (3) is

$$Jj_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x)$$

The Bessel function of half-integral order is used to define the important function:

Ref

4. Teboho A. Moloi, Spherical Bessel Functions, Nelson Mandela University, Port Elizabeth, South Africa, 2022.

$$J_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

J_n is called the spherical Bessel function of the first kind. For n=0 we can see that (1) becomes [5]

$$J_0(x) = \sqrt{\frac{\pi}{2x}} J_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \sin x = \frac{\sin x}{x}$$

III. Plotting with Maple

a) Plotting sin(x)/x

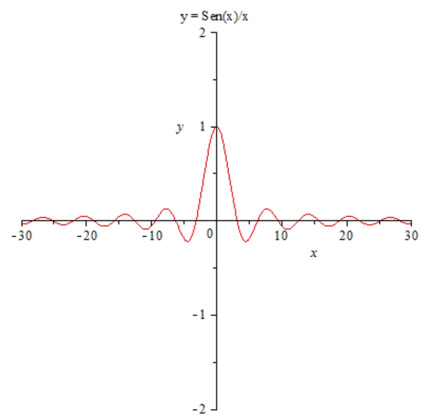

Figure 2: Plot of sin(x)/x

b) Plotting of J(-1/2)

$$plot\left(\sqrt{\frac{2}{\pi \cdot x}} \cdot \cos\left(x - \frac{1}{4}(1)\cdot\pi\right) + \sqrt{\frac{2}{\pi \cdot x}} \cdot \cos\left(x - \frac{1}{4}(2)\cdot\pi\right) + \sqrt{\frac{2}{\pi \cdot x}} \cdot \cos\left(x - \frac{1}{4}(3)\cdot\pi\right), x = 1..20, y = -1..5, discont = true, color = red\right)$$

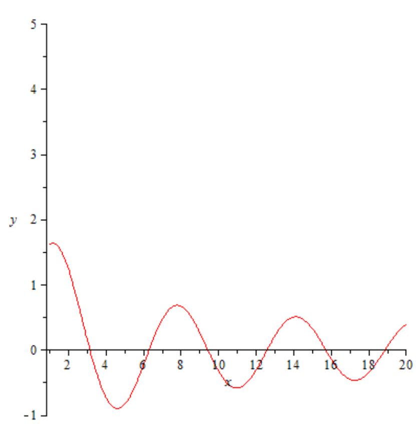

Figure 2: Plot of Sin (2/πx)^2*Sin(x)

Ref

5. Annie Cuyt1, Wen-shin Lee1; 2, and Min Wu3. High accuracy trigonometric approximations of the real Bessel functions of the First kind, University of Stirling, Scotland, UK,2019.

c) Plotting J(1/2) with variations

$$plot\left(\left[\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(1)\cdot\pi\right)+\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(2)\cdot\pi\right)+\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(3)\cdot\pi\right),2\cdot\left(\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(1)\cdot\pi\right)+\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(2)\cdot\pi\right)+\sqrt{\frac{2}{\pi\cdot x}}\cdot\sin\left(x-\frac{1}{4}(3)\cdot\pi\right)\right)\right],x=1..20,y=-5..5,discont=true,color=[red,blue]\right)$$

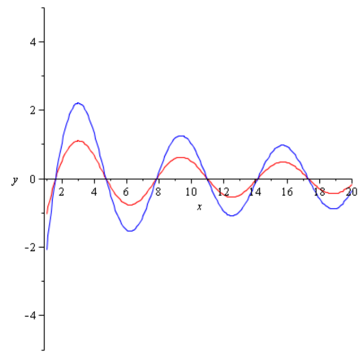

Figure 4: Plot of Two variations of J(1/2)

$$plot\left(2\cdot\left(\frac{\sin(x)}{8x}\right),x=-30..30,y=-2..2,discont=true,title="y = Sen(x)"\right);$$

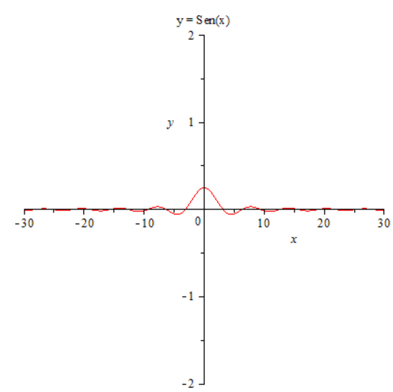

Figure 5: In this plot you can assume a change in the frequency

$$plot\left(2\cdot\left(\frac{\sin(x)}{x}\right),x=-30..30,y=-2..2,discont=true,title="y = Sen(x)"\right);$$

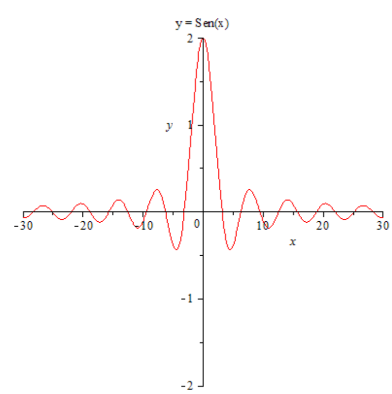

Figure 6: In this plot you can assume a change in Amplitude

Notes

Notes

IV. Comparing Bessel with a Linear Model Underdamped

Given that

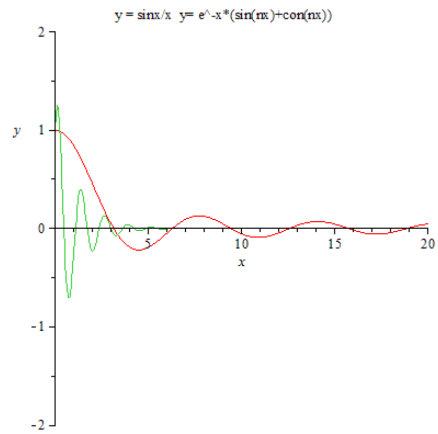

Figure 7: Bessel J(1/2) vs Over damped

V. Animation of Bessel on Maple Software

$$animate(wave, [uC(0,3)], t = 0..P(0,3));$$

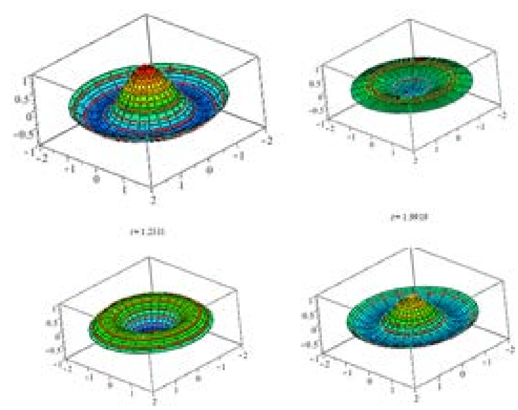

Figure 8: Four snapshots of the Bessel animation

VI. Conclusion

A study of Bessel equations can lead us to new applications in acoustics. By changing their parameters, we can determine the ratio between a membrane dimension and the acoustic power it can radiate or the way the membrane will vibrate. For example, we can find analogies between the elasticity of a trampoline jumper and a loudspeaker membrane. We can also study suspension, material thickness, and surfaces and compare them to computer animations which allow us to explore these analogies. Thus, educational institutions with low budgets can make physics experiments on vibration and mechanics at more affordable prices using computer animations. Knowing Bessel equation and its relationship with Helmholtz's formula we can extend the application to the resonance phenomena too.

Acknowledgment

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Professor in the department of Mathematics and Statistics. Dr. Sochacki is also co-author of the Parker-Sochacki's algorithm widely for solving systems of ordinary Differential Equations (https://en.wikipedia.org/wiki/Parker%E2%80%93Sochacki_method).

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6. Maple Software plot. https://www.maplesoft.com/

Notes