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Annotation- A spectral stability criterion is formulated for nonlinear control systems, continuous and discrete, whose matrices have a simple structure. The spectrum providing global and exponential stability of a continuous nonlinear system is called acceptable. The spectral stability criterion is formulated as a sufficient condition for the admissibility of the matrix spectrum. For continuous nonlinear systems, the elements of the matrix spectrum are represented by the sum of two components, the first (main) is an arbitrarily selected negative scalar common to the entire spectrum, the second (the so–called increment) is constructed as functions that differ for all elements of the spectrum. The conditions for the admissibility of the matrix spectrum are reduced to a restriction on the maximum absolute value of the increment. A formula has been developed that determines the exact upper bound of this value, which ensures the acceptability of the spectrum. The spectral stability criterion of discrete nonlinear systems is based on the developed criterion for continuous systems.

Keywords: continuous nonlinear control system, matrices of simple structure, matrix spectrum, lyapunov function, exact upper bound.

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Annotation- A spectral stability criterion is formulated for nonlinear control systems, continuous and discrete, whose matrices have a simple structure. The spectrum providing global and exponential stability of a continuous nonlinear system is called acceptable. The spectral stability criterion is formulated as a sufficient condition for the admissibility of the matrix spectrum. For continuous nonlinear systems, the elements of the matrix spectrum are represented by the sum of two components, the first (main) is an arbitrarily selected negative scalar common to the entire spectrum, the second (the so–called increment) is constructed as functions that differ for all elements of the spectrum. The conditions for the admissibility of the matrix spectrum are reduced to a restriction on the maximum absolute value of the increment. A formula has been developed that determines the exact upper bound of this value, which ensures the acceptability of the spectrum. The spectral stability criterion of discrete nonlinear systems is based on the developed criterion for continuous systems. A discrete system with a Lyapunov function in the form of a quadratic form with a constant matrix is compared by a given formula to a continuous system with a Lyapunov function of the same structure. In this case, the formulation of the spectral stability criterion for a nonlinear discrete system is reduced to the spectral stability criterion for a constructed correlated continuous system. The solution of the stabilization problem for a wide class of nonlinear control systems based on the formulated spectral stability criteria is obtained. The disadvantages of the proposed solution are noted.

Keywords: continuous nonlinear control system, matrices of simple structure, matrix spectrum, lyapunov function, exact upper bound.

I. Introduction

Are non-linear control systems

$$\dot{x} = Q(x)x \qquad x \in R^n,$$
$$x_{k+1} = G(x_k)x_k \quad x \in R^n \quad k > 0, \tag{1}$$

$Q(x)$ is a matrix of a simple structure with a spectrum $\Lambda(x)$.

The relationship between $\Lambda(x)$ and the stability of the system (1) is not fully established. The hypothesis of M.A. Aizerman [1], suggesting the proximity of the spectral criterion for systems of the form (1) to the corresponding criterion for a linear system, is refuted by counter examples [2, 3, 4]. In the future, numerous developments were carried out to determine the relationship between the spectrum of the matrix of the system and the stability of the system for much narrower and special types of the matrix of the system $Q(x)$. For example [4] the validity of the Aizerman hypothesis is asserted for $Q(x) = Q^T(x)$and the stability of the system (1) is checked when the matrix of the system has one eigenvalue of multiplicity n and n eigenvectors.In the works [5, 6,

Ref

1. Ayzerman M.A. About one problem of stability “in large” for dynamical systems. Successes of dynamical sciences. 1949. t. 4 edition 4. p. 186-188.

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7, 8, 9, 10, 11, 12] more complexly formed classes of control systems are considered, for which variants of the Aizerman hypothesis are valid, but the formation of these classes is set by cumbersome, difficult to test conditions, and the solution of the stabilization problem is extremely time-consuming.

The proposed article considers system (1), i.e. a wide class of nonlinear continuous control systems, and determines the conditions for choosing the spectrum of the matrix of the system that provides the system with the required stability properties. Similar problems are posed and solved for nonlinear discrete control systems with a matrix $G(x_k)$ of a simple structure.

The spectrum of the matrix of the continuous part of system (1) $\Lambda(x)$ is formed as

$$\Lambda(x) = \{\lambda_i(x)\} i = \overline{1,n},$$

$$\lambda_i(x) = \lambda_0(x) + \rho_i\big(\lambda_0(x)\big), \qquad (2),$$

where the smooth scalar function $\lambda_0(x) < -\frac{1}{2}$ is given arbitrarily, the functions $\rho_i\big(\lambda_0(x)\big)$ are defined for each a priori given $\lambda_0(x)$ under assumptions.

$$\rho_i\big(\lambda_0(x)\big) \neq 0, \qquad \rho_i\big(\lambda_0(x)\big) \neq \rho_j\big(\lambda_0(x)\big), \text{где } i \neq j, \quad x \in R^n.$$

The problem of determining sufficient conditions for the choice of functions —$\left|\rho_i\big(\lambda_0(x)\big)\right|$ is posed and solved, providing exponential and global stability to the continuous part of system (1), (2) and the Lyapunov function of this system in the form

$$V(x) = x^T I x \qquad (3)$$

The spectral stability criterion for a discrete system with a matrix $G(x_k)$ is reduced to the spectral stability criterion of a continuous system with a matrix $Q(x) = (G(x) + I)(G(x) - I)^{-1}$

The problem of stabilization of nonlinear control systems of the class under consideration is solved on the basis of the spectral stability criteria.

II. Formation of a Spectral Stability Criterion for a Nonlinear Continuous Control System

Let's consider the system (1), (2) and its assumed Lyapunov function (3). We will consider the Lyapunov function as a quadratic form with a unit matrix, since a quadratic form with an arbitrary positive definite matrix is reduced to such a form by a similarity transformation that does not change the spectrum of the matrix of a closed system.

Consider the spectral decomposition of the matrix $Q(x)$.

$$Q(x) = \sum_{i=1}^{n} \lambda_i(x) d_i(x)\, g_i^T(x),$$

where $\lambda_i(x) \in \Lambda(x), i = \overline{1,n}$, $d_i(x)$ are the eigenvectors of the matrix $Q(x)$, $g_i(x)$ are the eigenvectors of the matrix $Q^T(x)$. At the same time [13]

Ref

13. Voevodin V.V., Kuznetsov Yu. A. Matrixes and computations. M., Science 1984, p. 317.

$$d_i^T(x)g_j(x) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (4)$$

We fix an arbitrary number $\alpha > 0$and we will require the fulfillment of the central condition for the function $V(x)$

$$\dot{V}(x) = x^T L(x)x < -\alpha V(x), \quad (5)$$

where

$$L(x) = Q^T(x) + Q(x) = \sum_{i=1}^n \lambda_i(x)\, d_i(x) g_i^T(x) + \sum_{i=1}^n \overline{\lambda_i}\,(x) g_i(x) d_i^T(x). \quad (6)$$

We introduce the following matrices into consideration

$$R_i(x) = R_i^T(x) = d_i(x)d_i^T(x). \quad (7)$$

$$D(x) = D^T(x) = \sum_{i=1}^n R_i(x). \quad (8)$$

$$\mathrm{P}(x) = D^T(x)L(x)D(x), \quad (9)$$

i. e.

$$signP(x) = signL(x).$$

Let's write out $P(x)$in more detail and replace the central requirement (5) with an equivalent requirement

$$P(x) = \sum_{i=1}^n \lambda_i(x)R_i(x) \sum_{j=1}^n R_j + \sum_{i=1}^n R_i(x) \sum_{j=1}^n \lambda_j(x)\, R_j(x) < -\alpha\, D^T(x)\, D(x) \quad (10)$$

Taking into account (2), (4), (9) we have

$$P(x) = 2\,\lambda_0(x)D^T(x)D(x) + D^T(x)S(x) + S^T(x)D(x). \quad (11)$$

Note the fairness of [13] inequality

$$D^T(x)\, S(x) + S^T(x)\, D(x) \le D^T\, D(x) + S^T(x)\, S(x),$$

at the same time

$$S^T(x)S(x) < \tau D^T(x)D(x), \text{where} \tau = \max_i \left|\rho_i\big(\lambda_0(x)\big)\right|^2$$

Now replace requirement (10) with a stronger requirement

$$P(x) < (2\,\lambda_0(x) + 1 + \alpha + \tau\,)D^T\, D(x) < 0$$

Thus proved

Ref

13. Voevodin V.V., Kuznetsov Yu. A. Matrixes and computations. M., Science 1984, p. 317.

Theorem 1

Let the spectrum of the matrix of system (1) have the form (2), where $\lambda_0(x) < -\frac{1}{2}$ is an arbitrarily a priori defined function, $\rho_i(\lambda_0(x))$, satisfy the conditions

$$\rho_i(\lambda_0(x)) \neq 0, \quad \rho_i(\lambda_0(x)) \neq \rho_j(\lambda_0(x)), \quad \text{где } i \neq j \;\; x \in R^n,$$

$$\max_i \rho_i^{\,2}(\lambda_0(x)) < - \; 2\,\lambda_0(x) - 1 - \alpha \qquad (12)$$

Then the system (1) – (4) is exponentially and globally stable and has a Lyapunov function (3), where $\dot{V}(x)$ satisfies condition (5).

III. Solution of the Spectral Stabilization Problem for a Nonlinear Continuous Control System

Consider the system

$$\dot{x} = A(x)x + b(x)u(x) \qquad u(x) = s^T(x)x \qquad x \in R^n \qquad (13)$$

The pair $(A(x)x,\; b(x))$is given and satisfies the condition of uniform controllability.

We assume $\alpha > 0$to be a priori a given number. It is required to determine the feedback vector $s(x)$,at which the closed system (13) has a Lyapunov function (3) satisfying condition (5), i.e. the global and exponential stability of the closed system (13).

The solution of the problem is carried out by the same operations that would be used to solve spectral stabilization for a linear control system.
The spectral decomposition of a matrix in a closed system (13) is considered

$$Q(x) = A(x) + b(x)s^T(x) = \sum_{i=1}^{n} \lambda_i(x)\, d_i(x) g_i^T(x),$$

where $\lambda_i(x)$, satisfies the condition $\lambda_i(x) \neq \lambda_j(x) \quad i \neq j$,a priori defined spectrum of the matrix $Q(x)$,the eigenvectors of the matrix $Q(x)$are determined by the formula $d_i(x) = (A - \lambda_i(x)I)^{-1}b(x)$,the eigenvectors of the matrix $Q^T(x)$ satisfy condition (4), and the desired feedback vector is determined by the formula

$$s^T(x)d_i(x) = -1 \qquad i = \overline{1,n} \qquad (14)$$

Let us select the spectrum of the considered closed system (13) in accordance with the conditions of Theorem (1), i.e. with the formula (12). Then it is obvious that the closed system (13) has the required stability properties. The main difference between this result and the result obtained for a linear control system is the solution of the system (14). For linear control systems, the choice of an arbitrarily set negative spectrum of the matrix of a closed system allows a priori to set the norm of the feedback vector $s(x)$. For the considered variant of spectral stabilization, the question of the possibility of obtaining the norm of the vector $s(x)$permissible under the preconditions of the problem under consideration remains open.

Notes

IV. Spectral Stability Criterion for Nonlinear Discrete Control Systems

The control system is considered

$$x_{k+1} = G(x_k)x_k, \qquad x_k \in R^n, \qquad k > 0 \tag{15}$$

where, without detracting from the generality, we believe

$$\inf_x |det(G(x) - I)| > \alpha > 0.$$

We will call the continuous system

$$\dot{x} = G_1(x)x \tag{16}$$

pair system (16), if

$$G_1\ (x) = (G(x) + I)(G(x) - I)^{-1}\ . \tag{17}$$

Note that the presence of the Lyapunov function (3) in the system (16) ensures the presence of the Lyapunov function

$$W(x_k) = x_k^T I x_k \tag{18}$$

for the system (15). Indeed, the conditions $\dot{W}(x) < 0$,by virtue of (16), (17) takes the form

$$(G^T(x) - I)^{-1}(G^T(x) + I) + (G(x) + I)(G(x) - I)^{-1} < 0$$

Multiplying the last inequality on the left by $(G^T(x) - I)$, and on the right by $(G(x) - I)$, we are convinced of the validity of the condition

$$G^T(x)G(x) - I < 0, \qquad x \in R^n,$$

the implementation of which ensures the existence of a Lyapunov function of the form (18) for the system (15).

Note that the eigenvalues of the matrix $G(x)$,denoted by $\mu_i(x)$ and the eigenvalues of the matrix $G_I(x)$, denoted by $\vartheta_i(x)$, are related [14] by the relations

$$\mu_i\ (x) = \frac{\vartheta_i(x) + 1}{\vartheta_i(x) - 1},$$

i. e.

$$\vartheta_i(x) \quad = \frac{\mu_i(x) + 1}{\mu_i(x) - 1}$$

The stability of the system (16) is ensured by the fulfillment of the conditions of Theorem (1) for the spectrum of the matrix $G_1(x)$Thus, it is proved

Ref

14. Gantmaher R.F. Theory of matrixes. M., Science 1966, p. 570.

Theorem 2

The global stability of the system (15) and the presence of the Lyapunov function (18) in this system is ensured by the implementation on the spectrum of the matrix $G_1(x_k)$

$$\mu_i(x) = \frac{\vartheta_i(x_k)+1}{\vartheta_i(x_k)-1}, \qquad k>0, \quad x\in R^n,$$

then the conditions that the eigen values of the matrix of the system (16), the paired system (15), $\vartheta_i(x) i=\overline{1,n}$ correspond to the formulation of Theorem (1).

V. Spectral Stabilization of Nonlinear Discrete Control Systems

The solution of the problem of spectral stabilization of a nonlinear discrete control system is performed by the same sequence of operations that takes place when solving this problem for a linear control system. The difference lies in the fact that in the case of a nonlinear system, the a priori selected spectrum of the matrix of a closed nonlinear system is formed in accordance with the conditions of Theorem (2).

VI. Conclusion

The main result of the proposed article is the formation of a spectral criterion for the stability of nonlinear control systems

$$\dot{x}=Q(x)x, \quad x_{k+1}=G(x_k)x_k, \qquad x\in R^n \qquad k>0,$$

where $Q(x), G(x_k)$are matrices of a simple structure.

The spectral stability criterion is reduced to determining sufficient conditions for the elements of the matrix spectrum of the considered closed systems, which determine the required stability properties of the system and the presence of the Lyapunov function in the form of a quadratic form with a single matrix.

For continuous control systems we have:

$$\Lambda(x)=\{\lambda_i(x)\} i=\overline{1,n}, \lambda_i(x)=\lambda_0(x)+\rho_i\big(\lambda_0(x)\big),$$

where the smooth scalar function $\lambda_0(x)<-\frac{1}{2}$ is given arbitrarily, the functions $\rho_i\big(\lambda_0(x)\big)$are defined for each a priori given $\lambda_0(x)$under the assumptions

$$\rho_i\big(\lambda_0(x)\big)\neq 0, \rho_i\big(\lambda_0(x)\big)\neq \rho_j\big(\lambda_0(x)\big), \qquad \text{где } i\neq j, \ x \in R^n$$

and the exact upper bound for $\left|\rho_i\big(\lambda_0(x)\big)\right|$, defined by formula (12).

For discrete control systems, a similar, somewhat more cumbersome result is obtained.

The solution of the problem of spectral stabilization of the systems under consideration is performed by the same operations based on the spectral decomposition of the matrix of a closed system, which this problem is solved for linear control systems. At the same time, the selection of the a priori spectrum of the matrix of a closed system is made in accordance with the developed spectral criterion for the stability of nonlinear control systems.

Notes

The developed technique has a significant drawback, when using it, the norm of the feedback vector here may be significantly greater than when solving the stabilization problem based on a priori linearization of the object matrix.

Notes

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