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Approximation of the Localized Szász-Mirakjan-Durrmeyer Operators

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17. S. M. Mazhar, V. Totik, Approximation by modified Szász operators, Acta Sci. Math 49 (1985), 257–269.

Approximation of the Localized Szász-Mirakjan-Durrmeyer Operators

Ma Yingdian α & Sun Yue σ

Abstract- In the present paper, we localize the Szász-Mirakjan-Durrmeyer operators. For which we discuss the convergence and the approximation and obtain theorems of convergence, approximation and rate of approximation.

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I. Introduction and Background

The famous Szasz-Mirakjan perator is a positive linear operator on $C[0,\infty)$ (continuous function on $[0,\infty)$), defined as follows

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)\qquad x\in[0,\infty).$$

It is easy to obtain by calculation

$$S_n(1;x)=1\ ;\ S_n(t;x)=x\ ;\quad S_n(t^2;x)=x^2+\frac{x}{n}\tag{1}$$

$$S_n\left((t-x)^2;x\right)=\frac{x}{n}\tag{2}$$

There are a lot of in-depth researches on its approximation, and the research results are very rich.[1-21]

The literature[17] introduce The Szasz-Mirakjan-Durrmeyer operator by Durrmeyer transformation of Szasz-Mirakjan operator

$$\tilde{S}_n(f;x)=\sum_{k=0}^{\infty}n\int_0^{+\infty}f(t)S_{n,k}(t)\,dtS_{n,k}(x)\qquad x\in[0,\infty)$$

Of which

$$S_{n,k}(x)=\frac{(nx)^k}{k!}e^{-nx},\ k=0,1,\ldots,$$

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It is easily obtained by (1) and (2)

$$\tilde{S}_n(1;x)=1\ ;\ \tilde{S}_n(t;x)=x+\frac{1}{n}\ ;\ \tilde{S}_n(t^2;x)=x^2+\frac{4x}{n}+\frac{2}{n^2} \tag{3}$$

$$\tilde{S}_n((t-x);x)=\frac{1}{n}\ ;\qquad \tilde{S}_n((t-x)^2;x)=\frac{2x}{n}+\frac{2}{n^2} \tag{4}$$

Notes

Obviously this is a positive linear operator on $L[0,\infty)$ (integrable function on $[0,\infty)$), and there has been a lot of research on it[21-26].

Operator localization is one of the methods to transform operators. In order to be convenient and feasible, the localization of some operators is necessary. The idea of operator localization originates from the truncation of infinite sum. In 1980, literature [27] firstly introduced Szasz-Mirakjan localization operator as follows

$$S_{n,N}(f;x)=e^{-nx}\sum_{k=0}^{N}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)\quad x\in[0,\infty)$$

In this paper, the same method is used to introduce the localized Szasz-Mirakjan-Durrmeyer operator as follows

$$\tilde{S}_{n,N}(f;x)=\sum_{k=0}^{N}n\int_0^{+\infty}f(t)S_{n,k}(t)dtS_{n,k}(x)\quad x\in[0,\infty),$$

Of which

$$S_{n,k}(x)=\frac{(nx)^k}{k!}e^{-nx},\ k=0,1,\ldots.$$

Here we mainly study the approximation problem of localized Szasz-Mirakjan-Durrmeyer operator. The following part is used to discuss the convergence and approximation of $\tilde{S}_{n,N}(f)$ operator, resulting in theorems 1 and 2. The last part is used to study the approximation velocity of $\tilde{S}_{n,N}(f)$ operator, and theorem 3 is obtained.

II. The Convergence 0f Localized Szasz-Mirakjan-Durrmeyer Operator

Let's start with the following lemma.

Lemma 1[28] Let $x>0$ and $y\in(-\infty,+\infty)$, Then

$$\left|e^{-nx}\sum_{k=0}^{[nx+y\sqrt{n}]}\frac{(nx)^k}{k!}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{y}{\sqrt{x}}}e^{-\frac{1}{2}u^2}du\right|\le\frac{Ax\sqrt{3x+1}}{\sqrt{n}\left(\sqrt{x}+|y|\right)^3}$$

Where A is a constant. If $x=0,y>0$, change $\frac{y}{\sqrt{x}}$ to ∞, then the conclusion is valid.

Notes

Using this lemma, we come to the following conclusion

Theorem 1: Let $f \in C[0,\infty)$ and $N = N(n,x)$, make

1) $(N-nx)/n$ uniformly bounded on $[x_1, x_2]$, where $0 < x_1 < x_2 < \infty$

2) $\lim\limits_{n\to\infty} \dfrac{N-nx}{\sqrt{n}} = C(x)$ preserving uniform convergence in $[x_1, x_2]$, Then, it is established that there is

$$\lim_{n\to\infty} \tilde{S}_{n,N}(f;x) = \frac{f(x)}{\sqrt{2\pi}} \int_{-\infty}^{\frac{C(x)}{\sqrt{x}}} e^{-\frac{1}{2}u^2} du \tag{5}$$

uniformly in $[x_1, x_2]$.

Proof: Let

$$\delta_n = (N-nx)/n , \quad G = \sup_{n\ge1} \{|\delta_n|\},$$

For $\eta > 0$, define the optical sliding mode

$$\omega(f;\eta) = \max_{x,y\in[0,x_2+G],|x-y|<\eta} |f(x) - f(y)|$$

According to the property of the optical sliding mode, if $\lambda > 0$, then

$$\omega(f;\lambda\eta) \le (1+\lambda)\omega(f,\eta) \tag{6}$$

If $x \in [x_1, x_2]$, then

$$\tilde{S}_{n,N}(f;x) = \sum_{k=0}^{N} n\int_0^{+\infty} (f(t) - f(x)) \frac{(nt)^k}{k!} e^{-nt} dt S_{n,k}(x) + \sum_{k=0}^{N} f(x) S_{n,k}(x)$$

$$\equiv I_1 + I_2$$

According to (6) and (4)

$$|I_1| \le \sum_{k=0}^{N} n\int_0^{+\infty} |f(t) - f(x)| S_{n,k}(t) dt S_{n,k}(x)$$

$$\le \omega\left(f;\frac{1}{\sqrt{n}}\right) \sum_{k=0}^{N} n\int_0^{+\infty} \left(1+\sqrt{n}|t-x|\right) S_{n,k}(t) dt S_{n,k}(x)$$

$$\le \omega\left(f;\frac{1}{\sqrt{n}}\right) \left(\sum_{k=0}^{N} S_{n,k}(x) + \sum_{k=0}^{N} n\int_0^{+\infty} \sqrt{n}|t-x| S_{n,k}(t) dt S_{n,k}(x)\right)$$

$$\leq \omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+\left(\sum_{k=0}^{\infty} n\int_0^{+\infty} n(t-x)^2 S_{n,k}(t)\,dt S_{n,k}(x)\right)^{1/2}\right)$$

$$\leq \omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{\frac{2nx+2}{n}}\right) \tag{7}$$

From lemma 1, we have

$$I_2 = f(x)e^{-nx}\sum_{k=0}^{nx+\sqrt{n}\cdot\sqrt{n}\delta_n}\frac{(nx)^k}{k!}$$

$$= f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\sqrt{n}\delta_n}{\sqrt{x}}} e^{-\frac{1}{2}u^2}du + O\left(\frac{x_2\sqrt{3x_2+1}}{\sqrt{n}\left(\sqrt{x_1}+\sqrt{n}|\delta_n|\right)^3}\right)\right\} \tag{8}$$

From (7) and (8), it can be concluded that (5) converges uniformly in $[x_1, x_2]$
In a similar way to Theorem 1, we get the following conclusion.

Theorem 2: Let $f \in C[0,\infty)$ and $N = N(n,x)$, make

1) $(N-nx)/n$ is uniformly bounded on $[x_1, x_2]$, where, $0 \leq x_1 < x_2 < \infty$

2) $\lim\limits_{n\to\infty}\frac{N-nx}{\sqrt{n}} = C(x), |C(x)| \geq \rho > 0$ maintain consistent convergence within $[x_1, x_2]$, then $\lim\limits_{n\to\infty}\tilde{S}_{n,N}(f;x) = \frac{f(x)}{\sqrt{2\pi}}\int_{-\infty}^{\frac{C(x)}{\sqrt{x}}} e^{-\frac{1}{2}u^2}du$ is uniformly true on $[x_1, x_2]$. When $x = 0$, think of $\frac{C(x)}{\sqrt{x}}$ as ∞.

Corollary 1: Let $f \in C[0,\infty)$ and $N = N(n,x)$, make

1) $(N-nx)/n$ is uniformly bounded on $[x_1, x_2]$, where, $0 \leq x_1 < x_2 < \infty$

(ii) $\lim\limits_{n\to\infty}\frac{N-nx}{\sqrt{n}} = \infty$ preserving uniform convergence in $[x_1, x_2]$, then $\lim\limits_{n\to\infty}\tilde{S}_{n,N}(f;x) = f(x)$ is uniformly true on $[x_1, x_2]$.

Corollary 2: Let $f \in C[0,\infty)$ and $x_0 \in (0,\infty)$, If $(N-nx_0)/n$ is bounded, and when $n \to \infty$, $(N-nx_0)/n$ does not converge to ∞, Then $\lim\limits_{n\to\infty}\tilde{S}_{n,N}(f;x_0) = f(x_0)$ must result in $f(x_0) = 0$

Corollary 3: Let $x_0 \in [0,\infty)$, If $(N-nx_0)/n$ is bounded, then

$$\lim_{n\to\infty}\tilde{S}_{n,N}(f;x_0) = f(x_0)$$

is true for any $f \in C[0,\infty)$,

If and only if

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$$\lim_{n\to\infty}\frac{N-nx_0}{\sqrt{n}}=\infty \text{ is true.}$$

III. Approximation Speed of Szász-Mirakja-Durrmeyer Deformation Localization Operator

Notes

A similar result is obtained for the localization Szász-Mirakjan-Durrmeyer operator $\tilde{S}_{n,N}(f,x)$ as follows.

Theorem 3: Let $f\in C_\alpha$ for fixed $x>0$, there is $\delta>0$,

Make f conform to

$$\left|\frac{f(t)-f(x)}{t-x}\right|\leq C(x,\delta), \tag{9}$$

in which $|t-x|\leq\delta$, and $t\geq 0$. and $N=N(n,x)$,

$$\liminf_{n\to\infty}\frac{N-nx}{\sqrt{nx\ln(n)}}>1 \tag{10}$$

is true, then

$$\sup_{n\geq 1}\sqrt{n}\left|\tilde{S}_{n,N}(f;x)-f(x)\right|<\infty \tag{11}$$

If $f'(x)$ exists, and $N=N(n,x)$, (13) is true, then

$$\lim_{n\to\infty}\sqrt{n}\left(\tilde{S}_{n,N}(f;x)-f(x)\right)=0. \tag{12}$$

Proof: (11) first. We note that

$$\tilde{S}_{n,N}(f;x)-f(x)=\sum_{k=0}^{N}n\int_0^{+\infty}(f(t)-f(x))S_{n,k}(t)dtS_{n,k}(x)-f(x)\sum_{k=N+1}^{\infty}S_{n,k}(x)$$

$$=I_1+I_2 \tag{13}$$

$$I_1=\sum_{k=0}^{N}n\int_0^{+\infty}(f(t)-f(x))S_{n,k}(t)dtS_{n,k}(x)$$

$$=\sum_{k=0}^{N}n\int_{|t-x|<\delta}(f(t)-f(x))S_{n,k}(t)dtS_{n,k}(x)$$

$$+\sum_{k=0}^{N}n\int_{|t-x|\geq\delta}(f(t)-f(x))S_{n,k}(t)dtS_{n,k}(x)$$

$$\equiv I_{11} + I_{12} \tag{14}$$

From (9) and (4) we have

$$|I_{11}| \le \sum_{k=0}^{N} n\int_{|t-x|<\delta} |f(t) - f(x)| S_{n,k}(t)dtS_{n,k}(x)$$

$$\le C(x,\delta)\sum_{k=0}^{N} n\int_{|t-x|<\delta} |t-x| S_{n,k}(t)dtS_{n,k}(x)$$

$$\le C(x,\delta)\left(\sum_{k=0}^{\infty} n\int_0^{\infty} (t-x)^2 S_{n,k}(t)dtS_{n,k}(x)\right)^{1/2}$$

$$\le \frac{C(x,\delta)}{\sqrt{n}}\sqrt{\frac{2nx+2}{n}} \tag{15}$$

$$|I_{12}| = \sum_{k=0}^{N} n\int_{|t-x|\ge\delta} |f(t) - f(x)| S_{n,k}(t)dtS_{n,k}(x)$$

$$\le \sum_{k=0}^{N} n\int_{|t-x|\ge\delta} |f(t)| S_{n,k}(t)dtS_{n,k}(x) + \sum_{k=0}^{N} n\int_{|t-x|\ge\delta} |f(x)| S_{n,k}(t)dtS_{n,k}(x)$$

$$= I_{12}^{'} + I_{12}^{''} \tag{16}$$

▪ $n > \alpha$， 由 $f \in C_\alpha$， 可得

$$I_{12}^{'} \le A\sum_{k=0}^{N} n\int_{|t-x|\ge\delta} e^{\alpha t} \cdot e^{-nt}\frac{(nt)^k}{k!}dtS_{n,k}(x)$$

$$\le \frac{A}{\delta^2}\sum_{k=0}^{N} n\left[\int_0^{\infty} t^2 e^{-(n-\alpha)t}\frac{(nt)^k}{k!}dt - 2x\int_0^{\infty} te^{-(n-\alpha)t}\frac{(nt)^k}{k!}dt\right.$$

$$\left. + x^2\int_0^{\infty} e^{-(n-\alpha)t}\frac{(nt)^k}{k!}dt\right]S_{n,k}(x)$$

$$= \frac{A}{\delta^2}\sum_{k=0}^{N} n\left[\frac{(k+2)(k+1)n^k}{(n-\alpha)^{k+3}} - \frac{2x(k+1)n^k}{(n-\alpha)^{k+2}} + \frac{x^2 n^k}{(n-\alpha)^{k+1}}\right]S_{n,k}(x)$$

Notes

$$=\frac{A}{\delta^2}\sum_{k=0}^{N}\left(\frac{n}{n-\alpha}\right)^{k+1}\left[\frac{(k+2)(k+1)}{(n-\alpha)^2}-\frac{2x(k+1)}{(n-\alpha)}+x^2\right]S_{n,k}(x)$$

$$=\frac{A}{\delta^2}\sum_{k=0}^{N}\left(\frac{n}{n-\alpha}\right)^{k+1}\left[\left(\frac{k}{n}-\frac{nx}{n-\alpha}\right)^2+\frac{\alpha(2n-\alpha)}{(n-\alpha)^2}\left(\frac{k}{n}\right)^2-\frac{2x}{n-\alpha}\right.$$

$$\left.-\frac{\alpha(2n-\alpha)}{(n-\alpha)^2}x^2+\frac{3k}{(n-\alpha)^2}+\frac{1}{(n-\alpha)^2}\right]S_{n,k}(x)$$

$$\leq\frac{A}{\delta^2}\sum_{k=0}^{N}\left(\frac{n}{n-\alpha}\right)^{k+1}\left[\left(\frac{k}{n}-\frac{nx}{n-\alpha}\right)^2+\frac{\alpha(2n-\alpha)}{(n-\alpha)^2}\left(\frac{k}{n}\right)^2\right.$$

$$\left.+\frac{3n}{(n-\alpha)^2}\frac{k}{n}+\frac{1}{(n-\alpha)^2}\right]S_{n,k}(x)$$

$$\equiv\frac{A}{\delta^2}[J_1+J_2+J_3+J_4]$$

According to (2)

$$J_1=\sum_{k=0}^{N}\left(\frac{n}{n-\alpha}\right)^{k+1}\left(\frac{k}{n}-\frac{nx}{n-\alpha}\right)^2\frac{(nx)^k}{k!}e^{-nx}$$

$$=\frac{n}{n-\alpha}e^{-nx+\frac{nnx}{n-\alpha}}\sum_{k=0}^{N}\left(\frac{k}{n}-\frac{nx}{n-\alpha}\right)^2\frac{\left(n\cdot\frac{nx}{n-\alpha}\right)^k}{k!}e^{-n\cdot\frac{nx}{n-\alpha}}$$

$$\leq\frac{n}{n-\alpha}e^{\frac{\alpha nx}{n-\alpha}}\left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-\frac{nx}{n-\alpha}\right)^2\frac{\left(n\cdot\frac{nx}{n-\alpha}\right)^k}{k!}e^{-n\cdot\frac{nx}{n-\alpha}}\right)$$

$$\leq\frac{1}{n-\alpha}\frac{nx}{n-\alpha}e^{\frac{\alpha nx}{n-\alpha}}$$

so

$$J_1=O_{x,\alpha}\left(\frac{1}{n}\right). \tag{17}$$

According to (1)

$$J_2=\sum_{k=0}^{N}\left(\frac{n}{n-\alpha}\right)^{k+1}\frac{\alpha(2n-\alpha)}{(n-\alpha)^2}\left(\frac{k}{n}\right)^2\frac{(nx)^k}{k!}e^{-nx}$$

$$=\frac{\alpha(2n-\alpha)n}{(n-\alpha)^3}e^{\frac{\alpha nx}{n-\alpha}}\sum_{k=0}^{N}\left(\frac{k}{n}\right)^2\frac{\left(n\cdot\frac{nx}{n-\alpha}\right)^k}{k!}e^{-n\cdot\frac{nx}{n-\alpha}}$$

$$\leq\frac{\alpha(2n-\alpha)n}{(n-\alpha)^3}e^{\frac{\alpha nx}{n-\alpha}}\left[\left(\frac{nx}{n-\alpha}\right)^2+\frac{x}{n-\alpha}\right]$$

so

$$J_2=O_{x,\alpha}\left(\frac{1}{n}\right)\tag{18}$$

By the same token, we can obtain from (1)

$$J_3=\frac{3n^3x}{(n-\alpha)^4}e^{\frac{\alpha nx}{n-\alpha}}=O_{x,\alpha}\left(\frac{1}{n}\right)\tag{19}$$

$$J_4=\frac{n}{(n-\alpha)^3}e^{\frac{\alpha nx}{n-\alpha}}=O_{x,\alpha}\left(\frac{1}{n}\right)\tag{20}$$

Combined with (17)-(20), we can get

$$I_{12}^{'}=O_{x,\alpha}\left(\frac{1}{n}\right)\tag{21}$$

According to (4)

$$I_{12}^{''}=|f(x)|\sum_{k=0}^{N}n\int_{|t-x|\geq\delta}S_{n,k}(t)dtS_{n,k}(x)$$

$$\leq|f(x)|\sum_{k=0}^{N}n\int_{|t-x|\geq\delta}\frac{(t-x)^2}{\delta^2}S_{n,k}(t)dtS_{n,k}(x)$$

$$\leq\frac{|f(x)|}{\delta^2}\frac{2nx+2}{n^2}$$

$$\leq\frac{Ae^{\alpha x}(2nx+2)}{\delta^2n^2}\tag{22}$$

According to (21) and (22),

Notes

$$|I_{12}| = O_{x,\alpha}\left(\frac{1}{n}\right) \tag{23}$$

Then, from (15) and (23),

Notes

$$|I_1| = O_{x,\alpha}\left(\frac{1}{\sqrt{n}}\right) \tag{24}$$

According to literature [28]

$$|I_2| \le \frac{|f(x)|}{\sqrt{2\pi}\sqrt{n\ln(n)}} \cdot + O\left(\frac{|f(x)|\sqrt{3x+1}}{\sqrt{nx}\left(\sqrt{\ln(n)}\right)^3}\right) \tag{25}$$

Combine (24) with (25) to get, existence n_0 such that when $n > n_0$, there is

$$\left|\tilde{S}_{n,N}(f,x) - f(x)\right| \le C^*(x,\delta,\alpha)\frac{1}{\sqrt{n}}$$

In which $C^*(x,\delta,\alpha)$ is a constant that depends only on x, δ and α, So (11) is true. Next, we prove (12). Only certificate

$$|I_{11}| = o_x\left(\frac{1}{\sqrt{n}}\right) \tag{26}$$

In fact, if there is $f'(x)$, then for any $\varepsilon > 0$, there is $\delta > 0$, and exist

$$\left|f(t) - f(x) - f'(x)(t-x)\right| < \varepsilon|t-x|, \quad |t-x| < \delta, \tag{27}$$

from (27)

$$|I_{11}| \le \left|f'(x)\sum_{k=0}^{N} n\int_{|t-x|<\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right|$$

$$+\varepsilon\sum_{k=0}^{N} n\int_{|t-x|<\delta}|t-x|S_{n,k}(t)dtS_{n,k}(x)$$

$$\equiv H_1 + H_2 \tag{28}$$

from (4)

$$H_1 = \left| f'(x)\sum_{k=0}^{N} n\int_{|t-x|<\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right|$$

$$= \left| f'(x)\sum_{k=0}^{\infty} n\int_{|t-x|<\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right.$$

$$\left. -f'(x)\sum_{k=N+1}^{\infty} n\int_{|t-x|<\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right|$$

$$= \left| -f'(x)\sum_{k=0}^{\infty} n\int_{|t-x|\geq\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)+\frac{1}{n}\right.$$

$$\left. -f'(x)\sum_{k=N+1}^{\infty} n\int_{|t-x|<\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right|$$

$$= \left| -f'(x)\sum_{k=0}^{N} n\int_{|t-x|\geq\delta}(t-x)S_{n,k}(t)dtS_{n,k}(x)+\frac{1}{n}\right.$$

$$\left. -f'(x)\sum_{k=N+1}^{\infty} n\int_{0}^{\infty}(t-x)S_{n,k}(t)dtS_{n,k}(x)\right|$$

$$\leq \frac{|f'(x)|}{\delta}\sum_{k=0}^{N} n\int_{0}^{\infty}(t-x)^2 S_{n,k}(t)dtS_{n,k}(x)$$

$$+|f'(x)|\sum_{k=N+1}^{\infty} n\int_{0}^{\infty}|t-x|S_{n,k}(t)dtS_{n,k}(x)+\frac{1}{n}$$

$$\equiv K_1 + K_2 + \frac{1}{n}$$

from（4）

$$K_1 \leq \frac{|f'(x)|}{\delta}\frac{2nx+2}{n^2} \tag{29}$$

$$K_2 \leq |f'(x)|\sum_{k=N+1}^{\infty} n\int_{0}^{\infty} tS_{n,k}(t)dtS_{n,k}(x)+x|f'(x)|\sum_{k=N+1}^{\infty}S_{n,k}(x)$$

$$= |f'(x)|\sum_{k=N+1}^{\infty}\frac{k+1}{n}S_{n,k}(x)+x|f'(x)|\sum_{k=N+1}^{\infty}S_{n,k}(x)$$

Notes

$$\leq 2\left|f'(x)\right|\sum_{k=N+1}^{\infty}\frac{k}{n}S_{n,k}(x)+x\left|f'(x)\right|\sum_{k=N+1}^{\infty}S_{n,k}(x)$$

$$\leq 2x\left|f'(x)\right|\sum_{k=N}^{\infty}S_{n,k}(x)+x\left|f'(x)\right|\sum_{k=N+1}^{\infty}S_{n,k}(x)$$

Notes

And then we know from (25)

$$K_2 = o_x\left(\frac{1}{\sqrt{n}}\right) \tag{30}$$

According to (4)

$$H_2 = \varepsilon\sum_{k=0}^{N} n\int_{|t-x|<\delta}|t-x|S_{n,k}(t)dtS_{n,k}(x)$$

$$\leq \varepsilon\left(\sum_{k=0}^{\infty} n\int_{|t-x|<\delta}(t-x)^2 S_{n,k}(t)dtS_{n,k}(x)\right)^{1/2}$$

$$\leq \frac{\varepsilon}{\sqrt{n}}\sqrt{\frac{2nx+2}{n}} \tag{31}$$

(26) is established by combining (29) with (31).
Then, from (23), (25) and (26), when n is sufficiently large, there is

$$\left|\widetilde{S}_{n,N}(f,x)-f(x)\right|\leq C^{**}(x,\delta,\alpha)\frac{\varepsilon}{\sqrt{n}}$$

in which $C^{**}(x,\delta,\alpha)$ is a constant that depends only on x, δ and α, so (12) is true. so the theorem is proved.

References Références Referencias

1. Otto Szász, Generalization of Bernstein's Polynomials to the infinite interval, J. Res. Nat. Bur. Standards. 45(1950); Collected Mathematical Works (Cincinnati, 1955), 1401–1407.
2. Z. Ditzian and V. Totik, Moduli of Smoothness, Springer–Verlag, New York, 1987.
3. Z.Finta, N.K. Govil, Vijay Gupta, Some results on modified Szász–Mirakjan operators, J. Math. Anal. Appl 327 (2007), 1284-1296.
4. A. Aral, A generalization of Szász–Mirakyan operators based on q-integers, Mathematical and Computer Modelling 47 (9-10) (2008), 1052-1062.
5. Naokant Deo, Direct result on exponential-type operators, Applied Mathematics and Computation 204 (2008), 109–115.

6. Z. Finta, On converse approximation theorems, J. Math. Anal. Appl. 312 (2005), 159–180.
7. Jesús de la Cal and J.Cárcamo, On uniform approximation by someclassical Bernstein-type operators , J. Math. Anal. Appl. 279 (2003), 625–638.
8. J. A. Adell , A. Lekuona, Best constants in preservation of global smoothness for Szász–Mirakyan operators, Journal of Mathematical Analysis and Applications 338 (2) (2008), 753-757.
9. N. K. Govil, Z.Finta, Vijay Gupta, Some results on modified Szász–Mirakjan operators, J. Math. Anal. Appl. 327 (2007), 1284-1296.
10. Z. Walczak, On Certain Positive Linear Operators in Polynomial Weight Spaces, Acta Mathematica Hungarica 101 (3) (2003), 179- 191.
11. R. Y. Yang, J. Y. Xiong, F. L. Cao, Strong Converse Inequality for Modified Szász Operators, Journal of mathematical research and exposition 24 (3) (2003), 437-444.
12. Y.L.Hu, Point saturation of Szász operator, Journal of Hebei Normal University (Natural Science Edition) 21 (1) (1997), 9-10.
13. O. Dumana, M. A. Özarslan, Szjasz–Mirakjan type operators providing a better error estimation, Applied Mathematics Letters 20 (2007), 1184–1188.
14. J.Tian, W.Q.Chen, CH.Q.Liu, On modifying the Local saturation Theorem of Szász-Gamma type operators, Journal of Chengdu University of Technology 27 (4) (2000).
15. Jesús de la Cal and J.Cárcamo, On the approximation of convex functions by Bernstein-type operators, J. Math. Anal. Appl. 334 (2007), 1106–1115.
16. G. Z. Zhou, S.P.Zhou, A remark on a modified Szász–Mirakjan operator, Colloq. Math. 79 (1999), 157–160.
17. S. M. Mazhar, V. Totik, Approximation by modified Szász operators, Acta Sci. Math 49 (1985), 257–269.
18. H. G. Lehnhoff, On a modified Szász–Mirakjan operator, J. Approx. Theory 42 (1984), 278–282.
19. E. Omey, Note on operators of Szász–Mirakjan type, J. Approx. Theory 47 (1986), 246–254.
20. X. H. Sun, On the convergence of the modified Szász–Mirakjan operator, Approx. Theory Appl. 10 (1) (1994), 20–25.
21. W.ZH.CHen, Operator Approximation Theory, Xiamen University Press, Xiamen, 1989.
22. H. S. Kasana, G. Prasad, P.N. Agrawal, A. Sahai, Modified Szász operators, Conf. on Math. Anal. Appl., Kuwait, 1985, Pergamon, Oxford, 1988, 29–42.
23. C. G. Zhang and Q-E Wang .The complete asymptotic expansions for the Szász–Mirakjan–Type operators, International Journal of Wavelets Multiresolution and Information Processing (6) (2009), 851-858.
24. V. Gupta, Simultaneous approximation by Szász–Durrmeyer operators, The Math. Studia 64 (1–4) (1995), 27–36.

Notes

25. V. Gupta, M.A. Noor, Convergence of derivatives for certain mixed Szász–Beta operators, J. Math. Anal. Appl. 321 (1) (2006), 1–9.
26. V. Gupta, J. Sinha, Direct results on certain Szász–Mirakjan operators, Applied Mathematics and Computation 204 (2008), 109–115.
27. J. Grof, über approximation durch polynome mit Belegfunktionen, Acta Math. Acad. Sci. Hungar 35 (1980), 109–116.
28. L. Xie, T. Xie, Approximation theorems for localized Szász–Mirakjan operator, Journal of Approximation Theory 152 (2008), 125-134.
29. Krech, Grażyna. "Some approximation results for operators of Szász-Mirakjan-Durrmeyer type" Mathematica Slovaca, vol. 66, no. 4, 2016, pp. 945-958.
30. Lin Zhi Peng; Cheng Wen Tao; Xu Xiao Wei. Approximation Properties of -Szász-Mirakjan-Durrmeyer Operators, Journal of Function Spaces Volume 2021, 2021.

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