On Lie Symmetry Analysis and Analytical Solutions of the Time-Fractional Modified ZKB Equation in Mathematical Physics

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Abstract- In this article, we explore the time-fractional modified Zakharov-Kuznetsov-Burgers (MZKB) equation of (3+1) dimensions. The Lie symmetry analysis is used to identify the symmetries and vector fields for the equation understudy with the assistance of the Riemann-Liouville derivatives. These symmetries are then employed to build a transformation that reduces the above equation into a nonlinear ordinary differential equation of fractional order with the aiding of ErdLélyi-Kober fractional operator. Further, two sets of new analytical solutions are constructed by the fractional sub-equation method and the extended Kudryashov method. Subsequently, we graphically represent these results in the 2D and 3D plots with physical interpretation for the behavior of the obtained solutions. The conservation laws that associate with the symmetries of the equation are also constructed by considering the new conservation theorem and the formal Lagrangian *L*. As a final result, we anticipate that this study will assist in the discovery of alternative evolutionary processes for the considered equation.

Keywords: lie symmetry analysis; conservation laws; time fractional modified schamel-zakharov kuznetsov burgers equation; riemann-liouville derivatives; fractional sub-equation method; extended kudryashov method.

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I. Introduction

The partial differential equations of fractional order (FPDEs) have been widely employed in recent years to explain a wide range of physical effects and complicated nonlinear phenomena. This is because they accurately describe nonlinear phenomena in the fields of fluid mechanics, viscoelasticity, electrical chemistry, quantum biology, physics, and engineering mechanics [1]-[6] as well as other scientific domains. As a result, the research of PDEs has received a lot of interest as many physical events may be explained using the idea of fractional derivatives and integrals [7]-[9]. Add to that, when the exact solutions to the majority of FPDEs are difficult to find, analytical and numerical methods [10]-[29] which are proposed and developed by many authors must be used.

Lie symmetry analysis is extremely important in many fields of science, particularly in integrable systems with an infinite number of symmetries. Thus, Lie symmetry analysis is regarded as one of the most effective methods for obtaining analytical solutions to nonlinear partial differential equations (NLPDEs). Also, many FPDEs have been studied using this analysis [30]-[36]. Add to that, this analysis is used to build conservation laws, which are crucial in the study of nonlinear physical phenomena. Conservation laws are mathematical formulations state that the total amount of a certain physical quantity remains constant as a physical system evolves. Furthermore, conservation laws are used in the development of numerical methods to establish the existence and uniqueness of a solution. There are many studies that discuss conservation laws for time FPDEs, which are mentioned in the references [37]-[43].

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In this article, we focus on the following time-fractional MZKB equation of the form:

$$\partial_t^\alpha q + \kappa_1 \sqrt{q}\, q_x + \kappa_2 q_{xxx} + \kappa_3(q_{xyy} + q_{xzz}) + \kappa_4 q_{xx} = 0, \tag{1}$$

where ∂_t^α is the fractional derivative of order α (with $0 < \alpha < 1$), $\kappa_i (i = 0, 1, ..., 4)$ are respectively the dispersion, non linearity, mixed derivative, and dissipation. The $q(x, y, z, t)$ is the potential function of space x, y, z and time t. If $\alpha = 1$, Eq. (1) is reduced to the classical MZKB equation [44, 45], which describes the nonlinear plasma dust ion acoustic waves DIAWs in a magnetized dusty plasma and it is derived using the standard reductive perturbation technique in small amplitude.

The article is organized as follows: The introduction is presented in Section 1. In Section 2, some definitions and description of Lie symmetry analysis for fractional partial differential equations (FPDEs) are briefly presented. Lie symmetry analysis and similarity reduction of the Eq. (1) are obtained In Section 3. We construct two sets of analytical solutions for Eq. (1) by using fractional sub-equation method and extended Kudryashov method in Section 4 and 5 respectively. In Section 6, the conservation laws of the Eq. (1) are obtained. Finally, the discussions and conclusions of this article are presented in Section 7.

II. Preliminaries

Here in this section, we focus on some of the concepts that revolve around the subject of our article
Definition 1: Let $\alpha > 0$. The operator I^α defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds, \tag{2}$$

is called the Riema nn-Liouville (R-L) fractional integral operator of order α, and $\Gamma(.)$ denotes the gamma function.
Definition 2: Let $\alpha > 0$. The operator D_t^α is defined by

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds & if\ n - 1 < \alpha < n,\ \ n \in\ N, \\[3mm] \frac{d^n f(t)}{dt^n} & if\ \alpha = n,\ \ n \in\ N, \end{cases} \tag{3}$$

is called the R-L fractional partial derivative [7, 8].

a) Description of Lie symmetry analysis

Let's consider the symmetry analysis for a FPDE of the form

$$D_t^\alpha q(x, y, z, t) = G(x, y, z, t, q, q_x, q_t, q_y, q_z, q_{xx}, q_{xy}, ...), \qquad 0 < \alpha < 1. \tag{4}$$

Now, let Eq. (4) is invariant under the following one-parameter Lie group of point transformation acting on both the dependent and independent variables, given as

$$\bar{x} = x + \varepsilon\xi(x, y, z, t) + O(\varepsilon^2),$$

$$\bar{y} = y + \varepsilon\zeta(x, y, z, t) + O(\varepsilon^2),$$

$$\bar{z} = z + \varepsilon\nu(x, y, z, t) + O(\varepsilon^2),$$

$$\bar{t} = t + \varepsilon\tau(x, y, z, t) + O(\varepsilon^2),$$

$$\bar{q} = q + \varepsilon\eta(x, y, z, t) + O(\varepsilon^2),$$

$$D_t^\alpha \bar{q} = D_t^\alpha q + \varepsilon\eta_\alpha^0(x, y, z, t) + O(\varepsilon^2),$$

$$\frac{\partial \bar{q}}{\partial \bar{x}} = \frac{\partial q}{\partial x} + \varepsilon\eta^x(x, y, z, t) + O(\varepsilon^2), \tag{5}$$

$$\frac{\partial^2 \bar{q}}{\partial \bar{x}^2} = \frac{\partial^2 q}{\partial x^2} + \varepsilon\eta^{xx}(x, y, z, t) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x}^3} = \frac{\partial^3 q}{\partial x^3} + \varepsilon \eta^{xxx}(x, y, z, t) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x} \partial \bar{y}^2} = \frac{\partial^3 q}{\partial x \partial y^2} + \varepsilon \eta^{xyy}(x, y, z, t) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x} \partial \bar{z}^2} = \frac{\partial^3 q}{\partial x \partial z^2} + \varepsilon \eta^{xzz}(x, y, z, t) + O(\varepsilon^2),$$

where $\varepsilon \ll 1$ is the Lie group parameter and ξ, ζ, ν, τ, η are the infinitesimals of the transformations for dependent and independent variables respectively. The explicit expressions of η^x, η^{xx}, η^{xxx}, η^{xyy}, η^{xzz} are given by

$$\eta^x = D_x(\eta) - q_x D_x(\xi) - q_y D_x(\zeta) - q_z D_x(\nu) - q_t D_x(\tau),$$

$$\eta^{xx} = D_x(\eta^x) - q_{xx} D_x(\xi) - q_{xy} D_x(\zeta) - q_{xz} D_x(\nu) - q_{xt} D_x(\tau),$$

$$\eta^{xxx} = D_x(\eta^{xx}) - q_{xxx} D_x(\xi) - q_{xxy} D_x(\zeta) - q_{xxz} D_x(\nu) - q_{xxt} D_x(\tau),$$

$$\eta^{xyy} = D_x(\eta^{yy}) - q_{xxy} D_x(\xi) - q_{xyy} D_x(\zeta) - q_{xyz} D_x(\nu) - q_{xyt} D_x(\tau),$$

$$\eta^{xzz} = D_x(\eta^{zz}) - q_{xxz} D_x(\xi) - q_{xyz} D_x(\zeta) - q_{xzz} D_x(\nu) - q_{xzt} D_x(\tau),$$

(6)

where D_x, D_y, D_z, and D_t are the total derivatives with respect to x, y, z, and t respectively that are defined for $x^1 = x$, $x^2 = y$, $x^3 = z$ as

$$D_{xj} = \frac{\partial}{\partial x^j} + q_j \frac{\partial}{\partial q} + q_{jk} \frac{\partial}{\partial q_k} + ..., \qquad j, k = 1, 2, 3, ...$$

where $q_j = \frac{\partial q}{\partial x^j}$, $q_{jk} = \frac{\partial^2 q}{\partial x^j \partial x^k}$ and so on.

The corresponding Lie algebra of symmetries consists of a set of vector fields of the form

$$V = \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}.$$

The invariance condition of Eq. (4) under the infinitesimal transformations is given as

$$Pr^{(n)} V(\Delta) |_{\Delta=0} = 0, \qquad n = 1, 2, 3, ...$$

where

$$\Delta := D_t^\alpha q(x, y, z, t) - G(x, y, z, t, q, q_x, q_t, q_y, q_z, q_{xx}, q_{xy}, ...).$$

Also, the invariance condition gives

$$\tau(x, y, z, t, u) |_{t=0} = 0. \tag{7}$$

The αth extended infinitesimal related to RL fractional time derivative with Eq. (7) can be represented as follows

$$\eta_\alpha^0 = D_t^\alpha(\eta) + \xi\, D_t^\alpha(q_x) - D_t^\alpha(\xi\, q_x) + \zeta\, D_t^\alpha(q_y) - D_t^\alpha(\zeta\, q_y) + \nu\, D_t^\alpha(q_z)$$

$$- D_t^\alpha(\nu\, q_z) + D_t^\alpha(D_t(\tau)q) - D_t^{\alpha+1}(\tau\, q) + \tau\, D_t^{\alpha+1}(q),$$

(8)

where D_t^α is the total fractional derivative operator and by using the generalized Leibnitz rule

$$D_t^\alpha \left(f(t)g(t) \right) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

By applying the Leibnitz rule, Eq. (8) becomes

$$\eta_\alpha^0 = D_t^\alpha(\eta) - \alpha D_t^\alpha(\tau)\frac{\partial^\alpha q}{\partial t^\alpha} - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n} q_x - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n} q_y$$

$$- \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\nu) D_t^{\alpha-n} q_z - \sum_{n=1}^\infty \binom{\alpha}{n+1} D_t^{n+1}(\xi) D_t^{\alpha-n} q. \tag{9}$$

Now by using the chain rule for the compound function which is defined as follows

$$\frac{d^n \phi(h(t))}{dt^n} = \sum_{k=0}^n \sum_{r=0}^k \binom{k}{r} \frac{1}{k!} \big[-h(t)\big]^r \frac{d^n}{dt^n}\big[-h(t)^{k-r}\big] \times \frac{d^k \phi(h)}{dh^k}.$$

By applying this rule and the generalized Leibnitz rule with $f(t) = 1$, we have

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_q \frac{\partial^\alpha q}{\partial t^\alpha} - q\frac{\partial^\alpha \eta_q}{\partial t^\alpha} + \sum_{n=1}^\infty \binom{\alpha}{n} \frac{\partial^n \eta_q}{\partial t^n} D_t^{\alpha-n}(q) + \mu,$$

where

$$\mu = \sum_{n=2}^\infty \sum_{m=2}^n {\sim}_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n}\binom{n}{m}\binom{k}{r} \frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \big[-q\big]^r \frac{\partial^m}{\partial t^m}\big[q^{k-r}\big] \frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial q^k}.$$

Therefore, Eq. (9) yields

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_q - \alpha D_t^\alpha(\tau))\frac{\partial^\alpha q}{\partial t^\alpha} - q\frac{\partial^\alpha \eta_q}{\partial t^\alpha} + \mu + \sum_{n=1}^\infty \left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_q}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(q)$$

$$+ \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n} q_x - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n} q_y - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\nu) D_t^{\alpha-n} q_z. \tag{10}$$

Definition 3: The function $q = \theta(x,y,z,t)$ is an invariant solution of Eq. (4) associated with the vector field W, such that

1. $q = \theta(x,y,z,t)$ is an invariant surface of Eq. (4), this means

$$V\theta = 0 \Leftrightarrow \left(\xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}\right)\theta = 0,$$

2. $q = \theta(x,y,z,t)$ satisfies Eq. (4).

III. Lie Symmetry Analysis and Similarity Reduction of Eq. (1)

In this section, we implemented Lie group method for Eq. (1) and then, used these symmetries to reduce Eq. (1) to be a FODE as shown in the next two subsections

a) Lie symmetry analysis for Eq. (1)

Let us consider Eq. (1) is an invariant under Eq. (5), we get

$$\partial_t^\alpha \bar{q} + \kappa_1 \sqrt{\bar{q}}\ \bar{q}_x + \kappa_2 \bar{q}_{xxx} + \kappa_3(\bar{q}_{xyy} + \bar{q}_{xzz}) + \kappa_4 \bar{q}_{xx} = 0, \tag{11}$$

such that $q = q(x,y,z,t)$ satisfies Eq. (1), then using the point transformations Eq. (5) in Eq. (11), we get the invariant equation

$$\eta_\alpha^0 + \kappa_1 \sqrt{q} \, \eta^x + \frac{\kappa_1}{2\sqrt{q}} \eta \, q_x + \kappa_2 \eta^{xxx} + \kappa_3 (\eta^{xyy} + \eta^{xzz}) + \kappa_4 \eta^{xx} = 0, \tag{12}$$

By substituting Eq. (6) and Eq. (10) into Eq. (12), grouping the coefficients of all derivatives and various powers of u and equating them to zero we get an algebraic system of equations. Solving this system, we obtain a set of infinitesimal symmetries as below:

Case 1: When $\kappa_i \neq 0$, $i = 1, \ 2, \ 3, \ \ \kappa_4 = 0$

$$\tau = \frac{3}{2\alpha} c_1 t + c_2, \quad \xi = \frac{1}{2} c_1 x + c_3, \quad \zeta = \frac{1}{2} c_1 y + c_4, \quad \nu = \frac{1}{2} c_1 z + c_5, \quad \eta = -2 c_1 q, \tag{13}$$

where c_i, $i = 1, 2, 3, 4, 5$ are arbitrary constants. Thus, the infinitesimal generator of Eq. (1) can be expressed as follows

$$V = \left(\frac{3}{2\alpha} c_1 t + c_2 \right) \frac{\partial}{\partial t} + \left(\frac{1}{2} c_1 x + c_3 \right) \frac{\partial}{\partial x} + \left(\frac{1}{2} c_1 y + c_4 \right) \frac{\partial}{\partial y} + \left(\frac{1}{2} c_1 z + c_5 \right) \frac{\partial}{\partial z} - 2 c_1 q \frac{\partial}{\partial q}.$$

which can be spanned by the five vector fields listed below.

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial y}, \quad V_4 = \frac{\partial}{\partial z},$$

$$V_5 = \frac{3t}{2\alpha} \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + \frac{z}{2} \frac{\partial}{\partial z} - 2q \frac{\partial}{\partial q}. \tag{14}$$

Case 2: When $\kappa_i \neq 0$, $i = 1, \ 2, \ 3, \ 4$

$$\tau = c_6, \qquad \xi = c_7, \qquad \zeta = c_8, \qquad \nu = c_9, \qquad \eta = 0,$$

hence, there are four vector fields as below

$$V_6 = \frac{\partial}{\partial t}, \qquad V_7 = \frac{\partial}{\partial x}, \qquad V_8 = \frac{\partial}{\partial y}, \qquad V_9 = \frac{\partial}{\partial z}. \tag{15}$$

b) The similarity reduction for Eq. (1)

In this part of the article, we used the symmetries defined by Eq. (14) and Eq. (15) to construct the similarity reduction for Eq. (1) as presented in the next cases

Case 1: For $V_1 = \frac{\partial}{\partial t}$, $V_2 = \frac{\partial}{\partial x}$, $V_3 = \frac{\partial}{\partial y}$, $V_4 = \frac{\partial}{\partial z}$, $V_5 = \frac{3t}{2\alpha} \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + \frac{z}{2} \frac{\partial}{\partial z} - 2q \frac{\partial}{\partial q}$ with $\kappa_4 = 0$, we have a set of characteristic equations that arranged in the following subcases

Case 1.1: $V_1 = \frac{\partial}{\partial t}$ we have a characteristic equation of the form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{dq}{0}.$$

by integrating this equation and appoint the solutions q as function of the dependent variables $x, \ y, \ z$, that is

$$q(x, y, z, t) = \phi_1(x, y, z),$$

this implies $\frac{\partial^\alpha q}{\partial t^\alpha} = 0$ and Eq. (1) becomes

$$\kappa_1 \sqrt{\phi_1(x, y, z)} \frac{d\phi_1(x, y, z)}{dx} + \kappa_2 \frac{d^3 \phi_1(x, y, z)}{dx^3} + \kappa_3 \left(\frac{d^3 \phi_1(x, y, z)}{dxdydy} + \frac{d^3 \phi_1(x, y, z)}{dxdzdz} \right) = 0. \tag{16}$$

Case 1.2: For $V_2 = \frac{\partial}{\partial x}$ we have a characteristic equation of the form

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{0} = \frac{dq}{0},$$

by solving this equation we have $q(x, y, z, t) = \Phi_2(y, z, t)$ which makes all the derivatives of $u(x, y, z, t)$ with respect to x equal to zero and

$$\frac{\partial^\alpha q}{\partial t^\alpha} = 0, \qquad \text{this equation has the following solution}$$

$$q = \Phi_2(t) = \frac{B_o}{\Gamma(\alpha)}\, t^{\alpha-1}, \qquad \text{where } B_0 \text{ is a constant}$$

Case 1.3: For $V_3 = \frac{\partial}{\partial y}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{0} = \frac{dt}{0} = \frac{dq}{0},$$

thus $q(x,y,z,t) = \Phi_3(x,z,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to y equal to zero, therefore

$$\frac{\partial^\alpha \Phi_3(x,z,t)}{\partial t^\alpha} + \kappa_1 \sqrt{\Phi_3(x,z,t)}\frac{d\Phi_3(x,z,t))}{dx} + \kappa_2 \frac{d^3\Phi_3(x,z,t)}{dx^3} + \kappa_3 \frac{d^3\Phi_3(x,z,t)}{dxdzdz} = 0$$

Case 1.4: For $V_4 = \frac{\partial}{\partial z}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{dq}{0},$$

thus $q(x,y,z,t) = \Phi_4(x,y,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to z equal to zero, therefore

$$\frac{\partial^\alpha \Phi_4(x,y,t)}{\partial t^\alpha} + \kappa_1 \sqrt{\Phi_4(x,y,t)}\frac{d\Phi_4(x,y,t))}{dx} + \kappa_2 \frac{d^3\Phi_4(x,y,t)}{dx^3} + \kappa_3 \frac{d^3\Phi_4(x,y,t)}{dxdydy} = 0$$

Case 1.5: For $V_5 = \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z} - 2q\frac{\partial}{\partial q}$, the characteristic equation becomes

$$\frac{dx}{x/2} = \frac{dy}{y/2} = \frac{dz}{z/2} = \frac{dt}{3t/2\alpha} = \frac{dq}{-2q}.$$

Where solving this equation we can get the next similarity variables and similarity solution for Eq. (1) as below

$$\gamma_1 = xt^{-\frac{\alpha}{3}}, \qquad \gamma_2 = yt^{-\frac{\alpha}{3}}, \qquad \gamma_3 = zt^{-\frac{\alpha}{3}}, \qquad q = t^{-\frac{4}{3}\alpha}\phi(\gamma_1, \gamma_2, \gamma_3). \tag{17}$$

By using the above transformation, Eq. (1) can be turned into a nonlinear FODE with a set of independent variable $\gamma's$. Consequently, one can conclude the next theorem.

Theorem 1: The transformation Eq. (17) reduces the time-fractional generalized Z-K Eq. (1) to the following equation

$$\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3) + \kappa_1\sqrt{\phi}\,\phi_{\gamma_1} + \kappa_2\phi_{\gamma_1\gamma_1\gamma_1} + \kappa_3\left(\phi_{\gamma_1\gamma_2\gamma_2} + \phi_{\gamma_1\gamma_3\gamma_3}\right) = 0, \tag{18}$$

with the E-K fractional differential operator $\left(P_\beta^{\tau,\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)$ which is defined as

$$\left(P_{\beta_1,\beta_2,\beta_3}^{\tau,\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta_1}\gamma_1\frac{\partial}{\partial\gamma_2} - \frac{1}{\beta_2}\gamma_2\frac{\partial}{\partial\gamma_2} - \frac{1}{\beta_3}\gamma_3\frac{\partial}{\partial\gamma_3}\right)\left(K_{\beta_1,\beta_2,\beta_3}^{\tau+\alpha,n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3),$$

where

$$n = \begin{cases} |\alpha|+1, & n \notin N, \\ \alpha, & n \in N, \end{cases} \tag{19}$$

Notes

$$\left(K_{\beta_1,\beta_2,\beta_3}^{\tau+\alpha,n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (\Theta-1)^{\alpha-1}\Theta^{-(\tau+\alpha)}\phi(\gamma_1\Theta^{\frac{1}{\beta_1}},\gamma_2\Theta^{\frac{1}{\beta_2}},\gamma_3\Theta^{\frac{1}{\beta_3}})\,d\Theta, & \alpha>0, \\[2mm] \phi(\gamma_1,\gamma_2,\gamma_3), & \alpha=0, \end{cases} \tag{20}$$

is the E-K fractional integral operator.

The proof of theorem 1: Depending on the definition of the R-L fractional derivatives provided with $n-1<\alpha<1,\ \ n=1,2,3,\dots$, then we have

$$\partial_t^\alpha q = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^t (t-g)^{n-\alpha-1}g^{\frac{-1}{3}\alpha}\phi(\gamma_1 g^{-\frac{\alpha}{3}},\gamma_2 g^{-\frac{\alpha}{3}},\gamma_3 g^{-\frac{\alpha}{3}})\,dg\right]. \tag{21}$$

Let $\Lambda=\frac{t}{g}$, one can get $dg=-\frac{t}{\Lambda^2}$, thus Eq. (21) can be written as

$$\partial_t^\alpha q = \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\frac{4}{3}\alpha}}{\Gamma(n-\alpha)}\int_1^\infty (\Lambda-1)^{n-\alpha-1}\Lambda^{-(n+1-\frac{4}{3}\alpha)}\phi(\gamma_1\Lambda^{\frac{\alpha}{3}},\gamma_2\Lambda^{\frac{\alpha}{3}},\gamma_3\Lambda^{\frac{\alpha}{3}})\,d\Lambda\right], \tag{22}$$

following the definition of E-K fractional differential operator, then Eq. (22) becomes

$$\partial_t^\alpha q = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right], \tag{23}$$

it is time to deal with the right-hand side of Eq. (23). Where

$$t\frac{\partial}{\partial t}\wp(\gamma_1,\gamma_2,\gamma_3) = -\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}.$$

From that, we have

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right]$$

$$= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right)\right]$$

$$= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{4}{3}\alpha-1}\left(n-\frac{4}{3}\alpha-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right],$$

according to the above result provided with the same steps for $(n-1)$ times, we get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right]$$

$$= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{4}{3}\alpha-1}\left(n-\frac{4}{3}\alpha-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right]$$

$$\dots,$$

$$= t^{-\frac{4}{3}\alpha}\prod_{j=0}^{n-1}\left[\left(n-\frac{4}{3}\alpha+j-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right],$$

this implies

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right] = t^{-\frac{4}{3}\alpha}\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3), \tag{24}$$

thus

$$\partial_t^\alpha q = t^{-\frac{4}{3}\alpha}\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3). \tag{25}$$

At last, Eq. (1) can be reduced into the below equation and the proof is completed

$$\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3) + \kappa_1\sqrt{\phi}\,\phi_{\gamma_1} + \kappa_2\phi_{\gamma_1\gamma_1\gamma_1} + \kappa_3\left(\phi_{\gamma_1\gamma_2\gamma_2} + \phi_{\gamma_1\gamma_3\gamma_3}\right) = 0. \tag{26}$$

Therefore, the proof is completed.

Case 2: For $V_1 = \frac{\partial}{\partial t}$, $V_2 = \frac{\partial}{\partial x}$, $V_3 = \frac{\partial}{\partial y}$, $V_4 = \frac{\partial}{\partial z}$ with $\kappa_4 \neq 0$, we have the following sub-cases

Case 2.1: $V_6 = \frac{\partial}{\partial t}$ we have a characteristic equation of the form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{dq}{0}.$$

by integrating this equation and appoint the solutions q as function of the dependent variables x, y, z, that is

$$q(x,y,z,t) = \Psi_1(x,y,z),$$

this implies $\frac{\partial^\alpha q}{\partial t^\alpha} = 0$ and Eq. (1) becomes

$$\kappa_1\sqrt{\Psi_1(x,y,z)}\frac{d\Psi_1(x,y,z)}{dx} + \kappa_2\frac{d^3\Psi_1(x,y,z)}{dx^3} + \kappa_3\left(\frac{d^3\Psi_1(x,y,z)}{dxdydy} + \frac{d^3\Psi_1(x,y,z)}{dxdzdz}\right) + \kappa_4\frac{d^2\Psi_1(x,y,z)}{dx^2} = 0. \tag{27}$$

Case 2.2: For $V_7 = \frac{\partial}{\partial x}$ we have a characteristic equation of the form

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{0} = \frac{dq}{0},$$

by solving this equation we have $q(x,y,z,t) = \Psi_2(y,z,t)$ which makes all the derivatives of $q(x,y,z,t)$ with respect to x equal to zero and

$$\frac{\partial^\alpha q}{\partial t^\alpha} = 0, \qquad \text{this equation has the following solution}$$

$$q = \Psi_2(t) = \frac{C_o}{\Gamma(\alpha)}\,t^{\alpha-1}, \qquad \text{where } C_0 \text{ is a constant}$$

Case 2.3: For $V_3 = \frac{\partial}{\partial y}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{0} = \frac{dt}{0} = \frac{dq}{0},$$

thus $q(x,y,z,t) = \Psi_3(x,z,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to y equal to zero, therefore

$$\frac{\partial^\alpha \Psi_3(x,z,t)}{\partial t^\alpha} + \kappa_1\sqrt{\Psi_3(x,z,t)}\frac{d\Psi_3(x,z,t))}{dx} + \kappa_2\frac{d^3\Psi_3(x,z,t)}{dx^3} + \kappa_3\frac{d^3\Psi_3(x,z,t)}{dxdzdz} + \kappa_4\frac{d^2\Psi_3(x,y,z)}{dx^2} = 0$$

Case 2.4: For $V_4 = \frac{\partial}{\partial z}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{du}{0},$$

thus $q(x, y, z, t) = \Psi_4(x, y, t)$ and all the derivatives of $q(x, y, z, t)$ with respect to z equal to zero, therefore

$$\frac{\partial^\alpha \Psi_4(x, z, t)}{\partial t^\alpha} + \kappa_1 \sqrt{\Psi_4(x, z, t)} \frac{d\Psi_4(x, z, t))}{dx} + \kappa_2 \frac{d^3 \Psi_4(x, z, t)}{dx^3} + \kappa_3 \frac{d^3 \Psi_4(x, z, t)}{dxdydy} + \kappa_4 \frac{d^2 \Psi_4(x, y, z)}{dx^2} = 0.$$

IV. FRACTIONAL SUB-EQUATION METHOD FOR FPDES

In this section, we introduced a summary explanation of the fractional sub-equation method [15, 16] as shown in the following steps

Step 1: Let $q(x, y, z, t) = q(\xi)$, $\xi = x + y + z - \lambda\, t$ is the traveling wave transformation which can be used to reduce the below equation

$$F(q, q_x, q_y, q_z, q_{xx}, D_t^\alpha q, D_x^\alpha q, q_{xxx}, q_{xyy}, q_{xzz}, ...), \qquad 0 < \alpha < 1.$$

to be a non-linear FODE of the form

$$H(q, q', q'', \lambda^\alpha D_\xi^\alpha q, D_\xi^\alpha q, q''', ...), \qquad 0 < \alpha < 1. \tag{28}$$

Step 2: Assume that, the above equation has a solution of the form

$$q(\xi) = \sum_{i=0}^{n} A_i(\varphi(\xi))^i \tag{29}$$

where $A_i(i = 0, 1, ..., n)$ are constants to be detected and the positive integer n can be obtained by balancing the nonlinear terms and the highest order derivatives in Eq. (28). Also, the function $\varphi(\xi)$ satisfy the following fractional Riccati equation

$$D_\xi^\alpha \varphi(\xi) = \delta + \varphi^2(\xi) \tag{30}$$

where $\varphi(\xi)$ has a set of solutions as shown below

$$\varphi(\xi) = \begin{cases} -\sqrt{-\delta}\ tanh_\alpha(\sqrt{-\delta}\ \xi, \alpha), & \delta < 0, \\ -\sqrt{-\delta}\ coth_\alpha(\sqrt{-\delta}\ \xi, \alpha), & \delta < 0, \\ \sqrt{\delta}\ tan_\alpha(\sqrt{\delta}\ \xi, \alpha), & \delta > 0, \\ -\sqrt{\delta}\ cot_\alpha(\sqrt{\delta}\ \xi, \alpha), & \delta > 0, \\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha + v}, & v \text{ is a constant}, \ \delta = 0, \end{cases}$$

where all the previous trigonometric and hyperbolic functions are expressed by the following Mittag-Leffler function

$$E_\alpha(\xi) = \sum_{j=0}^{\infty} \frac{\xi^j}{\Gamma(1 + j\alpha)}, \quad \text{and}$$

$$sin_\alpha(\xi) = \frac{E_\alpha(i\xi^\alpha) - E_\alpha(-i\xi^\alpha)}{2i}, \quad cos_\alpha(\xi) = \frac{E_\alpha(i\xi^\alpha) + E_\alpha(-i\xi^\alpha)}{2i},$$

$$sinh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2}, \quad cosh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2}, \quad \text{where it is known that:}$$

$$tan_\alpha(\xi) = \frac{sin_\alpha(\xi)}{cos_\alpha(\xi)}, \quad cot_\alpha(\xi) = \frac{cos_\alpha(\xi)}{sin_\alpha(\xi)}, \quad tanh_\alpha(\xi) = \frac{sinh_\alpha(\xi)}{cosh_\alpha(\xi)}, \quad coth_\alpha(\xi) = \frac{cosh_\alpha(\xi)}{sinh_\alpha(\xi)}.$$

Step 3: Now, substituting Eq. (29) along with (30) into (28) and equate all the coefficients of all powers of $(\varphi(\xi))^i$ by zero. Then, we get a system of algebraic equations. Solving this system via the Mathematica program to determine the value of $A_i(i = 0, 1, ..., n)$. Consequently, we use these values with the solutions of Eq. (30) to construct the analytical solutions for Eq. (28) which is considered the main aim for this section.

16. Zhang S H, Zhang Q. Fractional sub-equation method and its applications to nonlinear fractional PDEs. *Phys. Lett. A*. 2011; 375: 1069-1073.

a) Fractional sub-equation method for Eq. (1)

In this section, we apply the fractional sub-equation method for Eq. (1) therefore, we rewrite this equation by using $q(x, y, z, t) = v^2(x, y, z, t)$ as follows:

$$v\frac{\partial^\alpha v}{\partial t^\alpha} + \kappa_1 v^2 v_x + (\kappa_2 + 2\kappa_3)(3v_x v_{xx} + vv_{xxx}) + \kappa_4 (v_x^2 + vv_{xx}) = 0. \tag{31}$$

Let us introduce an important transformation

$$v(x, y, z, t) = v(\xi), \quad \xi = x + y + z - \lambda t, \tag{32}$$

thus, Eq. (31) has the following form

$$\lambda^\alpha v D_\xi^\alpha v - \kappa_1 v^2 v' - (\kappa_2 + 2\kappa_3)(3v'v'' + vv''') - \kappa_4 (vv'' + v'^2) = 0. \tag{33}$$

According to the previous analysis of the considered method. We have the following solution for the reduced Eq. (33)

$$v(\xi) = A_0 + A_1 \varphi(\xi) + A_2 \varphi^2(\xi), \tag{34}$$

substituting Eq. (34) along with (30) into (33) and equate all the coefficients $(\varphi(\xi))^i$ by zero to get a system of algebraic equations. Solving this system with the aid of the Mathematica program we have

$$\text{Case 1:} \quad A_0 = \frac{3\lambda^\alpha}{8\kappa_1}, \quad A_1 = \pm\frac{3i\lambda^\alpha}{4\kappa_1\sqrt{\delta}}, \quad A_2 = -\frac{3\lambda^\alpha}{8\kappa_1\delta}, \quad \kappa_3 = -\frac{80\kappa_2\delta - \lambda^\alpha}{160\delta}, \quad \kappa_4 = -\frac{9i\lambda^\alpha}{40\sqrt{\delta}},$$

then, Eq. (1) has the below solutions

$$q_{11}(\xi) = \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1}tanh_\alpha\left(\sqrt{-\delta}\xi\right) + \frac{3\lambda^\alpha}{8\kappa_1}tanh_\alpha^2\left(\sqrt{-\delta}\xi\right)\right)^2, \qquad where \ \ \delta < 0,$$

$$q_{12}(\xi) = \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1}coth_\alpha\left(\sqrt{-\delta}\xi\right) + \frac{3\lambda^\alpha}{8\kappa_1}coth_\alpha^2\left(\sqrt{-\delta}\xi\right)\right)^2, \qquad where \ \ \delta < 0,$$

$$q_{13}(\xi) = \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3i\lambda^\alpha}{4\kappa_1}tan_\alpha\left(\sqrt{\delta}\xi\right) - \frac{3\lambda^\alpha}{8\kappa_1}tan_\alpha^2\left(\sqrt{\delta}\xi\right)\right)^2, \qquad where \ \ \delta > 0, \tag{35}$$

$$q_{14}(\xi) = \left(\frac{3\lambda^\alpha}{8\kappa_1} \mp \frac{3i\lambda^\alpha}{4\kappa_1}cot_\alpha\left(\sqrt{\delta}\xi\right) - \frac{3\lambda^\alpha}{8\kappa_1}cot_\alpha^2\left(\sqrt{\delta}\xi\right)\right)^2, \qquad where \ \ \delta > 0,$$

$$q_{15}(\xi) = \left(\frac{3\lambda^\alpha}{8\kappa_1} \mp \frac{3\kappa_4}{\kappa_1}\left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega_0}\right) - \frac{3\lambda^\alpha}{8\kappa_1}\left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega_0}\right)^2\right)^2, \qquad \delta = 0$$

$$\text{Case 2:} \quad A_0 = \frac{3\lambda^\alpha}{4\kappa_1}, \quad A_1 = \pm\frac{3i\lambda^\alpha}{4\kappa_1\sqrt{\delta}}, \quad A_2 = 0, \quad \kappa_3 = -\frac{\kappa_2}{2}, \quad \kappa_4 = \pm\frac{i\lambda^\alpha}{4\sqrt{\delta}},$$

thus, we have a set of analytical solutions for Eq. (1) which is presented as follows

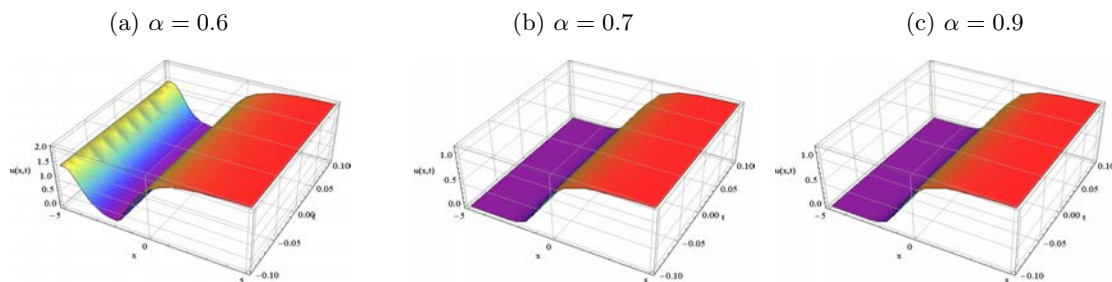$$q_{21}(\xi) = \left(\frac{3\lambda^\alpha}{4\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1}tanh_\alpha\left(\sqrt{-\delta}\xi\right)\right)^2, \qquad where \ \ \delta < 0,$$

$$q_{22}(\xi) = \left(frac3\lambda^\alpha4\kappa_1 \pm \frac{3\lambda^\alpha}{4\kappa_1}coth_\alpha\left(\sqrt{-\delta}\xi\right)\right)^2, \qquad where \ \ \delta < 0,$$

$$q_{23}(\xi) = \left(\frac{3\lambda^\alpha}{4\kappa_1} \pm \frac{3i\lambda^\alpha}{4\kappa_1}tan_\alpha\left(\sqrt{\delta}\xi\right)\right)^2, \qquad where \ \ \delta > 0, \tag{36}$$

Notes

$$q_{24}(\xi) = \left(\frac{3\lambda^\alpha}{4\kappa_1} \mp \frac{3i\lambda^\alpha}{4\kappa_1} cot_\alpha \left(\sqrt{\delta} \xi \right) \right)^2, \qquad where \ \ \delta > 0,$$
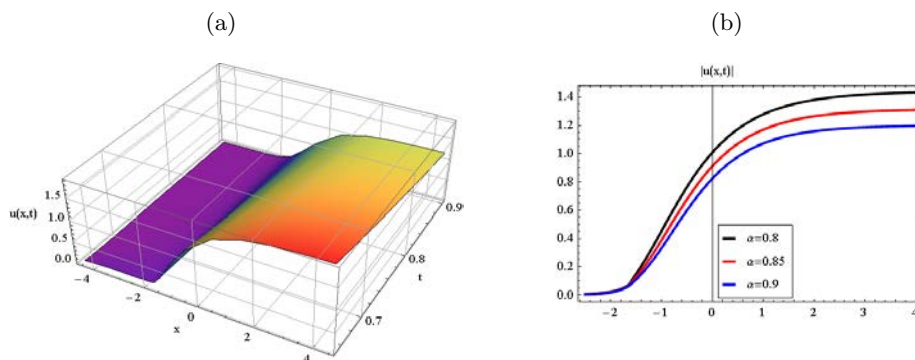
$$q_{25}(\xi) = \left(\frac{3\lambda^\alpha}{4\kappa_1} \mp \frac{3\kappa_4}{\kappa_1} \left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega_0} \right) \right)^2, \qquad \delta = 0$$

where v_0 is a constant and $\xi = x + y + z - \lambda t$.

The following figures show the 3D and 2D plots for the solution Eq. (31):

(a) $\alpha = 0.6$ (b) $\alpha = 0.7$ (c) $\alpha = 0.9$

Figure 1: The 3D double-layer solution (31). (a) The solution at $\alpha = 0.6$ with the parameters $\lambda = 0.4, \ \delta = -0.5, \ \kappa_1 = 0.6, \ \kappa_2 = 0.3$ (b) The solution at $\alpha = 0.7$ with the same parameters. (c) The solution at $\alpha = 0.9$ with the same parameters.

(a)   (b)

Figure 2: The effect of α on the analytical solution (36) (a) The 3D plot at a fixed time $t = 0.6$ and $0.65 \leq \alpha \leq 0.9$ with the parameters $\lambda = 0.4, \ \delta = -0.5, \ \kappa_1 = 0.6, \ \kappa_2 = 0.3$. (b) The 2D plot at $\alpha = 0.8, \ 0.85, \ 0.9$ and the same values of the others parameters.

V. The Methodology of the Extended Kudryashov Method

We briefly display the main steps of the extended Kudryashov method [46, 47]to construct analytical solutions for Eq. (1) as below.

Step 1: Consider a non-linear FODE Eq.(28) with the same traveling wave transformation as section 4 and assume that the solution of Eq.(28) can be expressed as follows:

$$q(\xi) = \sum_{i=0}^{M} B_i \varphi^i, \qquad (37)$$

where $a_i, \ \ i = 0, 1, 2, ..., n$ are constants to be determined, and $\varphi = \varphi(\xi)$ satisfies the following equation:

$$\varphi'(\xi) = \varphi(\xi)^3 - \varphi(\xi), \quad since \ \varphi(\xi) = \frac{\pm 1}{\sqrt{1 \pm e^{2\xi}}} \qquad (38)$$

Step 2: Determining the value of the positive integer M by balancing the highest order derivatives with the nonlinear terms which appear in Eq.(28) by using the relation $M = \frac{2(s-rp)}{r-l-1}$ since, $(q^{(p)}(\xi, \varphi))^r$ and $q^l(\xi, \varphi)q^{(s)}(\xi)$ are the balanced terms.

Step 3: Substituting Eq.(37) into Eq.(28) and using Eq.(38), collecting all terms with the same order of $\varphi(\xi)$ together to zero yields a set of algebraic equations. Solving the equations system and using Eq.(38) to construct a variety of analytical solutions for Eq.(28).

a) Extended Kudryashov method for Eq. (1)

In this section, we apply the extended Kudryashov method method for Eq. (33) which is considered a reduced form of Eq. (1) and according to the previous analysis of the considered method. We have the following solution

$$v(\xi) = B_0 + B_1\varphi(\xi) + B_2\varphi^2(\xi), \tag{39}$$

substituting Eq. (39) along with (38) into (33) and equate all the coefficients $(\varphi(\eta))^i$ by zero to get a system of algebraic equations. Solving this system with the aid of the Mathematica program we have

$$\textit{Case 1:} \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = \frac{3\lambda^\alpha}{2\kappa_1}, \quad \kappa_3 = -\frac{\kappa_2}{2}, \quad \kappa_4 = -\frac{\lambda^\alpha}{4},$$

then, Eq. (1) has the below solutions

$$q_{11}(\xi) = \frac{9\lambda^{2\alpha}}{16\kappa_1^2}e^{-2\xi}sech^2(\xi),$$

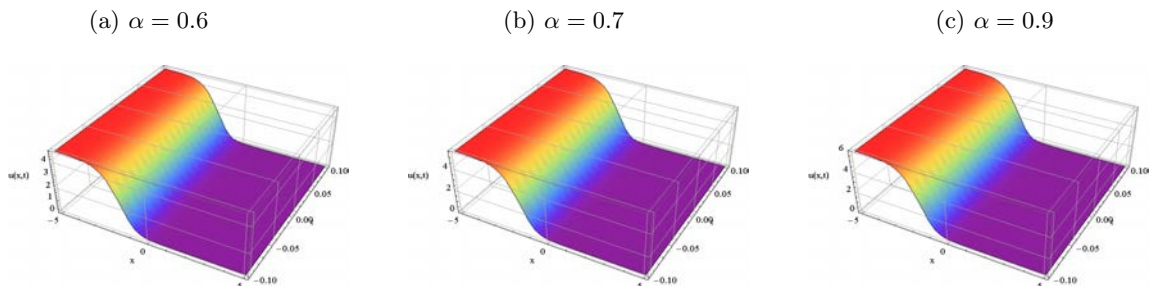$$q_{12}(\xi) = \frac{9\lambda^{2\alpha}}{16\kappa_1^2}e^{-2\xi}csch^2(\xi). \tag{40}$$

$$\textit{Case 2:} \quad B_0 = \frac{3\lambda^\alpha}{2\kappa_1}, \quad B_1 = 0, \quad B_2 = -\frac{3\lambda^\alpha}{2\kappa_1}, \quad \kappa_3 = -\frac{\kappa_2}{2}, \quad \kappa_4 = \frac{\lambda^\alpha}{4},$$

thus, we have the next solutions for Eq. (1) which is presented as follows

$$q_{21}(\xi) = \frac{9\lambda^{4\alpha}}{16\kappa_1^2}\left(1 - \frac{1}{2}e^{-2\xi}sech(\xi)\right)^2,$$

$$q_{22}(\xi) = \frac{9\lambda^{4\alpha}}{16\kappa_1^2}\left(1 - \frac{1}{2}e^{-2\xi}csch(\xi)\right)^2. \tag{41}$$

where $\xi = x + y + z - \lambda t$.

The 3D and 2D plots for the solution Eq. (41) are plotted in the following Figures:

(a) $\alpha = 0.6$ \qquad\qquad (b) $\alpha = 0.7$ \qquad\qquad (c) $\alpha = 0.9$

Figure 3: The 3D double-layer solution (41). (a) The solution at $\alpha = 0.6$ with the parameters $\lambda = 0.7,\ \kappa_1 = 0.5,\ \kappa_2 = 0.3$ (b) The solution at $\alpha = 0.7$ with the same parameters. (c) The solution at $\alpha = 0.9$ with the same parameters.
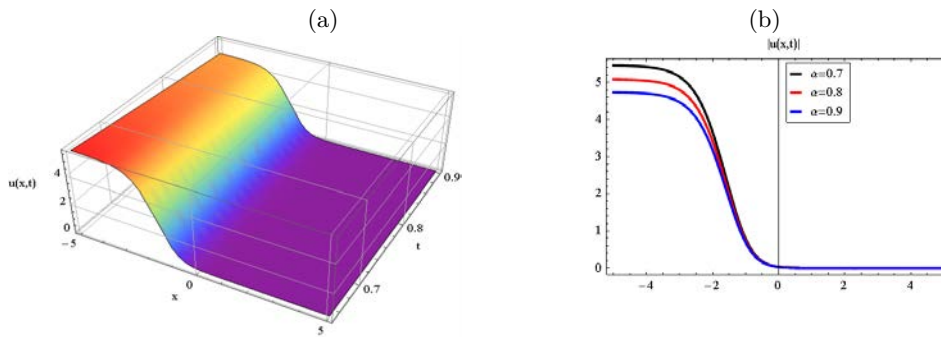
Figure 4: The effect of α on the analytical solution (41) (a) The 3D plot at a fixed time $t = 0.3$ and $0.65 \leq \alpha \leq 0.9$ with the parameters $\lambda = 0.7$, $\kappa_1 = 0.5$, $\kappa_2 = 0.3$. (b) The 2D plot at $\alpha = 0.7$, 0.8, 0.9 and the same values of the others parameters.

VI. Conservation Laws for Eq. (1)

In this section, the conservation laws of the time fractional MZKB equation (1) were derived, based on the formal lagrangian and Lie point symmetries as described in the following explanation:
Consider a vector $C = (C^t, C^x, C^y, C^z)$ admits the following conservation equation

$$[D_t(C^t) + D_x(C^x) + D_y(C^y) + D_z(C^z)]_{Eq.(1)} = 0, \tag{42}$$

where $C^t = C^t(x, y, z, t, u, ...)$, $C^x = C^x(x, y, z, t, u, ...)$, $C^y = C^y(x, y, z, t, u, ...)$, and $C^z = C^z(x, y, z, t, u, ...)$ are called the conserved vectors for Eq. (1). According to the new conservation theorem for Ibragimov [37], the formal Lagrangian for Eq. (1) can be given by

$$L = \omega((x, y, z, t) \left[\partial_t^\alpha q + \kappa_1 \sqrt{q}\, q_x + \kappa_2 q_{xxx} + \kappa_3(q_{xyy} + q_{xzz}) + \kappa_4 q_{xx}\right] = 0, \tag{43}$$

here $\omega((x, y, z, t)$ is a new dependent variable. Depending on the definition of the Lagrangian, we get an action integral as follows

$$\int_0^t \int_{\Omega_1} \int_{\Omega_2} \int_{\Omega_3} L\left(x, y, z, t, q, \omega, D_t^\alpha, q_x, q_{xxx}, q_{xyy}, q_{xzz}, q_{xx}\right) dx\, dy\, dz\, dt.$$

The Euler-Lagrange operator is defined as

$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha q} - D_x \frac{\partial}{\partial q_x} + D_x^2 \frac{\partial}{\partial q_{xx}} - D_x^3 \frac{\partial}{\partial q_{xxx}} - D_x D_y^2 \frac{\partial}{\partial q_{xyy}} - D_x D_z^2 \frac{\partial}{\partial q_{xzz}}$$

where $(D_t^\alpha)^*$ denotes to the adjoint operator of D_t^α, and the adjoint equation to the nonlinear by means of the Euler-Lagrange equation is given by

$$\frac{\delta L}{\delta u} = 0.$$

Adjoint operator $(D_t^\alpha)^*$ for R-L is defined by

$$(D_t^\alpha)^* = (-1)^n I_T^{n-\alpha}(D_t^n) \equiv_t^C D_T^\alpha,$$

where $I_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n - \alpha$ that is defined by

$$I_T^{n-\alpha} f(t, x) = \frac{1}{\Gamma(n - \alpha)} \int_t^T (\tau - t)^{n-\alpha-1} f(\tau, x)\, d\tau.$$

Considering the case of one dependent variable $u(x, y, z, t)$ with four independent variables $x, \ y, \ z, \ t$, we get

$$\bar{X} + D_t(\tau)I + D_x(\xi)I + D_y(\zeta)I + D_z(\nu)I = W\frac{\delta}{\delta u} + D_t(C^t) + D_x(C^x) + D_y(C^y) + D_z(C^z),$$

where $\bar{X}$ is defined by

$$\bar{X} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u} + \eta_\alpha^0\frac{\partial}{\partial D_t^\alpha u} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xyy}\frac{\partial}{\partial u_{xyy}} + \eta^{xzz}\frac{\partial}{\partial u_{xzz}},$$

and the Lie characteristic function W for case 1 and 2 in the subsection 3.1 is defined as

$$W = \eta - \tau u_t - \xi u_x - \zeta u_y - \nu u_z,$$

where W can be expanded to

$$W_1 = -\frac{\partial}{\partial t}, \quad W_2 = -\frac{\partial}{\partial x}, \quad W_3 = -\frac{\partial}{\partial y}, \quad W_4 = -\frac{\partial}{\partial z},$$

$$W_5 = -\frac{3t}{2\alpha}\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y} - \frac{z}{2}\frac{\partial}{\partial z} - 2q\frac{\partial}{\partial q}.$$

(44)

For the R-L time-fractional derivative, the density component C^t of conservation law is defined as:

$$C^t = \tau L + \sum_{k=0}^{n-1}(-1)^k \,_0D_t^{\alpha-1-k}(W_m)D_t^k\frac{\partial L}{\partial \,_0D_t^\alpha q} - (-1)^n J\left(W_m, D_t^n\frac{\partial L}{\partial \,_0D_t^\alpha q}\right),$$

(45)

where the operator $J(.)$ defined by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau, x, y, z)g(\mu, x, y, z)}{(\mu-\tau)^{\alpha+1-n}}\,d\mu\,d\tau,$$

and the other (flux) components are defined as

$$C^i = \xi^i L + W_m\left[\frac{\partial L}{\partial q_i^m} - D_j\left(\frac{\partial L}{\partial q_{ij}^m}\right) + D_jD_k\left(\frac{\partial L}{\partial q_{ijk}^m} - ...\right)\right] + D_j(W_m)\left[\frac{\partial L}{\partial q_{ij}^m} - D_k\left(\frac{\partial L}{\partial q_{ijk}^m}\right) + ...\right]$$

$$+ D_jD_k(W_m)\left(\frac{\partial L}{\partial q_{ijk}^m} - ...\right) + ...,$$

(46)

where $\xi^1 = \xi, \ \xi^2 = \zeta, \ \xi^3 = \nu$ and $m = 1, 2, ..., 5$.

Now by using Eq. (44) with the help of Eqs. (45) and (46), we obtain the components of conservation laws for the time-fractional MZKB equation as the follows

Case 1: When $\kappa_4 = 0$ we have the following subcases according to the vector fields Eq. (14)

Case 1.1: $W_1 = -q_x$ where $\xi^x = 1, \ \xi^t = 0, \ \xi^y = 0, \ \xi^z = 0$ and $\eta = 0$ we get

$$C_1^t = \omega \,_0D_t^{\alpha-1}(-q_x)\frac{\partial L}{\partial \,_0D_t^\alpha q} - J\left(-q_x, D_t^n\frac{\partial L}{\partial \,_0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_x) - q_x D_t^\alpha(\omega),$$

$$C_1^x = \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - q_x\left[\kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right]$$

$$+ \kappa_2 q_{xx}\omega_x + \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{xz}\omega_z),$$

$$C_1^y = -\frac{\kappa_3}{3}q_x[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xx}\omega_y + q_{xy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyx} + q_{yxx}),$$

$$C_1^z = -\frac{\kappa_3}{3}q_x[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xx}\omega_z + q_{xz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzx} + q_{zxx}),$$

Case 1.2: $W_2 = -q_t$ where $\xi^x = 0$, $\xi^t = 1$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain

$$C_2^t = \omega L\omega \, _0D_t^{\alpha-1}(-q_t)\frac{\partial L}{\partial \, _0D_t^\alpha q} - J\left(-q_t, D_t^n\frac{\partial L}{\partial \, _0D_t^\alpha q}\right)$$

$$= \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \omega D_t^\alpha(q_t) - q_t D_t^\alpha(\omega),$$

$$C_2^x = -q_t\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xt}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yt}\omega_y + q_{zt}\omega_z)$$

$$- \kappa_2\omega q_{xxt} - \frac{\kappa_3}{3}\omega(q_{yyt} + q_{zzt}),$$

$$C_2^y = -\frac{\kappa_3}{3}q_t[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xt}\omega_y + q_{yt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyt} + q_{yxt}),$$

$$C_2^z = -\frac{\kappa_3}{3}q_t[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xt}\omega_z + q_{zt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzt} + q_{zxt})$$

Case 1.3: $W_3 = -q_y$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 1$, $\xi^z = 0$ and $\eta = 0$ we have

$$C_3^t = \omega \, _0D_t^{\alpha-1}(-q_y)\frac{\partial L}{\partial \, _0D_t^\alpha q} - J\left(-q_y, D_t^n\frac{\partial L}{\partial \, _0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_y) - q_y D_t^\alpha(\omega),$$

$$C_3^x = -q_y\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_3^y = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_3^z = -\frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}),$$

Case 1.4: $W_4 = -q_z$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 1$ and $\eta = 0$ we obtain

$$C_4^t = \omega \, _0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial \, _0D_t^\alpha q} - J\left(-q_z, D_t^n\frac{\partial L}{\partial \, _0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_z) - q_z D_t^\alpha(\omega),$$

$$C_4^x = -q_z\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_4^y = -\frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_4^z = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right]$$

$$- \frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}).$$

Case 1.5: $W_5 = -2q - \frac{3t}{2\alpha}\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y} - \frac{z}{2}\frac{\partial}{\partial z}$ then

$$C_5^t = \omega\; {}_0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial\; {}_0D_t^\alpha q} - J\left(-q_z, D_t^n\frac{\partial L}{\partial\; {}_0D_t^\alpha q}\right)$$

$$= \frac{3t}{2\alpha}\omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right]$$

$$- \omega D_t^\alpha\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)D_t^\alpha(\omega),$$

$$C_5^x = \frac{x}{2}\omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right]$$

$$- \left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\left[\kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right]$$

$$+ \kappa_2\omega_x\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial y}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right.$$

$$\left.+ \omega_z\frac{\partial}{\partial z}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right),$$

$$C_5^y = \frac{y}{2}\omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \frac{\kappa_3}{3}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \omega_x\frac{\partial}{\partial y}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right)$$

$$- \frac{\kappa_3}{3}\omega\frac{\partial}{\partial xy}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right),$$

$$C_5^z = \frac{z}{2}\omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \frac{\kappa_3}{3}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)[\omega_{xz} + \omega_{zx}]$$

$$+ \frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \omega_x\frac{\partial}{\partial z}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right)$$

$$- \frac{\kappa_3}{3}\omega\frac{\partial}{\partial xz}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right).$$

Case 2: When $\kappa_4 \neq 0$ we have the following subcases according to the vector fields Eq. (14)

Case 2.1: $W_1 = -q_x$ where $\xi^x = 1, \xi^t = 0, \xi^y = 0, \xi^z = 0$ and $\eta = 0$ we get

$$C_1^t = \omega\; {}_0D_t^{\alpha-1}(-q_x)\frac{\partial L}{\partial\; {}_0D_t^\alpha q} - J\left(-q_x, D_t^n\frac{\partial L}{\partial\; {}_0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_x) - q_x D_t^\alpha(\omega),$$

$$C_1^x = \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - q_x\left[-\kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right]$$

$$+ \kappa_2 q_{xx}\omega_x + \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{xz}\omega_z),$$

$$C_1^y = -\frac{\kappa_3}{3}q_x[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xx}\omega_y + q_{xy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyx} + q_{yxx}),$$

$$C_1^z = -\frac{\kappa_3}{3}q_x[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xx}\omega_z + q_{xz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzx} + q_{zxx}),$$

Notes

Case 2.2: $W_2 = -q_t$ where $\xi^x = 0$, $\xi^t = 1$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain

$$C_2^t = \omega L \omega\, _0D_t^{\alpha-1}(-q_t)\frac{\partial L}{\partial\, _0D_t^\alpha q} - J\left(-q_t, D_t^n \frac{\partial L}{\partial\, _0D_t^\alpha q}\right)$$

$$= \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \omega D_t^\alpha(q_t) - q_t D_t^\alpha(\omega),$$

$$C_2^x = -q_t\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xt}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yt}\omega_y + q_{zt}\omega_z)$$

$$- \kappa_2\omega q_{xxt} - \frac{\kappa_3}{3}\omega(q_{yyt} + q_{zzt}),$$

$$C_2^y = -\frac{\kappa_3}{3}q_t[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xt}\omega_y + q_{yt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyt} + q_{yxt}),$$

$$C_2^z = -\frac{\kappa_3}{3}q_t[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xt}\omega_z + q_{zt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzt} + q_{zxt})$$

Case 2.3: $W_3 = -u_y$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 1$, $\xi^z = 0$ and $\eta = 0$ we have

$$C_3^t = \omega\, _0D_t^{\alpha-1}(-q_y)\frac{\partial L}{\partial\, _0D_t^\alpha q} - J\left(-q_y, D_t^n \frac{\partial L}{\partial\, _0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_y) - q_y D_t^\alpha(\omega),$$

$$C_3^x = -q_y\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_3^y = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_3^z = -\frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}),$$

Case 2.4: $W_4 = -q_z$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain

$$C_4^t = \omega\, _0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial\, _0D_t^\alpha q} - J\left(-q_z, D_t^n \frac{\partial L}{\partial\, _0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_z) - q_z D_t^\alpha(\omega),$$

$$C_4^x = -q_z\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_4^y = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_4^z = -\frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}).$$

Remark: We have verified that all the cases satisfy the equation of conservation laws Eq. (42).

Notes

VII. Conclusion

In this article, we considered the time-fractional modified Zakharov-Kuznetsov-Burgers (MZKB) equation of (3+1) dimensions. With the help of the Riemann-Liouville derivatives, the Lie symmetry analysis was successfully applied to study this equation. This analysis generated the symmetries and vector fields, which aided us in constructing the similarity reductions of the considered equation. Consequently, we constructed two sets of new analytical solutions via two powerful methods which are the fractional sub-equation method and extended Kudryashov method. Furthermore, to gain a better understanding of the dynamics of these solutions, we graphed the 3D and 2D plots of obtained solutions using appropriate parameters. Figure 1 described the double-layer solution (31) at $\alpha = 0.6$, 0.7, 0.9 with the parameters $\lambda = 0.4$, $\delta = -0.5$, $\kappa_1 = 0.6$, $\kappa_2 = 0.3$ at $-5 \leq x \leq 5$. Figure 2 showed the effect of α on the solution (31) at a fixed time $t = 0.6$ with $\alpha = 0.8$, 0.85, 0.9 and the parameters $\lambda = 0.4$, $\delta = -0.5$, $\kappa_1 = 0.6$, $\kappa_2 = 0.3$ for $-2.5 \leq x \leq 4$. Figure 3 represented the double-layer solution (41) at $\alpha = 0.6$, 0.7, 0.9 with the parameters $\lambda = 0.7$, $\kappa_1 = 0.5$, $\kappa_2 = 0.3$ where this Figure was traced at $-5 \leq x \leq 5$. Figure 4 was graphed by taking suitable parameters as $\lambda = 0.7$, $\kappa_1 = 0.5$, $\kappa_2 = 0.3$ and described the stable behavior of the the solution (41) at $\alpha = 0.7$, 0.8, 0.9 for $0 \leq x \leq 5$ at a fixed time t and decreases as the fractional-order α increases outside this interval. As a consequence, we presume that the obtained results will be more useful in explaining the physical meaning of the time fractional MZKB equation. Furthermore, we obtained four kinds of conservation laws with independent variables laying the groundwork of Lie symmetries. Finally, because of accuracy, ease of application, and relevance of the used methods in this paper, they could be generalized to many FPDEs.

References Références Referencias

1. Ri M G, Yun, C H, Kim, M H. Construction of cubic spline hidden variable recurrent fractal interpolation function and its fractional calculus. *Chaos, Solitons & Fractals*. 2021; 150: 111177.
2. Guo B, Raza, A, Al-Khaled, K, et al. Fractional-order simulations for heat and mass transfer analysis confined by elliptic inclined plate with slip effects: A comparative fractional analysis. *Case Stud. Therm. Eng*. 2021; 28: 101359.
3. Khan M A, Akbar M A, binti Abd Hamid N N. Traveling wave solutions for space-time fractional Cahn Hilliard equation and space-time fractional symmetric regularized long-wave equation. *Alexandria Eng. J*. 2021; 60(1):1317-1324.
4. Hashemi M S, Ashpazzadeh E, Moharrami M, Lakestani M. Fractional order Alpert multiwavelets for discretizing delay fractional differential equation of pantograph type. *Appl. Numer. Math*. 2021; 170:1-13.]
5. Yang X J. General fractional derivatives: *Theory, methods and applications*. Chapman and Hall/CRC. 2019.
6. Baleanu D, Kumar D. Fractional calculus and its applications in physics. *Lausanne: Frontiers Media*. 2019.

7. Zhang X M. A new method for searching the integral solution of system of Riemann- Liouville fractional differential equations with non-instantaneous impulses. *J. Comput. Appl. Math*. 2021;388:113307.

8. Ji S, Yang D. Solutions to Riemann-Liouville fractional integro differential equations via fractional resolvents. *Adv. Differ. Equ*. 2019;1:1-17.

9. Graef J, Tunc C, Sevli H. Razumikhin qualitative analyses of Volterra integro-fractional delay differential equation with caputo derivatives. *Commun. Nonlinear Sci. Numer. Simul*. 2021;103:106037.

10. Wang K J. Variational principle and approximate solution for the generalized Burgers-Huxley equation with fractal derivative. *Fractals*. 2021;29(02):2150044.

11. He F, Li L. Time fractional modified KdV-type equations: Lie symmetries, exact solutions and conservation laws, *Open Phys*. 2019; 17(1): 480-488.

12. Postavaru O,Toma A. Numerical solution of two-dimensional fractional-order partial differential equations using hybrid functions. *Partial Differ. Equations Appl. Math*. 2021; 4:100099.

13. Farman M, Saleem M, Ahmad A, Ahmad M O. Analysis and numerical solution of SEIR epidemic model of measles with non-integer time fractional derivatives by using Laplace Adomian decomposition method. *Ain Shams Eng. J*. 2018; 9(4): 3391-3397.

14. Li W, Pang y. Application of Adomian decomposition method to nonlinear systems. *Adv. Differ. Equ*. 2020; 1:1-17.

15. Abdelkawy M A, EL-Kalaawy O H, Al-Denari R B, Biswas A. Application of fractional sub-equation method to nonlinear evolution equations. *Nonlinear Anal.-Model. Control*. 2018; 23: 710-723.

16. Zhang S H, Zhang Q. Fractional sub-equation method and its applications to nonlinear fractional PDEs. *Phys. Lett. A*. 2011; 375: 1069-1073.

17. Saberi E, Hejazi R, Lie symmetry analysis, conservation laws and exact solutions of the timefractional generalized Hirota-Satsuma coupled KdV system. *Physica A*. 2018; 492: 296-307.

18. Maitama S, Zhao W. Local fractional homotopy analysis method for solving non-differentiable problems on Cantor sets. *Adv. Differ. Equ*. 2019; 1:1-22.

19. Sahoo S, Ray S S, Abdou M A. New exact solutions for time-fractional Kaup-Kupershmidt equation using improved (G'/G)-expansion and extended (G'/G)-expansion methods. *Alex. Eng. J*. 2020;59(5):3105-3110.

20. Ali M N, Osman M S, Husnine S M. On the analytical solutions of conformable time-fractional extended Zakharov-Kuznetsov equation through (G'/G)-expansion method and the modified Kudryashov method. *J. SeMA*. 2019; 76:15-25.

21. Tayyan B A, Sakka A H. Lie symmetry analysis of some conformable fractional partial differential equations. *Arab. J. Math*. 2020; 9:201-212.

22. Prakash A, Goyal M, Gupta S. Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation. *Nonlinear Engineering*. 2019;9:164-171.

23. Angstmann C N, Henry B I. Generalized fractional power series solutions for fractional differential equations. *Appl. Math. Lett*. 2020; 102:106107.

24. Alaroud M. Application of Laplace residual power series method for approximate solutions of fractional IVP's. *Alexandria Eng. J.*. 2021; 61(2): 1585-1595.

25. Abuasad S, Hashim I, Abdul K, Samsul A. Modified fractional reduced differential transform method for the solution of multiterm time-fractional diffusion equations. *Adv. Math. Phys.* 2019.

26. Mukhtar S, Abuasad S, Hashim I, Abdul K. Effective method for solving different types of nonlinear fractional Burgers' equations. *Mathematics*. 2020; 8(5): 729.

27. Shi D, Zhang Y, Liu W, Liu J. Some exact solutions and conservation laws of the coupled timefractional Boussinesq-Burgers system. *Symmetry*. 2019; 11(1):77.

28. Kour B, Kumar S, Symmetry analysis, explicit power series solutions and conservation laws of the space-time fractional variant Boussinesq system. *Eur. Phys. J. Plus.* 2018; 133(12): 520.

29. Kumar S, Kour B, Yao S W, Inc M , Osman M S. Invariance Analysis, Exact Solution and Conservation Laws of (2+1) Dim Fractional Kadomtsev-Petviashvili (KP) System. *Symmetry*. 2021; 13(3):477.

30. Wang X B, Tian S F, Qin C Y, Zhang T T. Lie symmetry analysis, conservation laws and analytical solutions of a time-fractional generalized KdV-type equation. *J. Nonlinear Math. Phys.* 2017; 24:516-530.

31. Wang X B, Tian S F, Qin C Y, Zhang T T. Characteristics of the breathers, rogue waves and solitary waves in a generalized (2+1)-dimensional Boussinesq equation. *Europhys. Lett.* 2016; 115: 10002.

32. Liu J G, Yang X J, Feng Y Y, Cui P. On group analysis of the time fractional extended (2+1)-dimensional Zakharov-Kuznetsov equation in quantum magneto-plasmas. *Math. Comput. Simul.* 2020; 178:407-421.

33. Liu J G, Yang X J, Feng Y Y, Geng L L. Numerical solutions and conservation laws of the time fractional coupled WBK-type system. *Math. Methods Appl. Sci.* 2021; 44(5):4105-4116.

34. Liu J G, Yang X J, Feng Y Y, et al. On integrability of the higher-dimensional time fractional KdV-type equation. *J. Geom. Phys.* 2021; 160: 104000.

35. Liu J G, Yang X J, Geng L L, Fan Y R. group analysis of the time fractional (3+1)-dimensional kdv-type equation. *Fractals*. 2021; 29(6): 2150169.

36. Liu J G , Yang X J, Geng L L, et al. Fundamental analysis of the time fractional coupled Burgerstype equations. *J. Geom. Phys.* 2021; 169: 04334.

37. Wael S, Seadawy A R, Moawad S M, El-Kalaawy O H. Bilinear Backlund transformation, N-soliton, and infinite conservation laws for Lax-Kadomtsev-Petviashvili and generalized Korteweg-de Vries equations. *Math. Methods Appl. Sci.* 2021; 44: 11591-11612.

38. EL-Kalaawy O H. Modulational instability: Conservation laws and bright soliton solution of ionacoustic waves in electron-positron-ion-dust plasmas. *Eur. Phys. J. Plus.* 2018; 58: 1-12.

39. EL-Kalaawy O H. New: Variational principle-exact solutions and conservation laws for modifiedion-acoustic shock waves and double layers with electron degenerate in plasma. *Phys. Plasma*. 2017; 24:308-320.

40. EL-Kalaawy O H, Moawad S M, Tharwat M M, Al-Denari R B. Conservation laws, analytical solutions and stability analysis for the time-

Notes

fractional Schamel-Zakharov-Kuznetsov-Burgers equation. *Adv. Differ. Equ.* 2019; 445.

41. EL-Kalaawy O H, Variational principle, conservation laws and exact solutions for dust ion acoustic shock waves modeling modified Burger equation. *Comput. Math. Appl.* 2016; 72:1031-1041.

42. Tian Y I, Wang K L. Conservation laws for partial differential equations based on the polynomial characteristic method. *J. Therm. Sci.* 2020; 24: 2529-2534.

43. AL-Denari R B, Ibrahim R S, Tharwat M M, Moawad S M, EL-Kalaawy O H. Similarity reduction, conservation laws, and explicit solutions for the time-fractional coupled GI equation provided with convergence analysis and numerical simulation. *Alexandria Eng. J.* 2022; 61: 8763-8776.

44. EL-Kalaawy O H, Engy A A. Shock Waves, Variational principle and conservation Laws of a Schamel-Zakharov-Kuznetsov-Burgers equation in a magnetised dust plasma. *Z. Naturforsch., A: Phys. Sci.* 2018; 73: 693-704.

45. Hadjaz I. Mouloud Tribeche: Alternative dust-ion acoustic waves in a magnetized charge varying dusty plasma with nonthermal electrons having a vortex-like velocity distribution. *Astrophys Space Sci.* 2014; 351: 591-598.

46. Ege S M. Solitary wave solutions for some fractional evolution equations via new Kudryashov approach. *Rev. Mex. Fis.* 2022; 68(1):1-11.

47. Ege S M, Misirli E. Extended Kudryashov method for fractional nonlinear differential equations. *MSAEN.* 2018; 6(1):19-28.

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