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A Forced Pressure-Less Gas System via Sticky Particles Turbulences

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A Forced Pressure-Less Gas System via Sticky Particles Turbulences

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I. Introduction

We are interested in the turbulence of one dimensional fluid flows in one dimensional sticky dynamics. In [1], the authors considered, in Eulerian coordinates, the velocity field u of fluid particles and a probability field μ representing their mass or charge distribution. The particles are supposed accelerated between two successive shock times; the dynamics is then governed by a force (measure) field ν. For suitable initial data $(\mu, u)|_{t=0} = (\mu_0, u_0)$ and by discrete approximations, they solved the forced pressureless gas system

$$\begin{cases} \partial_t(\mu) + \partial_x(u\mu) = 0 \\ \partial_t(u\mu) + \partial_x(u^2\mu) = \nu \\ \mu_t \to \mu_0, \quad u(\cdot,t)\mu_t \to u_0\mu_0 \; weakly \;\; as \;\; t \longrightarrow 0 \end{cases} \tag{1}$$

where the force ν is absolutely continuous, in the space states, with respect to (w.r.t.) μ.

In this paper, we consider non accelerated fluid particles, so the force of [1] is null and the solution of (1) is thus the one of [2, 8, 3]. In this work, we concentrate our attention on turbulences which generate, for (1), a *new force* whose the support is included in the set of shock (and *pure turbulence*) sites, in space-time.

Let us first recall the constructions of [2, 8, 3]. They all rely on the sticky particle dynamics which was introduced, at a discrete level, by Zeldovich [9] in order to explain the formation of large structures in the universe. That is a finite number of particles which move with constant velocities while they

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Ref

1. Y. Brenier, W. Gangbo, G. Savare, and M. westdickenberg. Sticky particle dynamics with interactions. *Journal des mathématiques pures et Appliquées*, 99:577–617, 2013.

are not collided. All the shocks are inelastic following the conservation laws of mass and momentum.

At a continuous level, the initial state of particles is given by the support of a non negative measure μ_0. A particle starts from position x with velocity $u_0(x)$ and mass $\mu_0(\{x\})$. The particles move with constant velocities and masses while not collided. All the shocks are inelastic, following the conservation laws of mass and momentum. In their pioneering work, E et al [8] made this construction when the particles are every where in $\mathbb{R}$, u_0 is continuous and the mass of any interval $[a,b]$ is computed with a positive density f, i.e. $\mu_0([a,b]) = \int_a^b f(x)\mathrm{d}x$. At time t, a particle of position $x(t)$ has the mass $\mu(\{x(t)\},t)$ and the velocity $u(x(t),t)$, the momentum of any interval $[a(t),b(t)]$ is $\int_{a(t)}^{b(t)} u(x,t)\mu(\mathrm{d}x,t)$. The authors then solved (1) with $\nu \equiv 0$.

At the same time and independently, Brenier and Grenier [2] considered the case of particles confined in a in interval $[a,b]$, i.e. $\mu_0([a,b]^c) = 0$. By discretization of μ_0 and using discrete sticky particle dynamics, they solved the scalar conservation law $\partial_t M + \partial_x(A(M)) = 0$ by a weak solution (M, A), the unique which has some entropy condition. As a consequence, the Lebesgue-Stieltjes measure $\partial_x(A(M))$ is absolutely continuous w.r.t. $\partial_x M =: \mu(\cdot,t)$, of Randon-Nicodym derivative a function $u(\cdot,t)$. Then (μ, u) solves (1) with $\nu \equiv 0$.

In [3], Dermoune and Moutsinga constructed the sticky particles dynamics with an initial mass distribution μ_0, any probability measure, and a initial velocity function u_0, any continuous and locally integrable function such that $u_0(x) = o(x)$ as $x \to \infty$. The authors united and generalized previous works of [8, 2] with the arguments that the particles paths define a Markov process $t \mapsto X_t$ solution of the ODE

$$\mathrm{d}X_t = u(X_t,t)\mathrm{d}t, \tag{2}$$

and the velocity process $t \mapsto u(X_t,t)$ is a backward martingale. Moreover, $\mu(\cdot,t) = \mathrm{Law}(X_t)$.

In [6, 7], using suitable convex hulls, Moutsinga extended the construction when μ_0 is any non negative measure and u_0 has *no positive jump*. He gave the description of different kinds of clusters $[\alpha(x,t), \beta(x,t)]$, *i.e* the set of all the initial particles $y(0)$ which have the same position $y(t) = x$ at time t.

Following the preoccupation of Eyink and Drivas ([4]) about turbulences, Nzissila, Moutsinga and Eyi Obiang [5] defined a *turbulent interval* as a set $[a,b]$ of initial positions of sticky particles from which rise a turbulence. This means that for all $y \in [a,b]$, the interval $[a,b]$ is the widest among the intervals $[a',b'] \ni y$ which have the same position $y + \tau(y)u_0(y)$ at their common first shock time $\tau(y)$. The term "turbulence" (instead of "shock") is justified by the description of a degenerated turbulent interval $[a,b] = \{a\}$. In this case, at its *mathematical* first shock time $\tau(a)$, the particle a does not enter in a real shock but it begins a *coagulation* process; it enters in a *pure turbulence* without beginning by a real shock.

Ref

2. Y. Brenier and E. Grenier. Sticky particles and scalar conservation laws. *Siam J. Numer. Anal.*, 35:2317–2328, 2008.

At time of turbulence $\tau(a)$, the *turbulent interval* $[a, b]$ is part of a cluster $[\alpha, \beta]$ ($a, b \in [\alpha, \beta]$). The initial positions a, b, α, β are called *turbulent particles*. The motions of these particles are given by four backward Markov processes, respectively, Z^1, Z^2, Z^3 and Z^4 solutions of (2) and whose the velocity processes (the derivatives) are semi-martingales.

In this paper, we consider a process Z of more general form than in [5]. The gas system (1) is studied with a force generated at random turbulence time $\gamma = \tau(Z_0)$.

The paper is organized as follows. Section 2 is devoted to the sticky particles model. We recall its definition and the main properties used here. In section 3 we come back to the results of [5] according to the study of turbulence. These results were obtained when the support of μ_0 is an interval (i.e. there is no vacuum of matter). We generalize them to any type of support. The particularity, in presence of vacuum, is that traditional delta-shocks are transformed into *butterfly-shocks* (like in [3]). Section 4 is devoted to scalar conservations laws from the point of view of turbulent particles. First we give an entropy solution (N, A) with the same flux A as in [2], but with different initial data. Then, in subsection 4.1 we study the gas system. Considering the construction of [5], we define a process of more general form $t \mapsto Z_t = Z_t^1 \mathbb{1}_{A^1} + Z_t^2 \mathbb{1}_{A^2} + Z_t^3 \mathbb{1}_{A^3} + Z_t^4 \mathbb{1}_{A^4}$, with the help of any complete system of events A^1, A^2, A^3, A^4. A solution of (1), is given by $\mu(\cdot, t) := \mathrm{Law}(Z_t)$ and $u(Z_t, t) := \frac{\mathrm{d}Z_t}{\mathrm{d}t}$. The force ν is absolutely continuous w.r.t. the law of the couple (Z_γ, γ).

Although this solution is constructed from the sticky particles model, it does not have the properties of [1].

II. Flow and Velocity Field of Sticky Particles

a) *The sticky particle dynamics*

The definition of one dimensional sticky particle dynamics requires a mass distribution μ, any Radon measure (a measure finite on compact subsets) and a velocity function u, any real function such that the couple (μ, u) satisfies the *Negative Jump Condition* (NJC) defined in [6]. Precisely, consider the support $\mathcal{S} = \{x \in \mathbb{R} : \mu(x - \varepsilon, x + \varepsilon) > 0, \forall \varepsilon > 0\}$ of μ and the subsets $\mathcal{S}_- = \{x \in \mathbb{R} : \mu(x - \varepsilon, x) > 0\}$, $\mathcal{S}_+ = \{x \in \mathbb{R} : \mu(x, x + \varepsilon) > 0, \forall \varepsilon > 0\}$. Suppose that u is μ locally integrable and consider the generalized limits u^-, u^+ :

$$u^-(x) = \limsup_{\varepsilon \to 0} \frac{\int_{[x-\varepsilon, x)} u(\eta)\mu(d\eta)}{\mu[x - \varepsilon, x)}, \quad \forall x \in \mathcal{S}_-, \tag{3}$$

$$u^+(x) = \liminf_{\varepsilon \to 0} \frac{\int_{(x, x+\varepsilon]} u(\eta)\mu(d\eta)}{\mu(x, x + \varepsilon]}, \quad \forall x \in \mathcal{S}_+. \tag{4}$$

Ref

5. F. Eyi Obiang, O. Moutsinga, and F. Nzissila. Backward semi-martingale into burgers turbulence. *J. Math. Phys.*, 62:1–12, 2021.

The *Negative Jump Condition* requires that

$$u^-(x) \geq u(x)\ \forall\, x \in \mathcal{S}_-, \quad u(x) \geq u^+(x)\ \forall\, x \in \mathcal{S}_+. \tag{5}$$

In the whole paper, we mainly use $\mu_0 = \lambda$, the Lebesgue measure. That's why we always suppose that the support $\mathcal{S} = \mathbb{R}$.

Considering particles of initial mass distribution μ_0 and of initial velocity function u_0, their sticky dynamics is defined in [7], when the couple $(\mu_0,\, u_0)$ satisfies (5) and $x^{-1}u(x) \to 0$ as $|x| \to +\infty$. The dynamics is characterized by a forward flow $(x, s, t) \mapsto \phi_{s,t}(x)$ defined on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$.

b) Proposition (Forward flow)
For all x, s, t :

1. *$\phi_{s,s}(x) = x$ and $\phi_{s,t}(\cdot)$ is non-decreasing and continuous.*
2. *The value $\phi_{s,t}(x)$ is the position after supplementary time t of the particle which occupied the position x at time s. More precisely :*

$$\phi_{s,t}(\phi_{0,s}(y)) = \phi_{0,s+t}(y)\ , \quad \forall\, y. \tag{6}$$

3. *If $\phi_{0,t}^{-1}(\{x\}) =: [\alpha(x, 0, t),\, \beta(x, 0, t)]$ with $\alpha(x, 0, t) < \beta(x, 0, t)$, then*

$$x = \frac{\int_{[\alpha(x,0,t),\, \beta(x,0,t)]} (a + tu_0(a))\mathrm{d}\mu_0(a)}{\mu_0([\alpha(x, 0, t),\, \beta(x, 0, t)])}\ .$$

Else

$$x = \alpha(x, 0, t) + tu_0(\alpha(x, 0, t)) = \beta(x, 0, t) + tu_0(\beta(x, 0, t))\ .$$

4. *$\beta(x, 0, t) + tu_0(\beta(x, 0, t)) \leq x \leq \alpha(x, 0, t) + tu_0(\alpha(x, 0, t))$.*
If $\mu_0([\alpha(x, 0, t),\, y]) > 0$ and $\mu_0(]y,\, \beta(x, 0, t)]) > 0$, then

$$\frac{\int_{]y,\beta(x,0,t)]} (a + tu_0(a))\mathrm{d}\mu_0(a)}{\mu_0(]y, \beta(x, 0, t)])} \leq x \leq \frac{\int_{[\alpha(x,0,t),y]} (a + tu_0(a))\mathrm{d}\mu_0(a)}{\mu_0([\alpha(x, 0, t), y])}\ .$$

5. *The function $[0, t] \ni x \longmapsto \phi_{0,s}(\alpha(x, 0, t))$ is concave. It is a straight line if and only if $x = \alpha(x, 0, t) + tu_0(\alpha(x, 0, t))$.*
The function $[0, t] \ni x \longmapsto \phi_{0,s}(\alpha(x, 0, t))$ is convex. It is a straight line if and only if $x = \beta(x, 0, t) + tu_0(\beta(x, 0, t))$.
6. *For any compact subset $K = [a, b]\times[0, T]$, consider $A_T = \alpha(\phi_{s,T}(a), s, T)$, $B_T = \beta(\phi_{s,T}(b), s, T)$ and the probability $\mu_s^K = \frac{\mathrm{I}_{[A_T,B_T]}}{\mu_s([A_T,B_T])}\mu_s$. The sticky particle dynamics induced by (μ_s^K, u_s), during time interval $[0, T]$, is characterized by the restriction of the function $(y, t) \mapsto \phi_{s,t}(y)$ on $[A_T, B_T] \times [0, T]$.*

Ref
7. O. Moutsinga. Burgers' equation and the sticky particles model. *J. Math. Phys*, 53:1–12, 2012.

Notes

The latter means that the restriction of flow on a compact subset of space-time does not depend of the whole matter, but only on the restriction of the matter (distribution) on a compact subset of space states.

Remark that if $x = \alpha(x,0,t) + tu_0(\alpha(x,0,t)) = \beta(x,0,t) + tu_0(\beta(x,0,t))$, then the graphs $[0,t] \ni s \mapsto \phi_{0,s}(\alpha(x,0,t))$, $\phi_{0,s}(\beta(x,0,t))$ draw a delta-shock, well known in the literature (Figure 1). Otherwise, these graphs draw a kind of *butterfly-shock with foded wings* (Figure 2)

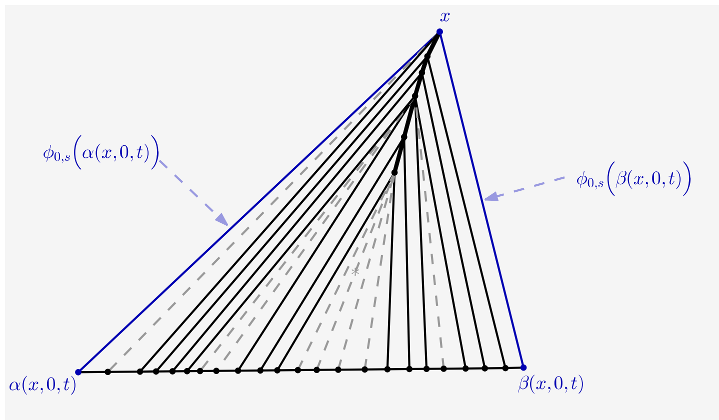

Figure 1: The blue line on the left (resp right) of the middle shock wave represents the trajectory of the particle which started from position $\alpha(x,0,t)$ (resp $\beta(x,0,t)$). It is trajectory $[0,t] \ni s \mapsto \phi_{0,s}(\alpha(x,0,t))$ (resp $[0,t] \ni s \mapsto \phi_{0,s}(\beta(x,0,t))$)

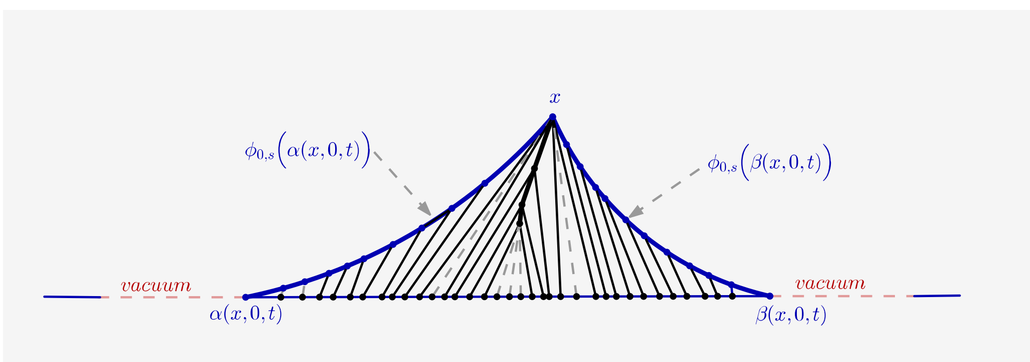

Figure 2: The blue curve on the left (resp right) represent the trajectory of particle which start at the position $\alpha(x,0,t)$ (resp $\beta(x,0,t)$) whch is the trajectory of $[0,t] \ni s \mapsto \phi_{0,s}(\alpha(x,0,t))$ (resp $[0,t] \ni s \mapsto \phi_{0,s}(\beta(x,0,t))$)

What about the velocity?

c) *Proposition (Flow derivative)*

1. *For all y, s, the function $t \mapsto \phi_{s,t}(y)$ has everywhere left hand derivatives. It has everywhere right hand derivatives, except when $\phi_{s,t}^{-1}(\phi_{s,t}(y)) =: [a, b]$ with $\mu_s([a, b]) = 0$ and $a < b$. Now and after, the notation $\frac{\partial}{\partial t}\phi_{s,t}(y)$ stands for the right hand derivative.*
2. *There exists a function $(x,t) \mapsto u_t(x)$ such that $\frac{\partial}{\partial t}\phi_{0,t}(y) = u_t(\phi_{0,t}(y))$ everywhere the right derivative exists.*

3. *For any compact subset* $K = [a, b] \times [0, T]$, *consider* $A_T = \alpha(\phi_{s,T}(a), s, T)$, $B_T = \beta(\phi_{s,T}(a), s, T)$ *and the probability* $\mu_s^K = \frac{\mathrm{I}_{[A_T, B_T]}}{\mu_s([A_T, B_T])}\mu_s$. *If the right hand derivative exits for* $(x, y) \in K$, *the using the conditional expectation under* μ_s^K, *we have*

$$\frac{\partial}{\partial t}\phi_{s,t}(y) = \mathbb{E}_{\mu_s^K}\left[u_s | \phi_{s,t}(\cdot) = \phi_{s,t}(y)\right]. \tag{7}$$

We call *a cluster at time t* all interval of the type $[\alpha(x, 0, t), \beta(x, 0, t)]$. The last assertion of proposition 2.1 implies an important property on the velocity of a cluster.

d) Corollary

1. *If* $[\alpha(x, 0, t), \beta(x, 0, t)]$ *has positive mass, then*

$$u_t(x) = \frac{\int_{[\alpha(x,0,t),\, \beta(x,0,t)]} u_0(a) \mathrm{d}\mu_0(a)}{\mu_0([\alpha(x, 0, t),\, \beta(x, 0, t)])}.$$

   If $\alpha(x, 0, t) = \beta(x, 0, t)$, *then* $u_t(x) = u_0(\alpha(x, 0, t))$.
   Else $u_t(x)$ *is not (well) defined.*

2. $u_0(\beta(x, 0, t)) \leq u_t(x) \leq u_0(\alpha(x, 0, t))$.
   If $\mu_0([\alpha(x, 0, t), y]) > 0$ *and* $\mu_0(]y, \beta(x, 0, t)]) > 0$, *then*

$$\frac{\int_{]y,\beta(x,0,t)]} u_0(a) \mathrm{d}\mu_0(a)}{\mu_0(]y, \beta(x, 0, t)])} \leq u_t(x) \leq \frac{\int_{[\alpha(x,0,t),y]} u_0(a) \mathrm{d}\mu_0(a)}{\mu_0([\alpha(x, 0, t), y])}.$$

3. *If* $\alpha(x, 0, t) \in \mathcal{S}_-$ *(resp.* $\beta(x, 0, t) \in \mathcal{S}_+$ *), then* $u_t^-(x) = u_0(\alpha(x, 0, t))$. *(resp.* $u_t^+(x) = u_0(\beta(x, 0, t))$*).*

4. *If* $u_0(\alpha(x, 0, t)) = u_t(x)$, *then* $\mu_0\left(]\alpha(x, 0, t),\, \beta(x, 0, t)]\right) = 0$ *and* $x = \alpha(x, 0, t) + tu_0(\alpha(x, 0, t)) = \beta(x, 0, t) + tu_0(\beta(x, 0, t))$.

5. *If* $u_0(\beta(x, 0, t)) = u_t(x)$, *then* $\mu_0\left([\alpha(x, 0, t),\, \beta(x, 0, t)[\right) = 0$ *and* $x = \alpha(x, 0, t) + tu_0(\alpha(x, 0, t)) = \beta(x, 0, t) + tu_0(\beta(x, 0, t))$.

6. *For all* $t \geq 0$, *we have* $u_t(x) = o(x)$ *as* $|x| \to +\infty$. *For all* $t > 0$, *if* $\alpha(x, 0, t) \in \mathcal{S}_-$ *(resp.* $\beta(x, 0, t) \in \mathcal{S}_+$ *), then* $\lim\limits_{\substack{y \to x \\ y < x}} u_t(y) = u_t^-(x) =$

$$u_0(\alpha(x, 0, t)) \text{ (resp. } \lim_{\substack{y \to x \\ y > x}} u_t(y) = u_t^+(x) = u_0(\beta(x, 0, t))).$$

e) Markov and martingale properties

Let (μ_0, u_0) be as in theorem 2.1. On abstract measure space $(\Omega, \mathcal{F}, P)$ we define a measurable function $X_0 : \Omega \longrightarrow \mathbb{R}$ with image-measure $P \circ X_0^{-1} =$

Notes

μ_0. In practice, $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_0)$ and X_0 is the identity function. For all $t \geq 0$, we set $X_t = \phi_{0,t}(X_0)$. As a consequence of theorem 2.1, we have the following :

f) Proposition (Markov and martingale property)

1. $\forall s, t$, *we have*

$$X_{s+t} = \phi_{s,t}(X_s) \tag{8}$$

2. If u_0 *is* μ_0 *integrable, then under the measure* μ_0 *(or* P*) :*

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = \mathbb{E}[u_0(X_0)|X_t] = u_t(X_t). \tag{9}$$

Else, for any compact $K = [a, b] \times [0, t+s]$, *if* $\phi_{0,t+s}(a) \leq X_{t+s} \leq \phi_{0,t+s}(b)$, *then under the conditional probability* μ_0^K, *we get (9).*

3. If u_0 *is* μ_0 *integrable, then under the measure* μ_0 *(or* P*) :*

$$u_{t+s}(X_{t+s}) = \mathbb{E}[u_t(X_t)|\mathcal{F}_{t+s}] \,, \quad \textit{with} \quad \mathcal{F}_t = \sigma(X_u, u \geq t). \tag{10}$$

Else, for any compact $K = [a, b] \times [0, t]$, $\phi_{0,t+s}(a) \leq X_{t+s} \leq \phi_{0,t+s}(b)$, *then we get (10) under the probability* μ_0^K *(or under the conditional probability knowing* $\alpha(\phi_{0,t+s}(a),\, 0,\, t+s) \leq X_0 \leq \beta(\phi_{0,t+s}(a),\, 0,\, t+s)$.

III. Turbulence

In this section, inspired by a preocupation from [4], we study the sticky particles dynamics from the point of view of turbulence. Generalizing the results of [5], we get a class of Markov processes solution (2). The velocities fields are backward semi-martingales.

a) Flow, delta-shock and butterfly-shock

In [5], was defined the first turbulence (or shock) time of the particle initial position a :

$$\tau(a) = \inf\left\{t :\, u^-(\phi_{0,t}(a), t) \neq u^+(\phi_{0,t}(a), t)\right\}. \tag{11}$$

Let X_0 be of image-measure μ_0. Define $\gamma = \tau(X_0)$ and the cluster $[Z_0^3,\, Z_0^4] = [\alpha(X_\gamma,\, 0,\, \gamma),\, \beta(X_\gamma,\, 0,\, \gamma)]$ in which belongs X_0 at time γ. The turbulent interval $[Z_0^1,\, Z_0^2]$ is defined as the greatest interval containing X_0 on which τ is constant. It was shown in [5] that the velocities of these variables are semi-martingales, when $\mu_0 = \lambda$ the Lebesgue measure. The same result was obtained for the combination $Z_0^5 = Z_0^5 \mathbb{1}_A + Z_0^5 \mathbb{1}_{A^c}$, with the event A : " the particle enters in the shock from the left ". The interesting variable Z_0^5 was introduced [4] in order to study the Burgers turbulence.

Ref

5. F. Eyi Obiang, O. Moutsinga, and F. Nzissila. Backward semi-martingale into burgers turbulence. *J. Math. Phys.*, 62:1–12, 2021.

Our goal is to generalize the results of [5] to any non-negative measure μ_0 and any function u_0 with *negative jumps* (w.r.t. μ_0). For $i = 1, 2, 3, 4, 5,$ we consider the process $t \longmapsto Z^i_t = \phi_{0,t}(Z^i_0)$. But one could have other preoccupations than the above event A of [5]. We are led to defined the process of more of more general form $t \longmapsto Z_t = Z^1_t \mathbb{1}_{A^1} + Z^2_t \mathbb{1}_{A^2} + Z^3_t \mathbb{1}_{A^3} + Z^4_t \mathbb{1}_{A^4}$, with the help of any partition A^1, A^2, A^3, A^4 of Ω, events of $\sigma(X_0)$. Following the implication the application of the Z^i_0's, we have fifteen $(2^4 - 1)$ types of processes . (If $A^i = \Omega$; then $Z = Z^i$).

i. *Proposition (Random butterfly-shock)*

1. *Let Z stand independently for Z^1, Z^2, Z^3 or Z^4.*

$$\forall\, t, s \geq 0, \quad Z_{s+t} = \phi_{s,t}(Z_s)\ , \quad \frac{\mathrm{d}}{\mathrm{d}t} Z_t = u(Z_t, t)\ .$$

2. $\tau(Z^1_0) = \tau(Z^2_0) = \tau(X_0) = \gamma$ *and*

$$\forall\, t \leq \gamma, \quad Z^1_t = Z^1_0 + t u_0(Z^1_0) \leq X_t = X_0 + t u_0(X_0) \leq Z^1_t = Z^2_0 + t u_0(Z^2_0)\ ,$$

$$\forall\, t \geq \gamma, \qquad X_t = Z^1_t = Z^2_t = Z^3_t = Z^4_t\ .$$

3. $\tau(Z^3_0) \leq \gamma$ *and* $\tau(Z^4_0) \leq \gamma$.

   $[0,\, \gamma] \ni\, t \longmapsto Z^3_t$ *is concave and* $[0,\, \gamma] \ni\, t \longmapsto Z^3_t$ *is convex.*

$$\forall\, t \leq \tau(Z^3_0), \quad Z^3_t = Z^3_0 + t u_0(Z^3_0)$$

$$\forall\, t \leq \tau(Z^4_0), \qquad Z^4_t = Z^4_0 + t u_0(Z^4_0);.$$

The segment $[Z^1_0,\, Z^2_0]$ and the paths $[0,\, \gamma] \ni\, t \longmapsto Z^1_t,\, Z^2_t$ draw a *prime delta-shock* (so called in [5] because of the first shock time of turbulence).

If $\tau(Z^3_0) = \tau(Z^4_0) = \gamma$, then the paths $[0,\, \gamma] \ni\, t \longmapsto Z^3_t,\, Z^4_t$ are linear; and the draw, with the segment $[Z^3_0,\, Z^4_0]$, *delta shock* (well known in the literature) (see figure 1).

If $\tau(Z^3_0) < \gamma$ (resp. $\tau(Z^4_0 < \gamma)$), then the path $[0,\, \gamma] \ni\longmapsto Z^3_t$ (resp. $[0,\, \gamma] \ni\longmapsto Z^3_t$) is linear; this can occur o,ly when $Z^3_0 \notin \mathcal{S}^-$ (resp. $Z^4_0 \notin \mathcal{S}^+$). If $\max\left(\tau(Z^3_0),\, \tau(Z^4_0)\right) < \gamma$, then the paths $[0,\, \gamma] \ni\longmapsto Z^3_t,\, Z^3_t$ draw, not a delta-shock, but *butterfly-shock with folded wings* (see Figure 3).

Ref

5. F. Eyi Obiang, O. Moutsinga, and F. Nzissila. Backward semi-martingale into burgers turbulence. *J. Math. Phys.*, 62:1–12, 2021.

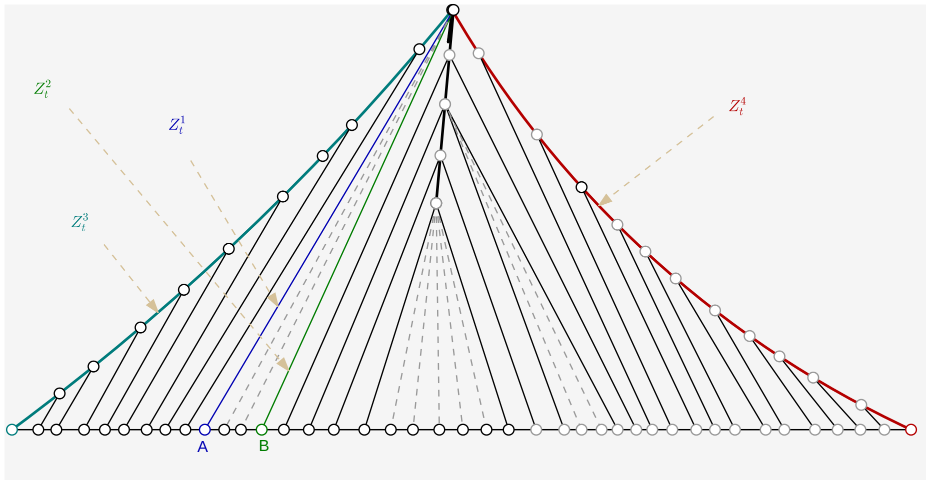

Figure 3: Delta-shock and butterfly-shock

b) *Velocity process as semi-martingale*

i. *Proposition*

1. *$t \longmapsto u(Z_t, t)\mathbb{1}_{t<\gamma}$ is bounded variational process adapted to the natural non increasing filtration $\mathcal{F}^X$ of X.*
2. *For all t, $u(Z_t, t) = [u(Z_t, t) - u_0(X_0)]\mathbb{1}_{t<\gamma} + M_t$, with $M_t = E\left[u_0(X_0)|\mathcal{F}_t^X\right]$. Hence, $t \longmapsto u(Z_t, t)$ is a backward càdlàg semi-martingale of $\mathcal{F}^X$.*
3. *If $\gamma = \tau(Z_0)$, then for all t, $u(Z_t, t) = [u_0(Z_0) - M_0]\mathbb{1}_{t<\gamma} + M_t$, with $M_t = E\left[u_0(X_0)|\mathcal{F}_t^Z\right]$. Hence, $t \longmapsto u(Z_t, t)$ is a backward càdlàg semi-martingale of $\mathcal{F}^Z$.*
4. *If γ is an optional time of $\mathcal{F}^Z$, then $t \longmapsto u(Z_t, t)$ is a backward càdlàg semi-martingale of the completed filtration $\overline{\mathcal{F}^Z}$. Moreover $t \longmapsto u(Z_t, t) - [u(Z_t, t) - M_\gamma^-]\mathbb{1}_{t<\gamma}$*

We recall that for any non increasing filtration $\mathcal{F}$, the filtration $\overline{\mathcal{F}}$ is defined by $\overline{\mathcal{F}_t} = \sigma(F_t \cup \mathcal{N})$, where $\mathcal{N}$ is the set of negligible events of $\mathcal{F}_0$.

Before the proof, we recall some properties well known in the theory of stochastic processes.

c) *Lemma*

Let a process Z be adapted to a non increasing filtration $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$. Let Γ be an optional time with respect to $\mathcal{G}$, i.e. for all $t \geq 0$, the event $\{\Gamma > t\} \in \mathcal{G}_t$. The following holds.

1. *The set $\mathcal{G}_\Gamma := \{A \in \mathcal{G}_0 \ : A \cap \{\Gamma > t\} \in \mathcal{G}_t\}$ is a sigma-algebra.*
2. *If all the paths of Z are either continuous on the right or on the left, then the r.v. $Z_\Gamma \mathbb{1}_{\Gamma<\infty}$ is $\mathcal{G}_\Gamma$ measurable.*
3. *Suppose that $\mathcal{G}$ is continuous on the right; that is, for all t, $\mathcal{G}_t = \sigma\left(\underset{s>t}{\cup}\mathcal{G}_s\right)$. If Z is a backward martingale with respect to $\mathcal{G}$, then for all*

t, the right hand and left hand limits Z_{t^+}, Z_{t^-} exist a.s. Moreover, the process $t \mapsto Z_{(\Gamma\vee t)^+} - \Delta_\Gamma \mathbb{1}_{\Gamma>t}$ is a backward martingale with respect to the completed filtration $\overline{\mathcal{G}}$, with $\Delta_\Gamma = Z_{\Gamma^+} - Z_{\Gamma^-}$.

d) Lemma

1. *If a process Z is such that $Z_{s+t} = \phi_{s,t}(Z_s)$ for all $t, s \geq 0$, then $\tau(Z_0) =: \Gamma$ is an optional time with respect to the natural non increasing filtration $\mathcal{F}^Z$ of Z. Moreover, $\mathcal{F}_0^Z = \mathcal{F}_\Gamma^Z$.*
2. *Suppose that $\{\Gamma \leq t\} \in \mathcal{F}^Z \cap \mathcal{F}^{Z'}$ for some $t \geq 0$. If $Z'_t \mathbb{1}_{\Gamma\leq t} = Z_t \mathbb{1}_{\Gamma\leq t}$, then $\mathrm{E}[F|Z'_t]\mathbb{1}_{\Gamma\leq t} = \mathrm{E}[F|Z_t]\mathbb{1}_{\Gamma\leq t}$ for all integrable r.v. F.*

The second assertion is satisfied by $(Z, Z') = (X, Z^1)$ and $(Z, Z') = (X, Z^2)$, with $\Gamma = \gamma$. Both Z^3 and Z^4 satisfy only the first assertion.

Proof. We begin with the first assertion. $u^-(\cdot, t), u^+(\cdot, t)$ are Borel functions and it is well known that if u is discontinuous in (Z_t, t), it is also discontinuous in $(Z_{t+s}, t+s)$. Then,

$$\{\Gamma \leq t\} = \{u^-(Z_t, t) \neq u^+(Z_t, t)\} \cup [\{u^-(Z_t, t) = u^+(Z_t, t)\} \cap \{\Gamma = t\}] \, .$$

Since

$$\{u^-(Z_t, t) = u^+(Z_t, t)\} \cap \{\Gamma = t\} = \{u^-(Z_t, t) = u^+(Z_t, t)\} \cap \left[\bigcap_{n\geq 1} \{u^-(Z_{t+1/n}, t+1/n) \neq u^+(Z_{t+1/n}, t+1/n)\}\right],$$

the proof of the first assertion is done.
Remark that $Z_{t+1/n} = \phi_{t,1/n}(Z_t)$. So $\{\Gamma \leq t\} = Z_t^{-1}(A_t)$, with

$$A_t = \{u^-(\cdot, t) \neq u^+(\cdot, t)\} \cup \Bigg(\{u^-(\cdot, t) = u^+(\cdot, t)\} \cap \left[\bigcap_{n\geq 1} \{u^-(\phi_{t,1/n}, t+1/n) \neq u^+(\phi_{t,1/n}, t+1/n)\}\right] \Bigg)$$

Now we show that $\mathcal{F}_0^Z = \mathcal{F}_\Gamma^Z$. First remark that if $\{b\} \neq \phi_{0,t}^{-1}(\phi_{0,t}(b))$, then $\tau(b) \leq t$. Thus for all Borel subset B and $t \geq 0$, we have $B \cap \{\tau > t\} = \phi_{0,t}^{-1}(\phi_{0,t}(B)) \cap \{\tau > t\}$ and

$$Z_0^{-1}(B) \cap \{\tau(Z_0) > t\} = Z_t^{-1}(\phi_{0,t}(B)) \cap \{\tau(Z_0) > t\} \, ,$$

$$Z_0^{-1}(B) \cap \{\Gamma > t\} = Z_t^{-1}(\phi_{0,t}(B)) \cap \{\Gamma > t\} \in \mathcal{F}_t^Z \, .$$

This means that $Z_0^{-1}(B) \in \mathcal{F}_\Gamma^Z$.

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For the second assertion, since $Z_t\mathbb{1}_{\gamma\leq t} = Z'_t\mathbb{1}_{\Gamma\leq t}$, it is easy to see that $\mathrm{E}[F|Z'_t]\mathbb{1}_{\Gamma\leq t}$ is $\sigma(Z'_t)\cap\sigma(Z_t)$ measurable; for all bounded Borel function h,

$$\mathrm{E}\big(h(Z_t)\mathrm{E}[F|Z'_t]\mathbb{1}_{\Gamma\leq t}\big) = \mathrm{E}\big(h(Z'_t)\mathrm{E}[F|Z'_t]\mathbb{1}_{\Gamma\leq t}\big) = \mathrm{E}\big(h(Z'_t)F\mathbb{1}_{\Gamma\leq t}\big)$$

$$= \mathrm{E}\big(h(Z_t)F\mathbb{1}_{\Gamma\leq t}\big) = \mathrm{E}\big(h(Z_t)\mathrm{E}[F|Z_t]\mathbb{1}_{\Gamma\leq t}\big).$$

Hence, $\mathrm{E}[F|Z'_t]\mathbb{1}_{\Gamma\leq t} = \mathrm{E}[F|Z_t]\mathbb{1}_{\Gamma\leq t}$ a.s.

Proof of proposition 3.2

1) The restriction $[0,\gamma[\ni t\longmapsto u(Z_t,t)$ is monotone. Thus, the process $\mathbb{R}_+\ni t\longmapsto u(Z_t,t)\mathbb{1}_{t<\gamma}$ is a bounded variational process. It is adapted to $\mathcal{F}^X$ since γ is an optional time of this filtration.

2) We have $\mathcal{F}_0^X = \mathcal{F}_\gamma^X$. So for all t, the r.v. $u_0(X_0)\mathbb{1}_{t<\gamma}$ is $\mathcal{F}_t^X$-measurable. Since $\mathcal{F}_t^X = \sigma(X_t)$, we get

$$\begin{aligned} u(Z_t,t)\mathbb{1}_{\gamma\leq t} &= u(X_t,t)\mathbb{1}_{\gamma\leq t} = \underbrace{E\left[u_0(X_0)|X_t\right]}_{M_t}\mathbb{1}_{\gamma\leq t} \\ &= M_t - E\left[u_0(X_0)\mathbb{1}_{t<\gamma}|X_t\right] = M_t - u_0(X_0)\mathbb{1}_{t<\gamma}. \end{aligned}$$

Then for all t, $u(Z_t,t) = \left[u(Z_t,t)-u_0(X_0)\right]\mathbb{1}_{t<\gamma} + M_t$.

3) Same proof as previous, using the fact that $\mathcal{F}_0^Z = \mathcal{F}_\gamma^Z$ and $E\left[u_0(X_0)|X_t\right]\mathbb{1}_{\gamma\leq t} = E\left[u_0(X_0)|Z_t\right]\mathbb{1}_{\gamma\leq t}$ (lemma 3.4)

4) Simple application of lemma 3.3. For all t,

$$\begin{aligned} u(Z_t,t)\mathbb{1}_{\gamma\leq t} &= u(X_t,t)\mathbb{1}_{\gamma\leq t} = E\left[u_0(X_0)|X_t\right]\mathbb{1}_{\gamma\leq t} = \underbrace{E\left[u_0(X_0)|Z_t\right]}_{M_t}\mathbb{1}_{\gamma\leq t} \\ &= M_{\gamma\vee t} - \Delta_\gamma\mathbb{1}_{t<\gamma} - M_\gamma^-\mathbb{1}_{t<\gamma} \end{aligned}$$

with $\Delta_\gamma = M_\gamma - M_\gamma^-$.

Remark that assertion 3) is a consequence of 4). Indeed, if $\gamma = \tau(Z_0)$, then $\mathcal{F}_0^Z = \mathcal{F}_\gamma^Z$ (lemma 3.4). So M_γ^- and M_γ are $\mathcal{F}_\gamma^Z$ measurable and the processes $t\longmapsto M_{\gamma\vee t}$, $M_\gamma\mathbb{1}_{t<\gamma}$, $\Delta_\gamma\mathbb{1}_{t<\gamma}$ are adapted to $\mathcal{F}^Z$. Thus, the process $M_{\gamma\vee t} - \Delta_\gamma\mathbb{1}_{t<\gamma}$ is a backward martingale of $\mathcal{F}^Z$. Hence the process $t\longmapsto u(Z_{t,},t)$ is a semi-martingale of $\mathcal{F}^Z$.

In fact, $, M_\gamma\mathbb{1}_{t<\gamma} =, M_0\mathbb{1}_{t<\gamma} =, M_\gamma^-\mathbb{1}_{t<\gamma}$. So the martingale part is M.

Now we precise, under more general assumptions, when the velocity of turbulence is a martingale.

c) *Martingales and soft turbulence*

In this part, we show that the martingality of the velocity turbulence implies that all mass of any turbulent interval is concentrated in at most one point (*single turbulent point*). Let $\mathcal{T}$ be the set of turbulent intervals which are not reduced to single points.

d) Corollary (Turbulence martingales and prime-delta-shocks)

1. *The process* $t \longmapsto u(Z_t, t)$ *is a martingale of* $\mathcal{F}^X$ *iff a.s.* $Z \equiv X$.
2. *Suppose that* γ *is an optional time of* $\mathcal{F}^Z$ *(which is effectively the case when* $\mathcal{S}$ *is an interval). The process* $t \longmapsto u(Z_t, t)$ *is a martingale of* $\mathcal{F}^X$ *iff a.s.* $Z_0 = E[X_0|Z_0]$. *Furthermore, if* $A_i = \Omega$, *then a.s.* $Z \equiv Z^i \equiv X$.

The following describes the turbulent intervals and clusters when the velocity of their borders are martingales.

e) Proposition

If $Z_0 = X_0$ *a.s., then* $\mathcal{T}$ *is at most countable and the interior of all turbulent interval is a vacuum.*

1. *Case* $Z \equiv Z^3$ $(A_3 = \Omega)$: *we have a.e.* $Z^3 \equiv Z^1 \equiv X$ *and* $Z^2 \equiv Z^4$.

$$\forall [\alpha, \beta] \in \mathcal{T}, \ \mu_0(]\alpha, \beta]) = 0; \qquad P(Z_0^3 \neq (Z_0^4) = \sum_{[\alpha,\beta]\in\mathcal{T}} \mu_0(\{\alpha\})$$

2. *Case* $Z \equiv Z^4$ $(A_4 = \Omega)$: *we have a.e.* $Z^4 \equiv Z^2 \equiv X$ *and* $Z^1 \equiv Z^3$.

$$\forall [\alpha, \beta] \in \mathcal{T}, \ \mu_0([\alpha, \beta[) = 0; \qquad P(Z_0^3 \neq (Z_0^4) = \sum_{[\alpha,\beta]\in\mathcal{T}} \mu_0(\{\beta\})$$

any turbulent interval is also a cluster at turbulent time.

3. *Case* $A_1 = A_2 = \emptyset$: *we have a.e.* $Z^1 \mathbb{1}_{A_3} \equiv Z^2 \mathbb{1}_{A_3}$ *and* $Z^2 \mathbb{1}_{A_4} \equiv Z^4{}_{A_4}$.

$$\forall [\alpha, \beta] \in \mathcal{T}, \ \mu_0([\alpha, \beta]) = \mu_0(\{\alpha\}) + \mu_0(\{\beta\});$$

$$P(Z_0^3 \neq (Z_0^4) = \sum_{[\alpha,\beta]\in\mathcal{T}} [\mu_0(\{\alpha\}) + \mu_0(\{\beta\})].$$

4. *Case* $Z \equiv Z^1$ $(A_1 = \Omega)$: *we have a.e.* $Z^3 \equiv Z^1 \equiv X$ *and* $Z^2 \equiv Z^4$.

$$\forall [\alpha, \beta] \in \mathcal{T}, \ \mu_0(]\alpha, \beta]) = 0; \qquad P(Z_0^1 \neq (Z_0^2) = \sum_{[\alpha,\beta]\in\mathcal{T}} \mu_0(\{\alpha\})$$

5. *Case* $Z \equiv Z^2$ $(A_2 = \Omega)$: *we have a.e.* $Z^4 \equiv Z^2 \equiv X$ *and* $Z^1 \equiv Z^3$.

$$\forall [\alpha, \beta] \in \mathcal{T}, \ \mu_0([\alpha, \beta[) = 0; \qquad P(Z_0^1 \neq (Z_0^2) = \sum_{[\alpha,\beta]\in\mathcal{T}} \mu_0(\{\beta\})$$

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Proof: Let us study each semi-martingale.

1. For $Z = Z^3$: A necessary condition is that $E\left[u_0(Z_0^3)\right] = E\left[u(Z_\gamma^3)\right] = E\left[u(X_\gamma, \gamma)\right] = E\left[u_0(X_0)\right]$. But $X_\gamma = X_0 + \gamma u_0(X_0) \leq Z_0 + \gamma u_0(Z_0)$ and $Z_0^3 \mathbb{1}_{\gamma>0} = X_0 \mathbb{1}_{\gamma>0}$. Then $\gamma^{-1}(X_0 - Z_0^3)\mathbb{1}_{\gamma>0} = u_0(Z_0^3) - u_0(X_0) \geq 0$ and $E\left[\gamma^{-1}(X_0 - Z_0^3)\mathbb{1}_{\gamma>0}\right] = 0$. So $X_0 = Z_0^3 = Z_0^1$ a.e. Thus $X \equiv Z^3 \equiv Z^1$ a.e.

   In the other hand, we have a.e. $u_0(Z_0^3) \geq u(X_\gamma, \gamma)$ and $E\left[u_0(Z_0^3) - u(X_\gamma, \gamma)\right] = 0$. So $u_0(Z_0^3) = u(X_\gamma, \gamma)$ a.e.
   Now we show that $Z_0^2 \equiv Z_0^4$. If $Z_0^3 = Z_0^4$, then $Z_0^2 = Z_0^4$. If $Z_0^3 \neq Z_0^4$, then $\exists\, \alpha < \beta$ s.t. $[Z_0^3, Z_0^4] = [\alpha, \beta]$. If moreover $\mu_0([\alpha, \beta]) = 0$, then $[\alpha, \beta] \in \mathcal{T}$ and $Z_0^2 = Z_0^4 = \beta$. If $\mu_0([\alpha, \beta]) > 0$, then

   $$u_0(\alpha) = u_0(Z_0) = u(X_\gamma, \gamma) = \frac{\int_{[\alpha, \beta]} u_0(x)\mu_0(\mathrm{d}x)}{\mu_0([\alpha, \beta])}.$$

   From corollary 2.3, we get $\mu_0([\alpha, \beta]) = 0$ and $[\alpha, \beta] \in \mathcal{T}$. And again $Z_0^2 = Z_0^4 = \beta$. Thus a.e. : $Z_0^2 = Z_0^4$ and $Z_0^2 \equiv Z_0^4$.

   Moreover $\{Z_0^3 \neq Z_0^4\} \subset \bigcup_{[\alpha, \beta] \in \mathcal{T}} \{\alpha \leq X_0 \leq \beta\}$. Conversely, if $[\alpha, \beta] \in \mathcal{T}$, $\exists\, [\alpha', \beta'] = [Z_0^3, Z_0^4] \supset [\alpha, \beta]$, with $\mu_0(]\alpha', \beta']) = 0$. This implies $[\alpha, \beta] = [\alpha', \beta']$. So $\{Z_0^3 \neq Z_0^4\} = \bigcup_{[\alpha, \beta] \in \mathcal{T}} \{\alpha \leq X_0 \leq \beta\}$ and $\mu_0(]\alpha, \beta]) = 0$ for all $[\alpha, \beta] \in \mathcal{T}$. But the set of vacuums is at most countable. So is $\mathcal{T}$. We then get the result $P(\{Z_0^3 \neq Z_0^4\})$.

2. For $Z = Z^4$: Analogous to previous case.

3. For $Z = Y$: The process $t \mapsto u(Y_t, t)$ is a martingale if only if its bounded variational part vanishes :

   $$a.e\ \left[u(Y_t, t) - u(Y_\gamma, \gamma)\right] \mathbb{1}_{t<\gamma},\ \forall\, t.$$

   This equivalent to $u_0(Y_0) = u(Y_t, t) = u(Y_\gamma, \gamma) = u(X_\gamma, \gamma)$ for all $t < \gamma$. Then a.e.

   $$\tau(Y_0) = \gamma, \qquad u_0(Y_0) = u(X_\gamma, \gamma), \qquad Y_0 + t\gamma u_0(Y_0) = X_0 + t\gamma u_0(X_0).$$

   If $Z_0^3 \neq Z_0^4$, then $\exists \alpha < \beta$ s.t. $[Z_0^3, Z_0^4] = [\alpha, \beta]$ and $Y_0 = \alpha$ or $Y_0 = \beta$. If moreover $\mu_0([\alpha, \beta]) > 0$, then

   $$u_0(\alpha) = u_0(Y_0) = u(X_\gamma, \gamma) = \frac{\int_{[\alpha, \beta]} u_0(x)\mu_0(\mathrm{d}x)}{\mu_0([\alpha, \beta])}.$$

In this case, If $Y_0 = \alpha$, then from the corollary 2.3, we get $\mu_0([\alpha, \beta]) = 0$ and $Y_0 = Z_0^1 = Z_0^3 = \alpha$. Then $P(Y_0 = \alpha \neq X_0) = \mu_0(]\alpha, \beta]) = 0$. In the same way, if $Y_0 = \beta$, we get $Y_0 = Z_0^2 = Z_0^4 = \beta$ and $P(Y_0 = \beta \neq X_0) = \mu_0([\alpha, \beta[) = 0$. In any case, $[\alpha, \beta] \in \mathcal{T}$ and $P(Y_0 \neq X_0,\, \alpha \leq X_0 \leq \beta) = 0$.

We conclude that $\mathcal{T}$ is at most countable and

$$\begin{array}{rcl} \forall\, [\alpha, \beta] \in \mathcal{T} & , & \mu_0([\alpha, \beta]) = \max\left(\mu_0(\{\alpha\}),\, \mu_0(\{\beta\})\right); \\ \left\{Z_0^3 \neq Z_0^4\right\} & = & \bigcup_{[\alpha, \beta] \in \mathcal{T}} \{\alpha \leq X_0 \leq \beta\} \end{array}$$

This give the results.

4. For $Z = Z^1$: $E[u_0(Z_0^1)] = E[u(Z_\gamma^1, \gamma)] = E[u_0(X_0)]$. But $u_0(Z_0^1) - u_0(X_0) = \gamma^{-1}(X_0 - Z_0^1)\mathbb{1}_{0<\gamma} \geq 0$. Then a.e. : $Z_0^1 = X_0$ and $Z_0^1 \equiv X_0$. Furthermore,

$$\left\{Z_0^2 \neq Z_0^1\right\} = \bigcup_{\substack{[\alpha, \beta] \in \mathcal{T} \\ \alpha < \beta}} \{\alpha \leq X_0 \leq \beta\}$$

and for all $[\alpha, \beta] \in \mathcal{T}$, we have

$$\mu_0(]\alpha, \beta]) = P(\alpha < X_0 \leq \beta) \leq P(Z_0^1 \neq X_0) = 0.$$

So $\mathcal{T}$ is at most countable and we get the result for $P(Z_0^2 \neq Z_0^1)$.

5. $Z = Z^4$: Analogous to previous case.

Now, we are interested in the conservation laws.

IV. Conservation Laws

In this section, we investigate if a process of type of Z can provide solutions to the scalar conservation law

$$\partial_t M + \partial_x \left(A(M)\right) = 0 \tag{12}$$

and to the pressure-less gas system

$$\begin{cases} \partial_t(\mu) + \partial_x(u\mu) = 0 \\ \partial_t(u\mu) + \partial_x(u^2\mu) = 0 \\ \mu_t \to \mu_0, \qquad u(\cdot, t)\mu_t \to u_0\mu_0 \; weakly \; as \; t \to 0 \end{cases} \tag{13}$$

It is well known, from the sticky particles model, that the mass distribution μ_t of the matter and their velocity functions $u(\cdot, t)$ provide a weak solution (in the sense of distributions) to the system (13). The first line of

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(13) is usually called *conservation law* of mass and the second is a conservation law of *momentum*. Moreover, the couple c.d.f and the momentum function provide an entropy solution to (12). Precisely, $\forall (x, t) \in \mathbb{R} \times \mathbb{R}_+$, $M(x,t) = \mu_t(]-\infty, x])$

$$\forall m \in (0,1), \qquad A(m) = \int_0^m u_0(M_0^{-1}(z))dz, \tag{14}$$

where $M_0 = M(\cdot, 0)$. The equation (12) is *conservation law of mass and momentum*. Can we have the same thing for the function $N \ : (x,t) \mapsto P(Z_t \leq x)$ with the same flux (14) ?

a) Proposition

Consider the real function

$$v_0 \ : \ a \mapsto \frac{\int_{Z_0=a} u_0(X_0)\mathrm{d}P}{P(Z_0 = a)} = E\left[u_0(X_0)|Z_0 = a\right]. \tag{15}$$

The couple (N, A) *is a weak solution of the conservation law (12) if only if* Z *coincides with the sticky particles process defined from* (N_0, v_0).

Before the proof, let us describe what happens in our investigation. From the point of view of the matter, our investigation consists in a change of distributions. We recall that the paths of Z are "extracted" from significant paths of X on which rise turbulences. The extraction procedure redistributes the mass. If τ is constant on $[a, b]$ and $[\alpha, \beta]$ is the cluster which contains $[a, b]$ at time $\tau(a)$, one of the four particles α, a, b or β are extracted. We call them "turbulent particles". All the mass of $[a, b]$ is initially re-affected to these particles. In order to expect the preservation of the conservation law, one can also re-affect the momentum as follows. First remark that each event A_i of section 3.1 is of type "$X_0 \in E_i$".

- The mass $\mu_0\left([a, b] \cap E_3\right)$ and the momentum $\int_{[a,b]\cap E_3} u_0(x)d\mu_0(x)$ are affected to α.
- The mass $\mu_0\left([a, b] \cap E_1\right)$ and the momentum $\int_{[a,b]\cap E_1} u_0(x)d\mu_0(x)$ are affected to a.
- The mass $\mu_0\left([a, b] \cap E_2\right)$ and the momentum $\int_{[a,b]\cap E_2} u_0(x)d\mu_0(x)$ are affected to b.
- The mass $\mu_0\left([a, b] \cap E_4\right)$ and the momentum $\int_{[a,b]\cap E_4} u_0(x)d\mu_0(x)$ are affected to β.
- The total mass and total momentum of α and β are aggregations of the masses and momenta extracted from turbulent intervals inside $[\alpha, \beta]$.

Algorithm : Extraction along the time and aggregation of mass and momentum to α *(resp.* β*) until it is hinted from the left (resp. the right).*

The momentum transferred to turbulent particles can also be computed from the flux A (14) and c.d.f $N_0 := N(\cdot, 0)$ of Z_0. Indeed, the turbulent particles in $[a, b]$ have the momentum

$$\int_{\{a\le Z_0\le b\}} u_0(X_0)\mathrm{d}P \;=\; \int_{\{a\le N_0^{-1}\le b\}} u_0(M_0^{-1}(z))\mathrm{d}z = \int_{N_0(a)}^{N_0(b)} u_0(M_0^{-1}(z))\mathrm{d}z$$

$$= \; A(N_0(b)) - A(N_0(a^-))$$

However, the velocity function induced by this momentum is not the correct one (u_0) for the real dynamics of Z. Indeed, each turbulent particle of initial position a' received the mass $P(Z_0 = a')$ and the momentum $\int_{Z_0=a'} u_0(X_0)\mathrm{d}P$. This induces the velocity

$$\frac{\int_{Z_0=a'} u_0(X_0)\mathrm{d}P}{P(Z_0 = a')} = E\left[u_0(X_0)|Z_0 = a'\right] = v_0(a')$$

So, $A(N_0(b)) - A(N_0(a^-)) = \int_{N_0(a)}^{N_0(b)} u_0(N_0^{-1}(z))\mathrm{d}z$ is the momentum of another sticky particles dynamics, the one from $(N_0,\, v_0)$.

Proof of proposition 4.1: Such a weak solution is an entropy solution which is unique once imposed the initial datum N_0. Let be the flow constructed from $(N_0,\, v_0)$ and define $N_t = N(\cdot,\, t)$. One has $N_t^{-1} = \;(N_0^{-1},\, t)$ for all t. Since $Z_t = \phi(Z_0,\, t)$, one has also $N_t^{-1} = \phi(N_0^{-1},\, t)$. So $\;(\cdot,\, t) = \phi(\cdot,\, t)$ on the support of the law of Z_0, and one gets $Z_t = \;(Z_0,\, t)$ for all t.

Now we consider the momentum which corresponds to the dynamics of Z, the function $B\,:\,(0,\,1)\ni\mapsto \int_0^m v_0(N_0^{-1}(z))\mathrm{d}Z$. It is also the momentum function of the sticky particles dynamics defined from $(N_0,\, v_0)$.

b) Proposition

Suppose that $\gamma = \tau\,(Z_0)$ a.e. We have

$$\partial_t N + \partial_x\,(A(N)) \;=\; -\partial_x\,(C(N,t)) \tag{16}$$

$$\partial_t N + \partial_x\,(B(N)) \;=\; \partial_x\,(D(N,t)) \tag{17}$$

with, $\forall\,(m,t)\in(0,1)\times\mathbb{R}_+$,

$$C(m,t) \;=\; \int_0^m \Delta_0\left(N_0^{-1}(z)\right)\mathbb{1}_{t<\tau\left(N_0^{-1}(z)\right)}dz$$

$$D(m,t) \;=\; \int_0^m \Delta_0\left(N_0^{-1}(z)\right)\mathbb{1}_{t\ge\tau\left(N_0^{-1}(z)\right)}dz$$

and $\Delta_0 = u_0 - v_0$.

Surprisingly, as shown in the sequel, these results lead to the homogeneous conservation law of the momentum. For all t, let ν_t be the distribution of Z_t, i.e. $\nu_t(B) = P(Z_t\in\, B)$ for all Borel set B.

Notes

c) Gas system with turbulence force

For all t, let ν_t be the distribution of Z_t, i.e. $\nu_t(B) = P(Z_t \in B)$ for all Borel set B.

d) Corollary

Let us define $\delta(x, t) = E\left[\Delta_0(Z_0)|Z_\gamma = x, \gamma = t\right]$ and consider the law $P_{Z_t,\gamma}$ of (Z_t, γ). If $\gamma = \tau(Z_0)$ a.e, then we have

$$\begin{cases} \partial_t(\nu) + \partial_x(u\nu) = 0 \\ \partial_t(u\nu) + \partial_x(u^2\nu) = -\delta P_{Z_\gamma,\gamma} \\ \nu_t \to \mu_0, \qquad u(\cdot, t)\nu_t \to u_0\nu_0 \text{ weakly as } t \to 0 \end{cases} \tag{18}$$

The couple (ν, u) is thus a weak solution of a pressure gas system of initial datum is (ν_0, u_0).

Proof of proposition 4.2: $u(Z_t, t) = E\left[v_0(Z_0) + \Delta_0(Z_0)\mathbb{1}_{t<\tau(Z_0)}|Z_t\right]$. Using $w(Z_0, t) := v_0(Z_0) + \Delta_0(Z_0)\mathbb{1}_{t<\tau(Z_0)}$, we have, for any test function f on $\mathbb{R} \times \mathbb{R}_+^*$:

$$\int\int f_t(x, t)N(x, t)\mathrm{d}t\mathrm{d}x = E\int\int f_t(x, t)H(x - Z_t)\mathrm{d}t\mathrm{d}x$$

$$= E\int\int f(x, t)u(Z_t, t)\delta_{Z_t}(\mathrm{d}x)\mathrm{d}t = E\int f(Z_t, t)u(Z_t, t)\mathrm{d}t$$

$$= E\int f(Z_t, t)w(Z_0, t)\mathrm{d}t = -E\int\int f_x(x, t)H(x - Z_t)w(Z_0, t)\mathrm{d}t\mathrm{d}x$$

$$= -\int\int f_x(x, t)E\left[H(x - Z_t)w(Z_0, t)\right]\mathrm{d}t\mathrm{d}x$$

and

$$E\left[H(x - Z_t)w(Z_0, t)\right] = \int_0^{N(x,t)} w(N_0^{-1}(z))\mathrm{d}z$$

$$= A(N(x, t)) + C(N(x, t), t) = B(N(x, t)) - D(N(x, t), t).$$

Proof of corollary 4.3

$$E\left[H(x - Z_t)w(Z_0, t)\right] = E\left[H(x - Z_t)u(Z_t, t)\right] = \int_{-\infty}^{x} u(y, t)\mathrm{d}\nu_t(y),$$

$$\partial_x\left[A(N(x, t)) + C(N(x, t), t)\right] = u(x, t)\mathrm{d}\nu_t(x).$$

From previous proposition, one gets $\partial_t M + u(x, t)\mathrm{d}\nu_t(x) = 0$. Then, in order to have the first equation of gas system, use the fact that

Notes

$$\partial_t\partial_x M = \partial_x\partial_t M.$$

It remains the last equation. For any test function f on $\mathbb{R}_+^*$ and any test function g on $\mathbb{R}$,

$$\int\int f'(t)g(x)u(x,\, t)\mathrm{d}\nu_t(x)\mathrm{d}t = E\int f'(t)g(Z_t)u(Z_t,\, t)\mathrm{d}t$$

$$= E\int f'(t)g(Z_t)v_0(Z_0)\mathrm{d}t + E\int f'(t)g(Z_t)\Delta_0(Z_0)\mathbb{1}_{t<\tau(Z_0)}\mathrm{d}t$$

$$= -E\int f'(t)g(Z_t)u(Z_t,\, t)v_0(Z_0)\mathrm{d}t$$

$$+E\left[f(\gamma)g(Z_\gamma)\Delta_0(Z_0)\right] - E\int\int f'(t)g(Z_t)\Delta_0(Z_0)\mathbb{1}_{t<\tau(Z_0)}\mathrm{d}t$$

$$= -E\int f(t)g'(Z_t)u^2(Z_t,\, t)\mathrm{d}t + E\left[f(\gamma)g(Z_\gamma)\Delta_0(Z_0)\right]$$

This ends the proof.

Notes

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