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Convergence Theorems of Multi-Valued Generalized Nonexpansive Mappings in Banach Spaces

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Keywords: *generalized nonexpansive mappings, multi-valued mapping, fixed point, iteration process, uniformly convex banach spaces.*

GJSFR-F Classification: *47H010; 54H25*

Ref

8. J. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc. 74 (1968), 639–640.

Convergence Theorems of Multi-Valued Generalized Nonexpansive Mappings in Banach Spaces

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I. Introduction and Preliminaries

Fixed point theory for multi-valued mappings has many useful applications in various fields, control theory, convex optimization, game theory and mathematical economics.Therefore, it is natural to extend the known fixed point results for single-valued mappings to the setting of multi-valued mappings.The theory of multi-valued nonexpansive mappings is more difficult than the corresponding theory of single-valued nonexpansive mappings. The convergence of a sequence of fixed points of a convergent sequence of set valued contractions was investigated by [8] and [9]. In the last few decades, the numerous numbers of researchers attracted in these direction and developed the study of multi-valued version of iterative processes have been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings. Iterative techniques for approximating fixed points of nonexpansive multi-valued mappings have been investigated by various authors using the Mann iteration scheme or the Ishikawa iteration scheme (see [11], [14], [16] and so on).

We assume throughout this paper that $(X, \|.\|)$ is a Banach space and K is a nonempty subset of X. The set K is called proximinal if for each $x \in X$, there exists some $y \in K$ such that $d(x, y) = d(x, K)$, where $d(x, K) = inf\,\{d(x, y) : y \in K\}$. In the sequel, the notations $\mathcal{P}_{px}(K), \mathcal{P}_{cb}(K), \mathcal{P}_{cp}(K)$ and $\mathcal{P}(K)$ will denote the families of nonempty proximinal subsets, closed and bounded subsets, compact subsets and all subsets of K , respectively. A point $y \in K$ is said to be a fixed point of $T : K \to \mathcal{P}(K)$ if $y \in T(y)$. The set of fixed points of T will be denote by $F(T)$. Let $H(,)$ be the Hausdorff distance on $\mathcal{P}_{cb}(K)$ is defined by

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$H(A,B) = max\{sup_{x\in A} d(x,B), sup_{y\in B} d(y,A), \forall A,B \in \mathcal{P}_{cb}(K), x\in A, y \in B\}$.

Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping. An element $p \in K$ is said to be a fixed point of T, if $p \in T(p)$. The set of fixed points of T will be denoted by $F(T)$. A multivalued mapping $T : K \to \mathcal{P}(K)$ is said to be nonexpansive, if $H(Tx,Ty) \le \|x-y\|$, for all $x,y \in K$, quasi-nonexpansive, if $F(T) \neq \emptyset$ and $H(Tx,Tp)) \le \|x-p\|$, for all $x \in K$, and all $p \in F(T)$. It is well known that if K is a nonempty closed, bounded and convex subset of a uniformly convex Banach space X, then a multivalued nonexpansive mapping $T : K \to \mathcal{P}(K)$ has a fixed point has a fixed point [7]. Shahzad and Zegeye [14] presented the set $P_T(x) = \{y \in Tx : d(x,Tx) = \|x-y\|\}$ for a multivalued mapping, $T : K \to \mathcal{P}(K)$ and showed that Mann and the Ishikawa iteration processes for multi-valued mappings are well defined. They proved the convergence of these iteration processes for multivalued mappings in a uniformly convex Banach space. In 2011, Abkar and Eslamian [1] extended the notion of condition (C) to the case of multi-valued mappings. In 2012, Abkar and Eslamian [2] introduced an iterative process for a finite family of generalized nonexpansive multivalued mappings and proved Δ-convergence and strong convergence theorems in CAT(0) spaces.

In this paper, we introduce multi-valued version iterative scheme presented in [4] for multi-valued mappings as follows: for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} v_n = (1-c_n)x_n + c_n\tau_n, \\ s_n = (1-b_n)\tau_n + b_n\eta_n, \\ w_n \in P_T(s_n), \\ x_{n+1} = (1-a_n)t_n + a_n u_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\} \in (0,1)$, $\tau_n \in P_T(x_n), \eta_n \in P_T(z_n), t_n \in P_T(w_n), u_n \in P_T(\eta_n)$.

One can find in the literature that there are important studies about generalized nonexpansive mappings that are weaker nonexpansive mappings and stronger than quasi-nonexpansive mappings. For instance, in 2008, Suzuki [15] defined a class of generalized nonexpansive mappings on a nonempty subset K of a Banach space X. Such type of mappings was called the class of mappings satisfying the condition (C) (also referred as Suzuki generalized nonexpansive mapping), which properly includes the class of nonexpansive mappings. Another one of the generalized nonexpansive mappings , in 2011, García-Falset et al. [6] introduced two new conditions on single-valued mappings, are called condition (E)(also referred as Garsia-Falset generalized nonexpansive mapping) and (C_λ) which are weaker than nonexpansive and stronger than quasi-nonexpansive.

A single-valued mapping $T : K \to X$ satisfies condition (E_μ) on K, if there exists $\mu \ge 1$ such that for all $x,y \in K$,

$$\|x-Ty\| \le \mu\|x-Tx\| + \|x-y\|.$$

Moreover, it is said that T satisfies condition (E) on K, whenever T satisfies condition (E_μ), for some $\mu \ge 1$. It is obvious that if $T : K \to X$ is nonexpansive, then it satisfies condition (E_1) and from Lemma 7 in [15] we know that if $T : K \to K$ satisfies condition (C) on K, then T satisfies condition (E_3) (see [6]). Proposition 1 in [6], we know also that if $T : K \to X$ a mapping which satisfies condition (E) on K has some fixed point, then T is quasi-nonexpansive. The converse is not true (see example in [6]). Thus the class of Garcia-Falset generalized nonexpansive mappings exceeds the class of Suzuki generalized nonexpansive mappings

Ref

1. Abkar, M. Eslamian, *A fixed point theorem for generalized nonexpansive multivalued mappings*, Fixed Point Theory 12(2) (2011), 241–246.

(and therefore the class of nonexpansive mappings), but still remains stronger than quasi-nonexpansiveness.

Motivated by the above, we prove some weak and strong convergence results using (1.1) iteration process for multi-valued Garsia-Falset generalized nonexpansive mappings(generalized nonexpansive mappings satisfying condition (E)) in uniformly convex Banach spaces. Moreover, we present an illustrative numerical example of approximating fixed point of multi-valued generalized nonexpansive mappings satisfying condition (E) considering the iteration process (1.1).

Now we recall some notations to be used in main results:

A Banach space X is said to satisfy *Opial's condition* [10] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ converges weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|.$$

In the following we shall give some preliminaries on the concepts of asymptotic radius and asymptotic center which are due to [3].

Let $\{x_n\}$ be a bounded sequence in a Banach space X. Then

(1) *The asymptotic radius of* $\{x_n\}$ *at point* $x \in X$ *is the number*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

(2) *The asymptotic radius of* $\{x_n\}$ *relative to* K is defined by

$$r(K, \{x_n\}) = \inf\{r(x, \{u_n\}) : x \in K\}.$$

(3) *The asymptotic center of* $\{x_n\}$ *relative to* K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is well known that, in uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point.

Lemma 1.1. *([12]) Suppose that X is a uniformly convex Banach space and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{x_n\}$ be two sequence of X such that $\limsup_{n\to\infty}\|x_n\| \leq \xi$, $\limsup_{n\to\infty}\|y_n\| \leq \xi$ and $\limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = \xi$ hold for $\xi \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Definition 1.2. Let $T : K \to \mathcal{P}_{cb}(K)$. A sequence $\{x_n\}$ in K is called an approximate fixed point sequence (or a.f.p.s) for T provided that $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Definition 1.3. A multivalued mapping $T : K \to \mathcal{P}(K)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K weakly convergent to x and $y_n \in Tx_n$ strongly convergent to y, we have $y \in Tx$.

The following is the multi-valued version of condition (I) of Senter and Dotson [13].

Definition 1.4. A multivalued mapping $T : K \to \mathcal{P}(K)$ is said to satisfy *condition(I)*, if there is a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(\xi) > 0$ for all $\xi \in (0, \infty)$ such that $d(x, Tx) \geq \varphi(d(x, F(T))$ for all $x \in K$.

Ref

3. M. Edelstein, *Fixed point theorems in uniformly convex Banach spaces*, Proc. Amer. Math. Soc., 44(2) (1974), 369–374.

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Lemma 1.5. *([16]) Let $T : K \to \mathcal{P}_{px}(K)$ and $P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}$. Then the following are equivalent.*

(1) $x \in F(T)$.

(2) $P_T(x) = \{x\}$.

(3) $x \in F(P_T)$.

Moreover, $F(T) = F(P_T)$.

Now we give the definition of multi-valued generalized nonexpansive mapping satisfying condition (E):

Definition 1.6. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to \mathcal{P}_{cb}(K)$ is called a multi-valued generalized nonexpansive mapping satisfying condition (E) if there exists an $\mu \geq 1$ such that for each $x, y \in K$,

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y)$$

We say that $T : K \to \mathcal{P}_{cb}(K)$ satisfies condition (E) on K whenever T satisfies (E_μ) for some $\mu \geq 1$.

Every multi-valued nonexpansive mapping satisfies condition (E_1) (see [2]). Moreover, if $z \in K$ is a fixed point of the mapping $T : K \to \mathcal{P}_{cb}(K)$, and this mapping satisfies condition (E_μ) on K, then for all $x \in K$, $d(z, Tx) \leq \|z - x\|$. In other words, T is a quasi-nonexpansive mapping.

Proposition 1.7. [5] *Let $T : K \to \mathcal{P}_{cb}(K)$ be a mapping satisfying condition $(C_{1/2})$. Then, T satisfies condition (E_3).*

II. Convergence of Multi-Valued Generalized Nonexpansive Mappings

In this section, we prove weak and strong convergence theorems for (1.1) iterative scheme of multi-valued generalized nonexpansive mappings satisfying condition (E) in uniformly convex Banach space.

Lemma 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping such that $F(T) \neq \emptyset$ and P_T is a Garsia-Falset generalized nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by (1.1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. By Lemma 1.5, $P_T(p) = \{p\}$ and $F(T) = F(P_T)$. Since P_T is a Garsia-Falset generalized nonexpansive mapping, then P_T is a quasi-nonexpansive mapping. Now, for any $p \in F(T)$, we have

$$\begin{aligned} H(P_T(v_n), P_T(p)) &\leq \|v_n - p\|, \\ H(P_T(s_n), P_T(p)) &\leq \|s_n - p\|, \\ H(P_T(w_n), P_T(p)) &\leq \|w_n - p\|, \\ H(P_T(\tau_n), P_T(p)) &\leq \|\tau_n - p\|, \\ H(P_T(\eta_n), P_T(p)) &\leq \|\eta_n - p\|. \end{aligned}$$

Next by (1.1), we have

$$\begin{aligned} \|\tau_n - p\| &\leq d(\tau_n, P_T(p)) \\ &\leq H(P_T(x_n), P_T(p)) \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.1}$$

Ref

2. A. Abkar, M. Eslamain, *Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT(0) spaces*, Nonlinear Anal.,75 (2012) 1895–1903.

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By (2.1), we have

$$\begin{aligned}\|v_n - p\| &= \|(1-c_n)x_n + c_n\tau_n - p\| \\ &\le (1-c_n)\|x_n - p\| + c_n\|\tau_n - p\| \\ &\le (1-c_n)\|x_n - p\| + c_n d(\tau_n, P_T(p)) \\ &\le (1-c_n)\|x_n - p\| + c_n H(P_T(x_n), P_T(p)) \\ &\le (1-c_n)\|x_n - p\| + c_n\|x_n - p\| = \|x_n - p\| \qquad (2.2)\end{aligned}$$

and also we have

$$\begin{aligned}\|\eta_n - p\| &\le d(\eta_n, P_T(p)) \\ &\le H(P_T(v_n), P_T(p)) \\ &\le \|v_n - p\|. \qquad (2.3)\end{aligned}$$

By (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\|s_n - p\| &= \|(1-b_n)\tau_n + b_n\eta_n - p\| \\ &\le (1-b_n)\|x_n - p\| + b_n\|v_n - p\| = \|x_n - p\|. \qquad (2.4)\end{aligned}$$

By (2.4), we have

$$\begin{aligned}\|w_n - p\| &\le d(w_n, P_T(p)) \\ &\le H(P_T(s_n), P_T(p)) \\ &\le \|s_n - p\| \le \|x_n - p\|. \qquad (2.5)\end{aligned}$$

By (2.1)-(2.5), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-a_n)t_n + a_n u_n - p\| \\ &\le (1-a_n)\|t_n - p\| + a_n\|u_n - p\| \\ &\le (1-a_n)d(t_n, P_T(p)) + a_n d(u_n, P_T(p)) \\ &\le (1-a_n)H(P_T(w_n), P_T(p)) + a_n H(P_T(\eta_n), P_T(p)) \\ &\le (1-a_n)\|w_n - p\| + b_n\|\eta_n - p\| \\ &\le (1-a_n)\|s_n - p\| + b_n\|v_n - p\| \\ &\le (1-a_n)\|x_n - p\| + a_n\|x_n - p\| = \|x_n - p\|. \qquad (2.6)\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. It follows that $\lim_{n\to\infty}\|x_n - p\|$ exists.

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping and P_T is a generalized nonexpansive mapping satisfying condition (E). Let $\{x_n\}$ be a sequence generated by (1.1). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - \tau_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty}\|x_n - p\| = \xi. \qquad (2.7)$$

From (2.2)-(2.5), we have

$$\limsup_{n\to\infty}\|v_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le \xi,$$

$$\limsup_{n\to\infty}\|\tau_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le \xi. \qquad (2.8)$$

Also

$$\limsup_{n\to\infty}\|\eta_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le \xi,$$

$$\limsup_{n\to\infty}\|s_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le \xi$$

and

$$\limsup_{n\to\infty}\|w_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le \xi.$$

Also we have the following inequalities

$$\|u_n - p\| \le H(P_T(\eta_n), P_T(p)) \le \|\eta_n - p\|$$

and

$$\|t_n - p\| \le H(P_T(w_n), P_T(p)) \le \|w_n - p\|.$$

On taking $\limsup\limits_{n\to\infty}$ on both sides of the all above inequalities, we obtain that

$$\limsup_{n\to\infty}\|t_n - p\| \le \xi, \qquad (2.9)$$

$$\limsup_{n\to\infty}\|u_n - p\| \le \xi \qquad (2.10)$$

and so

$$\begin{aligned}\xi &= \limsup_{n\to\infty}\|x_{n+1} - p\| \\ &= \limsup_{n\to\infty}\|(1-a_n)t_n + a_n u_n - p\|\end{aligned}$$

By Lemma 1.1, we have

$$\limsup_{n\to\infty}\|t_n - u_n\| = 0. \qquad (2.11)$$

Now

$$\begin{aligned}\|x_{n+1} - p\| &= \|((1-a_n)t_n + a_n u_n - p\| \\ &= \|(t_n - p) + a_n(u_n - t_n)\| \\ &\le \|t_n - p\| + a_n\|u_n - t_n\|.\end{aligned}$$

Making $n \longrightarrow \infty$ and from (2.11) we get

$$\xi = \limsup_{n\to\infty}\|x_{n+1} - p\| \le \limsup_{n\to\infty}\|t_n - p\|.$$

So by from (2.9) we have

$$\limsup_{n\to\infty}\|t_n - p\| = \xi.$$

Then

$$\|t_n - p\| \le \|t_n - u_n\| + \|u_n - p\|$$

Making $n \longrightarrow \infty$ and from (2.11), we get

$$\xi \le \limsup_{n\to\infty}\|u_n - p\|.$$

Hence together with (2.10) we have

$$\xi = \lim_{n\to\infty}\|u_n - p\|.$$

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Thus

$$\begin{aligned}\xi &= \lim_{n\to\infty}\|u_n-p\| \le \lim_{n\to\infty} H(P_T(\eta_n),P_T(p))\\ &\le \lim_{n\to\infty}\|\eta_n-p\| \le \lim_{n\to\infty} H(P_T(v_n),P_T(p))\\ &\le \lim_{n\to\infty}\|v_n-p\|\\ &\le \lim_{n\to\infty}\|(1-c_n)x_n+c_n\tau_n-p\|\\ &\le \lim_{n\to\infty}(1-c_n)\|x_n-p\|+c_n\|\tau_n-p\|\\ &\le \xi.\end{aligned}$$

Consequently

$$\lim_{n\to\infty}\|(1-c_n)(x_n-p)+c_n(\tau_n-p)\| = \xi. \tag{2.12}$$

Thus from (2.7), (2.8), (2.12) and by Lemma 1.1 we have

$$\lim_{n\to\infty}\|x_n-\tau_n\| = 0$$

which implies that $\lim_{n\to\infty} d(x_n,P_T(x_n)) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,P_T(x_n)) = 0$. Let $p \in A(K,\{x_n\})$. Then we have

$$d(x_n,P_T(p)) \le \mu d(x_n,P_T(x_n)) + \|x_n-p\|$$

Using the definition of asymptotic center we have

$$\begin{aligned}r(Tp,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,P_T(p))\\ &\le \mu\limsup_{n\to\infty} d(P_T(x_n),x_n) + \limsup_{n\to\infty}\|x_n-p\|\\ &= \limsup_{n\to\infty}\|x_n-p\| = r(p,\{x_n\}).\end{aligned}$$

This implies that for $Tp = p \in A(K,\{x_n\})$. Since X is uniformly Banach space, $A(K,\{x_n\})$ is consists of a unique element. Thus, we have $Tp = p$. Hence $F(T) \neq \emptyset$.

In the next result, we prove our strong convergence theorems as follows.

Theorem 2.3. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Let $T : K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping such that $F(T) \neq \emptyset$ and P_T is a generalized nonexpansive mapping satisfying condition (E). Let $\{x_n\}$ be a sequence generated by (1.1). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. $F(T) \neq \emptyset$, so by Theorem 2.2, we have $\lim_{n\to\infty} d(x_n,P_T(x_n)) = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow q$ as $k \to \infty$ for some $q \in K$. Because P_T is a generalized nonexpansive mapping satisfying condition (E), one can find some real constant $\mu \ge 1$, such that

$$d(x_{n_k},P_T(q)) \quad \le \quad \mu d(x_{n_k},P_T(x_{n_k})) + \|x_{n_k}-q\|.$$

As $F(T) = F(P_T)$, on taking limit as $k \to \infty$, we get $q \in P_T(q)$ i.e. $q \in F(T)$. So $\{x_n\}$ converges strongly to a fixed point of T. □

The proof of the following result is elementary and hence omitted.

Theorem 2.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T: K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping such that P_T is a generalized nonexpansive mapping satisfying condition (E). $\{x_n\}$ be a sequence generated by (1.1). If $F(T) \neq \emptyset$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 2.5. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping satisfying condition (I) such that $F(T) \neq \emptyset$. $\{x_n\}$ be a sequence generated by (1.1). If P_T is a generalized nonexpansive mapping satisfying condition (E), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists and for all $p \in F(T)$. Put $\xi = \lim_{n\to\infty} \|x_n - p\|$ for some $\xi \geq 0$. If $\xi = 0$ then the result follows. Suppose that $\xi > 0$. Then

$$\lim_{n\to\infty} \|x_{n+1} - p\| \leq \lim_{n\to\infty} \|x_n - p\|.$$

It follows that

$$\lim_{n\to\infty} d(x_{n+1}, F(T)) \leq \lim_{n\to\infty} d(x_n, F(T)).$$

$\lim_{n\to\infty} d(x_n, F(T))$ exists. We show that it follows $\lim_{n\to\infty} d(x_n, F(T)) = 0$. From Theorem 2.2 $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. As $F(T) = F(P_T)$, by Theorem 2.2 and condition (I) we have $\lim_{n\to\infty} \varphi(d(x_n, F(T)) \leq \lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. That is, $\lim_{n\to\infty} \varphi(d(x_n, F(T))) = 0$. Since $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\varphi(0) = 0$ and $\varphi(\xi) > 0$ for all $\xi \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. All the conditions of Theorem 2.4 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of T. The proof is completed. □

Finally, we prove the weak convergence of the iterative scheme (1.1) for multi-valued generalized nonexpansive mappings satisfying condition (E) in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T : K \to \mathcal{P}_{px}(K)$ be a multi-valued mapping such that $F(T) \neq \emptyset$. Suppose P_T is a generalized nonexpansive mapping satisfying condition (E) and $I - P_T$ is demi-closed with respect to zero. Then $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T.*

Proof. Let $p \in F(T) = F(P_T)$.By Lemma 2.1, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Since X is uniformly convex, X is reflexive. By the reflexiveness of X , there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $\omega_1 \in K$. Since $I - P_T$ is demi-closed with respect to zero, $\omega_1 \in F(P_T) = F(T)$. We prove that ω_1 is the unique weak limit of $\{x_n\}$. Let one can find another weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with weak limit say $\omega_2 \in K$ and $\omega_2 \neq \omega_1$. Again $\omega_1 \in F(P_T) = F(T)$. From the Opial's property and Lemma 2.1, we obtain

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \omega_1\| &= \lim_{j\to\infty} \|x_{n_j} - \omega_1\| < \lim_{j\to\infty} \|x_{n_j} - \omega_2\| = \lim_{n\to\infty} \|x_n - \omega_2\| \\ &= \lim_{k\to\infty} \|x_{n_k} - \omega_2\| < \lim_{k\to\infty} \|x_{n_k} - \omega_1\| = \lim_{n\to\infty} \|x_n - \omega_1\|,\end{aligned}$$

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Notes

which is a contradiction. So, $\omega_1 = \omega_2$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. $\square$

III. Example

Example 3.1. *Let $K = [0, \infty) \subset \mathbb{R}$ endowed with usual norm in $\mathbb{R}$ and $T : K \to \mathcal{P}(K)$ be defined by*

$$Tx = \begin{cases} (0), & 0 \leq x < \frac{1}{400} \\ [0, \frac{3x}{4}], & \frac{1}{400} \leq x \leq 1 \end{cases}$$

If $x \in [0, \frac{1}{400})$, then $P_T(x) = 0$. For $x \in [\frac{1}{400}, 1]$, then $P_T(x) = \{\frac{3x}{4}\}$. We show that P_T is generalized nonexpansive mapping satisfying condition $(E_{\mu=4})$ with $F(T)$. We consider the following cases:

Case I: *Let $x \in [0, \frac{1}{400})$ and $y \in [0, \frac{1}{400})$. We have*

$$d(x, P_T(y)) = |x| \leq \mu|x| \leq \mu d(x, P_T(x)) + |x - y|.$$

Case II: *Let $x \in [\frac{1}{400}, 1]$ and $y \in [\frac{1}{400}, 1]$. We have*

$$\begin{aligned} d(x, P_T(y)) &\leq d(x, P_T(x)) + H(P_T(x), P_T(y)) \\ &= d(x, P_T(x)) + |\frac{3x}{4} - \frac{3y}{4}| \\ &\leq d(x, P_T(x)) + \frac{3}{4}|x - y| \leq d(x, P_T(x)) + |x - y| \\ &\leq \mu d(x, P_T(x)) + |x - y| \end{aligned}$$

Case III: *Let $x \in [\frac{1}{400}, 1]$ and $y \in [0, \frac{1}{400})$. One has*

$$d(x, P_T(y)) = |x| = \frac{4|x|}{4} = \mu d(x, P_T(x)) \leq \mu d(x, P_T(x)) + |x - y|$$

Thus, P_T is generalized nonexpansive mapping satisfying condition $(E_{\mu=4})$ with $p = 0$ fixed point.

Finally, let us prove that T does not satisfy condition (C). Indeed, if we take $x = \frac{1}{1200}, y = \frac{1}{400}$ then

$$\frac{1}{2}d(x, P_T(x)) = \frac{1}{2}|\frac{1}{1200} - 0| = 0.000416 < 0.001666 = |x - y|.$$

$$H(P_T(x), P_T(y)) = |\frac{3}{4} \times \frac{1}{400} - 0| = 0.001875 > 0.001666 = |x - y|.$$

Thus P_T does not satisfy Suzuki's condition (C).

Let $a_n = b_n = c_n = 0.75$ for all $n \in \mathbb{N}$ and be $x_1 = 0.5$. We compute that the sequence $\{x_n\}$ generated by iterative scheme (1.1) converge to fixed point 0 of the multi-valued generalized nonexpansive mapping satisfying condition (E) defined in Example 3.1 which is shown by the Figure 1.

IV. Conclusions

We study the convergence of (1.1)-iteration process to fixed for the multi-valued generalized nonexpansive mapping satisfying condition (E) in uniformly convex Banach space. Moreover, we give an illustrative numerical example that is multi-valued generalized nonexpansive mapping satisfying condition (E) but is not Suzuki generalized nonexpansive mapping, as in Example 3.1 of this paper.

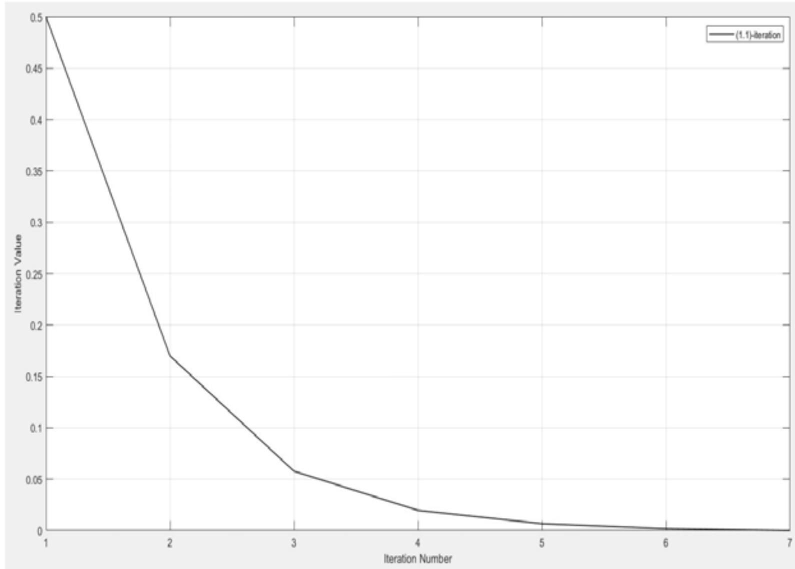

Figure 1: Convergence of (1.1)-iteration to the fixed point 0 of the multi-valued generalized nonexpansive mapping satisfying condition (E) defined in Example 3.1.

Notes

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