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Solve the 3x+1 Problem by the Multiplication and Division of Binary Numbers

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I. Introduction

The $3x+1$ problem, also known as the Collatz conjecture, $3x+1$ mapping, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, or Syracuse problem [1], is one of the unsolved problems in mathematics. Paul Erdos (1913-1996) commented on the intractability of the $3x+1$ problem [2], stating that "Mathematics is not ready for those problems yet".

The $2x+1$ problem states that, for any positive integer x, if x is even, divide it by 2; if x is odd, multiply it by 3 and add 1. Repeating this process continuously leads to the conjecture that no matter which number is initially chosen, the result will always reach 1 eventually.

We use the notations as in [4,7], and describe a Collatz function as follows:

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \quad (1)$$

Let N denote the set of positive integers. For $n \in N$, and $k = 0, 1, 2, 3, \cdots$, $T^0(n)$ and $T^{k+1}(n)$ denote n and $T(T^k(n))$, respectively. Concerning the behavior of the iteration of the Collatz function, for any integer n, there must exist an integer r so that

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1. Jeffrey C. Lagarias. The 3x + 1 Problem and Its Generalizations. American Mathematical Monthly, Vol. 92, No. 1, pp 3-23.(1985).

$$T^r(n) = 1. \tag{2}$$

a) *The modified Sarkovskii ordering and integer lattice*

We convert the last row of numbers into the first column to get a modified Sarkovskii ordered integer lattice[6] as the following,

$$\begin{array}{lllllllllll}
1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & 19, & \cdots \\
2, & 2\cdot 3, & 2\cdot 5, & 2\cdot 7, & 2\cdot 9, & 2\cdot 11, & 2\cdot 13, & 2\cdot 15, & 2\cdot 17, & 2\cdot 19, & \cdots \\
2^2, & 2^2\cdot 3, & 2^2\cdot 5, & 2^2\cdot 7, & 2^2\cdot 9, & 2^2\cdot 11, & 2^2\cdot 13, & 2^2\cdot 15, & 2^2\cdot 17, & 2^2\cdot 19, & \cdots \\
2^3, & 2^3\cdot 3, & 2^3\cdot 5, & 2^3\cdot 7, & 2^3\cdot 9, & 2^3\cdot 11, & 2^3\cdot 13, & 2^3\cdot 15, & 2^3\cdot 17, & 2^3\cdot 19, & \cdots \\
2^4, & 2^4\cdot 3, & 2^4\cdot 5, & 2^4\cdot 7, & 2^4\cdot 9, & 2^4\cdot 11, & 2^4\cdot 13, & 2^4\cdot 15, & 2^4\cdot 17, & 2^4\cdot 19, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

In the first row, they are odd numbers from left to right, that are $1, 3, 5, 7, 9, 11, 13, \cdots$. From the second row, each number is two times the number in its previous row, and so on.

b) *The algebraic formula and Collatz graph*

If we draw a line segment with an arrow between two digits in the lattice of integers in the modified Sarkovskii ordering, one being the original value x and the other being its value of the Collatz function $T(x)$, and then connect $T(x)$ to $T^2(x)$, and so on $T^2(x)$ to $T^3(x), \cdots$, we get a graph which can be called a *Collatz graph*.

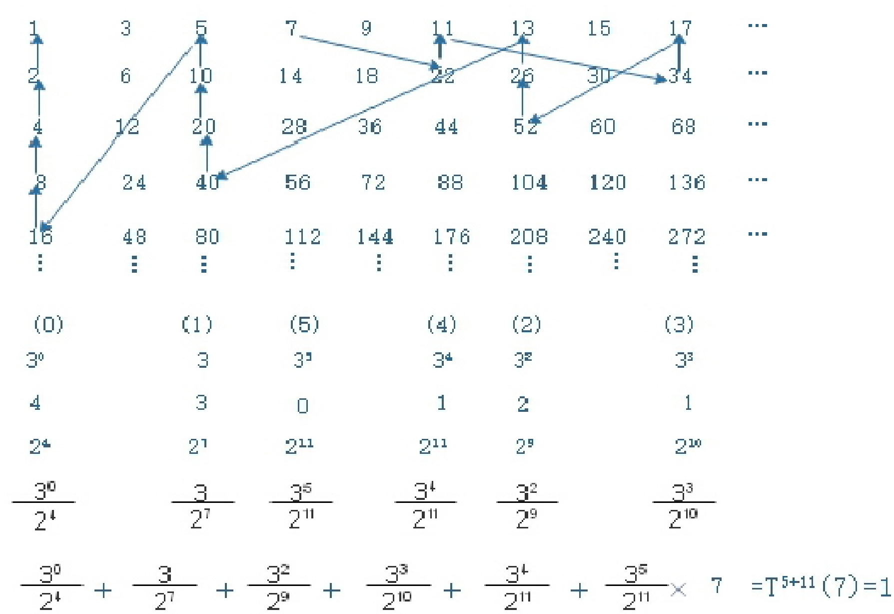

Fig. 1: The Collatz graph of $T^{16}(7) = T(5, 11, 7) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

Ref

6. Jishe FENG, Xiaomeng WANG, Xiaolu GAO, Zhuo PAN. The research and progress of the enumeration of lattice paths. Frontiers of Mathematics in China, 2022, 17(5): 747-766.

Notes

For different integers k and l, if there is a common vertex in their Collatz graphs, their graphs will overlap from that point onwards until they reach the minimum value of 1. Using the Collatz function $T(x)$, we can obtain an algebraic formula of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r} \cdot x$, where r is the number of vertical segments and m is the number of oblique segments in the Collatz graph,

$$T^{m+r}(n) = T(m, r, n) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \cdots + \frac{3^m}{2^r} \cdot x = 1. \tag{3}$$

For example, $n = 7$, $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{16}(7) = T(5, 11, 7) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{11}} \cdot 7 = 1,$$

and the Collatz graph is Fig. 1.

And $n = 36$, $36 \to 18 \to 9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{21}(36) = T(6, 15, 36) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{15}} \cdot 36 = 1$$

and the Collatz graph is Fig. 2.

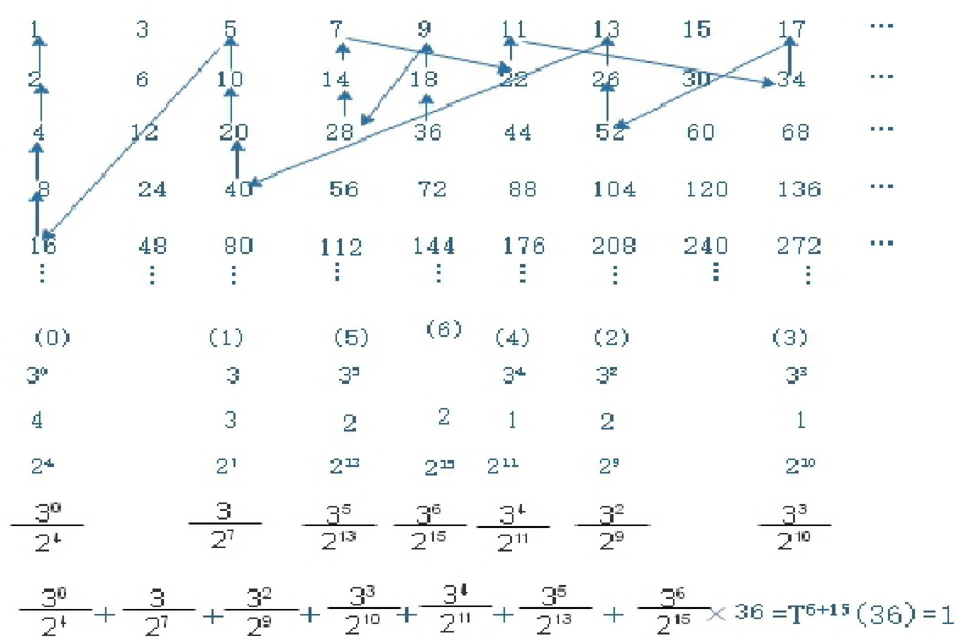

Fig. 2: The Collatz graph of $T^{21}(36) = T(6, 15, 36) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

We observe that there is a property present,

Proposition 1 *For positive integers i, j, k, l, and $l_k, l_{k-1}, \cdots, l_1$, if $i > j$, then there is a recurrence relation*

$$T^i(n) = \frac{3^k}{2^l}T^j(n) + \frac{3^{k-1}}{2^{l_k}} + \cdots + \frac{3^2}{2^{l_3}} + \frac{3}{2^{l_2}} + \frac{1}{2^{l_1}}$$

where $k + l = i - j$, and $l \geq l_k \geq l_{k-1} \geq \cdots \geq l_1$.

For example, there are

$$T^3(97) = \frac{3}{2^2} \cdot 97 + \frac{1}{2^2} = 73$$

$$T^{18}(97) = \frac{3^7}{2^{11}} \cdot 97 + \frac{3^6}{2^{11}} + \frac{3^5}{2^9} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^2} + \frac{1}{2} = 107$$

We can get the recurrence formula about the Collatz function,

$$T^{26}(97) = \frac{3^3}{2^5} \cdot T^{18}(97) + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2}$$

$$= \frac{3^{10}}{2^{16}} \cdot 97 + \frac{3^9}{2^{16}} + \frac{3^8}{2^{14}} + \frac{3^7}{2^{12}} + \frac{3^6}{2^{11}} + \frac{3^5}{2^{10}} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2}$$

$$= 91,$$

II. Numerical Example

Using the above Collatz graphs of the integer lattice of the modified Sarkovskii ordering, we give the following algebraic formulas,

$$T^{19}(9) = T(6,13,9) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{13}} \cdot 9 = 1, \quad (4)$$

$$T^{15}(23) = T(4,11,23) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{11}} \cdot 23 = 1, \quad (5)$$

$$T^{17}(15) = T(5,12,15) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{12}} + \frac{3^5}{2^{12}} \cdot 15 = 1, \quad (6)$$

$$T^{12}(17) = T(3,9,17) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^9} \cdot 17 = 1, \quad (7)$$

$$T^{19}(61) = T(5,14,61) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{14}} + \frac{3^5}{2^{14}} \cdot 61 = 1. \quad (8)$$

$$T^{16}(397) = T(5,11,397) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{17}} + \frac{3^6}{2^{20}} + \frac{3^7}{2^{20}} \cdot 397 = 1, \quad (9)$$

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Example 2 For the formula

$$T(6,14,18) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{14}} \cdot 18 = 1,$$

we rewrite it as an integer equation,

$$3^6 \cdot 18 + 3^5 \cdot 2 + 3^4 \cdot 2^3 + 3^3 \cdot 2^4 + 3^2 \cdot 2^5 + 3 \cdot 2^7 + 2^{10} = 2^{14}.$$

Proof. To calculate the power of 3 and the value of 18 using powers of 2,

$$\begin{aligned} 3 &= 2+1 \\ 3^2 &= 2^3+1 \\ 3^3 &= 2^4+2^3+2+1 \\ 3^4 &= 2^6+2^4+1 \\ 3^5 &= 2^7+2^6+2^5+2^4+2+1 \\ 3^6 &= 2^9+2^7+2^6+2^4+2^3+1 \\ 18 &= 2^4+2 \end{aligned}$$

substituting these expressions into the left-hand side of the above equation, one obtains,

$$\begin{aligned} &3^6 \cdot 18 + 3^5 \cdot 2 + 3^4 \cdot 2^3 + 3^3 \cdot 2^4 + 3^2 \cdot 2^5 + 3 \cdot 2^7 + 2^{10} \\ &= (2^9+2^7+2^6+2^4+2^3+1) \cdot (2^4+2) + (2^7+2^6+2^5+2^4+2+1) \cdot 2 \\ &+(2^6+2^4+1) \cdot 2^3 + (2^4+2^3+2+1) \cdot 2^4 + (2^3+1) \cdot 2^5 + (2+1) \cdot 2^7 + 2^{10}, \end{aligned}$$

and get the value 2^{14} which is equal to the right value of the equation.

III. Convert the Integer Number from Decimal to Binary

Be inspired by the above, we use binary to rewrite the Collatz function (1) as the following formulas (2) and (3). We denote a binary number, which is a string of 0s and 1s, as $n = (1\times\cdots\times)_2$, where $\times$ is either 1 or 0, e.g. $3 = (11)_2$,

$$T(n) = T((1\times\cdots\times)_2) = \begin{cases} (11)_2 \cdot (1\times\cdots\times 1)_2 + 1, & \text{if } n \text{ is odd number,} \\ \frac{(1\times\cdots\times 10\cdots 00)_2}{(10)_2}, & \text{if } n \text{ is even number.} \end{cases} \quad (10)$$

The result is

$$T(n) = T((1\times\cdots\times)_2) = \begin{cases} (1\times\times\times 10\cdots 0)_2, & \text{if } n \text{ is odd number,} \\ (1\times\cdots\times 10\cdots 0)_2, & \text{if } n \text{ is even number.} \end{cases} \quad (11)$$

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Namely, when n is an odd number, we multiply it with $(11)_2$ and add 1 to the end of the binary number. For example, $T(97) = T(1100001)$ in Fig. 3. When n is an even number, the division is equal to deleting the zero at the end in the binary number. We give the iteration of the Collatz function for $1, 5, 7, 9$, 97 in binary as the following five tables.

We convert the modified Sarkovskii ordered integer lattice[6] from decimal to binary as the follows,

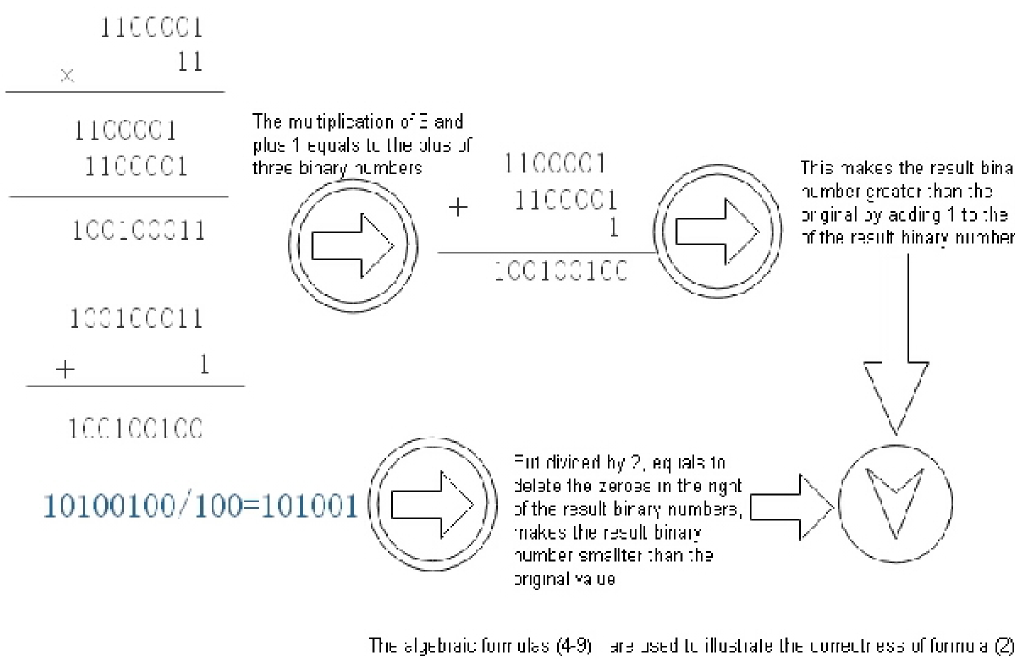

Fig. 3: For the Collatz function $T(97)$ in binary, the first step is the multiplication in left, the second step is division in the right bottom.

1,	11,	101,	111,	1001,	1011,	1101,	1111,	10001,	$\cdots$
10,	110,	1010,	110,	10010,	10110,	11010,	11110,	100010,	$\cdots$
100,	1100,	10100,	1100,	100100,	101100,	110100,	111100,	1000100,	$\cdots$
1000,	11000,	101000,	11000,	1001000,	1011000,	1101000,	1111000,	10001000,	$\cdots$
10000,	110000,	1010000,	110000,	10010000,	10110000,	11010000,	11110000,	100010000,	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Example 3 For positive integer 1, we manipulate the iteration of the Collatz function in both decimal and binary numbers,

ith	*0*	*1*	*2*	*3*
decimal	*1*	*4*	*2*	*1*
binary	*1*	*100*	*10*	*1*

Notes

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Example 4 *For positive integer* $5 = (101)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers,*

ith	*0*	*1*	*2*	*3*	*4*	*5*
decimal	*5*	*16*	*8*	*4*	*2*	*1*
binary	*101*	*10000*	*1000*	*100*	*10*	*1*

Example 5 *For* $7 = (111)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*7*	*8*	*11*	*12*	*16*
decimal	*7*	*22*	*11*	*34*	*17*	*52*	*13*	*40*	*5*	*16*	*1*
binary	*111*	*10110*	*1011*	*100010*	*10001*	*110100*	*1101*	*101000*	*101*	*10000*	*1*

Example 6 *For* $9 = (1001)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*14*	*15*	*19*
decimal	*9*	*28*	*14*	*7*	*22*	*11*	*34*	*17*	*5*	*16*	*1*
binary	*1001*	*11100*	*1110*	*111*	*10110*	*1011*	*100010*	*10001*	*101*	*10000*	*1*

Example 7 *For* $97 = (1100001)_2$, *we manipulate the iteration of the Collatz function in binary as the following table,*

97	**1100001**	206	11001110	**425**	**110101001**	866	1101100010
292	100100100	**103**	**1100111**	1276	10011111100	**433**	**110110001**
146	10010010	310	100110110	638	1001111110	1300	10100010100
73	**1001001**	**155**	**10011011**	**319**	**100111111**	650	1010001010
220	11011100	466	111010010	958	1110111110	**325**	**101000101**
110	1101110	**233**	**11101001**	**479**	**111011111**	976	1111010000
55	**110111**	700	1010111100	1438	10110011110	488	111101000
166	10100110	350	101011110	719	**1011001111**	244	11110100
83	**1010011**	**175**	**10101111**	2158	100001101110	122	1111010
250	11111010	526	1000001110	**1079**	**10000110111**	**61**	**111101**
125	**1111101**	**263**	**100000111**	3238	110010100110	184	10111000
376	101111000	790	1100010110	**1619**	**11001010011**	92	1011100
188	10111100	**395**	**110001011**	4858	1001011111010	46	101110
94	1011110	1186	10010100010	**2429**	**100101111101**	**23**	**10111**
47	**101111**	**593**	**1001010001**	7288	1110001111000	70	1000110
142	10001110	1780	11011110100	3644	111000111100	**35**	**100011**
71	**1000111**	890	1101111010	1822	11100011110	106	1101010
214	11010110	**445**	**110111101**	**911**	**1110001111**	**53**	**110101**
107	**1101011**	1336	10100111000	2734	101010101110	160	10100000
322	101000010	668	1010011100	**1367**	**10101010111**	80	1010000
161	**10100001**	334	101001110	4102	1000000000110	40	101000
484	111100100	**167**	**10100111**	2051	**100000000011**	20	10100
242	11110010	502	111110110	6154	1100000001010	10	1010
121	**1111001**	**251**	**11111011**	**3077**	**110000000101**	**5**	**101**
364	101101100	754	1011110010	9232	10010000010000	16	10000
182	10110110	**377**	**101111001**	4616	1001000001000	8	1000
91	**1011011**	1132	10001101100	2308	100100000100	4	100
274	100010010	566	1000110110	1154	10010000010	2	10
137	**10001001**	**283**	**100011011**	**577**	**1001000001**	1	1
412	110011100	850	1101010010	1732	11011000100		

Notes

Corollary 8 *The Collatz function makes an odd integer number in binary bigger by adding 1 or 2 bits to the left of the sequence of 1s and 0s, and an even integer number in binary smaller by deleting all zeros at the end of the sequence of 1s and 0s. Thus, although in some cases the value of The Collatz function $T(x)$ may be bigger than x in decimal, in general, the iteration of the Collatz function will make an integer number smaller and smaller, eventually reaching the smallest positive integer number 1.*

Notes

We can rewrite the Collatz conjecture in binary, which makes it an easier problem to solve, thus allowing us to completely solve the Collatz conjecture.

Fact 9 *For any positive integer, under the Collatz function, the sequence of integer numbers in binary will eventually reach 1.*

Proof. *For an odd binary integer, we multiply it by $(11)_2$ and add 1 in the last bit, the result number must be an even number in binary which at least one zero at the end. We delete all these zeros, namely it is the division. This is the above corollary 8. Thus, we repeat this process as long as we can, because the bits of the sequence in binary of a positive integer number is finite. Eventually, we must in finitely steps reach the smallest positive integer number 1.*

Remark 10 *We can say that the 3x+1 problem is a converse proposition of "period three implies chaos" [4], and it is also an example of any one positive integer number having a period of 3 in the Collatz function.*

IV. Conclusions

We rewrite the Collatz function in binary, which makes the $3x + 1$ problem easier. The multiplications of $(11)_2$ and divisions of $(10)_2$ make the positive integer number smaller and smaller with the iterations of the Collatz function. In some cases, the value of the Collatz function $T(x)$ may be bigger than x, thus allowing us to completely solve the Collatz conjecture.

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