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An Extension of 'In-Radius Property' of Pythagorean Triangles

By K. B Subramaniam & Aji Thomas
Regional Institute of Education

Introduction- It is a well-known fact that the in- radius of a Pythagorean triangle (A right-triangle whose sides form a Pythagorean triple) is always an integer [1]. The purpose of this note is to extent this result in the following sense.

If in any Pythagorean triangle a string of a finite number (say, k) of equal circles, inside the triangle, are so taken that

i. each of the k circles touches a given side (other than the hypotenuse)
ii. each of the (k-2) non-extreme circles also touch the two neighbouring circles.
iii. the extreme two circles touch the nearest other side also.

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An Extension of 'In-Radius Property' of Pythagorean Triangles

K. B Subramaniam [α] & Aji Thomas [σ]

Introduction

It is a well-known fact that the in- radius of a Pythagorean triangle (A right-triangle whose sides form a Pythagorean triple) is always an integer [1]. The purpose of this note is to extent this result in the following sense.

If in any Pythagorean triangle a string of a finite number (say, k) of equal circles, inside the triangle, are so taken that

i. each of the k circles touches a given side (other than the hypotenuse)
ii. each of the (k-2) non-extreme circles also touch the two neighbouring circles.
iii. the extreme two circles touch the nearest other side also.

We claim that these circles will have a rational radius for all k. We also work out the value of r explicitly.

Before proceeding for the proof, we need to use the following facts

a. A special category of **Pythagorean triples** is that of primitive Pythagorean triples which are merely Pythagorean triples having no common factors.

b. Every Pythagorean triple is of the form $2ab$, a^2-b^2, a^2+b^2, where a and b are positive coprime integers and $a>b$ [2].

Proof:

Let ΔABC be right angled at B.

Without loss of generality, we can assume that the sides of ΔABC form a primitive pythagorean triple. Let $AB=2ab$, $BC=a^2-b^2$ and $AC=a^2+b^2$, where a and b are coprime with $a>b$.

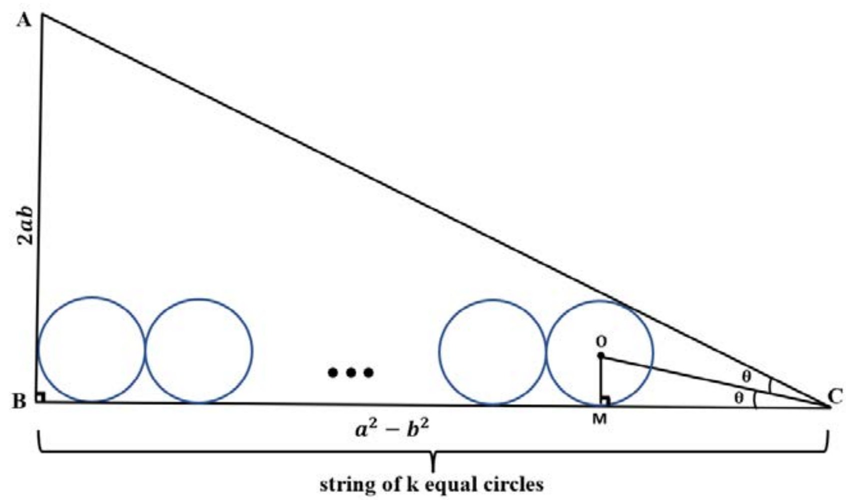
Author α: 135, 'Ganit Ashram' Fine Avenue, Phase 1, Nayapura, Kolar Road, Bhopal- 462042, India.
e-mail: kbsubramaniam.1950@gmail.com
Author σ: Department of Education in Science and Mathematics, Regional Institute of Education, Bhopal, (National Council of Educational Research and Training), India. e-mail: aji.thomas20@gmail.com

Ref

1. Burton, David M. *Elementary Number Theory, -7th ed.* McGraw-Hill, *New York (2011)* p.250.

We need to consider two cases depending on whether the strings of circles are taken on BC or on AB. Accordingly, we have to prove our assertion considering both the cases.

Case 1: String of circles lying along BC



It may be noted here that in this case $(2k-1)r < (a^2 - b^2)$ (1)

Let O be the centre of the circle (nearest to AC) and OM ⊥ BC.

Let r be the radius of each of these circles.

Clearly, OC bisects ∠ACB. Let ∠ACB = 2θ so that ∠OCB = θ.

We have,

$$\tan\theta = \frac{\text{OM}}{\text{MC}} = \frac{r}{a^2 - b^2 - (2k-1)r}$$

Also,

$$\tan 2\theta = \frac{\text{AB}}{\text{BC}} = \frac{2ab}{a^2 - b^2}$$

$$\Rightarrow \frac{2ab}{a^2 - b^2} = \frac{\dfrac{2r}{(a^2 - b^2) - (2k-1)r}}{1 - \dfrac{r^2}{[(a^2 - b^2) - (2k-1)r]^2}}$$ (By the duplication formula for tangent function)

$$\Rightarrow \{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}r^2 - (a^2 - b^2)(4abk + a^2 - b^2 - 2ab)r + ab(a^2 - b^2)^2 = 0$$

Notes

Notes

i.e, $Ar^2 - Br + C = 0$

where,

$$A = 4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)$$

$$B = (a^2 - b^2)(4abk + a^2 - b^2 - 2ab)$$

$$C = ab(a^2 - b^2)^2$$

we have,

$$B^2 - 4AC = (a^2 - b^2)^2(4abk + a^2 - b^2 - 2ab)^2 - 4ab(a^2 - b^2)^2\{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}$$

$$= (a^4 - b^4)^2$$

which clearly asserts that r must be rational

now,

$$r = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{(a^2 - b^2)(4abk + a^2 - b^2 - 2ab) \pm (a^4 - b^4)}{2\{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}}$$

$$= \frac{(a^2 - b^2)(4abk + a^2 - b^2 - 2ab) \pm (a^4 - b^4)}{2(2ak - b - a)(2bk - b + a)}$$

Here we claim that we have to discard the plus sign.

Because if we take plus sign then r becomes

$$\frac{(a^2 - b^2)\{(4abk + a^2 - b^2 - 2ab) + (a^2 + b^2)\}}{2(2ak - b - a)(2bk - b + a)}$$

$$= \frac{a(a^2 - b^2)(2bk - b + a)}{(2ak - b - a)(2bk - b + a)}$$

$$= \frac{a(a^2 - b^2)}{(2ak - b - a)}$$

By the condition (1), we have

$$\frac{a^2 - b^2}{r} > 2k - 1$$

$$\Rightarrow \frac{2ak-b-a}{a} > 2k-1$$

$$\Rightarrow -b > 0$$

Which is absurd. This justifies our claim
If we take minus sign r becomes

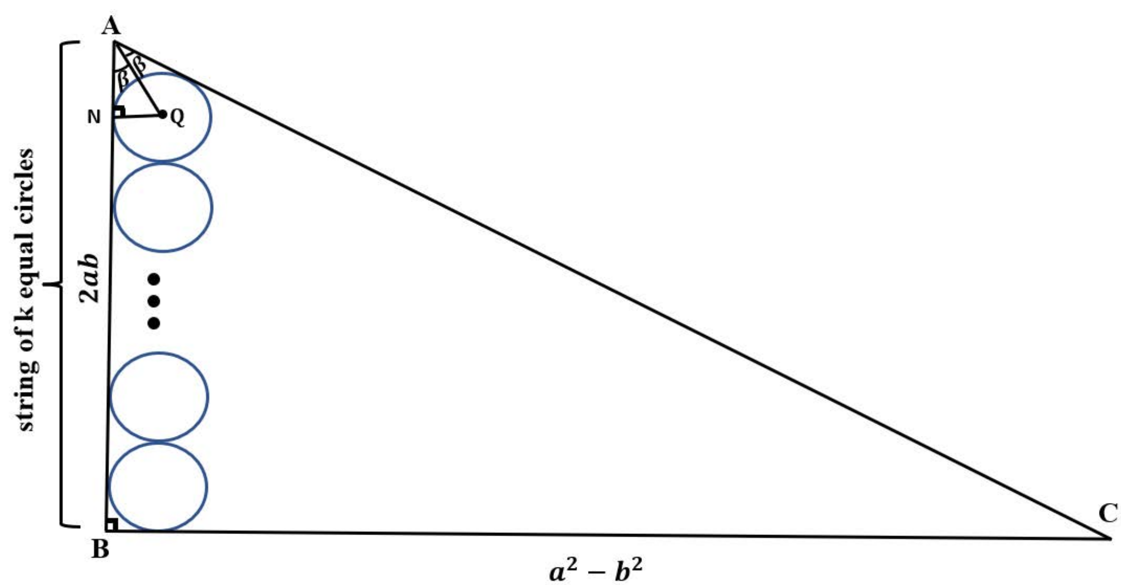
$$\frac{(a^2-b^2)\{(4abk+a^2-b^2-2ab)-(a^2+b^2)\}}{2(2ak-b-a)(2bk-b+a)}$$

$$=\frac{b(a^2-b^2)(2ak-b-a)}{(2bk-b+a)(2ak-b-a)}$$

$$=\frac{b(a^2-b^2)}{(2k-1)b+a}$$

Clearly, this option satisfies the condition (1)

Case 2: String of circles lying along BC



It may be noted here that **(2k-1)r < 2ab** (2)

Let Q be the centre of the circle (nearest to AC) and QN ⊥ AB.

Let r be the radius of each of these circles.

Notes

Notes

Clearly, QA bisects $\angle$BAC. Let $\angle$BAC = 2β so that $\angle$QAB = β.

We have,

$$\tan\beta = \frac{QN}{NA} = \frac{r}{2ab-(2k-1)r}$$

Also,

$$\tan 2\beta = \frac{BC}{AB} = \frac{a^2-b^2}{2ab}$$

$$\Rightarrow \frac{a^2-b^2}{2ab} = \frac{\frac{2r}{2ab-(2k-1)r}}{1-\frac{r^2}{\{2ab-(2k-1)r\}^2}}$$ (By the duplication formula for tangent function)

$$\Rightarrow \{(a^2-b^2)(k^2-k)+ab(2k-1)\}r^2-(ab)\{(a^2-b^2)(2k-1)+2(ab)\}r+(a^2-b^2)(ab)^2=0$$

i.e, $Ar^2-Br+C=0$, where

$$A=(a^2-b^2)(k^2-k)+ab(2k-1)$$

$$B=(ab)\{(a^2-b^2)(2k-1)+2(ab)\}$$

$$C=(a^2-b^2)(ab)^2$$

$$B^2-4AC=(ab)^2\{(a^2-b^2)(2k-1)+2(ab)\}^2-4\{(a^2-b^2)(k^2-k)+ab(2k-1)\}(a^2-b^2)(ab)^2$$

$$=(ab)^2(a^2+b^2)^2$$

which clearly asserts that r must be rational

now,

$$r=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

$$=\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\}\pm ab(a^2+b^2)}{2\{(a^2-b^2)(k^2-k)+ab(2k-1)\}}$$

$$= \frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} \pm ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

Here we claim that we have to discard the plus sign.

Because if we take plus sign then r becomes

$$\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} + ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{2(ab)(a+b)(ka-kb+b)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{(ab)(a+b)}{(ka+kb-a)}$$

By the condition (2), we have

$$\frac{(2k-1)r}{2} < ab$$

$$\Rightarrow \frac{r(ka+kb-a)}{(a+b)} > \frac{(2k-1)r}{2}$$

$$\Rightarrow k < \frac{b}{a+b}$$

Which is absurd (as k is an integer). This justifies our claim

If we take minus sign r becomes

$$\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} - ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{ab(a-b)(ka+kb-a)}{(ka-kb+b)(ka+kb-a)}$$

$$= \frac{ab(a-b)}{k(a-b)+b}$$

Clearly, this option satisfies the condition (2)

Notes

Remark

Interestingly, the values of r in in both the cases are free from k in the numerator. □

Notes

References Références Referencias

1. Burton, David M. *Elementary Number Theory, -7th ed.* McGraw-Hill, *New York (2011)* p.250.
2. Burton, David M. *Elementary Number Theory, -7th ed.* McGraw-Hill, *New York (2011)* p.248.