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Highlights

- Random Error Vector
- Holography of Traversing Flows
- Ordinary Differential Equations
- Formula of Stamp Folding Problem

Discovering Thoughts, Inventing Future

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Algebras of Smooth Functions and Holography of Traversing Flows

By Gabriel Katz

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Abstract- Let X be a smooth compact manifold and v a vector field on X which admits a smooth function f : X ! R such that df(v) > 0. Let @X be the boundary of X. We denote by C1(X) the algebra of smooth functions on X and by C1(@X) the algebra of smooth functions on @X. With the help of (v; f), we introduce two subalgebras A(v) and B(f) of C1(@X) and prove (under mild hypotheses) that C1(X) _ A(v) ^B(f), the topological tensor product. Thus the topological algebras A(v) and B(f), viewed as *boundary data*, allow for a reconstruction of C1(X). As a result, A(v) and B(f) allow for the recovery of the smooth topological type of the bulk X.

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Algebras of Smooth Functions and Holography of Traversing Flows

Gabriel Katz

Abstract- Let X be a smooth compact manifold and v a vector field on X which admits a smooth function $f: X \to \mathbb{R}$ such that $df(v) > 0$. Let ∂X be the boundary of X. We denote by $C^\infty(X)$ the algebra of smooth functions on X and by $C^\infty(\partial X)$ the algebra of smooth functions on ∂X. With the help of (v, f), we introduce two subalgebras $\mathcal{A}(v)$ and $\mathcal{B}(f)$ of $C^\infty(\partial X)$ and prove (under mild hypotheses) that $C^\infty(X) \approx \mathcal{A}(v)\hat{\otimes}\mathcal{B}(f)$, the topological tensor product. Thus the topological algebras $\mathcal{A}(v)$ and $\mathcal{B}(f)$, viewed as *boundary data*, allow for a reconstruction of $C^\infty(X)$. As a result, $\mathcal{A}(v)$ and $\mathcal{B}(f)$ allow for the recovery of the smooth topological type of the bulk X.

I. Introduction

It is classically known that the normed algebra $C^0(X)$ of continuous real-valued functions on a compact space X determines its topological type [GRS], [Ga], [Br]. In this context, X is interpreted as the space of maximal ideals of the algebra $C^0(X)$. In a similar spirit, the algebra $C^\infty(X)$ of smooth functions on a compact smooth manifold X (the algebra $C^\infty(X)$ is considered in the Whitney topology [W3]) determines the *smooth* topological type of X [KMS], [Na]. Again, X may be viewed as the space of maximal ideals of the algebra $C^\infty(X)$.

Recall that a harmonic function h on a compact connected Riemannian manifold X is uniquely determined by its restriction to the smooth boundary ∂X of X. In other words, the Dirichlet boundary value problem has a unique solution in the space of harmonic functions. Therefore, the vector space $\mathcal{H}(X)$ of harmonic functions on X is rigidly determined by its restriction (trace) $\mathcal{H}^\partial(X) := \mathcal{H}(X)|_{\partial X}$ to the boundary ∂X. As we embark on our journey, this fact will serve us as a beacon.

This paper revolves around the following question:

Which algebras of smooth functions on the boundary ∂X can be used to reconstruct the algebra $C^\infty(X)$ and thus the smooth topological type of X?

Remembering the flexible nature of smooth functions (in contrast with the rigid harmonic ones), at the first glance, we should anticipate the obvious answer "None!". However, when X carries an additional geometric structure, then the question, surprisingly, may have a positive answer. The geometric structure on X that does the trick is a vector field (i.e., an ordinary differential equation), drawn from a massive class of vector fields which we will introduce below.

Let X be a compact connected smooth $(n+1)$-dimensional manifold with boundary and v a smooth vector field admitting a Lyapunov function $f : X \to \mathbb{R}$ so that $df(v) > 0$. We call such vector fields traversing. We assume that v is in general position with respect to the boundary ∂X and call such vector fields boundary generic (see [K1] or [K3], Definition 5.1, for the notion of boundary generic vector fields). Temporarily, it will be sufficient to

Ref

[Br] Louis de Branges, L., *The Stone-Weierstrass theorem*, Proc. Amer. Math. Soc., (1959) 10 (5), 822-824.

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think of the boundary generic vector fields v as having only v-trajectories that are tangent to the boundary ∂X with the order of tangency less than or equal to $\dim(X)$. Section 3 contains a more accurate definition.

Informally, we use the term "holography" when some residual structures on the boundary ∂X are sufficient for a reconstruction of similar structures on the bulk X.

Given such a triple (X, v, f), in Section 3, we will introduce two subalgebras, $\mathcal{A}(v) = C^\infty(\partial X, v)$ and $\mathcal{B}(f) = (f^\partial)^*(C^\infty(\mathbb{R}))$, of the algebra $C^\infty(\partial X)$, which depend only on v and f, respectively. By Theorem 3.1, $\mathcal{A}(v)$ and $\mathcal{B}(f)$ will allow for a reconstruction of the algebra $C^\infty(X)$. In fact, the boundary data, generated by these subalgebras, lead to a unique (rigid) "solution"

$$C^\infty(X) \approx C^\infty(\partial X, v)\, \hat{\otimes}\, (f^\partial)^*(C^\infty(\mathbb{R})),$$

the topological tensor product of the two algebras. As a result, the pair $\mathcal{A}(v)$, $\mathcal{B}(f)$, "residing on the boundary", determines the smooth topological type of the bulk X and of the 1-dimensional foliation $\mathcal{F}(v)$, generated by the v-flow.

II. Holography on Manifolds with Boundary and the Causality Maps

Let X be a compact connected smooth $(n+1)$-dimensional manifold with boundary $\partial_1 X =_{\mathsf{def}} \partial X$ (we use this notation for the boundary ∂X to get some consistency with similar notations below), and v a smooth traversing vector field, admitting a smooth Lyapunov function $f : X \to \mathbb{R}$. We assume that v is boundary generic.

We denote by $\partial_1^+ X(v)$ the subset of $\partial_1 X$ where v is directed inwards of X or is tangent to $\partial_1 X$. Similarly, $\partial_1^- X(v)$ denotes the subset of $\partial_1 X$ where v is directed outwards of X or is tangent to $\partial_1 X$.

Let $\mathcal{F}(v)$ be the 1-dimensional oriented foliation, generated by the traversing v-flow.

We denote by γ_x the v-trajectory through $x \in X$. Since v is traversing and boundary generic, each γ_x is homeomorphic either a closed segment, or to a singleton [K1].

In what follows, we embed the compact manifold X in an open manifold $\hat{X}$ of the same dimension so that v extends to a smooth vector field $\hat{v}$ on $\hat{X}$, f extends to a smooth function $\hat{f}$ on $\hat{X}$, and $d\hat{f}(\hat{v}) > 0$ in $\hat{X}$. We treat $(\hat{X}, \hat{v}, \hat{f})$ as a germ in the vicinity of (X, v, f).

Definition 2.1. *We say that a boundary generic and traversing vector field v possesses* Property A, *if each v-trajectory γ is either transversal to $\partial_1 X$ at* some *point of the set $\gamma \cap \partial_1 X$, or $\gamma \cap \partial_1 X$ is a singleton x and γ is quadratically tangent to $\partial_1 X$ at x.* ♢

A traversing vector field v on X induces a structure of a partially-ordered set $(\partial_1 X, \succ_v)$ on the boundary $\partial_1 X$: for $x, y \in \partial_1 X$, we write $y \succ x$ if the two points lie on the same v-trajectory γ and y is reachable from x by moving in the v-direction.

We denote by $\mathcal{T}(v)$ the trajectory space of v and by $\Gamma : X \to \mathcal{T}(v)$ the obvious projection. For a traversing and boundary generic v, $\mathcal{T}(v)$ is a compact space in the topology induced by Γ. Since any trajectory of a traversing v intersects the boundary $\partial_1 X$, we get that $\mathcal{T}(v)$ is a quotient of $\partial_1 X$ modulo the partial order relation $\succ_v$.

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Ref

[K1] Katz, G., *Traversally Generic & Versal Flows: Semi-algebraic Models of Tangency to the Boundary* Asian J. of Math., vol. 21, No. 1 (2017), 127-168 (arXiv: 1407.1345v1 [mathGT] 4 July, 2014)).

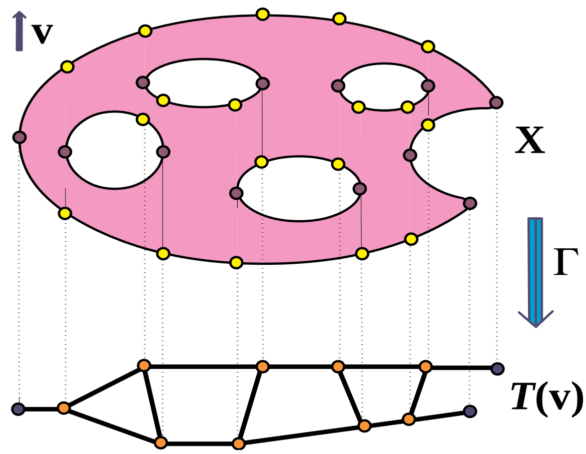

Figure 1: The map $\Gamma : X \to \mathcal{T}(v)$ for a traversally generic (vertical) vector field v on a disk with 4 holes. The trajectory space is a graph whose verticies are of valencies 1 and 3. The restriction of Γ to $\partial_1 X$ is a surjective map Γ^∂ with finite fibers of cardinality 3 at most; a generic fiber has cardinality 2.

A traversing and boundary generic v gives rise to the causality (scattering) map

$$C_v : \partial_1^+ X(v) \to \partial_1^- X(v) \tag{2.1}$$

that takes each point $x \in \partial_1^+ X(v)$ to the unique consecutive point $y \in \gamma_x \cap \partial_1^- X(v)$ that can be reached from x in the v-direction. If no such $y \neq x$ is available, we put $C_v(x) = x$. We stress that typically C_v is a *discontinuous* map (see Fig. 2).

We notice that, for any smooth positive function $\lambda : X \to \mathbb{R}_+$, we have $C_{\lambda \cdot v} = C_v$; thus the causality map depends only on the conformal class of a traversing vector field v. In fact, C_v depends only on the oriented foliation $\mathcal{F}(v)$, generated by the v-flow.

In the paper, we will discuss two kinds of intimately related holography problems. The first kind amounts to the question: To what extend given boundary data are sufficient for reconstructing the unknown bulk and the traversing v-flow on it, or rather, the foliation $\mathcal{F}(v)$? This question may be represented symbolically by the two diagrams:

- Holographic Reconstruction Problem

$$(\partial_1 X,\ \succ_v,) \stackrel{??}{\longrightarrow} (X,\ \mathcal{F}(v)), \tag{2.2}$$

$$(\partial_1 X,\ \succ_v,\ f^\partial) \stackrel{??}{\longrightarrow} (X,\ \mathcal{F}(v),\ f), \tag{2.3}$$

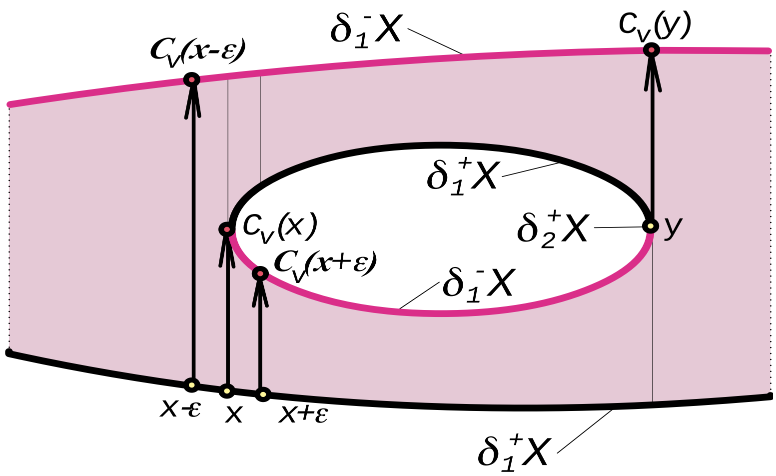

Figure 2: An example of the causality map $C_v : \partial_1^+ X(v) \to \partial_1^- X(v)$. Note the essential discontinuity of C_v in the vicinity of x.

where $\succ_v$ denotes the partial order on boundary, defined by the causality map C_v, and the symbol "$\stackrel{??}{\longrightarrow}$" points to the unknown ingredients of the diagrams.

The second kind of problem is: Given two manifolds, X_1 and X_2, equipped with traversing flows, and a diffeomorphism Φ^∂ of their boundaries, respecting the relevant boundary data, is it possible to extend Φ^∂ to a diffeomorphism/homeomorphism $\Phi : X_1 \to X_2$ that respects the corresponding flows-generated structures in the interiors of the two manifolds?
This problem may be represented by the commutative diagrams:

•Holographic Extension Problem

$$\begin{array}{ccc} (\partial_1 X_1,\ \succ_{v_1}) & \stackrel{\mathsf{inc}}{\longrightarrow} & (X_1,\ \mathcal{F}(v_1)) \\ \downarrow\ \Phi^\partial & & \downarrow ??\ \ \Phi \\ (\partial_1 X_2,\ \succ_{v_2}) & \stackrel{\mathsf{inc}}{\longrightarrow} & (X_2,\ \mathcal{F}(v_2)) \end{array} \tag{2.4}$$

$$\begin{array}{ccc} (\partial_1 X_1,\ \succ_{v_1},\ f_1^\partial) & \stackrel{\mathsf{inc}}{\longrightarrow} & (X_1,\ \mathcal{F}(v_1), f_1) \\ \downarrow\ \Phi^\partial & & \downarrow ??\ \ \Phi \\ (\partial_1 X_2,\ \succ_{v_2},\ f_2^\partial) & \stackrel{\mathsf{inc}}{\longrightarrow} & (X_2,\ \mathcal{F}(v_2),\ f_2), \end{array} \tag{2.5}$$

where inc denotes the inclusion of spaces, accompanied by the obvious restrictions of functions and foliations. The symbol "$\downarrow ??$ " indicates the unknown maps in the diagrams.

These two types of problems come in a big variety of flavors, depending on the more or less rich boundary data and on the anticipated quality of the transformations Φ (homeomorphisms, PD-homeomorphisms, Hölder homeomorphisms with some control of the Hölder exponent, and diffeomorphisms with different degree of smoothness).

Let us formulate the main result of [K4], Theorem 4.1, which captures the philosophy of this article and puts our main result, Theorem 3.1, in the proper context. Theorem 2.1 reflects the scheme depicted in (2.4).

Theorem 2.1. **(Conjugate Holographic Extensions)** *Let X_1, X_2 be compact connected oriented smooth $(n+1)$-dimensional manifolds with boundaries. Consider two traversing*

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Ref

[K4] Katz, G., *Causal Holography of Traversing Flows*, Journal of Dynamics and Differential Equations (2020) https://doi.org/10.1007/s10884-020-09910-y

boundary generic vector fields v_1, v_2 *on* X_1 *and* X_2*, respectively. In addition, assume that* v_1, v_2 *have Property* A *from Definition 2.1.*

Let a smooth orientation-preserving diffeomorphism $\Phi^\partial : \partial_1 X_1 \to \partial_1 X_2$ *commute with the two causality maps:*

$$C_{v_2} \circ \Phi^\partial = \Phi^\partial \circ C_{v_1}$$

Then Φ^∂ *extends to a smooth orientation-preserving diffeomorphism* $\Phi : X_1 \to X_2$ *such that* Φ *maps the oriented foliation* $\mathcal{F}(v_1)$ *to the oriented foliation* $\mathcal{F}(v_2)$*.*

Ref

[K4] Katz, G., *Causal Holography of Traversing Flows*, Journal of Dynamics and Differential Equations (2020) https://doi.org/10.1007/s10884-020-09910-y

Let us outline the spirit of Theorem 2.1's proof, since this will clarify the main ideas from Section 3. The reader interested in the technicalities may consult [K4].

Proof. First, using that v_2 is traversing, we construct a Lyapunov function $f_2 : X_2 \to \mathbb{R}$ for v_2. Then we pull-back, via the diffeomorphism Φ^∂, the restriction $f_2^\partial := f_2|_{\partial_1 X_2}$ to the boundary $\partial_1 X_2$. Since Φ^∂ commutes with the two causality maps, the pull back $f_1^\partial =_{\mathsf{def}} (\Phi^\partial)^*(f_2^\partial)$ has the property $f_1^\partial(y) > f_1^\partial(x)$ for any pair $y \succ x$ on the same v_1-trajectory, the order of points being defined by the v_1-flow. Equivalently, we get $f_1^\partial(C_{v_1}(x)) > f_1^\partial(x)$ for any $x \in \partial_1^+ X(v_1)$ such that $C_{v_1}(x) \neq x$. As the key step, we prove in [K4] that such f_1^∂ extends to a smooth function $f_1 : X_1 \to \mathbb{R}$ that has the property $df_1(v_1) > 0$. Hence, f_1 is a Lyapunov function for v_1.

Recall that each causality map C_{v_i}, $i = 1, 2$, allows to view the v_i-trajectory space $\mathcal{T}(v_i)$ as the quotient space $(\partial_1 X_i)/\{C_{v_i}(x) \sim x\}$, where $x \in \partial_1^+ X_i(v_i)$ and the topology in $\mathcal{T}(v_i)$ is defined as the quotient topology. Using that Φ^∂ commutes with the causality maps C_{v_1} and C_{v_2}, we conclude that Φ^∂ induces a homeomorphism $\Phi^{\mathcal{T}} : \mathcal{T}(v_1) \to \mathcal{T}(v_2)$ of the trajectory spaces, which preserves their natural stratifications.

For a traversing v_i, the manifold X_i carries two mutually transversal foliations: the oriented 1-dimensional $\mathcal{F}(v_i)$, generated by the v_i-flow, and the foliation $\mathcal{G}(f_i)$, generated by the constant level hypersurfaces of the Lyapunov function f_i. To avoid dealing the singularities of $\mathcal{F}(v_i)$ and $\mathcal{G}(f_i)$, we extend f_i to $\hat{f}_i : \hat{X}_i \to \mathbb{R}$ and v_i to $\hat{v}_i$ on $\hat{X}_i$ so that $d\hat{f}_i(\hat{v}_i) > 0$. This generates nonsingular foliations $\mathcal{F}(\hat{v}_i)$ and $\mathcal{G}(\hat{f}_i)$ on $\hat{X}_i$. By this construction, $\mathcal{F}(\hat{v}_i)|_{\hat{X}_i} = \mathcal{F}(v_i)$ and $\mathcal{G}(\hat{f}_i)|_{X_i} = \mathcal{G}(f_i)$. Note that the "leaves" of $\mathcal{G}(f_i)$ may be disconnected, while the leaves of $\mathcal{F}(v_i)$, the v_i-trajectories, are connected. The two smooth foliations, $\mathcal{F}(\hat{v}_i)$ and $\mathcal{G}(\hat{f}_i)$, will serve as a "coordinate grid" on X_i: every point $x \in X_i$ belongs to a *unique* pair of leaves $\gamma_x \in \mathcal{F}(v_i)$ and $L_x := \hat{f}_i^{-1}(f_i(x)) \in \mathcal{G}(\hat{f}_i)$.

Conversely, using the traversing nature of v_i, any pair (y, t), where $y \in \gamma_x \cap \partial_1 X_i$ and $t \in [f_i^\partial(\gamma_x \cap \partial_1 X_i)] \subset \mathbb{R}$, where $[f_i^\partial(\gamma_x \cap \partial_1 X_i)]$ denotes the minimal closed interval that contains the finite set $f_i^\partial(\gamma_x \cap \partial_1 X_i)$, determines a *unique* point $x \in X_i$. Note that some pairs of leaves L and γ may have an empty intersection, and some components of leaves L may have an empty intersection with the boundary $\partial_1 X_i$.

In fact, using that f_i is a Lyapunov function, the hyprsurface $L = f_i^{-1}(c)$ intersects with a v_i-trajectory γ if and only if $c \in [f_i^\partial(\gamma \cap \partial_1 X_i)]$. Since the two smooth leaves, $\hat{\gamma}_y$ and $\hat{f}_i^{-1}(f_i(z))$, depend smoothly on the points $y, z \in \partial_1 X_i$ and are transversal, their intersection point $\hat{\gamma}_y \cap \hat{f}_i^{-1}(f_i(z)) \in \hat{X}_i$ depends smoothly on $(y, z) \in (\partial_1 X_i) \times (\partial_1 X_i)$, as long as $f_i^\partial(z) \in [f_i^\partial(\gamma_y \cap \partial_1 X_i)]$. Note that pairs (y, z), where $y, z \in \partial_1 X_i$, with the property $f_i^\partial(z) \in f_i^\partial(\gamma_y \cap \partial_1 X_i)$ give rise to the intersections $\hat{\gamma}_y \cap \hat{f}_i^{-1}(f_i(z))$ that belong to $\partial_1 X_i$.

Now we are ready to extend the diffeomorphism Φ^∂ to a homeomorphism $\Phi : X_1 \to X_2$. In the process, following the scheme in (2.4), *we assume the the foliations* $\mathcal{F}(v_i)$ *and of the Lyapunov functions* f_i *on* X_i $(i = 1, 2)$ *do exist and are "knowable", although we have access only to their traces on the boundaries.*

Take any $x \in X_1$. It belongs to a unique pair of leaves $L_x \in \mathcal{G}(f_1)$ and $\gamma_x \in \mathcal{F}(v_1)$. We define $\Phi(x) = x' \in X_2$, where x' is the unique point that belongs to the intersection of $f_2^{-1}(f_1(x)) \in \mathcal{G}(f_2)$ and the v_2-trajectory $\gamma' = \Gamma_2^{-1}(\Phi^{\mathcal{T}}(\gamma_x))$. By its construction, $\Phi|_{\partial_1 X_1} = \Phi^\partial$. Therefore, Φ induces the same homeomorphism $\Phi^{\mathcal{T}} : \mathcal{T}(v_1) \to \mathcal{T}(v_2)$ as Φ^∂ does.

The leaf-hypersurface $\hat{f}_2^{-1}(f_1(x))$ depends smoothly on x, but the leaf-trajectory $\hat{\gamma}' = \Gamma_2^{-1}(\Phi^{\mathcal{T}}(\hat{\gamma}_x))$ may not! Although the homeomorphism Φ is a diffeomorphism along the v_1-trajectories, it is not clear that it is a diffeomorphism on X_1 (a priori, Φ is just a Hölder map with a Hölder exponent $\alpha = 1/m$, where m is the maximal tangency order of γ's to $\partial_1 X$). Presently, for proving that Φ is a diffeomorphism, we need Property A from Definition 2.1. Assuming its validity, we use the transversality of γ_x *somewhere* to $\partial_1 X$ to claim the smooth dependence of $\Gamma_2^{-1}(\Phi^{\mathcal{T}}(\hat{\gamma}_x))$ on x. Now, since the smooth foliations $\mathcal{F}(\hat{v}_i)$ and $\mathcal{G}(\hat{f}_i)$ are transversal, it follows that $x' = \Phi(x)$ depends smoothly on x. Conjecturally, Property A is unnecessary for establishing that Φ is a diffeomorphism. □

Note that this construction of the extension Φ is quite explicit, but not canonic. For example, it depends on the choice of extension of $f_1^\partial := (\Phi^\partial)^*(f_2^\partial)$ to a smooth function $f_1 : X_1 \to \mathbb{R}$, which is strictly monotone along the v_1-trajectories. The uniqueness (topological rigidity) of the extension Φ may be achieved, if one assumes *knowing fully* the manifolds X_i, equipped with the foliation grids $\mathcal{F}(v_i), \mathcal{G}(f_i)$ and the Lyapunov function f_i. In Theorem 3.1, we will reflect on this issue.

The next theorem (see [K4], Corollary 4.3) fits the scheme in (2.2). It claims that the *smooth topological type* of the triple $\{X, \mathcal{F}(v), \mathcal{G}(f)\}$ may be reconstructed from the appropriate boundary-confined data, provided that Property A is valid.

Corollary 2.1. **(Holography of Traversing Flows)** *Let X be a compact connected smooth $(n+1)$-dimensional manifold with boundary, and let v be a traversing boundary generic vector field, which possesses Property* A.

Then the following boundary-confined data:

- *the causality map $C_v : \partial_1^+ X(v) \to \partial_1^- X(v)$,*
- *the restriction $f^\partial : \partial_1 X \to \mathbb{R}$ of the Lyapunov function f,*

are sufficient for reconstructing the triple $(X, \mathcal{F}(v), f)$, up to diffeomorphisms $\Phi : X \to X$ which are the identity on the boundary $\partial_1 X$.

Proof. We claim that, in the presence of Property A, the data $\{C_v,\ f^\partial\}$ on the boundary $\partial_1 X$ allow for a reconstruction of the triple $(X, \mathcal{F}(v), f)$, up to a diffeomorphism that is the identity on $\partial_1 X$.

Assume that there exist two traversing flows $(X_1, \mathcal{F}(v_1), f_1)$ and $(X_2, \mathcal{F}(v_2), f_2)$ such that $\partial_1 X_1 = \partial_1 X_2 = \partial_1 X$,

$$\{C_{v_1},\ f_1^\partial\} = \{C_{v_2},\ f_2^\partial\} = \{C_v, f^\partial\}.$$

Applying Theorem 2.1 to the identity diffeomorphism $\Phi^\partial = \mathsf{id}_{\partial_1 \mathsf{X}}$, we conclude that it extends to a diffeomorphism $\Phi : X_1 \to X_2$ that takes $\{\mathcal{F}(v_1) \cap \partial_1 X_1,\ \ f_1^\partial\}$ to $\{\mathcal{F}(v_2) \cap \partial_1 X_2,\ \ f_2^\partial\}$. □

Remark 2.1. Unfortunately, Corollary 2.1 and its proof are not very constructive. They are just claims of existence: at the moment, it is not clear how to build the triple $(X, \mathcal{F}(v), f)$ only from the boundary data $(\partial_1 X, C_v, f^\partial)$. ♢

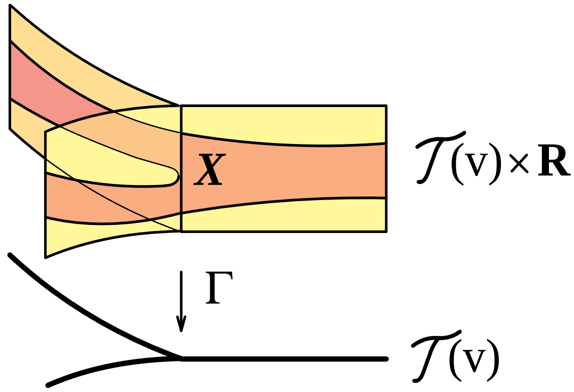

Figure 3: Embedding $\alpha : X \to \mathcal{T}(v) \times \mathbb{R}$, produced by $\Gamma : X \to \mathcal{T}(v)$ and $f : X \to \mathbb{R}$.

Fortunately, the following simple construction ([K4], Lemma 3.4), shown in Fig.3, produces an explicit recipe for recovering the triple $(X, \mathcal{F}(v), f)$ from the triple $(\partial_1 X, C_v, f^\partial)$, but only up to a *homeomorphism*.

As we have seen in the proof of Theorem 2.1, the causality map C_v determines the quotient trajectory space $\mathcal{T}(v)$ canonically. Let $f : X \to \mathbb{R}$ be a Lyapunov function for v.

The pair $(\mathcal{F}(v), f)$ gives rise to an embedding $\alpha : X \hookrightarrow \mathcal{T}(v) \times \mathbb{R}$, defined by the formula $\alpha(x) = ([\gamma_x], f(x))$, where $x \in X$ and $[\gamma_x] \in \mathcal{T}(v)$ denotes the point-trajectory through x.

The dependece $x \to [\gamma_x]$ is continuous by the definition of the quotient topology in $\mathcal{T}(v)$. Note that α maps each v-trajectory γ to the line $[\gamma] \times \mathbb{R}$, and, for any $c \in \mathbb{R}$, each (possibly disconnected) leaf $\mathcal{G}_c := f^{-1}(c)$ to the slice $\mathcal{T}(v) \times c$ of $\mathcal{T}(v) \times \mathbb{R}$. With the help of the embedding α, each trajectory $\gamma \in \mathcal{F}(v)$ may be identified with the closed interval $[f^\partial(\gamma \cap \partial_1 X)] \subset \mathbb{R}$, and the vector field $v|_\gamma$ with the constant vector field ∂_u on $\mathbb{R}$.

Consider now the restriction α^∂ of the embedding α to the boundary $\partial_1 X$. Evidently, the image of $\alpha^\partial : \partial_1 X \hookrightarrow \mathcal{T}(v) \times \mathbb{R}$ bounds the image $\alpha(X) = \coprod_{[\gamma] \in \mathcal{T}(v)} [f^\partial(\gamma \cap \partial_1 X)]$. Therefore, using the product structure in $\mathcal{T}(v) \times \mathbb{R}$, $\alpha^\partial(\partial_1 X)$ determines $\alpha(X)$ canonically. Hence, $\alpha(X)$ depends on C_v and f_1^∂ only! Note that α is a continuous 1-to-1 map on a compact space, and thus, a homeomorphism onto its image. Moreover, the topological type of X depends only on C_v: the apparent dependence of $\alpha(X)$ on f^∂ is not crucial, since, for a given v, the space $\mathsf{Lyap}(v)$ of Lyapunov functions for v is convex.

The standing issue is: How to make sense of the claim "α is a diffeomorphism"? Section 3 descibes our attempt to address this question (see Lemma 3.3 and Theorem 3.1).

III. Recovering the Algebra $C^\infty(X)$ in Terms of Subalgebras of $C^\infty(\partial_1 X)$

In what follows, we are inspired by the following classical property of functional algebras: for any compact smooth manifolds X, Y, we have an algebra isomorphism $C^\infty(X \times Y) \approx C^\infty(X) \hat{\otimes} C^\infty(Y)$, where $\hat{\otimes}$ denotes an appropriate completion of the algebraic tensor product $C^\infty(X) \otimes C^\infty(Y)$ [Grot].

The trajectory space $\mathcal{T}(v)$, although a singular space, carries a surrogate smooth structure [K3]. By definition, a function $h : \mathcal{T}(v) \to \mathbb{R}$ is smooth if its pull-back $\Gamma^*(h) : X \to \mathbb{R}$ is a smooth function on X. As a subspace of $C^\infty(X)$, the $C^\infty(\mathcal{T}(v))$ is formed exactly by the smooth functions $g : X \to \mathbb{R}$, whose directional derivatives $\mathcal{L}_v g$ vanish in X. If $\mathcal{L}_v(g) = 0$ and $\mathcal{L}_v(h) = 0$, then $\mathcal{L}_v(g \cdot h) = \mathcal{L}_v(g) \cdot h + g \cdot \mathcal{L}_v(h) = 0$. Thus, $C^\infty(\mathcal{T}(v))$ is indeed a subalgebra of $C^\infty(X)$.

Note that if we change v by a non-vanishing conformal factor λ, then $\mathcal{L}_v g = 0$ if and only if $\mathcal{L}_{\lambda\cdot v}\, g = 0$. Therefore, the algebra $C^\infty(\mathcal{T}(v))$ depends only on the conformal class of v; in other words, on the foliation $\mathcal{F}(v)$.

In the same spirit, we may talk about diffeomorphisms $\Phi^{\mathcal{T}} : \mathcal{T}(v) \to \mathcal{T}(v)$ of the trajectory spaces, as maps that induce isomorphisms of the algebra $C^\infty(\mathcal{T}(v))$.

If two (v-invariant) functions from $C^\infty(\mathcal{T}(v))$ take different values at a point $[\gamma] \in \mathcal{T}(v)$, then they must take different values on the finite set $\gamma \cap \partial_1 X \subset \partial_1 X$. Therefore, the obvious restriction homomorphism $res^{\partial}_{\mathcal{T}} : C^\infty(\mathcal{T}(v)) \to C^\infty(\partial_1 X)$, induced by the inclusion $\partial_1 X \subset X$, is a *monomorphism*. We denote its image by $C^\infty(\partial_1 X, v)$. Thus, we get an isomorphism $res^{\partial}_{\mathcal{T}} : C^\infty(\mathcal{T}(v)) \to C^\infty(\partial_1 X, v)$. We think of the subalgebra $C^\infty(\partial_1 X, v) \subset C^\infty(\partial_1 X)$ as an integral part of the boundary data for the holography problems we are tackling.

Let $\pi_k : J^k(X, \mathbb{R}) \to X$ be the vector bundle of k-jets of smooth maps from X to $\mathbb{R}$. We choose a continuous family semi-norms $| \sim |_k$ in the fibers of the jet bundle π_k and use it to define a sup-norm $\| \sim \|_k$ for the sections of π_k. We denote by jet^k the obvious map $C^\infty(X, \mathbb{R}) \to J^k(X, \mathbb{R})$ that takes each function h to the collection of its k-jets $\{jet^k_x(h)\}_{x \in X}$.

The Whitney topology [W3] in the space $C^\infty(X) = \{h : X \to \mathbb{R}\}$ is defined in terms of the countable family of the norms $\{\|jet^k(h)\|_k\}_{k \in \mathbb{N}}$ of such sections $jet^k(h)$ of π_k. This topology insures the uniform convergence, on the compact subsets of X, of functions and their partial derivatives of an arbitrary order. Note also that $\|jet^k(h_1 \cdot h_2)\|_k \leq \|jet^k(h_1)\|_k \cdot \|jet^k(h_2)\|_k$ for any $h_1, h_2 \in C^\infty(X)$.

Any subalgebra $\mathcal{A} \subset C^\infty(X)$ inherits a topology from the Whitney topology in $C^\infty(X)$. In particular, the subalgebra $C^\infty(\mathcal{T}(v)) \approx C^\infty(X, v)$ does.

As a locally convex vector spaces, $C^\infty(\mathcal{T}(v))$ and $C^\infty(\mathbb{R})$ are then nuclear ([DS], [Ga]) so that the topological tensor product $C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, C^\infty(\mathbb{R})$ (over $\mathbb{R}$) is uniquely defined as the completion of the algebraic tensor product $C^\infty(\mathcal{T}(v)) \otimes C^\infty(\mathbb{R})$ [Grot].

We interpret $C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, C^\infty(\mathbb{R})$ as the algebra of "smooth" functions on the product $\mathcal{T}(v) \times \mathbb{R}$ and denote it by $C^\infty(\mathcal{T}(v) \times \mathbb{R})$.

Lemma 3.1. The intersection $C^\infty(\mathcal{T}(v)) \cap (f)^*(C^\infty(\mathbb{R})) = \underline{\mathbb{R}}$, the space of constant functions on X.

Proof. If a smooth function $h : X \to \mathbb{R}$ is constant on each v-trajectory γ and belongs to $(f)^*(C^\infty(\mathbb{R}))$, then it must be constant on each connected leaf of $\mathcal{G}(f)$ that intersects γ. Thus, such h is constant on the maximal closed *connected* subset $A_\gamma \subseteq f^{-1}(f(\gamma))$ that contains γ. Each trajectory γ, homeomorphic to a closed interval, has an open neighborhood such that, for any trajectory γ' from that neighborhood, we have $A_\gamma \cap A_{\gamma'} \neq \emptyset$. Since X is connected, any pair γ, γ' of trajectories may be connected by a path $\delta \subset X$. Using the compactness of δ, we conclude that the function h must be a constant along δ. Therefore, h is a constant globally. □

Let us consider two subalgebras, $f^*(C^\infty(\mathbb{R})) \subset C^\infty(X)$ and $(f^\partial)^*(C^\infty(\mathbb{R})) \subset C^\infty(\partial_1 X)$, the second one is assumed to be a "known" part of the boundary data.

Lemma 3.2. The restriction operator $H^\partial_f : f^*(C^\infty(\mathbb{R})) \to (f^\partial)^*(C^\infty(\mathbb{R}))$ to the boundary $\partial_1 X$ is an epimorphism of algebras. If the range $f^\partial(\partial_1 X)$ of f^∂ is a connected closed interval of $\mathbb{R}$ (which is the case for a connected $\partial_1 X$), then H^∂_f is an isomorphism.

Proof. The restriction operator H^∂_f is an algebra epimorphism, since any composite function $\phi \circ f^\partial$, where $\phi \in C^\infty(\mathbb{R})$, is the restriction to $\partial_1 X$ of the function $\phi \circ f$.

Ref

[Grot] Grothendieck, A., *Produits tensoriels topologiques et espaces nucléaires* (French), Providence: American Mathematical Society, (1966), ISBN 0-8218-1216-5.

On the other hand, when $f^\partial(\partial_1 X)$ is a connected subset of $\mathbb{R}$, we claim that H^∂_f is a monomorphism. Indeed, take a function $\phi \in C^\infty(\mathbb{R})$, such that $\phi \circ f^\partial \equiv 0$, but $\phi \circ f$ is not identically zero on X. Then there is $x \in X$ such that $\phi \circ f(x) \neq 0$. On the other hand, by the hypothesis, $f(x) = f^\partial(y)$ for some $y \in \partial_1 X$. By the assumption, $f^\partial \circ \phi \equiv 0$, which implies that $\phi(f^\partial(y)) = 0$. This contradiction validates the claim about H^∂_f being a monomorphism. Therefore, when $f^\partial(\partial_1 X)$ is a connected interval, H^∂_f is an isomorphism of algebras. □

Ref

Consider the homomorphism of algebras

$$\mathsf{P} : C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R})) \to C^\infty(X)$$

that takes every finite sum $\sum_i h_i \otimes (f \circ g_i)$, where $h_i \in C^\infty(\mathcal{T}(v)) \subset C^\infty(X)$ and $g_i \in C^\infty(\mathbb{R})$, to the finite sum $\sum_i h_i \cdot (g_i \circ f) \in C^\infty(X)$.

Recall that, by Lemma 3.1, $C^\infty(\mathcal{T}(v)) \cap (f)^*(C^\infty(\mathbb{R})) = \underline{\mathbb{R}}$, the constants. For any linearly independent $\{h_i\}_i$, this lemma implies that if $\sum_i h_i \cdot (g_i \circ f) \equiv 0$, then $\{g_i \circ f \equiv 0\}_i$; therefore, P is a monomorphism.

Let us compare the, so called, projective crossnorms $\{\| \sim \|_k\}_{k \in \mathbb{Z}_+}$ (see (3.1)) of an element

$$\phi = \sum_i h_i \otimes (f \circ g_i)$$

and the norms of the element $\mathsf{P}(\phi) = \sum_i h_i \cdot (f \circ g_i)$. By comparing the Taylor polynomial of the product of two smooth functions with the product of their Taylor polynomials, we get that, for all $k \in \mathbb{Z}_+$,

$$\|\phi\|_k =_{\mathsf{def}} \inf\Big\{ \sum_i \|h_i\|_k \cdot \|(f \circ g_i)\|_k \Big\} \geq \|\mathsf{P}(\phi)\|_k, \tag{3.1}$$

where inf is taken over all the representations of the element $\phi \in C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$ as a sum $\sum_i h_i \otimes (f \circ g_i)$. Here we may assume that all $\{h_i\}_i$ are linearly independent elements and so are all $\{f \circ g_i\}_i$; otherwise, a simpler representation of ϕ is available.

By the inequality in (3.1), P is a bounded (continuous) operator. As a result, by continuity, P extends to an algebra homomorphism

$$\hat{\mathsf{P}} : C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, f^*(C^\infty(\mathbb{R})) \to C^\infty(X)$$

whose source is the completion of the algebraic tensor product $C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$.

Lemma 3.3. The embedding $\alpha : X \to \mathcal{T}(v) \times \mathbb{R}$ (introduced in the end of Section 2 and depicted in Fig. 3) induces an algebra epimorphism

$$\alpha^* : C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, C^\infty(\mathbb{R}) \stackrel{\mathsf{id}\,\hat{\otimes}\, f^*}{\longrightarrow} C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, f^*(C^\infty(\mathbb{R})) \stackrel{\hat{\mathsf{P}}}{\longrightarrow} C^\infty(X). \tag{3.2}$$

Moreover, the map $\hat{\mathsf{P}}$ is an isomorphism.

Proof. First, we claim that the subalgebra $\mathsf{P}\big(C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))\big) \subset C^\infty(X)$ satisfies the three hypotheses of Nachbin's Theorem [Na]. Therefore, by [Na], the P-image of $C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$ is *dense* in $C^\infty(X)$. Let us validate these three hypotheses.

[Na] Nachbin, L., *Sur les algèbres denses de fonctions differentiables sur une variète*, C.R. Acad. Sci. Paris 228 (1949) 1549-1551.

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(1) *For each $x \in X$, there is a function $q \in C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$ such that $q(x) \neq 0$.*

Just take $q = f \circ (t + c)$, where $c > \min_X f$ and $t : \mathbb{R} \to \mathbb{R}$ is the identity.

(2) *For each $x, y \in X$, there is a function $q \in C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$ such that $q(x) \neq q(y)$ (i.e., the algebra $C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$) separates the points of X)*

If $f(x) \neq f(y)$, $q = f$ will do. If $f(x) = f(y)$, but $[\gamma_x] \neq [\gamma_y]$, then there is a v-invariant function $h \in C^\infty(\mathcal{T}(v))$ such that $h(x) = 1$ and $h(y) = 0$. To construct this h, we take a transversal section $S_x \subset \hat{X}$ of the $\hat{v}$-flow in the vicinity of x such that all the $\hat{v}$-trajectories through S_x are distinct from the trajectory γ_y. We pick a smooth function $\tilde{h} : S_x \to \mathbb{R}$ such that h is supported in $\mathsf{int}(S_x)$, vanishes with all its derivatives along the boundary ∂S_x, and $\tilde{h}(x) = 1$. Let $\mathcal{S}$ denote the set of $\hat{v}$-trajectories through S_x. Of course, $\tilde{h}$ extends to a smooth function $h^\dagger : \mathcal{S} \to \mathbb{R}$ so that $h^\dagger$ is constant along each trajectory from $\mathcal{S}$. We denote by $h^\ddagger$ the obvious extension of $h^\dagger$ by the zero function. Finally, the restriction h of $h^\ddagger$ to X separates x and y.

(3) *For each $x \in X$ and $w \in T_xX$, there is a function $q \in C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R}))$ such that $dq_x(w) \neq 0$.*

Let us decompose $w = av + bw^\dagger$, where $a, b \in \mathbb{R}$ and the vector $w^\dagger$ is tangent to the hypersurfce $S_x = \hat{f}^{-1}(f(x))$. Then, if $a \neq 0$, then $df(w) \neq 0$. If $a = 0$, then the there is a function $\tilde{h} : S_x \to \mathbb{R}$ which, with all its derivatives, is compactly supported in the vicinity of x in S_x and such that $d\tilde{h}_x(w^\dagger) \neq 0$. As in the case (2), this function extends to a desired function $h \in C^\infty(\mathcal{T}(v))$. Now put $q = h \otimes 1$.

As a result, the image of $\mathsf{P} : C^\infty(\mathcal{T}(v)) \otimes f^*(C^\infty(\mathbb{R})) \longrightarrow C^\infty(X)$ is dense. Therefore, $\hat{\mathsf{P}}$ and, thus, $(\alpha)^* : C^\infty(\mathcal{T}(v)) \hat{\otimes} C^\infty(\mathbb{R}) \longrightarrow C^\infty(X)$ are epimorphisms.

Let us show that $\hat{\mathsf{P}}$ is also a monomorphism. Take a typical element

$$\theta = \sum_{i=1}^{\infty} h_i \otimes (f \circ g_i) \in C^\infty(X, v) \hat{\otimes} f^*(C^\infty(\mathbb{R})),$$

viewed as a sum that converges in all the norms $\| \sim \|_k$ from (3.1). We aim to prove that if $\hat{\mathsf{P}}(\theta) = \sum_{i=1}^{\infty} h_i \cdot (f \circ g_i)$ vanishes on X, then $\theta = 0$.

For each point $x \in \mathsf{int}(X)$, there is a small closed cylindrical solid $H_x \subset \mathsf{int}(X)$ that contains x and consists of segments of trajectories through a small n-ball $D^n \subset f^{-1}(f(x))$, transversal to the flow. Thus, the product structure $D^1 \times D^n$ of the solid H_x is given by the v-flow and the Lyapunov function $f : X \to \mathbb{R}$.

We localize the problem to the cylinder H_x. Consider the commutative diagram

$$\begin{array}{ccc} C^\infty(X, v) \hat{\otimes} f^*(C^\infty(\mathbb{R})) & \xrightarrow{\hat{\mathsf{P}}} & C^\infty(X). \\ \downarrow \mathsf{res}' \hat{\otimes} \mathsf{res}'' & & \downarrow \mathsf{res} \\ C^\infty(D^n) \hat{\otimes} C^\infty(D^1) & \xrightarrow{\approx \hat{\mathsf{Q}}} & C^\infty(H_x), \end{array} \tag{3.3}$$

where $\mathsf{res} : C^\infty(X) \to C^\infty(H_x)$ is the natural homomorphism,

$$(\mathsf{res}' \hat{\otimes} \mathsf{res}'')\Big(\sum_{i=1}^{\infty} h_i \otimes (f \circ g_i)\Big) =_{\mathsf{def}} \sum_{i=1}^{\infty} h_i|_{D^n} \otimes (f \circ g_i)|_{D^1},$$

and $\hat{\mathsf{Q}}\big(\sum_{i=1}^{\infty} \tilde{h}_i \otimes \tilde{g}_i\big) =_{\mathsf{def}} \sum_{i=1}^{\infty} \tilde{h}_i \cdot \tilde{g}_i$ for $\tilde{h}_i \in C^\infty(D^n)$, $\tilde{g}_i \in C^\infty(D^1)$.

Notes

Since $\hat{\mathsf{Q}}$ is an isomorphism [Grot] and $\hat{\mathsf{P}}(\theta) = 0$, it follows from (3.5) that $\theta \in \ker(\mathsf{res}' \hat{\otimes} \mathsf{res}'')$ for any cylinder H_x. After reshuffling terms in the sum, one may assume that all the functions $\{h_i|_{D^n}\}_i$ are linearly independent. Using that the functions $h_i|_{D^n}$ and $(f \circ g_i)|_{D^1}$ depend of the complementary groups of coordinates in H_x, we conclude that these functions must vanish for any $H_x \subset \mathsf{int}(X)$. As a result, $\theta = 0$ globally in $\mathsf{int}(X)$ and, by continuity, θ vanishes on X. □

Notes

Consider now the "known" homomorphism of algebras

$$(\alpha^\partial)^* : \; C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, C^\infty(\mathbb{R}) \xrightarrow{\approx \mathsf{res}^\partial_{\mathcal{T}} \,\hat{\otimes}\, (f^\partial)^*}$$

$$\longrightarrow C^\infty(\partial_1 X, v) \,\hat{\otimes}\, (f^\partial)^*(C^\infty(\mathbb{R})) \xrightarrow{\hat{\mathsf{R}}^\partial} C^\infty(\partial_1 X), \tag{3.4}$$

utilizing the boundary data. Here, by the definition of $C^\infty(\partial_1 X, v)$, $\mathsf{res}^\partial_{\mathcal{T}} : C^\infty(\mathcal{T}(v)) \to C^\infty(\partial_1 X, v)$ is an isomorphism, and $\hat{\mathsf{R}}^\partial$ denotes the completion of the bounded homomorphism R^∂ that takes each element $\sum_i h_i \otimes (f^\partial \circ g_i)$, where $h_i \in C^\infty(\partial_1 X, v)$ and $g_i \in C^\infty(\mathbb{R})$, to the sum $\sum_i h_i \cdot (g_i \circ f^\partial)$.

The next lemma shows that the hypotheses of Theorem 3.1 are not restrictive, even when $\partial_1 X$ has many connected components.

Lemma 3.4. Any traversing vector field v on a connected compact manifold X admits a Lyapunov function $f : X \to \mathbb{R}$ such that $f(X) = f(\partial_1 X)$.

Proof. Note that, for any Lyapunov function f, the image $f(\partial_1 X)$ is a disjoint union of finitely many closed intervals $\{I_k = [a_k, b_k]\}_k$, where the index k reflects the natural order of intervals in $\mathbb{R}$. We will show how to decrease, step by step, the number of these intervals by deforming the original function f. Note that the local extrema of any Lyapunov function on X occur on its boundary $\partial_1 X$ and away from the locus $\partial_2 X(v)$ where v is tangent to $\partial_1 X$. Consider a pair of points $A_{k+1}, B_k \in \partial_1 X \setminus \partial_2 X(v)$ such that $f(A_{k+1}) = a_{k+1}$ and $f(B_k) = b_k$, where $a_{k+1} > b_k$. Then we can increase f in the vicinity of its local maximum B_k so that the B_k-localized deformation $\tilde{f}$ of f has the property $\tilde{f}(B_k) > f(A_{k+1})$ and $\tilde{f}$ is a Lyapunov function for v. This construction decreases the number of intervals in $\tilde{f}(\partial_1 X)$ in comparison to $f(\partial_1 X)$ at least by one. □

We are ready to state the main result of this paper.

Theorem 3.1. Assuming that the range $f^\partial(\partial_1 X)$ is a connected interval of $\mathbb{R}$,[1] the algebra $C^\infty(X)$ is isomorphic to the subalgebra

$$C^\infty(\partial_1 X, v) \,\hat{\otimes}\, (f^\partial)^*(C^\infty(\mathbb{R})) \subset C^\infty(\partial_1 X) \hat{\otimes} C^\infty(\partial_1 X).$$

Moreover, by combining (3.2) with (3.4), we get a commutative diagram

$$\begin{array}{ccc} C^\infty(\mathcal{T}(v)) \,\hat{\otimes}\, f^*(C^\infty(\mathbb{R})) & \xrightarrow{\hat{\mathsf{R}}} & C^\infty(X) \\ \downarrow \mathsf{id} \,\hat{\otimes}\, H^\partial_f & & \downarrow \mathsf{res} \\ C^\infty(\partial_1 X, v) \,\hat{\otimes}\, (f^\partial)^*(C^\infty(\mathbb{R})) & \xrightarrow{\hat{\mathsf{R}}^\partial} & C^\infty(\partial_1 X), \end{array} \tag{3.5}$$

[1] which is the case for a connected $\partial_1 X$

whose vertical homomorphism $\mathsf{id}\,\hat{\otimes}\, H^\partial_f$ *and the horizontal homomorphism* $\hat{\mathsf{R}}$ *are isomorphisms, and the vertical epimorphism* res *is the obvious restriction operator.*

As a result, inverting $\mathsf{id}\,\hat{\otimes}\, H^\partial_f$*, we get an algebra isomorphism*

$$\mathcal{H}(v,f) : C^\infty(\partial_1 X, v)\,\hat{\otimes}\,(f^\partial)^*(C^\infty(\mathbb{R})) \approx C^\infty(X). \tag{3.6}$$

Proof. Consider the commutative diagram (3.5). Its upper-right conner is "unknown", while the lower row is "known" and represents the boundary data, and res is obviously an epimorphism. By Lemma 3.2, the left vertical arrow $\mathsf{id}\,\hat{\otimes}\, H^\partial_f$ is an isomorphism. Since, by Lemma 3.3, $\hat{\mathsf{R}}$ is an isomorphism, it follows that $\hat{\mathsf{R}} \circ (\mathsf{id}\,\hat{\otimes}\, H^\partial_f)^{-1}$ must be an isomorphism as well. In particular, $\hat{\mathsf{R}}^\partial$ is an epimorphism, whose kernel is isomorphic to the smooth functions on X whose restrictions to $\partial_1 X$ vanish. If $z \in C^\infty(X)$ is a smooth function such that zero is its regular value, $z^{-1}(0) = \partial_1 X$, and $z > 0$ in $\mathsf{int}(X)$, then the kernel of res is the principle ideal $\mathsf{m}(z)$, generated by z. Therefore, by the commutativity of (3.5), the kernel of the homomorphism $\hat{\mathsf{R}}^\partial$ must be also a principle ideal M_∂, generated by an element $(\hat{\mathsf{R}} \circ (\mathsf{id}\,\hat{\otimes}\, H^\partial_f))^{-1}(z)$. □

Corollary 3.1. If the range $f^\partial(\partial_1 X)$ *is a connected interval in* $\mathbb{R}$*, then the two topological algebras* $C^\infty(\partial_1 X, v) \subset C^\infty(\partial_1 X)$ *and* $(f^\partial)^*(C^\infty(\mathbb{R})) \subset C^\infty(\partial_1 X)$ *determine, up to an isomorphism, the algebra* $C^\infty(X)$*, and thus determine the smooth topological type of the manifold* X*.*

Proof. We call a maximal ideal of an algebra $\mathcal{A}$ **nontrivial** if it is different from $\mathcal{A}$.

By Theorem 3.1, the algebra $C^\infty(X)$ is determined by the two algebras on $\partial_1 X$, up to an isomorphism. In turn, the algebra $C^\infty(X)$ determines the smooth topological type of X, viewed as a ringed space. This fact is based on interpreting X as the space $\mathcal{M}(C^\infty(X))$ of nontrivial maximal ideals of the algebra $C^\infty(X)$ [KMS].

Let $\mathsf{m}^\partial_v \lhd C^\infty(\partial_1 X, v)$ and $\mathsf{m}^\partial_f \lhd (f^\partial)^*(C^\infty(\mathbb{R}))$ be a pair of nontrivial maximal ideals. Note that $\mathsf{m}^\partial_v = \mathsf{m}^\partial_v([\gamma])$ consists of functions from $C^\infty(\partial_1 X, v)$ that vanish on the locus $\gamma \cap \partial_1 X$, and $\mathsf{m}^\partial_f = \mathsf{m}^\partial_f(c)$ consists of functions from $(f^\partial)^*(C^\infty(\mathbb{R}))$ that vanish on the locus $\partial_1 X \cap f^{-1}(c)$, where $c \in f(\partial_1 X) \subset \mathbb{R}$. We denote by $\langle \mathsf{m}^\partial_v, \mathsf{m}^\partial_f\rangle$ the maximal ideal of $C^\infty(\partial_1 X, v)\,\hat{\otimes}\,(f^\partial)^*(C^\infty(\mathbb{R}))$ that contains both ideals $\mathsf{m}^\partial_v \hat{\otimes} 1$ and $1 \hat{\otimes} \mathsf{m}^\partial_f$. If the range $f^\partial(\partial_1 X)$ is a connected interval of $\mathbb{R}$ and $\langle \mathsf{m}^\partial_v, \mathsf{m}^\partial_f\rangle$ is a nontrivial ideal, then $\gamma \cap f^{-1}(c) \neq \emptyset$. Otherwise, $\gamma \cap f^{-1}(c) = \emptyset$. Therefore, with the help of the isomorphism $\mathcal{H}(v,f)$ from (3.6), the nontrivial maximal ideals of $C^\infty(X)$ (which by [KMS] correspond to points $x = \gamma \cap f^{-1}(c) \in X$) are of the form $\mathcal{H}(v,f)\big(\langle \mathsf{m}^\partial_v, \mathsf{m}^\partial_f\rangle\big)$. □

Corollary 3.2. Let the range $f^\partial(\partial_1 X)$ *be a connected interval of* $\mathbb{R}$*. With the isomorphism* $\mathcal{H}(v,f)$ *from (3.6) being fixed, any algebra isomorphism* $\Psi^\partial : C^\infty(\partial_1 X) \to C^\infty(\partial_1 X)$ *that preserves the subalgebras* $C^\infty(\partial_1 X, v)$ *and* $(f^\partial)^*(C^\infty(\mathbb{R}))$ *extends canonically to the algebra isomorphism* $\Psi : C^\infty(X) \to C^\infty(X)$*.*

Thus, an action of any group G *of such isomorphisms* Ψ^∂ *extends canonically to a* G*-action on the algebra* $C^\infty(X)$ *and, via it, to a* G*-action on* X *by smooth diffeomorphisms.*

Proof. By [Mr], any algebra isomorphism $\Psi : C^\infty(X_1) \to C^\infty(X_2)$ is induced by a unique smooth diffeomorphism $\Phi : X_1 \to X_2$. With this fact in hand, by Theorem 2.1 and Theorem 3.1, the proof is on the level of definitions. □

Ref

[KMS] Kriegl, A., Michhor, P., Schachermayer, W., *Characters of Algebras of Smooth Functions*, Ann. Global Anal. Geom. vol.7, No. 2 (1989), 85-92.

It remains to address the following crucial question: how to characterize intrinsically the trace $C^\infty(\partial_1 X, v)$ of the algebra $C^\infty(\mathcal{T}(v)) \approx \ker\{\mathcal{L}_v : C^\infty(X) \to C^\infty(X)\}$ in the algebra $C^\infty(\partial_1 X)$?

Evidently, functions from $C^\infty(\partial_1 X, v)$ are constant along each C_v-"trajectory" $\gamma^\partial := \gamma \cap \partial_1 X$ of the causality map. Furthermore, any smooth function $\boldsymbol{\psi}: \partial_1 X \to \mathbb{R}$ that is constant on each finite set γ^∂ gives rise to a unique *continuous* function ϕ on X that is constant along each v-trajectory γ. However, such functions ϕ may not be automatically *smooth* on X (a priory, they are just Hölderian with some control of the Hölder exponent that depends on the dimension of X only)! This potential complication leads to the following question.

Question 3.1. For a traversing and boundary generic (alternatively, traversally generic) vector field v on X, is it possible to characterize the subalgebra $C^\infty(\partial_1 X, v) \subset C^\infty(\partial_1 X)$ in terms of the causality map C_v and, perhaps, some additional v-generated data, residing in $\partial_1 X$? ♢

To get some feel for a possible answer, we need the notion of the Morse stratification of the boundary $\partial_1 X$ that a vector field v generates [Mo].

Let $\dim(X) = n + 1$ and v be a boundary generic traversing vector field on X.

Let us recall the definition of the Morse stratification $\{\partial_j^\pm X(v)\}_{j \in [1, n+1]}$ of $\partial_1 X$. We define the set $\partial_2 X(v)$ as the locus where v is tangent to $\partial_1 X$. It separates $\partial_1 X$ into $\partial_1^+ X(v)$ and $\partial_1^+ X(v)$. Let $\partial_3 X(v)$ be the locus where v is tangent to $\partial_2 X(v)$. For a boundary generic v, $\partial_2 X(v)$ is a smooth submanifold of $\partial_1 X(v)$ and $\partial_3 X(v)$ is a submanifold that divides $\partial_2 X(v)$ into two regions, $\partial_2^+ X(v)$ and $\partial_2^- X(v)$. Along $\partial_2^+ X(v)$, v points inside of $\partial_1^+ X(v)$, and along $\partial_2^- X(v)$, v points inside of $\partial_1^- X(v)$. This construction self-replicates until we reach finite sets $\partial_{n+1}^\pm X(v)$.

By definition, the boundary generic vector fields [K1] are the ones that satisfy certain nested transversality of v with respect to the boundary $\partial_1 X$, the transversality that guaranties that all the Morse strata $\partial_j X(v)$ are regular closed submanifolds and all the strata $\partial_j^\pm X(v)$ are compact submanifolds.

For a traversing boundary generic v, the map $C_v : \partial_1^+ X(v) \to \partial_1^- X(v)$ makes it possible to recover the Morse stratification $\{\partial_j^\pm X(v)\}_{j>0}$ ([K4]).

Let us describe now a good candidate for the subalgebra $C^\infty(\partial_1 X, v)$ in the algebra $C^\infty(\partial X)$.

We denote by $\mathcal{L}_v^{(k)}$ the k-th iteration of the Lie derivative $\mathcal{L}_v$. Let $\mathsf{M}(v)$ be the subalgebra of smooth functions $\boldsymbol{\psi}: \partial_1 X \to \mathbb{R}$ such that $(\mathcal{L}_v^{(k)}\boldsymbol{\psi})|_{\partial_{k+1}X(v)} = 0$ for all $k \leq n$ (by the Leibniz rule, $\mathsf{M}(v)$ is indeed a subalgebra). Let us denote by $\mathsf{M}(v)^{C_v}$ the subalgebra of functions from $\mathsf{M}(v)$ that are constant on each (finite) C_v-trajectory $\gamma^\partial := \gamma \cap \partial_1 X \subset \partial_1 X$.

Conjecture 3.1. Let v be a traversing and boundary generic vector field on a smooth compact $(n+1)$-manifold X. Then the algebra $C^\infty(\partial_1 X, v)$ coincides with the subalgebra $\mathsf{M}(v)^{C_v} \subset C^\infty(\partial_1 X)$.

In particular, $C^\infty(\partial_1 X, v)$ can be determined by the causality map C_v and the restriction of v to $\partial_2 X(v)$. ♢

It is easy to check that $C^\infty(\partial_1 X, v) \subset \mathsf{M}(v)^{C_v}$; the challenge is to show that the two algebras coincide.

The Holography Theorem (Corollary 2.1) has been established assuming Property A from Definition 2.1. If one assumes the validity of Conjecture 3.1, then, by Corollary 3.1,

Ref

[K4] Katz, G., *Causal Holography of Traversing Flows*, Journal of Dynamics and Differential Equations (2020) https://doi.org/10.1007/s10884-020-09910-y we may drop Property A from the hypotheses of the Holography Theorem. Indeed, the subalgebras $C^\infty(\partial_1 X, v)$ and $(f^\partial)^*(C^\infty(\mathbb{R}))$ would acquire a description in terms of C_v and f^∂. This would deliver an independent proof of a natural generalization of Corollary 2.1.

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Recurrence Formula of Stamp Folding Problem

By Shintaro Sakai
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Abstract- There is an unsolved problem that has plagued mathematicians for a long time, the "stamp folding problem" (strictly, is there a formula for counting the solutions to the stamp folding problem?). In this paper, I have succeeded in expressing the stamp-folding problem by a recurrence formula with an elegant idea.

Keywords: *stamp folding problem, discrete mathematics, discrete geometry, folding, crease.*

GJSFR-F Classification: *DDC Code: 510.92520973 LCC Code: QA27.5*

RECURRENCEFORMULAOFSTAMPFOLDINGPROBLEM

Strictly as per the compliance and regulations of:

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Recurrence Formula of Stamp Folding Problem

Shintaro Sakai

Abstract- There is an unsolved problem that has plagued mathematicians for a long time, the "stamp folding problem" (strictly, is there a formula for counting the solutions to the stamp folding problem?). In this paper, I have succeeded in expressing the stamp-folding problem by a recurrence formula with an elegant idea.

Keywords: stamp folding problem, discrete mathematics, discrete geometry, folding, crease.

I. Introduction

Until now, stamp folding, or in other words origami, was considered more of a child's pastime and hobby. However, in recent years, this folding technique has been applied in various fields and is being actively studied. In addition, many mathematicians have been working on the stamp-folding problem, which is addressed in this paper, for a long time, but it has not yet been solved.([1], [2],[3],[4])

Generally speaking, the stamp folding problem is "make n creases at equal intervals of length n + 1 and fold it to length 1. How many ways to fold it at this time?" In other words, the problem of stamp folding is to count the methods of folding to length 1 without worrying about the allocation of creases. A slightly smaller stamp folding problem is that the left edge of the paper must not be covered by the folded paper, that is, the left edge of the paper must be visible from the outside, and the rest of the paper must be able to be attached to the outside. This is the former divided by n (the cyclic permutation of a foldable stamp sequence is always foldable in itself), and in this paper, we consider this case and call it F(n).

This problem can also be rephrased as [the number of ways in which a curve with an infinite orientation on one side intersects a straight line n times]. Known methods for calculating these numbers take exponential time as a function of n. In short, there are no formulas or efficient algorithms that can extend this sequence to very large values of n.

The sequence of F(n) is as follows. By the way, the maximum value we know is F (43), which is 21 digits.

$$1, 1, 2, 4, 10, 24, 66, 174, 504, 1406, 4210, \ldots.$$

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In addition, although not dealt with here, new concept problems such as the problem of the folding complexity and the problem of crease width also arise, making this a very interesting field.([5],[6])

II. Extraction of Recurrence Formulas

The stamp folding problem is not represented by a recurrence formula, and it is thought that F(n) cannot be found to be related to F(n-1 or less). However, I now find that any large value of F(n) is related to F(n-1 or less).

a) Symmetry around the second stamp

First, let us consider a small value F(5) as an example. F(5) is 10 ways in the figure below (Figure 1). Vertical is not counted by connecting. The horizontal line has a different length on the figure, but it is length 1. For the sake of clarity, it is as follows.

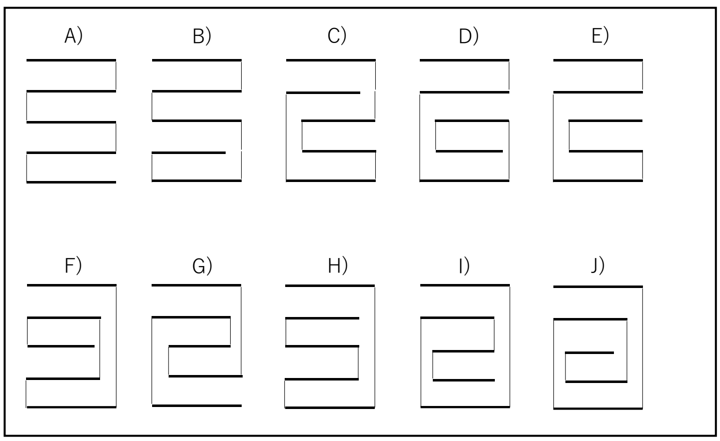

Figure 1: The stamp folding list of F(5)

From these 10 ways, you can see that A) ~ E) and F) ~ J) are symmetrical with the second stamp from the left as the boundary. A) and H). B) and F). D) and J). E) and I). It can be seen that C) and G) are also upside down, only the beginnings are upside down. Since this must not cover the leftmost stamp, the stamps A) to E) must be closed in the clouds in Figure 2, and so are F) to J). And since each of the remaining three stamps is placed, they have the same shape (Figure 2). Moreover, this idea is the same after n = 5. So you should consider either starting below or above the second stamp. From now on, I will consider only the case of starting from the bottom of the second stamp from the left (here A) to E))

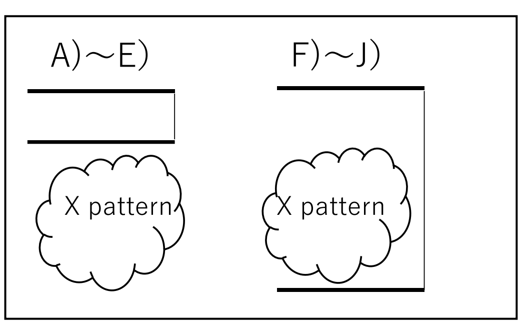

Figure 2: Schematic diagram of F (5)

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5. Ryuhei Uehara. On Stretch Minimization Problem on Unit Strip Paper, 22nd Canadian Conference on Computational Geometry, pp. 223-226, 2010/8/9-11.

b) Cases that end only under the second stamp

Next, I would like to look at the relationship between F (5) and F (4).

The stamp on the left end is the same, so I don't think about it. Considering the second and subsequent points, it can be seen that there are four F (4) inverted left and right other than C). As with the previous idea, the fact that the remaining three are always configured below the second from the left end is nothing but the placement of F (4). This is the same for F(5) and above, and can be considered in terms of stamps excluding the first leftmost stamp. In short, it means F(n) which is n+1 -1. In addition, in order to cover all stamp folding methods, there is always an inverted one at the boundary of the second stamp, so it is necessary to double it. From the above, the following equation is extracted (n is 5 or more). Here we can easily see that this sequence grows exponentially.

$$F(n+1) > 2F(n) \tag{1}$$

$$F(n) > 2^{n-2} \tag{2}$$

c) Case going from under the second stamp to above the second stamp only once

Next, I would like to consider how to fold F (n + 1) not represented by F (n), C) in F (5).

Here, I would like to consider F (7) as an example for a closer look. Figure 3 below is a list of things that started below the second stamp and moved to the top of the second stamp in the middle. First, consider c), d), f) and g). All of these have the same arrangement of the first four stamps. After that, it consists only above the second stamp (clouds in Figure 4). This means that there are only three clouds left, so we can see that F (4) = 4 ways. Here, the first stamp is fixed, so we have to do 3+1 ,or generally speaking n+1.

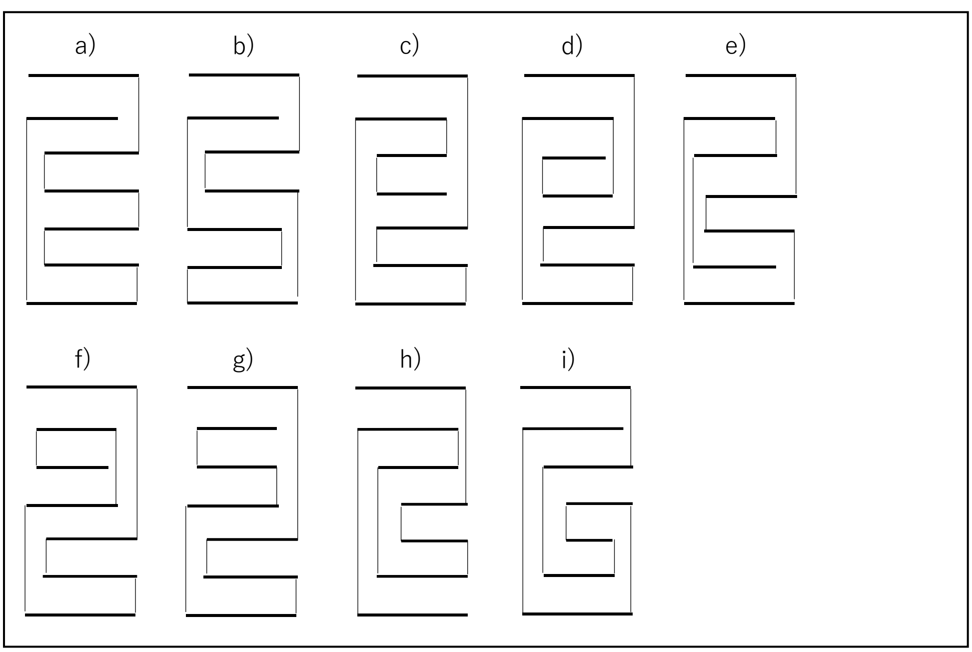

Figure 3: A list of things that started below the second stamp and moved to the top of the second stamp in the middle (F(7))

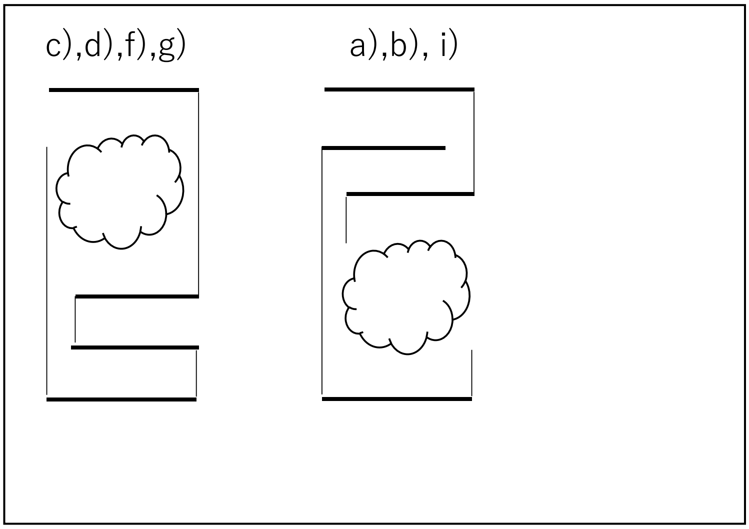

Figure 4: Schematic diagram of classification of F (7)

Next, consider a), b), and i). These three are the same that the last stamp is placed above the second stamp. This is the last stamp of F (5) facing outwards. Because if you subtract the first fixed stamp and the last stamp, 7-2=5. Also, in order to go from the penultimate stamp to the last stamp (by passing the second stamp), the penultimate stamp must always face outward as shown in Figure 5. In other words, if we think in terms of vertices rather than stamps, we have numbers whose last vertices are not in valleys and are not enclosed. This is due to the fact that the lines (surfaces) must not intersect (physically impossible) when folding stamps.

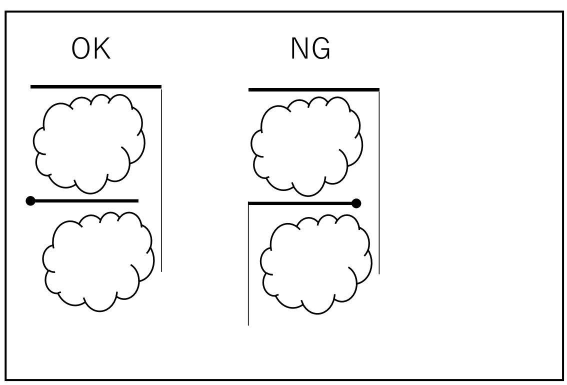

Figure 5: Judgment diagram that it is facing outward

From now on, put this number (the total number of points in F(x) where a given vertex value r faces outwards) as G (x, r_1, r_2, .. r_j). Indicates which vertex the second and subsequent vertices r_1 to r_j are facing outward, counting from the last vertex. If nothing is attached like G (x), it is the last vertex. Also, if r is 0, it agrees with F (x).

Patterns such as these a), b), j) that first start under the second stamp, come over the second stamp only once, and never again come under the second stamp be. These and the patterns that hold only under the second stamp discussed above are summarized in Table 1 below for each

Notes

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folding number F(n). The left side is the value of the stamp placed above the second stamp, and the right side is the value of the stamp placed below the second stamp. The row direction is arranged according to the number of times. The right side of Table 1 shows the final vertex facing outward in F() until the stamp placed below the second stamp goes up. In short, according to the definition mentioned above, it is represented by G(). Shown on the left side of Table 1 are the values placed above the second stamp, which are not returned below the second stamp anymore and are closed there, so represented by the remaining number F(). One thing to keep in mind is that it always takes two stamps to wrap, so you have to think about +2. It goes without saying that n=x+y if we assume F(x) and G(y) when looking at each row. From this, the following equation (3) can be derived. The brackets {} are multiplied by 2 because we are considering the case starting above the second stamp.

Table 1: Relationship diagram of F (n + 1) and F (n or less), G (n or less)

F(7)		F(8)		F(9)		F(10)		···	F(n+1)	
	F(6)		F(7)		F(8)		F(9)	···		F(n)
F(4)	G(3)	F(5)	G(3)	F(6)	G(3)	F(7)	G(3)	···	F(n-2)	G(3)
F(2)	G(5)	F(3)	G(5)	F(4)	G(5)	F(5)	G(5)	···	F(n-4)	G(5)
				F(2)	G(7)	F(3)	G(7)	···	F(n-6)	G(7)
								···	. . .	. . .
								···	F(n-2m-1)	G(2m+1)

$$F(n+1) > 2\left\{F(n) + \sum_{m=1}^{\left\lfloor\frac{n}{2}\right\rfloor}(F(n-2m-1) \times G(2m+1))\right\} \quad (3)$$

d) *All cases*

Finally, e) and i) are patterns that start from the bottom of the second stamp, move to the top of the second stamp, and then return to the bottom of the second stamp. Since it is complicated after this, I would like to think comprehensively rather than individually.

Again, consider the second stamp as a boundary. As shown in Figure 6 below, place the stamps that come and go at the second boundary as $D_1 \sim D_i$. For the sake of understanding only this Figure 6, the vertical line is the second stamp. The line connecting D? and D? is not counted as a connection. To conclude first, it can be seen that this is the product of the total number of folds of the stamps on the left side (lower side) and the number of folds of the stamp group on the right side (upper side) with the second stamp as the boundary. However, there is a condition that the bridge from left to right and from right to left must always face outward. If it meets that requirement, no matter how it is folded on the right or left side, it belongs to the total number of folding methods. This is because, as I said before, the lines (faces) of a stamp never cross. Then, pass the second stamp without covering (crossing) the already folded D?. In other words, the place where the second stamp passed is kept on the outside, and the number that will increase from now on from the stamps that have already been counted should be added. Since it will continue, it is sufficient to have information on the total number of right (lower) and left (upper) stamps on the second stamp and the number of F() facing outside of each total.

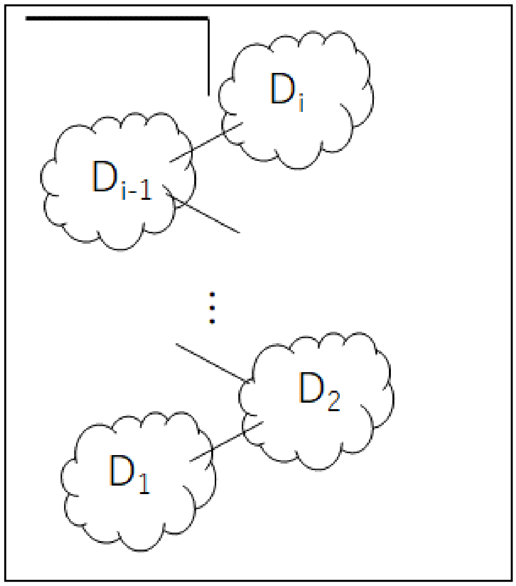

Figure 6: Schematic diagram of F (n)

Next, we have to think about the combination of the left side and the right side with the second stamp as the boundary. This can be done by considering the division of numbers and composition. There is already a theorem for this as in the following equation. It should be noted that both the right side and the left side are always even numbers except for the last D_1. This is because it doubles when folded. D_1 becomes odd when n is odd, and even when n is even. There is no need to wrap at the end, so there are both odd and even numbers.

For example, consider the case of F (12). The first two stamps are fixed, so subtract 2 to get 10. When 10 is divided, it becomes as shown in Table 2 below. b_j is the division number and c_k is the combination number.

Theorem 1

$$comp(n, j) = \binom{n-1}{j-1}$$

Table 2: Divided composition diagram of F (12)

	c_1	c_2	c_3	c_4	c_5	c_6
b_1	10					
b_2	8+2	2+8	6+4	4+6		
b_3	6+2+2	2+6+2	2+2+6	4+4+2	4+2+4	2+4+4
b_4	4+2+2+2	2+4+2+2	2+2+4+2	2+2+2+4		
b_5	2+2+2+2+2					

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Common cases are shown in Table 3 below.

The value of comp () is the number of terms in the composition and corresponds to the value of j in b_j.If one frame of Table 3 is placed as $B_{j,k}$, it will be expressed by the following formula (4).

Table 3: Divided composition diagram of F (n)

	c_1	c_2	…	c_{k-1}	c_k
b_1	D_1				
b_2	D_1+D_2	D_1+D_2			
…					
b_{j-1}	$D_1+D_2+\cdots+D_{i-2}+D_{i-1}$	$D_1+D_2+\cdots+D_{i-2}+D_{i-1}$			
b_j	$D_1+D_2+\cdots+D_{i-1}+D_i$				

$$B_{j,k} = G\left(\sum_{i=1}^{\left\lfloor\frac{j}{2}\right\rfloor} D_{2i}\left(b_j,c_k\right)+1, 1, D_2\left(b_j,c_k\right)+1, \ldots, \sum_{i=1}^{\left\lfloor\frac{j}{2}\right\rfloor-1} D_{2i}\left(b_j,c_k\right)+1\right)$$

$$\times G\left(\sum_{i=1}^{\left\lceil\frac{j}{2}\right\rceil} D_{2i-1}\left(b_j,c_k\right)+1, D_1\left(b_j,c_k\right)+1, .., \sum_{i=1}^{\left\lfloor\frac{j}{2}\right\rfloor-1} D_{2i-1}\left(b_j,c_k\right)+1\right) \quad (4)$$

Explain the formula. First, the inside of G () in the first line shown first is the one that does not own the last D_1. Therefore, i is on the even side. And the first row in G () is the total number of stamps on the even side. +1 is the addition of the leftmost stamp. The second row is the first outside, but after D_1 it does not return to the even side anymore, so it becomes 1. After that, the sum of D_2, D_4,… D_{2i} becomes the outside.

Next, in the second line G (), this is the one who owns D_1. Therefore, i is on the odd side. Similarly, the first column in G () is the total number of stamps on the odd side. The second column is D_1 at first, but the number of vertices is one larger than the number of stamps, so add +1. After that, the sum of D_3, D_5 ,... D_{2i-1} becomes the outside. Then, the product of each G () is $B_{j,k}$.

Then, by adding $B_{j,k}$ by the amount of b_j x c_k, F(n) can be obtained. By the way, this formula also applies to the cases b_1 and b_2 obtained earlier. As mentioned earlier, if G () is 0 in the second and subsequent columns, it will be F(n).

In addition, this time, we discussed the stamp folding problem in the case where the first stamp is fixed, but in the usual case where the first stamp is not fixed, simply multiply the equation (5) by n.

$$F(n) = 2\sum_{j=1}^{\left\lceil\frac{n-2}{2}\right\rceil} \sum_{k=1}^{\binom{\left\lceil\frac{n-2}{2}\right\rceil-1}{j-1}} B_{j,k} \quad (5)$$

III. Conclusion

Stamps never cross. Then, it moves independently on the boundary of the second stamp. Focusing on these two points, I clarified the stamp folding problem and devised a recurrence formula.

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Ordinary Differential Equations with an Approach in the Numerical Study of Malaria: SIR Model

By Anastácio Pascoal Epandi Canhanga
Instituto Superior Politécnico do Bié

Abstract- The present investigation aims to numerically predict cases of infections and recoveries from malaria in the city of Cuito, for which differential equations were use with which it was possible to study the behavior of the variables that affect the dynamics of malaria. Based on the infection and recovery variables, as well as the constant rates of infections, recoveries and deaths, analyzing the links betweens the same variables, the SIR endemic model was chosen, which allowed achieving the objective announced here. The study was based on data from a period when cases of this disease were already slowing down. The Runge-Kutta method was used to predict numbers of malaria nos. The results showed exctly what was expected to be the decrease in cases in this period an not only, the power of the model used was verified, as well as its usefulness.

Keywords: *ordinary differential equations, SIR model, malaria, runge-kutta method.*

GJSFR-F Classification: *MSC 2010: 12H20*

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Ordinary Differential Equations with an Approach in the Numerical Study of Malaria: SIR Model

Ecuaciones diferenciales ordinarias con enfoque en el estudio numérico de la malaria: modelo SIR

Anastácio Pascoal Epandi Canhanga

Resumen- La presente investigación tiene como objetivo predecir numericamente los casos de infeción y recuperación de la malaria en la ciudad del Cuito, utilizando ecuaciones diferenciales con los cuales se permita evaluar el comportamiento de las variables que afectan esta enfermedad. Teniendo en cuenta las variables de infección y recuperación, así como las tasas constantes de recuperación, infección y muerte, analizando la conexión entre estas variables así, como sus respectivas proporciones, se ha elegido el modelo endémico SIR que permitió lograr el dicho objetivo. Es-te estudio se basó en datos de un período en el que la disminución de infecciones palúdicas se registra. En ese contexto, se utilizó el método de Runge-Kutta para predecir números de infecciones en los meses de invierno en Angola, utilizando como datos de referencia inicial los de mayo en la ciudad de Cuito, capital de la provincial del Bié, Angola. Los resultados ilustran el poder y utilidad del modelo endémico SIR en los problemas de evoluciones, especialmente los epidemiológicos, bien como la sincronia en las variables.

Palabras-llave: ecuaciones diferenciales ordinarias, modelo SIR, malaria, méeto-do de runge-kutta.

Abstract- The present investigation aims to numerically predict cases of infections and recoveries from malaria in the city of Cuito, for which differential equations were use with which it was possible to study the behavior of the variables that affect the dynamics of malaria. Based on the infection and recovery variables, as well as the constant rates of infections, recoveries and deaths, analyzing the links betweens the same variables, the SIR endemic model was chosen, which allowed achieving the objective announced here. The study was based on data from a period when cases of this disease were already slowing down. The Runge-Kutta method was used to predict numbers of malaria nos. The results showed exctly what was expected to be the decrease in cases in this period an not only, the power of the model used was verified, as well as its usefulness.

Keywords: ordinary differential equations, SIR model, malaria, runge-kutta method.

I. Introducción

Las ecuaciones diferenciales, como se las conoce, nos ofrecen una amplia gama de aplicaciones en diferentes áreas[1], lo que trae cada vez más a los matemáticos y demás profesionales, desafíos para presentar soluciones a los diversos problemas que de vez en cuando o permanentemente enfrentará el mundo social. En esta ocasión, trajimos en este trabajo un enfoque de un modelo clásico de sistema de ecuaciones diferenciales ordinarias propues-

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to por Kermark y McKendrick[9], el modelo SIR (Susceptibles-Infectados-Recuperados) para estudios epidemiológicos que también nos puede ayudar, como es obvio en el estudio del comportamiento numérico de las variables que afectan el caracterización de la infección y recuperación de la malaria. La malaria es una enfermedad que predomina en áreas con clima tropical[4] como Angola y especialmente en regiones con saneamiento deficiente. Es una enfermedad infecciosa transmitida por mosquitos[2]. Estábamos interesados en modelar la malaria por el hecho de que es la mayor causa de muerte en Angola y, para presentar una vez más el poder de los modelos SIR que se pueden utilizar en el estudio del comportamiento, no solo de la malaria sino también en otras enfermedades infecciosas y en casos de enfermedades contagiosas.

Como el modelo SIR es EDO, entonces se eligió el método de Runge-Kutta de cuarto orden para presentar la demostración e implementación computacional y la consecuente ilustración gráfica y numérica del comportamiento de la enfermedad referida. obviamente, se puede utilizar cualquier otro método que nos permita resolver EDO[5],[7] y [3], pero como solo tuvimos que utilizar uno en este trabajo, elegimos el método sencillo y eficiente en sus resultados.

II. Modelo Endémico Clásico SIR

Para comprender el comportamiento numérico de la malaria en un intervalo de tiempo determinado, se decidió utilizar el modelo SIR(Susceptibles-Infectados-Recuperados), que es modelo epidemiológico clásico. Y como la malaria en Angola es una enfermedad endémica, hemos utilizado el modelo endémico clásico. El modelo endémico clásico es el modelo SIR con dinámica vital que incluye nacimientos y muerte, cuyo sistema de ecuaciones viene dado por[8]:

$$\begin{cases} \frac{dS}{dt} = \mu P - \mu S - \frac{\beta IS}{P}, \\ \frac{dI}{dt} = \frac{\beta IS}{P} - \gamma I - \mu I, \\ \frac{dR}{dt} = \gamma I - \mu R, \end{cases} \tag{2.1}$$

Donde S(t)+I(t)+R(t)=P. El modelo SIR (2.1) presentado es casi igual en relación al modelo epidémico SIR que podemos encontrar en [8] la diferencia es que el modelo endémico presenta un flujo de recién nacidos en la clase susceptible a tasa μP y muertes en todas las clases a tasas μS, μI y μR, $\mu > 0$. Dado que las muertes equilibran los nacimientos, entonces el tamaño de la población P es constante. S, denota el número de la población susceptible a la enfermedad, I es el número de infectados y R, el número de recuperados. También disponemos de parámetros β, μ y γ. Donde γ es la tasa de recuperación, β es la tasa de infección y μ, la tasa de muertes. Como el estudio se basa en la ciudad de Cuito, suponemos que la población es fija. Como el problema se analiza en un intervalo de tiempo, es necesario establecer un momento inicial de referencia del estudio para que se pueda abarcar la ingeniosidad del análisis hasta un tiempo final. Así, desde esta perspectiva, la ecuación (2.1) tendrá la forma:

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2. A. P. Gomes, R. R. Vitorino, A. P. Costa, E. G. Mendonça, M. G.A. Oliveira, R. Siqueira-Batista, Malária grave por plasmodium falciparum, Rev. Bras. Ter. Intensiva 23(3), Set, 2011, https://doi.org/10.1590/s0103-507X2011000300015.

$$\begin{cases} \frac{dS}{dt} = \mu P - \mu S - \frac{\beta IS}{P}, S(0) = S_0 > 0, \\ \frac{dI}{dt} = \frac{\beta IS}{P} - \gamma I - \mu I, I(0) = I_0 \geq 0, \\ \frac{dR}{dt} = \gamma I - \mu R, R(0) = R_0 \geq 0, \end{cases} \tag{2.2}$$

En este caso, ahora tenemos un problema de valor inicial (PVI) de ecuación diferencial ordinaria[6] de primer orden.

III. Forma Matricial

Siendo la ecuación (2.1) un sistema de ecuaciones diferenciales ordinarias de primer orden, puede escribirse como sigue:

$$\begin{cases} \frac{dS}{dt} = f_S(t, S, I, R) \\ \frac{dI}{dt} = f_I(t, S, I, R) \\ \frac{dR}{dt} = f_R(t, S, I, R) \end{cases}$$

Dado que t es la variable independiente, S, I y R son variables dependientes. Lo cual presupone que en forma vectorial tenemos

$$\frac{d\underline{Y}}{dt} = \underline{F}(t, \underline{Y})$$

Donde

$$\underline{Y} = S, I, R$$

ó

$$\underline{Y} = \begin{bmatrix} S \\ I \\ R \end{bmatrix}$$

y

$$\underline{F}(t, \underline{Y}) = \begin{bmatrix} f_S(t, S, I, R) \\ f_I(t, S, I, R) \\ f_R(t, S, I, R) \end{bmatrix}$$

Lo que podemos interpretar el problema (2.1) como:

Ref

6. N. S. Nedialkova, K. R. Jakson, G. F. Corlissb. Validated solutions of initial value problems for ordinary differential equations, Applied Mathematics and Coputation, vol.105, issue 1, October 1999, pages 21-68 hpps://doi.org/10.1016/s 0096- 3003(98)10083-8.

$$\underline{Y}(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix} \Rightarrow \underline{Y} = \begin{bmatrix} S \\ I \\ R \end{bmatrix}$$

Entonces, podemos escribir:

$$\underline{F}(t, \underline{Y}) = \begin{bmatrix} \mu P - \mu S - \beta IS/P \\ \beta IS/P - \gamma I - \mu I \\ \gamma I - \mu R \end{bmatrix} \underline{Y}(0) = \begin{bmatrix} S(0) \\ I(0) \\ R(0) \end{bmatrix}$$

La forma matricial presentada, siendo $\underline{Y}$ el vector de las variables S, I y R, lo que nos lleva a la derivada del vector con respecto al tiempo viene dada por $\frac{dY}{dt}$, lo que lleva a una función vectorial que depende S, I y R pues luego entonces tendremos $\underline{F}(t, \underline{Y})$ donde t es una variable independiente y el vector $\underline{Y}$ se queda como variable dependiente en la función vectorial.

IV. Métodos

La investigación se basó en datos proporcionados por el Departamento de Salud Pública de la Oficina Provincial de Salud de Bié, Angola, con los cuales fue posible realizar predieccíón basada en un sistema de ecuaciones diferenciales ordinarias, el modelo SIR y,consulta de artículos científicos y libros que abordan este tipo de ecuaciones y métodos con los que se buscan las respectivas soluciones. Una vez modelado el problema y identificado el modelo SIR que mejor se adapta al problema, el clásico modelo SIR endémico, cuyos parámetros muestran la dinámica vital. Para resolver el dicho sistema de ecuaciones que representa el problema se utilizó el método de Runge-Kutta de cuarto orden, el cual permitió la implementación computacional del problema. Fue utilizado el lenguaje de programación, Octave, que es un software muy bueno y es compatible con Matlab. La implentación fue teniendo en cuenta lo que sugiere la documentación de la versión 7.2.0 de Octave para ecuaciones diferenciales ordinarias, en la creación del respectivo Script.

V. Problema Modelado

Como se dice en la introdución, el problema que se presenta tiene como población de estudio a la ciudad de Cuito, Angola. Por ello, nos interesó estudiar el comportamiento numérico de la malaria en la citada ciudad en los meses de cacimbo (mayo, junio, julio y agosto), también conocido como periodo de invierno, ya que en este mismo periodo no llueve y en consecuencia bajan las temperaturas e infecciones de malaria. Así, a través del Departamento de Salud Pública de la Secretaría Provincial de Salud de Bié, habíamos solicitado datos sobre cifras de malaria. Sin embargo, nos proporcionaron la cantidad de personas infectadas con malaria (16448), la cantidad de recuperaciones (15139), la tasa de infección (28 %), la tasa de recuperaciones (92 %) y la tasa de muertes (8 %) para el mes de mayo de 2022. Como dato inicial para poder predecir a través del modelo (2.1), el comportamiento numérico de la malaria en los demás meses de invierno en la ciudad de Cuito.

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Interpretando los datos proporcionados para (2.2), tenemos:

$$\begin{cases} \frac{dS}{dt} = 0,08P - 0,08S - 0,28IS/P \\ \frac{dI}{dt} = 0,28IS/P - 0,92I - 0,08I \\ \frac{dR}{dt} = 0,92I - 0,08R \\ S(0) = 512706,\ t \in [0,4] \\ I(0) = 16448,\ t \in [0,4] \\ R(0) = 15139,\ t \in [0,4] \end{cases} \tag{5.3}$$

Para resolver este problema podemos utilizar entre varios métodos posibles el de Runge-Kutta de 4º y 2º orden, Euler, Euler Modificado, Euler Mejorado. Pero aquí preferimos usar el método de Runge-Kutta de 4º orden porque es más simple y eficiente.

VI. Método de Runge-Kutta 4º Orden

Los métodos de Runge-Kutta de orden superior se obtiene de forma similar a los de segundo orden. Los métodos de tercer orden, por ejemplo, tienen la función de incremento $\varnothing(x_j, y_j, h) = \alpha K_1 + \beta K_2 + \gamma K_3$ onde K_1, K_2 y K_3 se aproximan a las derivadas en varios puntos del intervalo $[x_1, x_{j+1}]$. Donde la serie de Taylor es fundamental para determinar los parámetros α, β y γ.

Entre los diversos métodos de Runge-Kutta, el más popular y eficiente es:

$$Y_{j+1} = Y_j + \frac{h}{6}(K_1 + 2K - 2 + 2K_3 + K_4),\ j = 0, 1, ..., m-1$$

$$K_1 = f(x_j, y_j)$$

$$K_2 = f(x_j + \tfrac{h}{2},\ y_j + \tfrac{h}{2}K_1) \tag{6.4}$$

$$K_3 = f(x_j + \tfrac{h}{2},\ y_j + \tfrac{h}{2}K_2)$$

$$K_4 = f(x_j + h,\ y_j + hK_3)$$

El método (6.4) es de Runge-Kutta de 4º orden. Los métodos Runge-Kutta son de arranque automático, ya que son de paso uno y no funcionan con derivada de $f(x, y)$[10].

VII. Los Resultados Numéricos

Con la aplicación del método de Runge-Kutta de 4º orden, se implementó el problema en Octave y resultó en los siguientes datos numéricos:

Ref

10. M. A. G. Ruggiero, V. L. da R. Lopes, Calculo Numérico: Aspectos teóricos e computacionais, 2ªed. Pearson, São Paulo, Pearson Makron Books, 1996, ISBN 978-85-346-0204-4.

Cuadro 1: Resultados numéricos de las variables S de la población susceptíble, I de infecciones y R de recuperación de la malaria.

$t(mensual)$	S	I	R
0	512 706.	16448.000	16448.000
1	509350.624	7475.339	26230.002
2	508093.560	3391.846	29221.579
3	507763.644	1538.129	29238.034

Los resultados de la tabla son mucho más claros en los gráficos seguientes, ya que ilustran el comportamiento visual de las variables de interés:

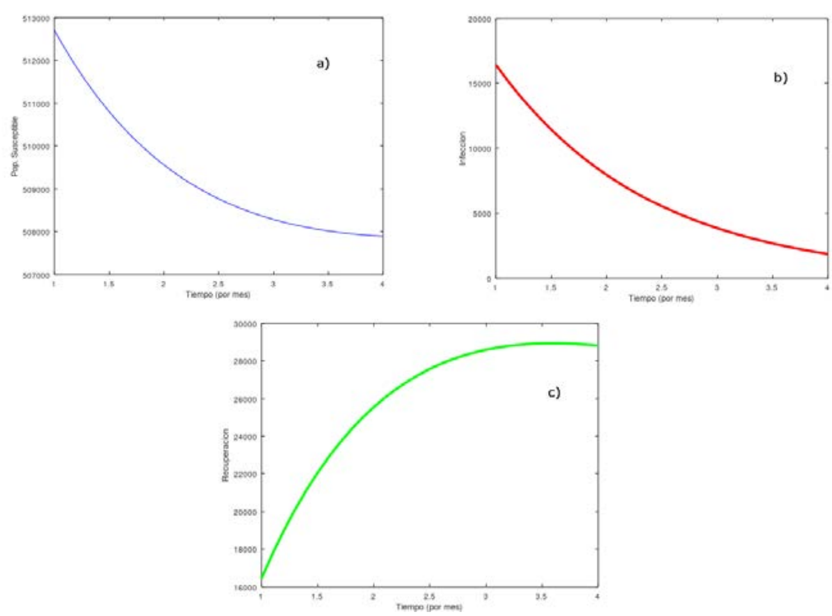

Figura 1: Gráficos de las variables Población Susceptible a), Infección b) y Recuperación c) de la malaria en invierno en la ciudad Cuito, 2022.

Como se puede observar del comportamiento de la recuperación c), se puede notar que existe un crecimiento en la gráfica que en un momento dado hace una inversión lenta y luego desciende, esto se justifica por la tasa alta de recuperación, 92 %, que contrasta con la tasa de infección que es de 28 %, es decir, debido a la redución de infecciones b), la subida de la gráfica de recuperación va disminuyendo.

Por otro lado, se nota que la gráfica de infecciones b), desciende en tiempo debido a su tasa relativamente baja. Estas infecciones no dependen de los recuperados, ya que la malaria no es una enfermedad contagiosa y su medicación no ofrece inmunidad, este último critério solo justifica el comportamiento del lento descenso de la gráfica de población susceptible a), los recuperados aparecen y también se convierten en poblacion susceptible. En este caso, se puede verificar en la gráfica seguiente que descienden las infecciones y crecen las recuperaciones hasta mas o menos al mes de Julio y después de eso comienza el descenso:

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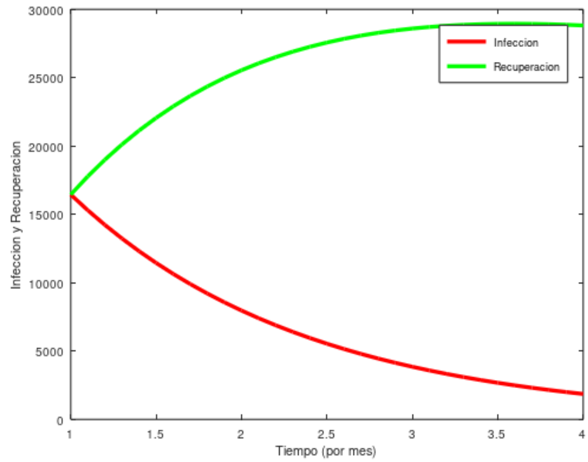

Figura 2: Gráficos de las Infecciones y Recuperaciones de la malaria en la ciudad de Cuito, Bié, Angola en invierno 2022.

VIII. Conclusión

Por lo tanto, el modelo del sistema de ecuaciones diferenciales ordinarias utilizado, modelo endémico SIR, su poder y utilidad en los problemas de evolución, por lo que nos ayuda a realizar experimentos para estudios de ambientes infecciosos, desde el punto de vista sanitario. Por otro lado, el método de Runge-Kutta muestra en su sencillez una eficiencia en los resultados. Y, los resultados del experimento demuestran una dependencia de las recuperaciones en relación a las infecciones, ya que como hemos visto en el comportamiento gráfico de las dos variables, hay un descenso en las recuperaciones porque hay también bajas en las infecciones.

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Decomposition of the Random Error Vector of a General Linear Model

By Jaesung Choi
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Abstract- This paper deals with the decomposition of an error vector to identify how the error vector is related to the expected value of an observation vector under a general linear sample model since the error vector is defined as the deviance of observation vector from the expected value. The main idea of the paper is in that a random error vector can be decomposed into two orthogonal components vectors; i.e., one is in a vector space generated by the coefficient matrix of the unknown parameter vector and the other is in orthogonal complement of it. As related topics to the decomposition, two things are discussed: partitioning an observation vector and constructing the covariance structure of it. It also shows the reason why a projection method would be preferred rather than a least squares method.

Keywords: *coefficient matrix; decomposition; least squares; orthogonal complement; projection; vector space.*

GJSFR-F Classification: *MSC 2010: 15A03*

DECOMPOSITIONOFTHERANDOMERRORVECTOROFAGENERALLINEARMODEL

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Decomposition of the Random Error Vector of a General Linear Model

Jaesung Choi

Abstract- This paper deals with the decomposition of an error vector to identify how the error vector is related to the expected value of an observation vector under a general linear sample model since the error vector is defined as the deviance of observation vector from the expected value. The main idea of the paper is in that a random error vector can be decomposed into two orthogonal components vectors; i.e., one is in a vector space generated by the coefficient matrix of the unknown parameter vector and the other is in orthogonal complement of it. As related topics to the decomposition, two things are discussed: partitioning an observation vector and constructing the covariance structure of it. It also shows the reason why a projection method would be preferred rather than a least squares method.

Keywords: coefficient matrix; decomposition; least squares; orthogonal complement; projection; vector space.

I. Introduction

When there are some nonrandom quantities affecting a response variable, the analysis of data can be done by using a general linear model. There are some notable references about linear models such as [1-4]. In a general linear model, the response variable is composed of two parts in general; one part is the deterministic portion as a linear function of the unknown parameters of the independent or predictor variables and the other is the random portion. When data are collected, the sample general linear model in matrix form is applicable to the data. However, the matrix equation seems not to be useful for catching any idea about the relationship between the matrix of predictors and error vector only with the assumptions about the error vector; i.e., $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $var(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. This requires the error vector be the line segment from the origin, which is the point $\mathbf{0}$, to the point $\boldsymbol{\epsilon}$. The general linear sample model in matrix form is

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where $\boldsymbol{y}$ denotes the $n \times 1$ vector of observations, $\boldsymbol{X}$ denotes the $n \times p$ matrix of known values, $\boldsymbol{\beta}$ denotes the $p \times 1$ vector of unknown parameters and $\boldsymbol{\epsilon}$ denotes the $n \times 1$ vector of unobserved random errors. The detailed discussion on random errors can be seen in Searle[1]. The matrix equation (1) shows that the vector of observations is composed of mean vector and error vector. When $E(\boldsymbol{y})$ is a $\mathbf{0}$ or a mean vector acting like an origin, the error vector satisfies the conditions of mean $\mathbf{0}$ and $var(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. However, if $E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta}$ then we might be interested in how to minimize the deviation vector, $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}$. The least square method can be used as one of available methods for minimizing the error vector to estimate the parameter vector $\boldsymbol{\beta}$. However, the decomposition of an random error vector can be a little bit more comfortable and effective than the minimization of error sum of

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squares by the method of least squares in perspective that the structure of an random error vector is primarily considered. This idea is applied to breaking up an error vector in model (1). Since $\boldsymbol{X\beta}$ is in a specific vector subspace generated by the column space of $\boldsymbol{X}$, and $\boldsymbol{\epsilon} = \boldsymbol{y} - \boldsymbol{X\beta}$ is nonzero, $\boldsymbol{\epsilon}$ is not in the column space of $\boldsymbol{X}$. The related topics about these can be seen in Graybill[5], Johnson[6] and so forth. Thus, when $\boldsymbol{X\beta}$ is used as a base for getting a error vector of $\boldsymbol{y}$ that is required to have one with the shortest distance from the origin among all the error vectors, we need to break up the error vector into a few component error vectors. This is the main idea of this paper. First, we discuss how to decompose the error vector. Secondly, we study the structure of error vector adjusted for the mean vector. Thirdly, we find the structure of $var(\boldsymbol{y})$ that is related to the error structure. Finally, we discuss a method that is useful to calculate the sum of squares associated with the error components.

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II. Decomposition of Random Error Vector

In the matrix equation (1) let the mean vector $\boldsymbol{X\beta}$ be an nonzero vector. Then the equation can be changed into

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{X\beta} + \boldsymbol{\epsilon} \\ &= \boldsymbol{X\beta} + \boldsymbol{\epsilon_m} + \boldsymbol{\epsilon_r} \end{aligned} \tag{2}$$

where $\boldsymbol{\epsilon_m}$ and $\boldsymbol{\epsilon_r}$ are denoting two component vectors of the error vector. Since $E(\boldsymbol{y})$ is $\boldsymbol{X\beta}$, $\boldsymbol{\epsilon}$ can be broken down into two types of error vector; one type of error vector is the one that is in column space of $\boldsymbol{X}$, and the other has the characteristic orthogonal to an every vector in a vector space generated by the columns of $\boldsymbol{X}$, which is an orthogonal complement of the column space of $\boldsymbol{X}$. In matrix equation (2), $\boldsymbol{y} - \boldsymbol{X\beta}$ defines the error vector based on the mean vector $\boldsymbol{X\beta}$ assumed to have mean $\boldsymbol{0}$ and $\sigma^2\boldsymbol{I}$ in n dimensional space. Since the error vector is defined as a vector of deviations from the mean vector of $E(\boldsymbol{y})$, we can decompose it into two component vectors depending on sources where the error vector is coming up; i.e., $\boldsymbol{\epsilon_m}$ or $\boldsymbol{\epsilon_r}$. Let's rewrite the matrix equation (2) in terms of error vector from the mean vector. Then,

$$\begin{aligned} \boldsymbol{y} - \boldsymbol{X\beta} &= \boldsymbol{\epsilon_m} + \boldsymbol{\epsilon_r} \\ &= \boldsymbol{XX^-\epsilon} + (\boldsymbol{I} - \boldsymbol{XX^-})\boldsymbol{\epsilon} \end{aligned} \tag{3}$$

where $\boldsymbol{\epsilon_m} = \boldsymbol{XX^-\epsilon}$, $\boldsymbol{\epsilon_r} = (\boldsymbol{I} - \boldsymbol{XX^-})\boldsymbol{\epsilon}$, and $\boldsymbol{X^-} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'}$ denotes the Moore-Penrose generalized inverse where $(\boldsymbol{X'X})^{-1}$ exists because $\boldsymbol{X}$ is a full column rank matrix. From the equation (3), we can know that there are two types of error vector when a vector space is decomposed into two orthogonal vector subspaces based on the mean vector, $\boldsymbol{X\beta}$, of $\boldsymbol{y}$. The above equation shows that $E(\boldsymbol{y} - \boldsymbol{X\beta}) = \boldsymbol{0}$ and $var(\boldsymbol{y} - \boldsymbol{X\beta}) = \sigma^2\boldsymbol{I}$. Here, the error vector $\boldsymbol{y} - \boldsymbol{X\beta}$ is assumed as a linear combination of two different types of error vector each of which coming from two orthogonal vector subspaces. This is completely different concept of an error vector from the one we have thought traditionally. A lot of stuffs can be developed by the newly idea of viewing the error vector as the sum of a mean component error vector and a residual component error vector. Here, two specific terms are used to differentiate the types of error component: a mean component error for $\boldsymbol{\epsilon_m}$ and a residual component error for $\boldsymbol{\epsilon_r}$.

III. Structure of Random Error Vector

Since the structure of a random error vector in matrix model (1) is changed depending on the structure of the mean vector, we are going to take a look at it with a bit simpler general linear models. Consider a situation where measurements are measured as deviations from the fixed size for products from a routine process. Let y be a random variable taking an observation on a randomly selected product from a population of products. Data collected from a sample of size n from the population can be arrayed in matrix form. The ith observation of y is expressed as $y_i = 0 + \epsilon_i$ for $i = 1, 2, \ldots, n$, where ϵ_i's are assumed to be independent with $E(\epsilon_i) = 0$ and $var(\epsilon_i) = \sigma^2$. Applying the sample general linear model (1) to the data, the model turns out to be

$$\boldsymbol{y} = \boldsymbol{0} + \boldsymbol{\epsilon} \tag{4}$$

where $\boldsymbol{y}$ is in R_n denoting a Euclidean n-space, and $E(\boldsymbol{y}) = \boldsymbol{0}$. Let V_0 be the vector space consisting only of $\boldsymbol{0}$ and let V_1 be the orthogonal complement of V_0. Since $\boldsymbol{y}$ is the sum of two vectors such that one in V_0 and the other in V_1, $\boldsymbol{0}'\boldsymbol{\epsilon} = 0$ which shows the relationship between mean vector and error vector of $\boldsymbol{y}$; i.e., an orthogonal property. To express the equation (4) as the second expression in (2), $\boldsymbol{\epsilon}$ can be divided into two terms, $\boldsymbol{\epsilon}_0$ and $\boldsymbol{\epsilon}_1$ where $\boldsymbol{\epsilon}_0$ denotes the error vector generated by $\boldsymbol{0}$ in a basis of V_0, and $\boldsymbol{\epsilon}_1$ denotes the error vector generated by a basis set of V_1. Now, the matrix equation can be expressed as

$$\boldsymbol{y} = \boldsymbol{0} + \boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}_1 \tag{5}$$

where $\boldsymbol{\epsilon}_0$ is in V_0 of dimension 0, while $\boldsymbol{\epsilon}_1$ is in V_1 of dimension n. Adding the information about the dimension of error component vectors, the equation (5) can be transformed into

$$\boldsymbol{y} - \boldsymbol{0} = \boldsymbol{O}\boldsymbol{\epsilon} + (\boldsymbol{I} - \boldsymbol{O})\boldsymbol{\epsilon} \tag{6}$$

where $\boldsymbol{O}$ represents $n \times n$ zero matrix and $\boldsymbol{I}$ is $n \times n$ identity matrix. The equation shows that $\boldsymbol{\epsilon}$ from the origin of rank 0 can be decomposed into two orthogonal vectors one of which being in the orthogonal complement of a vector space. In other words, this means that $\boldsymbol{\epsilon}$ is actually composed of a linear combination of two component vectors; i.e. $\boldsymbol{\epsilon}_0$ and $\boldsymbol{\epsilon}_1$. It seems such valuable to grasp the structure of a random error vector for finding the covariance structure of an observation vector. Now, we can study the error structure further with a little bit general but still simpler model having just one quantitative variable as an predictor. Consider the simple linear model with only one nonrandom independent variable in addition to an intercept term; i.e., $y_i = \beta_0 + \beta_1 X_{1i} + \epsilon_i$, for $i = 1, 2, \cdots, n$. We rewrite this in vector and matrix form as

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{j}\beta_0 + \boldsymbol{X}_1\beta_1 + \boldsymbol{\epsilon} \qquad (7)\\ &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \end{aligned}$$

where $\boldsymbol{X} = (\boldsymbol{j}, \boldsymbol{X}_1)$ is an $n \times 2$ coefficient matrix of $\boldsymbol{\beta}$, $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ is an $n \times 2$ parameter vector, $\boldsymbol{j}$ is an $n \times 1$ vector of ones, and $\boldsymbol{X}_1$ is an $n \times 1$ vector of quantitative values. $\boldsymbol{\epsilon}$ is an error vector assumed to have $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $var(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. The equation (7) is different from the one in (6) in that $E(\boldsymbol{y}) \neq \boldsymbol{0}$. This is not a surprising thing in a general linear model other than that the mean vector $\boldsymbol{X\beta}$ belongs to the column space of $\boldsymbol{X}$, $\boldsymbol{V}_m$ of dimension 2 which is thought to be a vector subspace in a Euclidean n-space, $\boldsymbol{R}_n$. Since the mean vector of $\boldsymbol{y}$, $\boldsymbol{X\beta}$, is in $\boldsymbol{V}_m$, $\boldsymbol{\epsilon}$ should be divided into two components: one in $\boldsymbol{V}_m$ and the other in $\boldsymbol{V}_m{}^{\perp}$ denoting an orthogonal complement of $\boldsymbol{V}_m$ in $\boldsymbol{R}_n$; $\boldsymbol{V}_m \oplus \boldsymbol{V}_m{}^{\perp} = \boldsymbol{R}_n$. The set of two vectors in the matrix of $\boldsymbol{X}$ can be regarded as a basis set of $\boldsymbol{V}_m$, which implies that $\boldsymbol{X\beta}$ is in $\boldsymbol{V}_m$. Hence, the error vector can be divided into two component vectors such that one component in $\boldsymbol{V}_m$ and the other in $\boldsymbol{V}_m{}^{\perp}$; that is, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_m + \boldsymbol{\epsilon}_r$. When we add this kind of information to the equation, the model will be

$$\boldsymbol{y} - \boldsymbol{X\beta} = \boldsymbol{\epsilon}_m + (\boldsymbol{I} - \boldsymbol{\epsilon}_m)\boldsymbol{\epsilon} \tag{8}$$

where $\boldsymbol{y} - \boldsymbol{X\beta} \in \boldsymbol{R}_n$, $\boldsymbol{\epsilon}_{\in}\boldsymbol{V}_m$ and $(\boldsymbol{I} - \boldsymbol{\epsilon}_m)\boldsymbol{\epsilon}$ in $\boldsymbol{V}_m{}^{\perp}$. Both the set of the columns of a matrix $\boldsymbol{X}$ of rank 2 equivalent to a basis for $\boldsymbol{V}_m$ and the set of the columns of a matrix $\boldsymbol{XX}^{-}$ can generate the same space, $\boldsymbol{V}_m$. Hence, the equation (8) can be changed into

$$\boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{XX}^{-}\boldsymbol{\epsilon} = (\boldsymbol{I} - \boldsymbol{XX}^{-})\boldsymbol{\epsilon} \tag{9}$$

where $\boldsymbol{XX}^{-}\boldsymbol{\epsilon}$ replaces $\boldsymbol{\epsilon}_m$ and denotes the projection of $\boldsymbol{\epsilon}_m$ onto a vector space, $\boldsymbol{V}_m$, generated by two vectors $\boldsymbol{j}$ and $\boldsymbol{x}$. The matrix equation (9) turns out to be

$$\begin{aligned}(\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{\epsilon} &= \boldsymbol{y}-(\boldsymbol{X\beta}+\boldsymbol{XX}^{-}\boldsymbol{\epsilon}) \\ &= (\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{y}\end{aligned} \tag{10}$$

where $\boldsymbol{X\beta}+\boldsymbol{XX}^{-}\boldsymbol{\epsilon}=\boldsymbol{XX}^{-}\boldsymbol{y}$.

IV. Covariance Structure of Vector of Observations

From the decomposition of a random error vector of a general linear sample model in matrix form, we can identify that the matrix model (1) can be transformed into

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{X\beta}+\boldsymbol{\epsilon} \\ &= \boldsymbol{XX}^{-}\boldsymbol{y}+(\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{y}\end{aligned} \tag{11}$$

where $\boldsymbol{y}$ is composed of two orthogonal vectors: i.e., $(\boldsymbol{XX}^{-}\boldsymbol{y})'(\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{y}=0$. The model equation (11) implies that all types of a general linear model can be represented by a sum of two orthogonal vectors where one vector belongs to an vector subspace and the other is in the orthogonal complement of the vector space generated by the coefficient matrix of $\boldsymbol{\beta}$: i.e., $\boldsymbol{XX}^{-}\boldsymbol{y}\in\boldsymbol{V}_m$, and $(\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{y}\in\boldsymbol{V}_m^{\perp}$. Here, the primary concern is actually in structural aspects of an assumed linear model while the least square method focuses only on getting the best approximate solution from a system of inconsistent equations such that $\boldsymbol{X\beta}-\boldsymbol{y}=\boldsymbol{\epsilon}$ by the method of minimizing the error sum of squares. Hence, they are different approaches developed from different view of points. Now, consider the calculation of $var(\boldsymbol{y})$. The covariance matrix of $\boldsymbol{y}$ is

$$\begin{aligned}var(\boldsymbol{y}) &= var(\boldsymbol{XX}^{-}\boldsymbol{y}+(\boldsymbol{I}-\boldsymbol{XX}^{-})\boldsymbol{y}) \\ &= \sigma^2\boldsymbol{XX}^{-}+\sigma^2(\boldsymbol{I}-\boldsymbol{XX}^{-}) \\ &= \sigma^2\boldsymbol{I}\end{aligned} \tag{12}$$

From the above equation (12), $var(\boldsymbol{y})$ can be obtained by identifying the linear transformations of $\boldsymbol{y}$; i.e., the covariance matrix of $\boldsymbol{y}$ can be partitioned as the sum of component covariance matrices, which can be done by ascertaining transformation matrices for component vectors of $\boldsymbol{y}$. There are some referable literature related to covariance matrix such as Milliken and Johnson[7], Hill[8], and Searle[9]. Hence, it is essential to figure out the coefficient matrices of component error vectors to find the projections of $\boldsymbol{y}$ onto the vector subspaces generated by the orthogonal coefficient matrices. Discussions on coefficient matrices are seen in Choi[10-12], where they are related to get nonnegative variance estimates.

V. Projection Method

As a result of the decomposition of $\boldsymbol{e}$, we see $\boldsymbol{y}$ can be represented by the sum of two orthogonal component vectors such as (11) where one is in a vector space covering the $E(\boldsymbol{y})$ and the other is in the orthogonal complement of it. This means that $\boldsymbol{XX}^{-}\boldsymbol{y}$ actually defines a projection of $\boldsymbol{y}$ onto the vector space spanned by the $\boldsymbol{XX}^{-}$ where $\boldsymbol{X}$ is coefficient matrix of $\boldsymbol{\beta}$ and given as $\boldsymbol{X}=(\boldsymbol{j},\boldsymbol{X}_1)$. For the estimation of parameter vector $\boldsymbol{\beta}$ we can use the mean part of the model in matrix form of (11). From the concept of a projection in a vector space the projection of $\boldsymbol{y}$ onto a column space of $\boldsymbol{X}$ is as follows:

$$\boldsymbol{X\beta}=\boldsymbol{XX}^{-}\boldsymbol{y} \tag{13}$$

where $E(\boldsymbol{y})=\boldsymbol{X\beta}$. When $\boldsymbol{XX}^{-}\boldsymbol{y}$ is viewed as the orthogonal projection of $\boldsymbol{y}$ onto a column space of $\boldsymbol{X}$ we can take $\hat{\boldsymbol{\beta}}_p=\boldsymbol{X}^{-}\boldsymbol{y}$ as the value of $\boldsymbol{\beta}$ where $\hat{\boldsymbol{\beta}}_p$ is an notation for differentiating from $\hat{\boldsymbol{\beta}}$ obtained from the normal equations. When the expression in (13) is viewed as the system of equations, the best approximate solution to the system can be

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obtained as $\hat{\boldsymbol{\beta}} = \boldsymbol{X}^{-}\boldsymbol{y}$ because the system of equations is inconsistent and $\boldsymbol{X}$ is $n \times 2$ matrix of rank 2. Although solutions of $\boldsymbol{\beta}$ can be obtained in different approaches, the results are actually same. In a similar way that $\boldsymbol{X}\boldsymbol{X}^{-}\boldsymbol{y}$ can be used for the estimation of $\boldsymbol{\beta}$, a quadratic form in $\boldsymbol{y}$ can also be used for the estimation of σ^2. Here, the required quadratic form is given as

$$\boldsymbol{Q}_r = \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})\boldsymbol{y} \tag{14}$$

where $(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})$ is a symmetric and idempotent matrix of rank $n-2$. Since $(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})\boldsymbol{y}$ is regarded as a linear transformation of $\boldsymbol{y}$, it has all the information about the residual random error component, $\boldsymbol{\epsilon}_r$. Hence, the quadratic form $\boldsymbol{Q}_r$ in $\boldsymbol{y}$ can be used to estimate the variance σ^2. Taking the expectation of $\boldsymbol{Q}_r$ is given as

$$\begin{aligned} E(\boldsymbol{Q}_r) &= E(\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})\boldsymbol{y}) \\ &= \sigma^2 tr(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-}) + (\boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})\boldsymbol{X}\boldsymbol{\beta} \\ &= \sigma^2(n-2) \end{aligned} \tag{15}$$

where $tr(\cdot)$ means trace of a square matrix denoted by $(\cdot)$, which is defined to be the sum of the diagonal elements of the square matrix. Some theorems and properties of trace can be seen in Graybill[2]. As an estimate of σ^2 from the equation (15), $\hat{\sigma_p^2}$ can be taken as $\boldsymbol{Q}_r/(n-2)$ which can also be obtained by the least square method when there is no normality assumption for $\boldsymbol{\epsilon}$. Even though those two procedures have the same result, it should be noticed they are basically approaching from different view of point; that is, one is from the decomposition of an error vector, and the other is from the minimization of error sum of squares.

VI. Example

As for an example of a simple linear model, we consider following data from Krumbein and Graybill[13]. The data are assumed to satisfy the model $y_i = \beta_0 + \beta X_{1i} + \epsilon_i$, for $i = 1, 2, \cdots, 10$, where ϵ_i are independent and identically distributed $N(0, \sigma^2)$.

Table 1: Krumbein and Graybill's Data[13].

x_i	550	200	280	340	410	475	160	380	510	510
y_i	200	50	60	140	130	180	20	120	190	160

For the estimation of two unknown parameters, β_0 and β, we can get $\hat{\boldsymbol{\beta}}_p$ by multiplying $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ on both sides of the equation (13), which is given as:

$$\begin{aligned} (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{X}^{-}\boldsymbol{y} \\ \hat{\boldsymbol{\beta}}_p &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} \\ &= \begin{pmatrix} 0.982060878 & -2.312086e-03 \\ -0.002312086 & 6.060514e-06 \end{pmatrix} \begin{pmatrix} 1250 \\ 550500 \end{pmatrix} = \begin{pmatrix} -45.2273450 \\ 0.4462054 \end{pmatrix} \end{aligned} \tag{16}$$

Denoting $\hat{\beta_{0p}}$, and $\hat{\beta_p}$ as estimates of β_0 and β respectively, $\hat{\beta_{0p}} = -45.2273450$ and $\hat{\beta_p} = 0.4462054$. Least squares estimates are given as $\hat{\beta_0} = -45.227$ and $\hat{\beta} = 0.446$. For the estimation of σ^2, we can get an estimate as:

$$\hat{\sigma_p^2} = \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})\boldsymbol{y}/8 = 2398.13/8 = 299.7663 \tag{17}$$

where $(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{-})$ is the 10×10 matrix of rank 8. As an estimate of σ^2 denoted by $\hat{\sigma_p^2}$ is given as 299.7663 which is the same as the least squares estimate $\hat{\sigma^2} = 299.766$; hence, $\hat{y} = -45.2273450 + 0.446205X_1$.

VII. Discussion

The primary concern of the study is on the decomposition of an error vector in matrix form of a general linear model. When $\boldsymbol{\epsilon}$ is $n \times 1$ vector, the usual assumptions for error vector are sometimes given as $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $Var(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. Under these assumptions, an idea for breaking up the error vector lies on the thought of which the mean vector is related to the error vector because the error vector is defined to be the deviation vector from the mean of the model. When the error vector is decomposed into two orthogonal components, it is shown that a projection can be defined from the decomposition of the error vector. Hence, a partition of the vector of observations can be seen as the sum of vectors which are orthogonal projections each other. The covariance matrix of $\boldsymbol{y}$ is partitioned into two covariance matrices; that is, one for $(\boldsymbol{X}\boldsymbol{X}^-)\boldsymbol{y}$, and the other for $(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^-)\boldsymbol{y}$. This implies that the covariance matrix of a vector of observations can always be partitioned into component matrices each of which corresponding to an orthogonal projection of $\boldsymbol{y}$ respectively. From the decomposition of an error vector of a general linear model, we derived two types of estimators; one is linear transformation of $\boldsymbol{y}$ for $\boldsymbol{X\beta}$ to estimate $\boldsymbol{\beta}$ and the other is quadratic form in $\boldsymbol{y}$ for σ^2. Partitioning of the covariance matrix can be useful to ascertain the covariance matrix of each component projection. It is worth to note that decomposition of an error vector is actually defines a projection of $\boldsymbol{y}$ onto a column space of $\boldsymbol{X}$ and which is quite different approach from the least square method in a point of view for an error vector.

VIII. Conclusions

Although the least squares method is very useful and accepted as one of well-known methods for estimating the unknown parameters included in a linear model, it seems not to be right for finding out whether there is any orthogonal property exists among errors. Since the least squares method concentrates only on minimizing the sum of squares of deviations of the observations from the expected values, it is not an appropriate method as a tool for getting the information on an orthogonal property between the groups of errors. The orthogonal property is extremely important in statistics especially in the analysis of variance for getting nonnegative estimates for variance components. There are lots of interesting papers[14-20] related to the negative estimates of variance components seemed to be caused by overlooking the orthogonality. So, it is emphasized that the orthogonal property can be found by the decomposition of the random error vector. Hence, the procedure discussed on this paper is distinct from any other methods for estimating the unknown parameters in a general linear model.

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The Wonders and Curiosities of the Number 9

By Hussein Ahmed Ayed Rababa

Abstract- Numbers are a world full of secrets and wonders that help explain many phenomena and enter the world of games to act as an engine for the human mind and a catalyst for the study of these numbers. For the numbers associated with the number 9, and help us build the number system so that it is divisible by the number 9, in addition to what is related to the result and its properties.

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I. Introduction

The world in which we live is unequal, with all its developments. Numbers played a major role in that, as they contributed to organization, discovery, invention, led the framework of technology, and achieved the pillars of the economy.

The numbers 1 to 9 are very important to the semantics of words, things, and businesses,[1] and these numbers vary in their meaning, contributions, and usage, so whether we see them in our number charts or wonder why they keep popping up in our lives, [2] it's important to know what they mean so that we can understand some of what these numbers are all about. of wonders.

This number is a human number at its core, and it requires a lot of research and more [3] effort from us to find out what it hides in terms of energy and strength, and it provides us with mathematical operations that we deal with as a new tributary to game theory and its multiple applications.

II. Pree Theory

Any number[4] whose sum of its digits is equal to a natural number. If the number representing the sum of the digits of that number is subtracted from it, the result becomes divisible by 9, this theory is expressed in the following way:

if a is natural number, and the sum of its digits equal q, then (a- q) $\sum a = q, a - q$, divisible by 9.

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Examples of the discovered theory, the application of the theory is shown in the following table (1)

Number(a)	sum of number of the digit number $\sum a$=q	$a-q$	$(a-q)\div 9$
678945	39	678945-39=678906	678906÷9=75434
45870234	33	45870234-33=45870201	45870231÷9=509689
98000111	20	98000111-20=98000091	98000091÷9=1088899
76	13	76-13=63	63÷9=7
3241	10	3241-10=3231	3231÷9=359
539	17	539-17=522	522÷9=58

From this theory, the following two branches can be reached:

Lemma 1: If any number consisting of two digits is divided after subtracting the sum of its digits by the number 9, the result of the division operation is the number in the ones place of the original number.

The application of the lemma (1) is shown in the following table (2)

Number(a)	sum of number of the digit number $\sum a$=q	$a-q$	$(a-q)\div 9$
32	5	27	27÷9=3
67	13	54	54÷9=6
98	17	81	81÷9=9
05	5	0	0÷9=0
12	3	9	9÷9=1

Lemma 2: If any number consisting of three digits is divided so that the tens digit is 0 and after subtracting the sum of its digits to the number 9, the result of the division process is the number in the ones place repeated.

The application of the lemma 2 is shown in the following table (3)

Number(a)	sum of number of the digit number $\sum a$=q	$a-q$	$(a-q)\div 9$
302	5	297	297÷9=33
609	15	594	594÷9=66
908	17	891	891÷9=99
405	9	396	396÷9=44
102	3	99	99÷9=11
801	9	792	792÷9=88

The two lemmas presented in this article contributes extensively to

1. Determine the divisibility of a number by 9
2. The theory is used in the applications of game theory
3. Create numbers that are divisible by 9

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Fellows may join as a member of the Editorial Board of Global Journals Incorporation (USA) after successful completion of three years as Fellow and as Peer Reviewer. Additionally, Fellows get a chance to nominate other members for Editorial Board.

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ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL is the membership of Global Journals awarded to individuals that the Open Association of Research Society judges to have made a 'substantial contribution to the improvement of computer science, technology, and electronics engineering.

> The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Associate membership can later be promoted to Fellow Membership. Associates are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Associate Members.

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Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

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Career Credibility Reputation

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To take future researches to the zenith, fellows receive access to almost all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

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Organize seminar/conference

Associates are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career Credibility Financial

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Exclusive Financial

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Get a remuneration of 15% of author fees

Associate members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

And Much More

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Associate	Fellow	Research Group	Basic
$4800 **lifetime designation**	$6800 **lifetime designation**	$12500.00 **organizational**	APC **per article**
Certificate, LoR and Momento 2 discounted publishing/year **Gradation** of Research 10 research contacts/day 1 **GB** Cloud Storage **GJ** Community Access	**Certificate,** LoR and Momento **Unlimited** discounted publishing/year **Gradation** of Research **Unlimited** research contacts/day 5 **GB** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**Certificates,** LoRs and Momentos **Unlimited** free publishing/year **Gradation** of Research **Unlimited** research contacts/day **Unlimited** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**GJ** Community Access

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

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Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

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2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
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It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

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2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

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Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

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Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, “What words would a source have to include to be truly valuable in a research paper?” Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

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Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

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This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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