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Highlights

- Solutions of the Time-Fractional
- Estimation of Matusita Overlapping
- Property' of Pythagorean Triangles
- Criterion of Nonlinear Control Systems

Discovering Thoughts, Inventing Future

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Spectral Stability Criterion of Nonlinear Control Systems

By I. E. Zuber

Annotation- A spectral stability criterion is formulated for nonlinear control systems, continuous and discrete, whose matrices have a simple structure. The spectrum providing global and exponential stability of a continuous nonlinear system is called acceptable. The spectral stability criterion is formulated as a sufficient condition for the admissibility of the matrix spectrum. For continuous nonlinear systems, the elements of the matrix spectrum are represented by the sum of two components, the first (main) is an arbitrarily selected negative scalar common to the entire spectrum, the second (the so–called increment) is constructed as functions that differ for all elements of the spectrum. The conditions for the admissibility of the matrix spectrum are reduced to a restriction on the maximum absolute value of the increment. A formula has been developed that determines the exact upper bound of this value, which ensures the acceptability of the spectrum. The spectral stability criterion of discrete nonlinear systems is based on the developed criterion for continuous systems.

Keywords: continuous nonlinear control system, matrices of simple structure, matrix spectrum, lyapunov function, exact upper bound.

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Spectral Stability Criterion of Nonlinear Control Systems

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Annotation- A spectral stability criterion is formulated for nonlinear control systems, continuous and discrete, whose matrices have a simple structure. The spectrum providing global and exponential stability of a continuous nonlinear system is called acceptable. The spectral stability criterion is formulated as a sufficient condition for the admissibility of the matrix spectrum. For continuous nonlinear systems, the elements of the matrix spectrum are represented by the sum of two components, the first (main) is an arbitrarily selected negative scalar common to the entire spectrum, the second (the so–called increment) is constructed as functions that differ for all elements of the spectrum. The conditions for the admissibility of the matrix spectrum are reduced to a restriction on the maximum absolute value of the increment. A formula has been developed that determines the exact upper bound of this value, which ensures the acceptability of the spectrum. The spectral stability criterion of discrete nonlinear systems is based on the developed criterion for continuous systems. A discrete system with a Lyapunov function in the form of a quadratic form with a constant matrix is compared by a given formula to a continuous system with a Lyapunov function of the same structure. In this case, the formulation of the spectral stability criterion for a nonlinear discrete system is reduced to the spectral stability criterion for a constructed correlated continuous system. The solution of the stabilization problem for a wide class of nonlinear control systems based on the formulated spectral stability criteria is obtained. The disadvantages of the proposed solution are noted.

Keywords: continuous nonlinear control system, matrices of simple structure, matrix spectrum, lyapunov function, exact upper bound.

I. Introduction

Are non-linear control systems

$$\begin{gathered} \dot{x} = Q(x)x \qquad\qquad x \in R^n, \\ x_{k+1} = G(x_k)x_k \quad x \in R^n \quad\quad k > 0, \end{gathered} \tag{1}$$

$Q(x)$ is a matrix of a simple structure with a spectrum $\Lambda(x)$.

The relationship between $\Lambda(x)$ and the stability of the system (1) is not fully established. The hypothesis of M.A. Aizerman [1], suggesting the proximity of the spectral criterion for systems of the form (1) to the corresponding criterion for a linear system, is refuted by counter examples [2, 3, 4]. In the future, numerous developments were carried out to determine the relationship between the spectrum of the matrix of the system and the stability of the system for much narrower and special types of the matrix of the system $Q(x)$. For example [4] the validity of the Aizerman hypothesis is asserted for $Q(x) = Q^T(x)$and the stability of the system (1) is checked when the matrix of the system has one eigenvalue of multiplicity n and n eigenvectors.In the works [5, 6,

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1. Ayzerman M.A. About one problem of stability “in large” for dynamical systems. Successes of dynamical sciences. 1949. t. 4 edition 4. p. 186-188.

7, 8, 9, 10, 11, 12] more complexly formed classes of control systems are considered, for which variants of the Aizerman hypothesis are valid, but the formation of these classes is set by cumbersome, difficult to test conditions, and the solution of the stabilization problem is extremely time-consuming.

The proposed article considers system (1), i.e. a wide class of nonlinear continuous control systems, and determines the conditions for choosing the spectrum of the matrix of the system that provides the system with the required stability properties. Similar problems are posed and solved for nonlinear discrete control systems with a matrix $G(x_k)$ of a simple structure.

The spectrum of the matrix of the continuous part of system (1) $\Lambda(x)$ is formed as

$$\Lambda(x) = \{\lambda_i(x)\} i = \overline{1, n},$$

$$\lambda_i(x) = \lambda_0(x) + \rho_i\big(\lambda_0(x)\big), \qquad (2),$$

where the smooth scalar function $\lambda_0(x) < -\frac{1}{2}$ is given arbitrarily, the functions $\rho_i\big(\lambda_0(x)\big)$ are defined for each a priori given $\lambda_0(x)$ under assumptions.

$$\rho_i\big(\lambda_0(x)\big) \neq 0, \qquad \rho_i\big(\lambda_0(x)\big) \neq \rho_j\big(\lambda_0(x)\big), \text{где } i \neq j\,, \quad x \in R^n.$$

The problem of determining sufficient conditions for the choice of functions —$\left|\rho_i\big(\lambda_0(x)\big)\right|$ is posed and solved, providing exponential and global stability to the continuous part of system (1), (2) and the Lyapunov function of this system in the form

$$V(x) = x^T I x \qquad (3)$$

The spectral stability criterion for a discrete system with a matrix $G(x_k)$ is reduced to the spectral stability criterion of a continuous system with a matrix $Q(x) = (G(x) + I)(G(x) - I)^{-1}$

The problem of stabilization of nonlinear control systems of the class under consideration is solved on the basis of the spectral stability criteria.

II. Formation of a Spectral Stability Criterion for a Nonlinear Continuous Control System

Let's consider the system (1), (2) and its assumed Lyapunov function (3). We will consider the Lyapunov function as a quadratic form with a unit matrix, since a quadratic form with an arbitrary positive definite matrix is reduced to such a form by a similarity transformation that does not change the spectrum of the matrix of a closed system.

Consider the spectral decomposition of the matrix $Q(x)$.

$$Q(x) = \sum_{i=1}^{n} \lambda_i(x) d_i(x)\, g_i^T(x),$$

where $\lambda_i(x) \in \Lambda(x), i = \overline{1, n}$, $d_i(x)$ are the eigenvectors of the matrix $Q(x)$, $g_i(x)$ are the eigenvectors of the matrix $Q^T(x)$. At the same time [13]

Ref

13. Voevodin V.V., Kuznecov Yu. A. Matrixes and computations. M., Science 1984, p. 317.

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$$d_i^T(x)g_j(x) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (4)$$

We fix an arbitrary number $\alpha > 0$and we will require the fulfillment of the central condition for the function $V(x)$

$$\dot{V}(x) = x^T L(x)x < -\alpha V(x), \quad (5)$$

where

$$L(x) = Q^T(x) + Q(x) = \sum_{i=1}^{n} \lambda_i(x)\, d_i(x) g_i^T(x) + \sum_{i=1}^{n} \overline{\lambda_i}(x) g_i(x) d_i^T(x). \quad (6)$$

We introduce the following matrices into consideration

$$R_i(x) = R_i^T(x) = d_i(x)d_i^T(x). \quad (7)$$

$$D(x) = D^T(x) = \sum_{i=1}^{n} R_i(x). \quad (8)$$

$$\mathrm{P}(x) = D^T(x)L(x)D(x), \quad (9)$$

i. e.

$$signP(x) = signL(x).$$

Let's write out $P(x)$in more detail and replace the central requirement (5) with an equivalent requirement

$$P(x) = \sum_{i=1}^{n} \lambda_i(x)R_i(x)\sum_{j=1}^{n} R_j + \sum_{i=1}^{n} R_i(x)\sum_{j=1}^{n} \lambda_j(x)\, R_j(x) < -\alpha\, D^T(x)\, D(x) \quad (10)$$

Taking into account (2), (4), (9) we have

$$P(x) = 2\,\lambda_0(x)D^T(x)D(x) + D^T(x)S(x) + S^T(x)D(x). \quad (11)$$

Note the fairness of [13] inequality

$$D^T(x)\, S(x) + S^T(x)\, D(x) \leq D^T\, D(x) + S^T(x)\, S(x),$$

at the same time

$$S^T(x)S(x) < \tau D^T(x)D(x), \text{where} \tau = \max_i \left|\rho_i\left(\lambda_0(x)\right)\right|^2$$

Now replace requirement (10) with a stronger requirement

$$P(x) < (2\,\lambda_0(x) + 1 + \alpha + \tau)D^T\, D(x) < 0$$

Thus proved

Ref

13. Voevodin V.V., Kuznetsov Yu. A. Matrixes and computations. M., Science 1984, p. 317.

Theorem 1

Let the spectrum of the matrix of system (1) have the form (2), where $\lambda_0(x) < -\frac{1}{2}$ is an arbitrarily a priori defined function, $\rho_i(\lambda_0(x))$, satisfy the conditions

$$\rho_i(\lambda_0(x)) \neq 0, \quad \rho_i(\lambda_0(x)) \neq \rho_j(\lambda_0(x)), \quad \text{где } i \neq j \quad x \in R^n,$$

$$\max_i \rho_i^{\,2}(\lambda_0(x)) < - \ 2\,\lambda_0(x) - 1 - \alpha \qquad (12)$$

Then the system (1) – (4) is exponentially and globally stable and has a Lyapunov function (3), where $\dot{V}(x)$ satisfies condition (5).

III. Solution of the Spectral Stabilization Problem for a Nonlinear Continuous Control System

Consider the system

$$\dot{x} = A(x)x + b(x)u(x) \qquad u(x) = s^T(x)x \qquad x \in R^n \qquad (13)$$

The pair $(A(x)x,\ b(x))$is given and satisfies the condition of uniform controllability.

We assume $\alpha > 0$to be a priori a given number. It is required to determine the feedback vector $s(x)$,at which the closed system (13) has a Lyapunov function (3) satisfying condition (5), i.e. the global and exponential stability of the closed system (13).

The solution of the problem is carried out by the same operations that would be used to solve spectral stabilization for a linear control system.
The spectral decomposition of a matrix in a closed system (13) is considered

$$Q(x) = A(x) + b(x)s^T(x) = \sum_{i=1}^{n} \lambda_i(x)\, d_i(x) g_i^T(x),$$

where $\lambda_i(x)$, satisfies the condition $\lambda_i(x) \neq \lambda_j(x) \quad i \neq j$,a priori defined spectrum of the matrix $Q(x)$,the eigenvectors of the matrix $Q(x)$are determined by the formula $d_i(x) = (A - \lambda_i(x)I)^{-1} b(x)$,the eigenvectors of the matrix $Q^T(x)$ satisfy condition (4), and the desired feedback vector is determined by the formula

$$s^T(x) d_i(x) = -1 \qquad i = \overline{1,n} \qquad (14)$$

Let us select the spectrum of the considered closed system (13) in accordance with the conditions of Theorem (1), i.e. with the formula (12). Then it is obvious that the closed system (13) has the required stability properties. The main difference between this result and the result obtained for a linear control system is the solution of the system (14). For linear control systems, the choice of an arbitrarily set negative spectrum of the matrix of a closed system allows a priori to set the norm of the feedback vector $s(x)$. For the considered variant of spectral stabilization, the question of the possibility of obtaining the norm of the vector $s(x)$permissible under the preconditions of the problem under consideration remains open.

Notes

IV. Spectral Stability Criterion for Nonlinear Discrete Control Systems

The control system is considered

$$x_{k+1} = G(x_k)x_k, \qquad x_k \in R^n, \qquad k > 0 \tag{15}$$

where, without detracting from the generality, we believe

$$\inf_x |det(G(x) - I)| > \alpha > 0.$$

We will call the continuous system

$$\dot{x} = G_1(x)x \tag{16}$$

pair system (16), if

$$G_1(x) = (G(x) + I)(G(x) - I)^{-1}. \tag{17}$$

Note that the presence of the Lyapunov function (3) in the system (16) ensures the presence of the Lyapunov function

$$W(x_k) = x_k^T I x_k \tag{18}$$

for the system (15). Indeed, the conditions $\dot{W}(x) < 0$,by virtue of (16), (17) takes the form

$$(G^T(x) - I)^{-1}(G^T(x) + I) + (G(x) + I)(G(x) - I)^{-1} < 0$$

Multiplying the last inequality on the left by $(G^T(x) - I)$, and on the right by $(G(x) - I)$, we are convinced of the validity of the condition

$$G^T(x)G(x) - I < 0, \qquad x \in R^n,$$

the implementation of which ensures the existence of a Lyapunov function of the form (18) for the system (15).

Note that the eigenvalues of the matrix $G(x)$,denoted by $\mu_i(x)$ and the eigenvalues of the matrix $G_I(x)$, denoted by $\vartheta_i(x)$, are related [14] by the relations

$$\mu_i(x) = \frac{\vartheta_i(x) + 1}{\vartheta_i(x) - 1},$$

i. e.

$$\vartheta_i(x) = \frac{\mu_i(x) + 1}{\mu_i(x) - 1}$$

The stability of the system (16) is ensured by the fulfillment of the conditions of Theorem (1) for the spectrum of the matrix $G_1(x)$Thus, it is proved

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14. Gantmaher R.F. Theory of matrixes. M., Science 1966, p. 570.

Theorem 2

The global stability of the system (15) and the presence of the Lyapunov function (18) in this system is ensured by the implementation on the spectrum of the matrix $G_1(x_k)$

$$\mu_i(x) = \frac{\vartheta_i(x_k) + 1}{\vartheta_i(x_k) - 1}, \qquad k > 0, \quad x \in R^n,$$

then the conditions that the eigen values of the matrix of the system (16), the paired system (15), $\vartheta_i(x) i = \overline{1, n}$ correspond to the formulation of Theorem (1).

V. Spectral Stabilization of Nonlinear Discrete Control Systems

The solution of the problem of spectral stabilization of a nonlinear discrete control system is performed by the same sequence of operations that takes place when solving this problem for a linear control system. The difference lies in the fact that in the case of a nonlinear system, the a priori selected spectrum of the matrix of a closed nonlinear system is formed in accordance with the conditions of Theorem (2).

VI. Conclusion

The main result of the proposed article is the formation of a spectral criterion for the stability of nonlinear control systems

$$\dot{x} = Q(x)x, \quad x_{k+1} = G(x_k)x_k, \qquad x \in R^n \qquad k > 0,$$

where $Q(x), G(x_k)$are matrices of a simple structure.

The spectral stability criterion is reduced to determining sufficient conditions for the elements of the matrix spectrum of the considered closed systems, which determine the required stability properties of the system and the presence of the Lyapunov function in the form of a quadratic form with a single matrix.

For continuous control systems we have:

$$\Lambda(x) = \{\lambda_i(x)\} i = \overline{1, n}, \lambda_i(x) = \lambda_0(x) + \rho_i\big(\lambda_0(x)\big),$$

where the smooth scalar function $\lambda_0(x) < -\frac{1}{2}$ is given arbitrarily, the functions $\rho_i\big(\lambda_0(x)\big)$are defined for each a priori given $\lambda_0(x)$under the assumptions

$$\rho_i\big(\lambda_0(x)\big) \neq 0, \rho_i\big(\lambda_0(x)\big) \neq \rho_j\big(\lambda_0(x)\big), \qquad \text{где } i \neq j, \ x \in R^n$$

and the exact upper bound for $\left|\rho_i\big(\lambda_0(x)\big)\right|$, defined by formula (12).

For discrete control systems, a similar, somewhat more cumbersome result is obtained.

The solution of the problem of spectral stabilization of the systems under consideration is performed by the same operations based on the spectral decomposition of the matrix of a closed system, which this problem is solved for linear control systems. At the same time, the selection of the a priori spectrum of the matrix of a closed system is made in accordance with the developed spectral criterion for the stability of nonlinear control systems.

Notes

The developed technique has a significant drawback, when using it, the norm of the feedback vector here may be significantly greater than when solving the stabilization problem based on a priori linearization of the object matrix.

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Notes

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On Lie Symmetry Analysis and Analytical Solutions of the Time-Fractional Modified ZKB Equation in Mathematical Physics

By Rasha. B. AL-Denari, Engy. A. Ahmed, S. M. Moawad & O. H. EL-Kalaawy

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Abstract- In this article, we explore the time-fractional modified Zakharov-Kuznetsov-Burgers (MZKB) equation of (3+1) dimensions. The Lie symmetry analysis is used to identify the symmetries and vector fields for the equation understudy with the assistance of the Riemann-Liouville derivatives. These symmetries are then employed to build a transformation that reduces the above equation into a nonlinear ordinary differential equation of fractional order with the aiding of ErdLélyi-Kober fractional operator. Further, two sets of new analytical solutions are constructed by the fractional sub-equation method and the extended Kudryashov method. Subsequently, we graphically represent these results in the 2*D* and 3*D* plots with physical interpretation for the behavior of the obtained solutions. The conservation laws that associate with the symmetries of the equation are also constructed by considering the new conservation theorem and the formal Lagrangian *L*. As a final result, we anticipate that this study will assist in the discovery of alternative evolutionary processes for the considered equation.

Keywords: *lie symmetry analysis; conservation laws; time fractional modified schamel-zakharov kuznetsov burgers equation; riemann-liouville derivatives; fractional sub-equation method; extended kudryashov method.*

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6. Baleanu D, Kumar D. Fractional calculus and its applications in physics. *Lausanne: Frontiers Media.* 2019.

On Lie Symmetry Analysis and Analytical Solutions of the Time-Fractional Modified ZKB Equation in Mathematical Physics

Rasha. B. AL-Denari [α], Engy. A. Ahmed [σ], S. M. Moawad [ρ] & O. H. EL-Kalaawy [ω]

Abstract- In this article, we explore the time-fractional modified Zakharov-Kuznetsov-Burgers (MZKB) equation of (3+1) dimensions. The Lie symmetry analysis is used to identify the symmetries and vector fields for the equation understudy with the assistance of the Riemann-Liouville derivatives. These symmetries are then employed to build a transformation that reduces the above equation into a nonlinear ordinary differential equation of fractional order with the aiding of ErdLélyi-Kober fractional operator. Further, two sets of new analytical solutions are constructed by the fractional sub-equation method and the extended Kudryashov method. Subsequently, we graphically represent these results in the $2D$ and $3D$ plots with physical interpretation for the behavior of the obtained solutions. The conservation laws that associate with the symmetries of the equation are also constructed by considering the new conservation theorem and the formal Lagrangian L. As a final result, we anticipate that this study will assist in the discovery of alternative evolutionary processes for the considered equation.

Keywords: *lie symmetry analysis; conservation laws; time fractional modified schamel-zakharov kuznetsov burgers equation; riemann-liouville derivatives; fractional sub-equation method; extended kudryashov method.*

I. Introduction

The partial differential equations of fractional order (FPDEs) have been widely employed in recent years to explain a wide range of physical effects and complicated nonlinear phenomena. This is because they accurately describe nonlinear phenomena in the fields of fluid mechanics, viscoelasticity, electrical chemistry, quantum biology, physics, and engineering mechanics [1]-[6] as well as other scientific domains. As a result, the research of PDEs has received a lot of interest as many physical events may be explained using the idea of fractional derivatives and integrals [7]-[9]. Add to that, when the exact solutions to the majority of FPDEs are difficult to find, analytical and numerical methods [10]-[29] which are proposed and developed by many authors must be used.

Lie symmetry analysis is extremely important in many fields of science, particularly in integrable systems with an infinite number of symmetries. Thus, Lie symmetry analysis is regarded as one of the most effective methods for obtaining analytical solutions to nonlinear partial differential equations (NLPDEs). Also, many FPDEs have been studied using this analysis [30]-[36]. Add to that, this analysis is used to build conservation laws, which are crucial in the study of nonlinear physical phenomena. Conservation laws are mathematical formulations state that the total amount of a certain physical quantity remains constant as a physical system evolves. Furthermore, conservation laws are used in the development of numerical methods to establish the existence and uniqueness of a solution. There are many studies that discuss conservation laws for time FPDEs, which are mentioned in the references [37]-[43].

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In this article, we focus on the following time-fractional MZKB equation of the form:

$$\partial_t^\alpha q + \kappa_1 \sqrt{q}\; q_x + \kappa_2 q_{xxx} + \kappa_3 (q_{xyy} + q_{xzz}) + \kappa_4 q_{xx} = 0, \tag{1}$$

where ∂_t^α is the fractional derivative of order α (with $0 < \alpha < 1$), $\kappa_i (i = 0, 1, ..., 4)$ are respectively the dispersion, non linearity, mixed derivative, and dissipation. The $q(x, y, z, t)$ is the potential function of space x, y, z and time t. If $\alpha = 1$, Eq. (1) is reduced to the classical MZKB equation [44, 45], which describes the nonlinear plasma dust ion acoustic waves DIAWs in a magnetized dusty plasma and it is derived using the standard reductive perturbation technique in small amplitude.

The article is organized as follows: The introduction is presented in Section 1. In Section 2, some definitions and description of Lie symmetry analysis for fractional partial differential equations (FPDEs) are briefly presented. Lie symmetry analysis and similarity reduction of the Eq. (1) are obtained In Section 3. We construct two sets of analytical solutions for Eq. (1) by using fractional sub-equation method and extended Kudryashov method in Section 4 and 5 respectively. In Section 6, the conservation laws of the Eq. (1) are obtained. Finally, the discussions and conclusions of this article are presented in Section 7.

Notes

II. Preliminaries

Here in this section, we focus on some of the concepts that revolve around the subject of our article
Definition 1: Let $\alpha > 0$. The operator I^α defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2}$$

is called the Riema nn-Liouville (R-L) fractional integral operator of order α, and $\Gamma(.)$ denotes the gamma function.
Definition 2: Let $\alpha > 0$. The operator D_t^α is defined by

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\; ds & if\ n-1 < \alpha < n, \quad n \in\ N, \\ \frac{d^n f(t)}{dt^n} & if\ \alpha = n, \quad n \in\ N, \end{cases} \tag{3}$$

is called the R-L fractional partial derivative [7, 8].

a) Description of Lie symmetry analysis

Let's consider the symmetry analysis for a FPDE of the form

$$D_t^\alpha q(x, y, z, t) = G(x, y, z, t, q, q_x, q_t, q_y, q_z, q_{xx}, q_{xy}, ...), \qquad 0 < \alpha < 1. \tag{4}$$

Now, let Eq. (4) is invariant under the following one-parameter Lie group of point transformation acting on both the dependent and independent variables, given as

$$\begin{aligned} \bar{x} &= x + \varepsilon \xi(x, y, z, t) + O(\varepsilon^2), \\ \bar{y} &= y + \varepsilon \zeta(x, y, z, t) + O(\varepsilon^2), \\ \bar{z} &= z + \varepsilon \nu(x, y, z, t) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon \tau(x, y, z, t) + O(\varepsilon^2), \\ \bar{q} &= q + \varepsilon \eta(x, y, z, t) + O(\varepsilon^2), \\ D_{\bar{t}}^\alpha \bar{q} &= D_t^\alpha q + \varepsilon \eta_\alpha^0(x, y, z, t) + O(\varepsilon^2), \\ \frac{\partial \bar{q}}{\partial \bar{x}} &= \frac{\partial q}{\partial x} + \varepsilon \eta^x(x, y, z, t) + O(\varepsilon^2), \\ \frac{\partial^2 \bar{q}}{\partial \bar{x}^2} &= \frac{\partial^2 q}{\partial x^2} + \varepsilon \eta^{xx}(x, y, z, t) + O(\varepsilon^2), \end{aligned} \tag{5}$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x}^3} = \frac{\partial^3 q}{\partial x^3} + \varepsilon \eta^{xxx}(x,y,z,t) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x}\partial \bar{y}^2} = \frac{\partial^3 q}{\partial x \partial y^2} + \varepsilon \eta^{xyy}(x,y,z,t) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{q}}{\partial \bar{x}\partial \bar{z}^2} = \frac{\partial^3 q}{\partial x \partial z^2} + \varepsilon \eta^{xzz}(x,y,z,t) + O(\varepsilon^2),$$

Notes

where $\varepsilon \ll 1$ is the Lie group parameter and $\xi,\ \zeta,\ \nu,\ \tau,\ \eta$ are the infinitesimals of the transformations for dependent and independent variables respectively. The explicit expressions of $\eta^x,\ \eta^{xx},\ \eta^{xxx},\ \eta^{xyy},\ \eta^{xzz}$ are given by

$$\begin{aligned}
\eta^x &= D_x(\eta) - q_x D_x(\xi) - q_y D_x(\zeta) - q_z D_x(\nu) - q_t D_x(\tau),\\
\eta^{xx} &= D_x(\eta^x) - q_{xx} D_x(\xi) - q_{xy} D_x(\zeta) - q_{xz} D_x(\nu) - q_{xt} D_x(\tau),\\
\eta^{xxx} &= D_x(\eta^{xx}) - q_{xxx} D_x(\xi) - q_{xxy} D_x(\zeta) - q_{xxz} D_x(\nu) - q_{xxt} D_x(\tau),\\
\eta^{xyy} &= D_x(\eta^{yy}) - q_{xxy} D_x(\xi) - q_{xyy} D_x(\zeta) - q_{xyz} D_x(\nu) - q_{xyt} D_x(\tau),\\
\eta^{xzz} &= D_x(\eta^{zz}) - q_{xxz} D_x(\xi) - q_{xyz} D_x(\zeta) - q_{xzz} D_x(\nu) - q_{xzt} D_x(\tau),
\end{aligned} \tag{6}$$

where $D_x,\ D_y,\ D_z$, and D_t are the total derivatives with respect to $x,\ y,\ z$, and t respectively that are defined for $x^1 = x,\ x^2 = y,\ x^3 = z$ as

$$D_{xj} = \frac{\partial}{\partial x^j} + q_j \frac{\partial}{\partial q} + q_{jk}\frac{\partial}{\partial q_k} + ..., \qquad j,k = 1,2,3,...$$

where $q_j = \frac{\partial q}{\partial x^j},\ \ q_{jk} = \frac{\partial^2 q}{\partial x^j\ \partial x^k}$ and so on.

The corresponding Lie algebra of symmetries consists of a set of vector fields of the form

$$V = \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}.$$

The invariance condition of Eq. (4) under the infinitesimal transformations is given as

$$Pr^{(n)}\ V(\Delta)\ |_{\Delta=0} = 0, \qquad n = 1,2,3,...$$

where

$$\Delta := D_t^\alpha q(x,y,z,t) - G(x,y,z,t,q,q_x,q_t,q_y,q_z,q_{xx},q_{xy},...).$$

Also, the invariance condition gives

$$\tau(x,y,z,t,u)\ |_{t=0} = 0. \tag{7}$$

The αth extended infinitesimal related to RL fractional time derivative with Eq. (7) can be represented as follows

$$\begin{aligned}
\eta_\alpha^0 = D_t^\alpha(\eta) + \xi\ D_t^\alpha(q_x) - D_t^\alpha(\xi\ q_x) + \zeta\ D_t^\alpha(q_y) - D_t^\alpha(\zeta\ q_y) + \nu\ D_t^\alpha(q_z)\\
- D_t^\alpha(\nu\ q_z) + D_t^\alpha(D_t(\tau)q) - D_t^{\alpha+1}(\tau\ q) + \tau\ D_t^{\alpha+1}(q),
\end{aligned} \tag{8}$$

where D_t^α is the total fractional derivative operator and by using the generalized Leibnitz rule

$$D_t^\alpha\Big(f(t)g(t)\Big) = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \binom{\alpha}{n} = \tfrac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

By applying the Leibnitz rule, Eq. (8) becomes

$$\eta_\alpha^0 = D_t^\alpha(\eta) - \alpha D_t^\alpha(\tau)\frac{\partial^\alpha q}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n} q_x - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n} q_y$$

$$- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\nu) D_t^{\alpha-n} q_z - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\xi) D_t^{\alpha-n} q. \tag{9}$$

Now by using the chain rule for the compound function which is defined as follows

$$\frac{d^n\phi(h(t))}{dt^n} = \sum_{k=0}^{n}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}\big[-h(t)\big]^r \frac{d^n}{dt^n}\big[-h(t)^{k-r}\big] \times \frac{d^k\phi(h)}{dh^k}.$$

By applying this rule and the generalized Leibnitz rule with $f(t)=1$, we have

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_q \frac{\partial^\alpha q}{\partial t^\alpha} - q\frac{\partial^\alpha \eta_q}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_q}{\partial t^n} D_t^{\alpha-n}(q) + \mu,$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sim_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\big[-q\big]^r\frac{\partial^m}{\partial t^m}\big[q^{k-r}\big]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial q^k}.$$

Therefore, Eq. (9) yields

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_q - \alpha D_t^\alpha(\tau))\frac{\partial^\alpha q}{\partial t^\alpha} - q\frac{\partial^\alpha \eta_q}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty}\Big[\binom{\alpha}{n}\frac{\partial^\alpha \eta_q}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\Big] D_t^{\alpha-n}(q)$$

$$+ \sum_{n=1}^{\infty}(\,\alpha\,) D_t^n(\xi) D_t^{\alpha-n} q_x - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n} q_y - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\nu) D_t^{\alpha-n} q_z. \tag{10}$$

Definition 3: The function $q=\theta(x,y,z,t\ )$ is an invariant solution of Eq. (4) associated with the vector field W, such that

1. $q=\theta(x,y,z,t)$ is an invariant surface of Eq. (4), this means

$$V\theta = 0 \Leftrightarrow \Big(\xi\tfrac{\partial}{\partial x} + \zeta\tfrac{\partial}{\partial y} + \nu\tfrac{\partial}{\partial z} + \tau\tfrac{\partial}{\partial t} + \eta\tfrac{\partial}{\partial u}\Big)\theta = 0,$$

2. $q=\theta(x,y,z,t)$ satisfies Eq. (4).

III. Lie Symmetry Analysis and Similarity Reduction of Eq. (1)

In this section, we implemented Lie group method for Eq. (1) and then, used these symmetries to reduce Eq. (1) to be a FODE as shown in the next two subsections

a) Lie symmetry analysis for Eq. (1)

Let us consider Eq. (1) is an invariant under Eq. (5), we get

$$\partial_t^\alpha \bar{q} + \kappa_1\sqrt{\bar{q}}\ \bar{q}_x + \kappa_2\bar{q}_{xxx} + \kappa_3(\bar{q}_{xyy} + \bar{q}_{xzz}) + \kappa_4\bar{q}_{xx} = 0, \tag{11}$$

such that $q=q(x,y,z,t)$ satisfies Eq. (1), then using the point transformations Eq. (5) in Eq. (11), we get the invariant equation

Notes

$$\eta_{\alpha}^{0}+\kappa_1\sqrt{q}\ \eta^{x}+\frac{\kappa_1}{2\sqrt{q}}\eta\ q_x+\kappa_2\eta^{xxx}+\kappa_3(\eta^{xyy}+\eta^{xzz})+\kappa_4\eta^{xx}=0, \tag{12}$$

By substituting Eq. (6) and Eq. (10) into Eq. (12), grouping the coefficients of all derivatives and various powers of u and equating them to zero we get an algebraic system of equations. Solving this system, we obtain a set of infinitesimal symmetries as below:

Case 1: When $\kappa_i\neq 0,\ i=1,\ 2,\ 3,\ \ \kappa_4=0$

$$\tau=\frac{3}{2\alpha}c_1t+c_2,\quad \xi=\frac{1}{2}c_1x+c_3,\quad \zeta=\frac{1}{2}c_1y+c_4,\quad \nu=\frac{1}{2}c_1z+c_5,\quad \eta=-2c_1q, \tag{13}$$

Notes

where $c_i,\ i=1,2,3,4,5$ are arbitrary constants. Thus, the infinitesimal generator of Eq. (1) can be expressed as follows

$$V=\left(\frac{3}{2\alpha}c_1t+c_2\right)\frac{\partial}{\partial t}+\left(\frac{1}{2}c_1x+c_3\right)\frac{\partial}{\partial x}+\left(\frac{1}{2}c_1y+c_4\right)\frac{\partial}{\partial y}+\left(\frac{1}{2}c_1z+c_5\right)\frac{\partial}{\partial z}-2c_1q\frac{\partial}{\partial q}.$$

which can be spanned by the five vector fields listed below.

$$\begin{aligned}&V_1=\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial x},\quad V_3=\frac{\partial}{\partial y},\quad V_4=\frac{\partial}{\partial z},\\ &V_5=\frac{3t}{2\alpha}\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial y}+\frac{z}{2}\frac{\partial}{\partial z}-2q\frac{\partial}{\partial q}.\end{aligned} \tag{14}$$

Case 2: When $\kappa_i\neq 0,\ i=1,\ 2,\ 3,\ 4$

$$\tau=c_6,\qquad \xi=c_7,\qquad \zeta=c_8,\qquad \nu=c_9,\qquad \eta=0,$$

hence, there are four vector fields as below

$$V_6=\frac{\partial}{\partial t},\qquad V_7=\frac{\partial}{\partial x},\qquad V_8=\frac{\partial}{\partial y},\qquad V_9=\frac{\partial}{\partial z}. \tag{15}$$

b) The similarity reduction for Eq. (1)

In this part of the article, we used the symmetries defined by Eq. (14) and Eq. (15) to construct the similarity reduction for Eq. (1) as presented in the next cases

Case 1: For $V_1=\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial x},\quad V_3=\frac{\partial}{\partial y},\quad V_4=\frac{\partial}{\partial z},\quad V_5=\frac{3t}{2\alpha}\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial y}+\frac{z}{2}\frac{\partial}{\partial z}-2q\frac{\partial}{\partial q}$ with $\kappa_4=0$, we have a set of characteristic equations that arranged in the following subcases

Case 1.1: $V_1=\frac{\partial}{\partial t}$ we have a characteristic equation of the form

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dz}{0}=\frac{dt}{1}=\frac{dq}{0}.$$

by integrating this equation and appoint the solutions q as function of the dependent variables $x,\ y,\ z,$ that is

$$q(x,y,z,t)=\phi_1(x,y,z),$$

this implies $\frac{\partial^{\alpha}q}{\partial t^{\alpha}}=0$ and Eq. (1) becomes

$$\kappa_1\sqrt{\phi_1(x,y,z)}\frac{d\phi_1(x,y,z)}{dx}+\kappa_2\frac{d^3\phi_1(x,y,z)}{dx^3}+\kappa_3\left(\frac{d^3\phi_1(x,y,z)}{dxdydy}+\frac{d^3\phi_1(x,y,z)}{dxdzdz}\right)=0. \tag{16}$$

Case 1.2: For $V_2=\frac{\partial}{\partial x}$ we have a characteristic equation of the form

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dz}{0}=\frac{dt}{0}=\frac{dq}{0},$$

by solving this equation we have $q(x,y,z,t)=\Phi_2(y,z,t)$ which makes all the derivatives of $u(x,y,z,t)$ with respect to x equal to zero and

$$\frac{\partial^\alpha q}{\partial t^\alpha} = 0, \qquad \text{this equation has the following solution}$$

$$q = \Phi_2(t) = \frac{B_o}{\Gamma(\alpha)}\, t^{\alpha-1}, \qquad \text{where } B_0 \text{ is a constant}$$

Case 1.3: For $V_3 = \frac{\partial}{\partial y}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{0} = \frac{dt}{0} = \frac{dq}{0},$$

thus $q(x,y,z,t) = \Phi_3(x,z,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to y equal to zero, therefore

$$\frac{\partial^\alpha \Phi_3(x,z,t)}{\partial t^\alpha} + \kappa_1\sqrt{\Phi_3(x,z,t)}\frac{d\Phi_3(x,z,t))}{dx} + \kappa_2\frac{d^3\Phi_3(x,z,t)}{dx^3} + \kappa_3\frac{d^3\Phi_3(x,z,t)}{dxdzdz} = 0$$

Case 1.4: For $V_4 = \frac{\partial}{\partial z}$ the characteristic equation is of the form

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{dq}{0},$$

thus $q(x,y,z,t) = \Phi_4(x,y,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to z equal to zero, therefore

$$\frac{\partial^\alpha \Phi_4(x,y,t)}{\partial t^\alpha} + \kappa_1\sqrt{\Phi_4(x,y,t)}\frac{d\Phi_4(x,y,t))}{dx} + \kappa_2\frac{d^3\Phi_4(x,y,t)}{dx^3} + \kappa_3\frac{d^3\Phi_4(x,y,t)}{dxdydy} = 0$$

Case 1.5: For $V_5 = \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z} - 2q\frac{\partial}{\partial q}$, the characteristic equation becomes

$$\frac{dx}{x/2} = \frac{dy}{y/2} = \frac{dz}{z/2} = \frac{dt}{3t/2\alpha} = \frac{dq}{-2q}.$$

Where solving this equation we can get the next similarity variables and similarity solution for Eq. (1) as below

$$\gamma_1 = xt^{-\frac{\alpha}{3}}, \qquad \gamma_2 = yt^{-\frac{\alpha}{3}}, \qquad \gamma_3 = zt^{-\frac{\alpha}{3}}, \qquad q = t^{-\frac{4}{3}\alpha}\phi(\gamma_1,\gamma_2,\gamma_3). \tag{17}$$

By using the above transformation, Eq. (1) can be turned into a nonlinear FODE with a set of independent variable $\gamma's$. Consequently, one can conclude the next theorem.

Theorem 1: The transformation Eq. (17) reduces the time-fractional generalized Z-K Eq. (1) to the following equation

$$\left(P^{1-\frac{7\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}\phi\right)(\gamma_1,\gamma_2,\gamma_3) + \kappa_1\sqrt{\phi}\,\phi_{\gamma_1} + \kappa_2\phi_{\gamma_1\gamma_1\gamma_1} + \kappa_3\left(\phi_{\gamma_1\gamma_2\gamma_2} + \phi_{\gamma_1\gamma_3\gamma_3}\right) = 0, \tag{18}$$

with the E-K fractional differential operator $\left(P^{\tau,\alpha}_{\beta}\phi\right)(\gamma_1,\gamma_2,\gamma_3)$ which is defined as

$$\left(P^{\tau,\alpha}_{\beta_1,\beta_2,\beta_3}\phi\right)(\gamma_1,\gamma_2,\gamma_3) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta_1}\gamma_1\frac{\partial}{\partial\gamma_2} - \frac{1}{\beta_2}\gamma_2\frac{\partial}{\partial\gamma_2} - \frac{1}{\beta_3}\gamma_3\frac{\partial}{\partial\gamma_3}\right)(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2,\beta_3}\phi)(\gamma_1,\gamma_2,\gamma_3),$$

where
$$n = \begin{cases} \mid\alpha\mid +1, & n \notin N, \\ \alpha, & n \in N, \end{cases} \tag{19}$$

$$(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2,\beta_3}\phi)(\gamma_1,\gamma_2,\gamma_3)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\Theta-1)^{\alpha-1}\Theta^{-(\tau+\alpha)}\phi(\gamma_1\Theta^{\frac{1}{\beta_1}},\gamma_2\Theta^{\frac{1}{\beta_2}},\gamma_3\Theta^{\frac{1}{\beta_3}})\,d\Theta, & \alpha>0,\\ \phi(\gamma_1,\gamma_2,\gamma_3), & \alpha=0,\end{cases} \tag{20}$$

is the E-K fractional integral operator.

The proof of theorem 1: Depending on the definition of the R-L fractional derivatives provided with $n-1<\alpha<1,\quad n=1,2,3,\dots$, then we have

$$\partial_t^{\alpha}q=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^t(t-g)^{n-\alpha-1}g^{\frac{-1}{3}\alpha}\phi(\gamma_1 g^{-\frac{\alpha}{3}},\gamma_2 g^{-\frac{\alpha}{3}},\gamma_3 g^{-\frac{\alpha}{3}})\,dg\right]. \tag{21}$$

Let $\Lambda=\frac{t}{g}$, one can get $dg=-\frac{t}{\Lambda^2}$, thus Eq. (21) can be written as

$$\partial_t^{\alpha}q=\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\frac{4}{3}\alpha}}{\Gamma(n-\alpha)}\int_1^{\infty}(\Lambda-1)^{n-\alpha-1}\Lambda^{-(n+1-\frac{4}{3}\alpha)}\phi(\gamma_1\Lambda^{\frac{\alpha}{3}},\gamma_2\Lambda^{\frac{\alpha}{3}},\gamma_3\Lambda^{\frac{\alpha}{3}})\,d\Lambda\right], \tag{22}$$

following the definition of E-K fractional differential operator, then Eq. (22) becomes

$$\partial_t^{\alpha}q=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right], \tag{23}$$

it is time to deal with the right-hand side of Eq. (23). Where

$$t\frac{\partial}{\partial t}\wp(\gamma_1,\gamma_2,\gamma_3)=-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}.$$

From that, we have

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{4}{3}\alpha}(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right)\right]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{4}{3}\alpha-1}\left(n-\frac{4}{3}\alpha-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right],\end{aligned}$$

according to the above result provided with the same steps for $(n-1)$ times, we get

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{4}{3}\alpha-1}\left(n-\frac{4}{3}\alpha-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right]\\ &\dots,\\ &=t^{-\frac{4}{3}\alpha}\prod_{j=0}^{n-1}\left[\left(n-\frac{4}{3}\alpha+j-\frac{\alpha}{3}\gamma_1\wp_{\gamma_1}-\frac{\alpha}{3}\gamma_2\wp_{\gamma_2}-\frac{\alpha}{3}\gamma_3\wp_{\gamma_3}\right)(K^{1-\frac{\alpha}{3},n-\alpha}_{\frac{3}{\alpha}}\phi)(\gamma_1,\gamma_2,\gamma_3)\right],\end{aligned}$$

Notes

this implies

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{4}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)\right]=t^{-\frac{4}{3}\alpha}\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3), \tag{24}$$

thus

$$\partial_t^\alpha q=t^{-\frac{4}{3}\alpha}\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3). \tag{25}$$

At last, Eq. (1) can be reduced into the below equation and the proof is completed

$$\left(P_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}\phi\right)(\gamma_1,\gamma_2,\gamma_3)+\kappa_1\sqrt{\phi}\,\phi_{\gamma_1}+\kappa_2\phi_{\gamma_1\gamma_1\gamma_1}+\kappa_3\left(\phi_{\gamma_1\gamma_2\gamma_2}+\phi_{\gamma_1\gamma_3\gamma_3}\right)=0. \tag{26}$$

Therefore, the proof is completed.

Case 2: For $V_1=\frac{\partial}{\partial t}$, $V_2=\frac{\partial}{\partial x}$, $V_3=\frac{\partial}{\partial y}$, $V_4=\frac{\partial}{\partial z}$ with $\kappa_4\neq 0$, we have the following sub-cases

Case 2.1: $V_6=\frac{\partial}{\partial t}$ we have a characteristic equation of the form

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dz}{0}=\frac{dt}{1}=\frac{dq}{0}.$$

by integrating this equation and appoint the solutions q as function of the dependent variables $x,\ y,\ z,$ that is

$$q(x,y,z,t)=\Psi_1(x,y,z),$$

this implies $\frac{\partial^\alpha q}{\partial t^\alpha}=0$ and Eq. (1) becomes

$$\kappa_1\sqrt{\Psi_1(x,y,z)}\frac{d\Psi_1(x,y,z)}{dx}+\kappa_2\frac{d^3\Psi_1(x,y,z)}{dx^3}+\kappa_3\left(\frac{d^3\Psi_1(x,y,z)}{dxdydy}+\frac{d^3\Psi_1(x,y,z)}{dxdzdz}\right)+\kappa_4\frac{d^2\Psi_1(x,y,z)}{dx^2}=0. \tag{27}$$

Case 2.2: For $V_7=\frac{\partial}{\partial x}$ we have a characteristic equation of the form

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dz}{0}=\frac{dt}{0}=\frac{dq}{0},$$

by solving this equation we have $q(x,y,z,t)=\Psi_2(y,z,t)$ which makes all the derivatives of $q(x,y,z,t)$ with respect to x equal to zero and

$$\frac{\partial^\alpha q}{\partial t^\alpha}=0,$$ this equation has the following solution

$$q=\Psi_2(t)=\frac{C_o}{\Gamma(\alpha)}\,t^{\alpha-1},$$ where C_0 is a constant

Case 2.3: For $V_3=\frac{\partial}{\partial y}$ the characteristic equation is of the form

$$\frac{dx}{0}=\frac{dy}{1}=\frac{dz}{0}=\frac{dt}{0}=\frac{dq}{0},$$

thus $q(x,y,z,t)=\Psi_3(x,z,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to y equal to zero, therefore

$$\frac{\partial^\alpha\Psi_3(x,z,t)}{\partial t^\alpha}+\kappa_1\sqrt{\Psi_3(x,z,t)}\frac{d\Psi_3(x,z,t))}{dx}+\kappa_2\frac{d^3\Psi_3(x,z,t)}{dx^3}+\kappa_3\frac{d^3\Psi_3(x,z,t)}{dxdzdz}+\kappa_4\frac{d^2\Psi_3(x,y,z)}{dx^2}=0$$

Case 2.4: For $V_4=\frac{\partial}{\partial z}$ the characteristic equation is of the form

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dz}{1}=\frac{dt}{0}=\frac{du}{0},$$

thus $q(x,y,z,t)=\Psi_4(x,y,t)$ and all the derivatives of $q(x,y,z,t)$ with respect to z equal to zero, therefore

$$\frac{\partial^\alpha \Psi_4(x,z,t)}{\partial t^\alpha}+\kappa_1\sqrt{\Psi_4(x,z,t)}\frac{d\Psi_4(x,z,t))}{dx}+\kappa_2\frac{d^3\Psi_4(x,z,t)}{dx^3}+\kappa_3\frac{d^3\Psi_4(x,z,t)}{dxdydy}+\kappa_4\frac{d^2\Psi_4(x,y,z)}{dx^2}=0.$$

IV. Fractional Sub-Equation Method for FPDES

In this section, we introduced a summary explanation of the fractional sub-equation method [15, 16] as shown in the following steps

Step 1: Let $q(x,y,z,t)=q(\xi)$, $\xi=x+y+z-\lambda\, t$ is the traveling wave transformation which can be used to reduce the below equation

$$F(q,q_x,q_y,q_z,q_{xx},D_t^\alpha q,D_x^\alpha q,q_{xxx},q_{xyy},q_{xzz},...),\qquad 0<\alpha<1.$$

to be a non-linear FODE of the form

$$H(q,q',q'',\lambda^\alpha D_\xi^\alpha q,D_\xi^\alpha q,q''',...),\qquad 0<\alpha<1. \tag{28}$$

Step 2: Assume that, the above equation has a solution of the form

$$q(\xi)=\sum_{i=0}^{n}A_i(\varphi(\xi))^i \tag{29}$$

where $A_i(i=0,1,...,n)$ are constants to be detected and the positive integer n can be obtained by balancing the nonlinear terms and the highest order derivatives in Eq. (28). Also, the function $\varphi(\xi)$ satisfy the following fractional Riccati equation

$$D_\xi^\alpha\varphi(\xi)=\delta+\varphi^2(\xi) \tag{30}$$

where $\varphi(\xi)$ has a set of solutions as shown below

$$\varphi(\xi)=\begin{cases}-\sqrt{-\delta}\ tanh_\alpha(\sqrt{-\delta}\ \xi,\alpha), & \delta<0,\\ -\sqrt{-\delta}\ coth_\alpha(\sqrt{-\delta}\ \xi,\alpha), & \delta<0,\\ \sqrt{\delta}\ tan_\alpha(\sqrt{\delta}\ \xi,\alpha), & \delta>0,\\ -\sqrt{\delta}\ cot_\alpha(\sqrt{\delta}\ \xi,\alpha), & \delta>0,\\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha+\upsilon}, \quad \upsilon \text{ is a constant}, & \delta=0,\end{cases}$$

where all the previous trigonometric and hyperbolic functions are expressed by the following Mittag-Leffler function

$$E_\alpha(\xi)=\sum_{j=0}^{\infty}\frac{\xi^j}{\Gamma(1+j\alpha)},\qquad \text{and}$$

$$sin_\alpha(\xi)=\frac{E_\alpha(i\xi^\alpha)-E_\alpha(-i\xi^\alpha)}{2i},\qquad cos_\alpha(\xi)=\frac{E_\alpha(i\xi^\alpha)+E_\alpha(-i\xi^\alpha)}{2i},$$

$$sinh_\alpha(\xi)=\frac{E_\alpha(\xi^\alpha)-E_\alpha(-\xi^\alpha)}{2},\qquad cosh_\alpha(\xi)=\frac{E_\alpha(\xi^\alpha)+E_\alpha(-\xi^\alpha)}{2},\qquad \text{where it is known that:}$$

$$tan_\alpha(\xi)=\frac{sin_\alpha(\xi)}{cos_\alpha(\xi)},\quad cot_\alpha(\xi)=\frac{cos_\alpha(\xi)}{sin_\alpha(\xi)},\quad tanh_\alpha(\xi)=\frac{sinh_\alpha(\xi)}{cosh_\alpha(\xi)},\quad coth_\alpha(\xi)=\frac{cosh_\alpha(\xi)}{sinh_\alpha(\xi)}.$$

Step 3: Now, substituting Eq. (29) along with (30) into (28) and equate all the coefficients of all powers of $(\varphi(\xi))^i$ by zero. Then, we get a system of algebraic equations. Solving this system via the Mathematica program to determine the value of $A_i(i=0,1,...,n)$. Consequently, we use these values with the solutions of Eq. (30) to construct the analytical solutions for Eq. (28) which is considered the main aim for this section.

Ref

16. Zhang S H, Zhang Q. Fractional sub-equation method and its applications to nonlinear fractional PDEs. *Phys. Lett. A.* 2011; 375: 1069-1073.

a) Fractional sub-equation method for Eq. (1)

In this section, we apply the fractional sub-equation method for Eq. (1) therefore, we rewrite this equation by using $q(x,y,z,t) = v^2(x,y,z,t)$ as follows:

$$v\frac{\partial^\alpha v}{\partial t^\alpha} + \kappa_1 v^2 v_x + (\kappa_2 + 2\kappa_3)\left(3v_x v_{xx} + v v_{xxx}\right) + \kappa_4\left(v_x^2 + v v_{xx}\right) = 0. \tag{31}$$

Let us introduce an important transformation

$$v(x,y,z,t) = v(\xi), \quad \xi = x + y + z - \lambda t, \tag{32}$$

thus, Eq. (31) has the following form

$$\lambda^\alpha v D_\xi^\alpha v - \kappa_1 v^2 v' - (\kappa_2 + 2\kappa_3)\left(3v'v'' + vv'''\right) - \kappa_4\left(vv'' + v'^2\right) = 0. \tag{33}$$

According to the previous analysis of the considered method. We have the following solution for the reduced Eq. (33)

$$v(\xi) = A_0 + A_1\varphi(\xi) + A_2\varphi^2(\xi), \tag{34}$$

substituting Eq. (34) along with (30) into (33) and equate all the coefficients $(\varphi(\xi))^i$ by zero to get a system of algebraic equations. Solving this system with the aid of the Mathematica program we have

$$\textit{Case 1:}\ A_0 = \frac{3\lambda^\alpha}{8\kappa_1}, \quad A_1 = \pm\frac{3i\lambda^\alpha}{4\kappa_1\sqrt{\delta}}, \quad A_2 = -\frac{3\lambda^\alpha}{8\kappa_1\delta}, \quad \kappa_3 = -\frac{80\kappa_2\delta - \lambda^\alpha}{160\delta}, \quad \kappa_4 = -\frac{9i\lambda^\alpha}{40\sqrt{\delta}},$$

then, Eq. (1) has the below solutions

$$\begin{aligned}
q_{11}(\xi) &= \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1} tanh_\alpha\left(\sqrt{-\delta}\xi\right) + \frac{3\lambda^\alpha}{8\kappa_1} tanh_\alpha^2\left(\sqrt{-\delta}\xi\right)\right)^2, && where\ \ \delta < 0,\\
q_{12}(\xi) &= \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1} coth_\alpha\left(\sqrt{-\delta}\xi\right) + \frac{3\lambda^\alpha}{8\kappa_1} coth_\alpha^2\left(\sqrt{-\delta}\xi\right)\right)^2, && where\ \ \delta < 0,\\
q_{13}(\xi) &= \left(\frac{3\lambda^\alpha}{8\kappa_1} \pm \frac{3i\lambda^\alpha}{4\kappa_1} tan_\alpha\left(\sqrt{\delta}\xi\right) - \frac{3\lambda^\alpha}{8\kappa_1} tan_\alpha^2\left(\sqrt{\delta}\xi\right)\right)^2, && where\ \ \delta > 0,\\
q_{14}(\xi) &= \left(\frac{3\lambda^\alpha}{8\kappa_1} \mp \frac{3i\lambda^\alpha}{4\kappa_1} cot_\alpha\left(\sqrt{\delta}\xi\right) - \frac{3\lambda^\alpha}{8\kappa_1} cot_\alpha^2\left(\sqrt{\delta}\xi\right)\right)^2, && where\ \ \delta > 0,\\
q_{15}(\xi) &= \quad \frac{3\lambda^\alpha}{8\kappa_1} \mp \frac{3\kappa_4}{\kappa_1}\left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega_0}\right) - \frac{3\lambda^\alpha}{8\kappa_1}\left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega_0}\right)^2\Bigg)^2, && \delta = 0
\end{aligned} \tag{35}$$

$$\textit{Case 2:}\ \ A_0 = \frac{3\lambda^\alpha}{4\kappa_1}, \quad A_1 = \pm\frac{3i\lambda^\alpha}{4\kappa_1\sqrt{\delta}}, \quad A_2 = 0, \quad \kappa_3 = -\frac{\kappa_2}{2}, \quad \kappa_4 = \pm\frac{i\lambda^\alpha}{4\sqrt{\delta}},$$

thus, we have a set of analytical solutions for Eq. (1) which is presented as follows

$$\begin{aligned}
q_{21}(\xi) &= \left(\frac{3\lambda^\alpha}{4\kappa_1} \pm \frac{3\lambda^\alpha}{4\kappa_1} tanh_\alpha\left(\sqrt{-\delta}\xi\right)\right)^2, && where\ \ \delta < 0,\\
q_{22}(\xi) &= \left(frac3\lambda^\alpha 4\kappa_1 \pm \frac{3\lambda^\alpha}{4\kappa_1} coth_\alpha\left(\sqrt{-\delta}\xi\right)\right)^2, && where\ \ \delta < 0,\\
q_{23}(\xi) &= \left(\frac{3\lambda^\alpha}{4\kappa_1} \pm \frac{3i\lambda^\alpha}{4\kappa_1} tan_\alpha\left(\sqrt{\delta}\xi\right)\right)^2, && where\ \ \delta > 0,
\end{aligned} \tag{36}$$

Notes

$$q_{24}(\xi) = \left(\frac{3\lambda^{\alpha}}{4\kappa_1} \mp \frac{3i\lambda^{\alpha}}{4\kappa_1} cot_{\alpha}\left(\sqrt{\delta}\xi\right)\right)^2, \qquad where \;\; \delta > 0,$$

$$q_{25}(\xi) = \left(\frac{3\lambda^{\alpha}}{4\kappa_1} \mp \frac{3\kappa_4}{\kappa_1}\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+\omega_0}\right)\right)^2, \qquad \delta = 0$$

where v_0 is a constant and $\xi = x + y + z - \lambda\, t$.

The following figures show the 3D and 2D plots for the solution Eq. (31):

(a) $\alpha = 0.6$ (b) $\alpha = 0.7$ (c) $\alpha = 0.9$

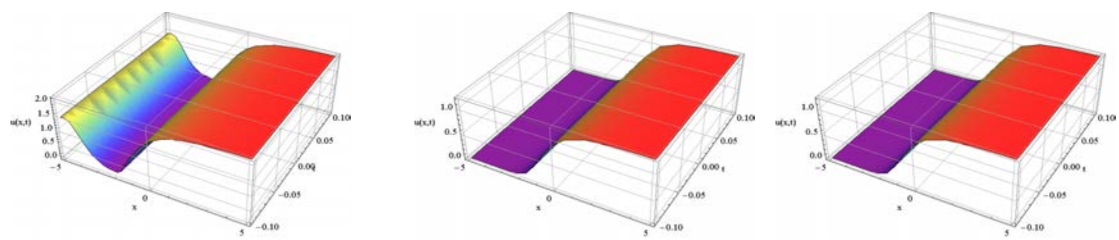

Figure 1: The 3D double-layer solution (31). (a) The solution at $\alpha = 0.6$ with the parameters $\lambda = 0.4,\ \delta = -0.5,\ \kappa_1 = 0.6,\ \kappa_2 = 0.3$ (b) The solution at $\alpha = 0.7$ with the same parameters. (c) The solution at $\alpha = 0.9$ with the same parameters.

(a) (b)

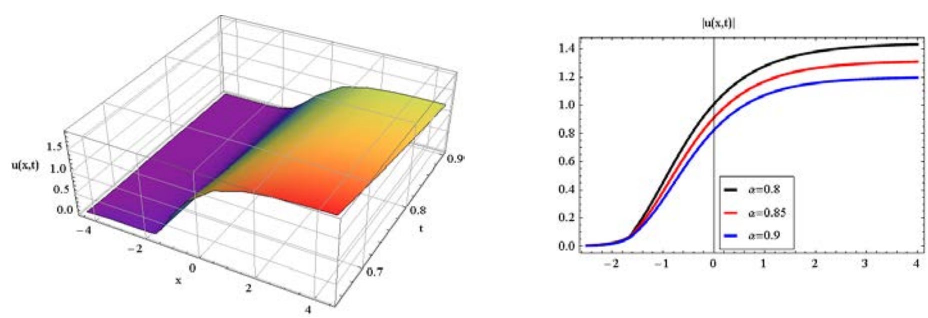

Figure 2: The effect of α on the analytical solution (36) (a) The 3D plot at a fixed time $t = 0.6$ and $0.65 \leq \alpha \leq 0.9$ with the parameters $\lambda = 0.4,\ \delta = -0.5,\ \kappa_1 = 0.6,\ \kappa_2 = 0.3.$ (b) The 2D plot at $\alpha = 0.8,\ 0.85,\ 0.9$ and the same values of the others parameters.

V. The Methodology of the Extended Kudryashov Method

We briefly display the main steps of the extended Kudryashov method [46, 47]to construct analytical solutions for Eq. (1) as below.

Step 1: Consider a non-linear FODE Eq.(28) with the same traveling wave transformation as section 4 and assume that the solution of Eq.(28) can be expressed as follows:

$$q(\xi) = \sum_{i=0}^{M} B_i \varphi^i, \tag{37}$$

where $a_i,\ \ i = 0, 1, 2, ..., n$ are constants to be determined, and $\varphi = \varphi(\xi)$ satisfies the following equation:

$$\varphi'(\xi) = \varphi(\xi)^3 - \varphi(\xi), \quad \text{since } \varphi(\xi) = \frac{\pm 1}{\sqrt{1 \pm e^{2\xi}}} \tag{38}$$

Step 2: Determining the value of the positive integer M by balancing the highest order derivatives with the nonlinear terms which appear in Eq.(28) by using the relation $M = \frac{2(s-rp)}{r-l-1}$ since, $(q^{(p)}(\xi,\varphi))^r$ and $q^l(\xi,\varphi)q^{(s)}(\xi)$ are the balanced terms.

Ref

46. Ege S M. Solitary wave solutions for some fractional evolution equations via new Kudryashov approach. *Rev. Mex. Fis.* 2022; 68(1):1-11.

Step 3: Substituting Eq.(37) into Eq.(28) and using Eq.(38), collecting all terms with the same order of $\varphi(\xi)$ together to zero yields a set of algebraic equations. Solving the equations system and using Eq.(38) to construct a variety of analytical solutions for Eq.(28).

a) Extended Kudryashov method for Eq. (1)

In this section, we apply the extended Kudryashov method method for Eq. (33) which is considered a reduced form of Eq. (1) and according to the previous analysis of the considered method. We have the following solution

$$v(\xi) = B_0 + B_1\varphi(\xi) + B_2\varphi^2(\xi), \tag{39}$$

substituting Eq. (39) along with (38) into (33) and equate all the coefficients $(\varphi(\eta))^i$ by zero to get a system of algebraic equations. Solving this system with the aid of the Mathematica program we have

$$\textit{Case 1:}\quad B_0 = 0, \qquad B_1 = 0, \qquad B_2 = \frac{3\lambda^\alpha}{2\kappa_1}, \qquad \kappa_3 = -\frac{\kappa_2}{2}, \qquad \kappa_4 = -\frac{\lambda^\alpha}{4},$$

then, Eq. (1) has the below solutions

$$\begin{gathered} q_{11}(\xi) = \frac{9\lambda^{2\alpha}}{16\kappa_1^2} e^{-2\xi} sech^2(\xi), \\ q_{12}(\xi) = \frac{9\lambda^{2\alpha}}{16\kappa_1^2} e^{-2\xi} csch^2(\xi). \end{gathered} \tag{40}$$

$$\textit{Case 2:}\quad B_0 = \frac{3\lambda^\alpha}{2\kappa_1}, \qquad B_1 = 0, \qquad B_2 = -\frac{3\lambda^\alpha}{2\kappa_1}, \qquad \kappa_3 = -\frac{\kappa_2}{2}, \qquad \kappa_4 = \frac{\lambda^\alpha}{4},$$

thus, we have the next solutions for Eq. (1) which is presented as follows

$$\begin{gathered} q_{21}(\xi) = \frac{9\lambda^{4\alpha}}{16\kappa_1^2} \left(1 - \frac{1}{2} e^{-2\xi} sech(\xi)\right)^2, \\ q_{22}(\xi) = \frac{9\lambda^{4\alpha}}{16\kappa_1^2} \left(1 - \frac{1}{2} e^{-2\xi} csch(\xi)\right)^2. \end{gathered} \tag{41}$$

where $\xi = x + y + z - \lambda\, t$.

The 3D and 2D plots for the solution Eq. (41) are plotted in the following Figures:

(a) $\alpha = 0.6$ (b) $\alpha = 0.7$ (c) $\alpha = 0.9$

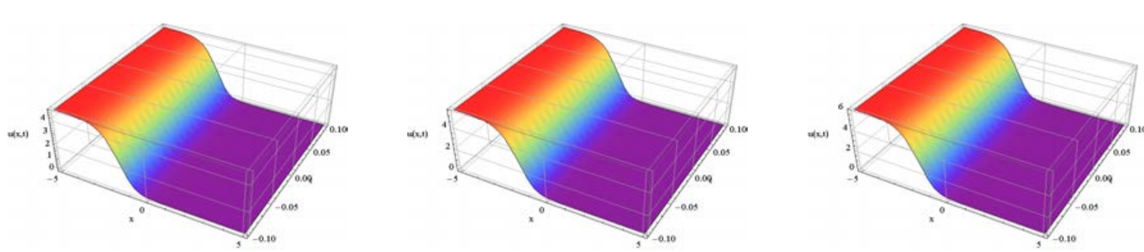

Figure 3: The 3D double-layer solution (41). (a) The solution at $\alpha = 0.6$ with the parameters $\lambda = 0.7,\ \kappa_1 = 0.5,\ \kappa_2 = 0.3$ (b) The solution at $\alpha = 0.7$ with the same parameters. (c) The solution at $\alpha = 0.9$ with the same parameters.

Notes

Notes

(a) (b)

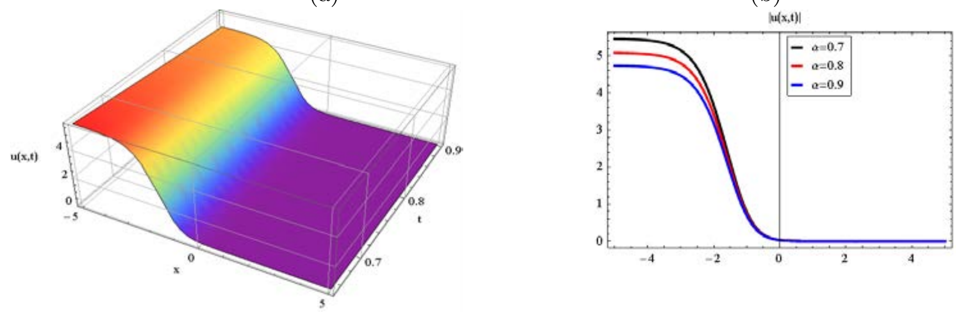

Figure 4: The effect of α on the analytical solution (41) (a) The 3D plot at a fixed time $t = 0.3$ and $0.65 \leq \alpha \leq 0.9$ with the parameters $\lambda = 0.7,\ \kappa_1 = 0.5,\ \kappa_2 = 0.3$. (b) The 2D plot at $\alpha = 0.7,\ 0.8,\ 0.9$ and the same values of the others parameters.

VI. Conservation Laws for Eq. (1)

In this section, the conservation laws of the time fractional MZKB equation (1) were derived, based on the formal lagrangian and Lie point symmetries as described in the following explanation:
Consider a vector $C = (C^t, C^x, C^y, C^z)$ admits the following conservation equation

$$[D_t(C^t) + D_x(C^x) + D_y(C^y) + D_z(C^z)]_{Eq.(1)} = 0, \tag{42}$$

where $C^t = C^t(x, y, z, t, u, ...)$, $C^x = C^x(x, y, z, t, u, ...)$, $C^y = C^y(x, y, z, t, u, ...)$, and $C^z = C^z(x, y, z, t, u, ...)$ are called the conserved vectors for Eq. (1). According to the new conservation theorem for Ibragimov [37], the formal Lagrangian for Eq. (1) can be given by

$$L = \omega((x, y, z, t)\,[\partial_t^\alpha q + \kappa_1\sqrt{q}\ q_x + \kappa_2 q_{xxx} + \kappa_3(q_{xyy} + q_{xzz}) + \kappa_4 q_{xx}] = 0, \tag{43}$$

here $\omega((x, y, z, t)$ is a new dependent variable. Depending on the definition of the Lagrangian, we get an action integral as follows

$$\int_0^t \int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} L\,(x, y, z, t, q, \omega, D_t^\alpha, q_x, q_{xxx}, q_{xyy}, q_{xzz}, q_{xx})\,dx\ dy\ dz\ dt.$$

The Euler-Lagrange operator is defined as

$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha q} - D_x\frac{\partial}{\partial q_x} + D_x^2\frac{\partial}{\partial q_{xx}} - D_x^3\frac{\partial}{\partial q_{xxx}} - D_x D_y^2\frac{\partial}{\partial q_{xyy}} - D_x D_z^2\frac{\partial}{\partial q_{xzz}}$$

where $(D_t^\alpha)^*$ denotes to the adjoint operator of D_t^α, and the adjoint equation to the nonlinear by means of the Euler-Lagrange equation is given by

$$\frac{\delta L}{\delta u} = 0.$$

Adjoint operator $(D_t^\alpha)^*$ for R-L is defined by

$$(D_t^\alpha)^* = (-1)^n I_T^{n-\alpha}(D_t^n) \equiv_t^C D_T^\alpha,$$

where $I_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n - \alpha$ that is defined by

$$I_T^{n-\alpha} f(t, x) = \frac{1}{\Gamma(n-\alpha)}\int_t^T (\tau - t)^{n-\alpha-1} f(\tau, x)\ d\tau.$$

Considering the case of one dependent variable $u(x,y,z,t)$ with four independent variables $x,\ y,\ z,\ t$, we get

$$\bar{X}+D_t(\tau)I+D_x(\xi)I+D_y(\zeta)I+D_z(\nu)I=W\frac{\delta}{\delta u}+D_t(C^t)+D_x(C^x)+D_y(C^y)+D_z(C^z),$$

where $\bar{X}$ is defined by

$$\bar{X}=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\nu\frac{\partial}{\partial z}+\eta\frac{\partial}{\partial u}+\eta^0_\alpha\frac{\partial}{\partial D^\alpha_t u}+\eta^x\frac{\partial}{\partial u_x}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xxx}\frac{\partial}{\partial u_{xxx}}+\eta^{xyy}\frac{\partial}{\partial u_{xyy}}+\eta^{xzz}\frac{\partial}{\partial u_{xzz}},$$

and the Lie characteristic function W for case 1 and 2 in the subsection 3.1 is defined as

$$W=\eta-\tau u_t-\xi u_x-\zeta u_y-\nu u_z,$$

where W can be expanded to

$$\begin{aligned}&W_1=-\frac{\partial}{\partial t},\quad W_2=-\frac{\partial}{\partial x},\quad W_3=-\frac{\partial}{\partial y},\quad W_4=-\frac{\partial}{\partial z},\\&W_5=-\frac{3t}{2\alpha}\frac{\partial}{\partial t}-\frac{x}{2}\frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial y}-\frac{z}{2}\frac{\partial}{\partial z}-2q\frac{\partial}{\partial q}.\end{aligned}\tag{44}$$

For the R-L time-fractional derivative, the density component C^t of conservation law is defined as:

$$C^t=\tau L+\sum_{k=0}^{n-1}(-1)^k\ {}_0D_t^{\alpha-1-k}(W_m)D_t^k\frac{\partial L}{\partial\ {}_0D_t^\alpha q}-(-1)^nJ\left(W_m,D_t^n\frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right),\tag{45}$$

where the operator $J(.)$ defined by

$$J(f,g)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau,x,y,z)g(\mu,x,y,z)}{(\mu-\tau)^{\alpha+1-n}}\,d\mu\,d\tau,$$

and the other (flux) components are defined as

$$\begin{aligned}C^i=\ &\xi^iL+W_m\left[\frac{\partial L}{\partial q_i^m}-D_j\left(\frac{\partial L}{\partial q_{ij}^m}\right)+D_jD_k\quad\frac{\partial L}{\partial q_{ijk}^m}-...\right)\right]+D_j(W_m)\left[\frac{\partial L}{\partial q_{ij}^m}-D_k\quad\frac{\partial L}{\partial q_{ijk}^m}\right)+...\right]\\&+D_jD_k(W_m)\quad\frac{\partial L}{\partial q_{ijk}^m}-...\right)+...,\end{aligned}\tag{46}$$

where $\xi^1=\xi,\ \xi^2=\zeta,\ \xi^3=\nu$ and $m=1,2,...,5$.

Now by using Eq. (44) with the help of Eqs. (45) and (46), we obtain the components of conservation laws for the time-fractional MZKB equation as the follows

Case 1: When $\kappa_4=0$ we have the following subcases according to the vector fields Eq. (14)

Case 1.1: $W_1=-q_x$ where $\xi^x=1,\ \xi^t=0,\ \xi^y=0,\ \xi^z=0$ and $\eta=0$ we get

$$C_1^t=\omega\ {}_0D_t^{\alpha-1}(-q_x)\frac{\partial L}{\partial\ {}_0D_t^\alpha q}-J\left(-q_x,D_t^n\frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right)=-\omega D_t^\alpha(q_x)-q_xD_t^\alpha(\omega),$$

$$\begin{aligned}C_1^x=\ &\omega\left[D_t^\alpha q+\kappa_1\sqrt{q}q_x+\frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\right]-q_x\left[\kappa_2\omega_{xx}+\frac{\kappa_3}{3}(\omega_{yy}+\omega_{zz})\right]\\&+\kappa_2q_{xx}\omega_x+\frac{\kappa_3}{3}(q_{xy}\omega_y+q_{xz}\omega_z),\end{aligned}$$

Notes

$$C_1^y = -\frac{\kappa_3}{3} q_x[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xx}\omega_y + q_{xy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyx} + q_{yxx}),$$

$$C_1^z = -\frac{\kappa_3}{3} q_x[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xx}\omega_z + q_{xz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzx} + q_{zxx}),$$

Case 1.2: $W_2 = -q_t$ where $\xi^x = 0,\ \xi^t = 1,\ \xi^y = 0,\ \xi^z = 0$ and $\eta = 0$ we obtain

$$C_2^t = \omega L \omega\ {}_0D_t^{\alpha-1}(-q_t)\frac{\partial L}{\partial\ {}_0D_t^\alpha q} - J\left(-q_t, D_t^n \frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right)$$

$$= \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \omega D_t^\alpha(q_t) - q_t D_t^\alpha(\omega),$$

$$C_2^x = -q_t\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xt}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yt}\omega_y + q_{zt}\omega_z)$$

$$- \kappa_2\omega q_{xxt} - \frac{\kappa_3}{3}\omega(q_{yyt} + q_{zzt}),$$

$$C_2^y = -\frac{\kappa_3}{3} q_t[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xt}\omega_y + q_{yt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyt} + q_{yxt}),$$

$$C_2^z = -\frac{\kappa_3}{3} q_t[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xt}\omega_z + q_{zt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzt} + q_{zxt})$$

Case 1.3: $W_3 = -q_y$ where $\xi^x = 0,\ \xi^t = 0,\ \xi^y = 1,\ \xi^z = 0$ and $\eta = 0$ we have

$$C_3^t = \omega\ {}_0D_t^{\alpha-1}(-q_y)\frac{\partial L}{\partial\ {}_0D_t^\alpha q} - J\left(-q_y, D_t^n \frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_y) - q_y D_t^\alpha(\omega),$$

$$C_3^x = -q_y\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_3^y = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right] - \frac{\kappa_3}{3} q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_3^z = -\frac{\kappa_3}{3} q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}),$$

Case 1.4: $W_4 = -q_z$ where $\xi^x = 0,\ \xi^t = 0,\ \xi^y = 0,\ \xi^z = 1$ and $\eta = 0$ we obtain

$$C_4^t = \omega\ {}_0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial\ {}_0D_t^\alpha q} - J\left(-q_z, D_t^n \frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_z) - q_z D_t^\alpha(\omega),$$

$$C_4^x = -q_z\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz})\right] - q_{xy}[-\kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_4^y = -\frac{\kappa_3}{3} q_y[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_4^z = \omega\left[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx})\right]$$

$$- \frac{\kappa_3}{3} q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}).$$

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Case 1.5: $W_5 = -2q - \frac{3t}{2\alpha}\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y} - \frac{z}{2}\frac{\partial}{\partial z}$ then

$$C_5^t = \omega\ {}_0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial\ {}_0D_t^\alpha q} - J\left(-q_z, D_t^n\frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right)$$

$$= \frac{3t}{2\alpha}\omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\right]$$

$$-\omega D_t^\alpha\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)D_t^\alpha(\omega),$$

$$C_5^x = \frac{x}{2}\omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\right]$$

$$-\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\left[\kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy}+\omega_{zz})\right]$$

$$+\kappa_2\omega_x\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial y}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right.$$

$$\left.+\omega_z\frac{\partial}{\partial z}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right),$$

$$C_5^y = \frac{y}{2}\omega\Big[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\Big] - \frac{\kappa_3}{3}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)[\omega_{xy}+\omega_{yx}]$$

$$+\frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \omega_x\frac{\partial}{\partial y}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right)$$

$$-\frac{\kappa_3}{3}\omega\frac{\partial}{\partial xy}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right),$$

$$C_5^z = \frac{z}{2}\omega\Big[D_t^\alpha q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\Big] - \frac{\kappa_3}{3}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)[\omega_{xz}+\omega_{zx}]$$

$$+\frac{\kappa_3}{3}\left(\omega_y\frac{\partial}{\partial x}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right) + \omega_x\frac{\partial}{\partial z}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right)\right)$$

$$-\frac{\kappa_3}{3}\omega\frac{\partial}{\partial xz}\left(2q + \frac{3t}{2\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}\right).$$

Case 2: When $\kappa_4 \neq 0$ we have the following subcases according to the vector fields Eq. (14)

Case 2.1: $W_1 = -q_x$ where $\xi^x = 1$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we get

$$C_1^t = \omega\ {}_0D_t^{\alpha-1}(-q_x)\frac{\partial L}{\partial\ {}_0D_t^\alpha q} - J\left(-q_x, D_t^n\frac{\partial L}{\partial\ {}_0D_t^\alpha q}\right) = -\omega D_t^\alpha(q_x) - q_x D_t^\alpha(\omega),$$

$$C_1^x = \omega\left[D_t^\alpha q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy}+q_{yyx}+q_{zxz}+q_{zzx})\right] - q_x\left[-\kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy}+\omega_{zz}\right]$$

$$+\kappa_2 q_{xx}\omega_x + \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{xz}\omega_z),$$

$$C_1^y = -\frac{\kappa_3}{3}q_x[\omega_{xy}+\omega_{yx}] + \frac{\kappa_3}{3}(q_{xx}\omega_y + q_{xy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyx}+q_{yxx}),$$

$$C_1^z = -\frac{\kappa_3}{3}q_x[\omega_{xz}+\omega_{zx}] + \frac{\kappa_3}{3}(q_{xx}\omega_z + q_{xz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzx}+q_{zxx}),$$

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Case 2.2: $W_2 = -q_t$ where $\xi^x = 0$, $\xi^t = 1$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain

$$C_2^t = \omega L \omega \; {}_0D_t^{\alpha-1}(-q_t)\frac{\partial L}{\partial \; {}_0D_t^{\alpha} q} - J\left(-q_t, D_t^n \frac{\partial L}{\partial \; {}_0D_t^{\alpha} q}\right)$$

$$= \omega\left[D_t^{\alpha} q + \kappa_1\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \omega D_t^{\alpha}(q_t) - q_t D_t^{\alpha}(\omega),$$

$$C_2^x = -q_t\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz}\right] - q_{xt}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yt}\omega_y + q_{zt}\omega_z)$$

$$- \kappa_2\omega q_{xxt} - \frac{\kappa_3}{3}\omega(q_{yyt} + q_{zzt}),$$

$$C_2^y = -\frac{\kappa_3}{3}q_t[\omega_{xy} + \omega_{yx}] + \frac{\kappa_3}{3}(q_{xt}\omega_y + q_{yt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyt} + q_{yxt}),$$

$$C_2^z = -\frac{\kappa_3}{3}q_t[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xt}\omega_z + q_{zt}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzt} + q_{zxt})$$

Case 2.3: $W_3 = -u_y$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 1$, $\xi^z = 0$ and $\eta = 0$ we have

$$C_3^t = \omega \; {}_0D_t^{\alpha-1}(-q_y)\frac{\partial L}{\partial \; {}_0D_t^{\alpha} q} - J\left(-q_y, D_t^n \frac{\partial L}{\partial \; {}_0D_t^{\alpha} q}\right) = -\omega D_t^{\alpha}(q_y) - q_y D_t^{\alpha}(\omega),$$

$$C_3^x = -q_y\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz}\right] - q_{xy}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_3^y = \omega\left[D_t^{\alpha} q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_3^z = -\frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}),$$

Case 2.4: $W_4 = -q_z$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain

$$C_4^t = \omega \; {}_0D_t^{\alpha-1}(-q_z)\frac{\partial L}{\partial \; {}_0D_t^{\alpha} q} - J\left(-q_z, D_t^n \frac{\partial L}{\partial \; {}_0D_t^{\alpha} q}\right) = -\omega D_t^{\alpha}(q_z) - q_z D_t^{\alpha}(\omega),$$

$$C_4^x = -q_z\left[\kappa_1\omega\sqrt{q} - \kappa_4\omega_x + \kappa_2\omega_{xx} + \frac{\kappa_3}{3}(\omega_{yy} + \omega_{zz}\right] - q_{xy}[\kappa_4\omega_x - \kappa_2\omega_x] + \frac{\kappa_3}{3}(q_{yy}\omega_y + q_{yz}\omega_z)$$

$$- \kappa_2\omega q_{xxy} - \frac{\kappa_3}{3}\omega(q_{yyy} + q_{zzy}),$$

$$C_4^y = \omega\left[D_t^{\alpha} q + a\sqrt{q}q_x + \frac{\kappa_3}{3}(q_{yxy} + q_{yyx} + q_{zxz} + q_{zzx}) + \kappa_4 q_{xx}\right] - \frac{\kappa_3}{3}q_y[\omega_{xy} + \omega_{yx}]$$

$$+ \frac{\kappa_3}{3}(q_{xy}\omega_y + q_{yy}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xyy} + q_{yxy}),$$

$$C_4^z = -\frac{\kappa_3}{3}q_y[\omega_{xz} + \omega_{zx}] + \frac{\kappa_3}{3}(q_{xy}\omega_z + q_{yz}\omega_x) - \frac{\kappa_3}{3}\omega(q_{xzy} + q_{zxy}).$$

Remark: We have verified that all the cases satisfy the equation of conservation laws Eq. (42).

VII. Conclusion

In this article, we considered the time-fractional modified Zakharov-Kuznetsov-Burgers (MZKB) equation of (3+1) dimensions. With the help of the Riemann-Liouville derivatives, the Lie symmetry analysis was successfully applied to study this equation. This analysis generated the symmetries and vector fields, which aided us in constructing the similarity reductions of the considered equation. Consequently, we constructed two sets of new analytical solutions via two powerful methods which are the fractional sub-equation method and extended Kudryashov method. Furthermore, to gain a better understanding of the dynamics of these solutions, we graphed the 3D and 2D plots of obtained solutions using appropriate parameters. Figure 1 described the double-layer solution (31) at $\alpha = 0.6,\ 0.7,\ 0.9$ with the parameters $\lambda = 0.4,\ \delta = -0.5,\ \kappa_1 = 0.6,\ \kappa_2 = 0.3$ at $-5 \leq x \leq 5$. Figure 2 showed the effect of α on the solution (31) at a fixed time $t = 0.6$ with $\alpha = 0.8,\ 0.85,\ 0.9$ and the parameters $\lambda = 0.4,\ \delta = -0.5,\ \kappa_1 = 0.6,\ \kappa_2 = 0.3$ for $-2.5 \leq x \leq 4$. Figure 3 represented the double-layer solution (41) at $\alpha = 0.6,\ 0.7,\ 0.9$ with the parameters $\lambda = 0.7,\ \kappa_1 = 0.5,\ \kappa_2 = 0.3$ where this Figure was traced at $-5 \leq x \leq 5$. Figure 4 was graphed by taking suitable parameters as $\lambda = 0.7,\ \kappa_1 = 0.5,\ \kappa_2 = 0.3$ and described the stable behavior of the the solution (41) at $\alpha = 0.7,\ 0.8,\ 0.9$ for $0 \leq x \leq 5$ at a fixed time t and decreases as the fractional-order α increases outside this interval. As a consequence, we presume that the obtained results will be more useful in explaining the physical meaning of the time fractional MZKB equation. Furthermore, we obtained four kinds of conservation laws with independent variables laying the groundwork of Lie symmetries. Finally, because of accuracy, ease of application, and relevance of the used methods in this paper, they could be generalized to many FPDEs.

Data availability

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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Competing interests

The author has declared that no competing interests exist.

Contributions

The authors declare that the work was realized in collaboration with a specified responsibility for each author. All authors read and approved the final paper.

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Plane Wave Propagation and Fundamental Solution for Nonlocal Homogenous Isotropic Thermoelastic Media with Diffusion

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Abstract- In the present problem, we study plane wave propagation and establish fundamental solution in the theory of nonlocal homogenous isotropic thermoelastic media with diffusion. We observe that there exists a set of three coupled waves namely longitudinal wave(P), thermal wave(T) and mass diffusion wave(MD) and one uncoupled transverse wave(SV) with different phase velocities. The effects of nonlocal parameter and diffusion on phase velocity, attenuation coefficient, penetration depth and specific loss have been studied numerically and presented graphically with respect to angular frequency. It is observed that characteristics of all the waves are influenced by the diffusion and nonlocal parameter. Fundamental solution of differential equations of motion in case of steady oscillations has been investigated and basic properties have also been discussed. Particular case of interest is also deduced from the present work and compared with the established result.

Keywords: *nonlocal, diffusion, fundamental solution, steady oscillations.*

GJSFR-F Classification: *DDC Code: 684.08 LCC Code: TT180*

PLANEWAVEPROPAGATIONANDFUNDAMENTALSOLUTIONFORNONLOCALHOMOGENOUSISOTROPICTHERMOELASTICMEDIAWITHDIFFUSION

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Plane Wave Propagation and Fundamental Solution for Nonlocal Homogenous Isotropic Thermoelastic Media with Diffusion

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Abstract- In the present problem, we study plane wave propagation and establish fundamental solution in the theory of nonlocal homogenous isotropic thermoelastic media with diffusion. We observe that there exists a set of three coupled waves namely longitudinal wave(P), thermal wave(T) and mass diffusion wave(MD) and one uncoupled transverse wave(SV) with different phase velocities. The effects of nonlocal parameter and diffusion on phase velocity, attenuation coefficient, penetration depth and specific loss have been studied numerically and presented graphically with respect to angular frequency. It is observed that characteristics of all the waves are influenced by the diffusion and nonlocal parameter. Fundamental solution of differential equations of motion in case of steady oscillations has been investigated and basic properties have also been discussed. Particular case of interest is also deduced from the present work and compared with the established result. The analysis of fundamental solution is very useful to investigate various problems of nonlocal thermoelastic solid with diffusion. The graphical analysis of current study is also very beneficial in order to investigate the different fields of geophysics, aerospace and electronics like seismology, manufacturing of aircrafts, volcanology, telecommunication etc.

Keywords: *nonlocal, diffusion, fundamental solution, steady oscillations.*

I. Introduction

It is well known that linear theory of elasticity describes the effective properties of various materials like steel, wood and concrete etc. But this theory is unable to explore the nano mechanical applications like nano structure vibrations, nano device stability etc. The theory of nonlocal elasticity is of great importance in determinig the properties of nano structure and wave propagation. The nonlocal theory of elasticity takes account of remote action between atoms because in nonlocal elasticity, stresses at a point not only depend on strain at that point but also on all points of the body. Eringen[1-3]elaborated the concept of nonlocality to elasicity and proposed the theory of nonlocal elasticity. Eringen and Edelen[4] obtained constitutive equations for the nonlinear theory. Gurtin[5] gave linear thermoelastic model to investigate the stresses produced to temperature field and distribution of temperature

Ref

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due to action of internal forces. Nowacki[6-7] constructed asymptotic solution of boundary value problems of three dimensional micropolar theory of elasticity with free field of rotations and displacements. Green and Naghdi[8-9] introduced a new thermodynamical theory which uses a general entropy balance and discussed thermoelastic behaviour without energy dissipation. Kupradze et.al.[10] discussed three dimensional problem of the mathematical theory of elasticity and thermoelasticity. Kumar and Kumar[11] studied plane wave propagation in nonlocal micropolar thermoelastic material with voids. Kaur and Singh[12] studied propagation of plane wave in a nonlocal magneto-thermoelastic semiconductor solid with rotation and identified four types of reflected coupled longitudinal waves.

Diffusion is the spontaneous movement of anything generally from a region of higher concentration to that of lower concentration and thermal diffusion makes use of heat transfer. The thermoelastic diffusiion in elastic solids is due to coupling of mass diffusion field of temperature and that of strain in addition to mass and heat exchange with environment. Auoadi[13-16] derived equation of motion and constitutive equations for a generalized thermoelastic diffusion with one relaxation time and obtained variation principle for the governing equations. He proved uniqueness theorem for these equations by using Laplace transform. Free vibration of a thermoelastic diffusive cylinder was investigated by Sharma et al.[17]. Hörmander[18-19] contained analysed the partial differential operators which are very useful in order to find fundamental solution in the thermoelastic diffusion solid. To examine boundary value problem of thermoelasticity, it is mandatory to evaluate the fundamental solution of the system of partial differential equation and to discuss their basic properties. Fudamental solution in the classical theory of coupled thermoelasticity was firstly studied by Hetnarski[20-21]. Svanadze[22-25] obtained fundamental solution of equations of steady oscillations in different types of thermoelastic solids. Scarpetta[26], Ciarletta et al.[27], Svanadze et al.[28] found fundamental solution in the theory of micropolar elasticity. Fundamental solution in the theory of thermoelastic diffusion is established by Kumar and Kansal[29-30]. Many problems related to plane wave propagation and fundamental solution have been studied by some of other researchers like Sharma and Kumar[31], Kumar[32], Kumar et.al.[33], Kumar and Devi[34], Biswas[35], Kumar and Batra[36], Biswas[37-38], Kumar et al.[39], Poonam et al.[40], Kumar and Batra[41]. However, from the best of author's knowledge, no study has been done for investigating the combining effect of nonlocal and diffusion on fundamental solution of homogenous isotropic thermoelastic solid. In current problem, we have discussed plane wave propagation and established the fundamental solution of differential equations in case of steady oscillations in terms of elementary functions for nonlocal homogenous isotropic thermoelastic solid with diffusion. Some basic properties and special case are also discussed.

Ref

10. Kupradze V.D., Gegelia T.G., Basheleishvili MO, Burchuladze TV, Three dimensional problems of the mathematical theory of elasticity and thermoelasticity. North-Holand pub. company: amsterdam New-York.; 1979.

Notes

II. Basic Equations

In three dimensional Euclidean space E^3, let $\mathbf{X} = (x, y, z)$ be a point, t represents the time variable and $\mathbf{D}_x \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Following Eringen [1-3], the constitutive relations for nonlocal generalised thermoelastic solid with diffusion are given by

$$(1 - \varepsilon^2 \nabla^2)\, \sigma_{ij} = \sigma'_{ij} = 2\mu e_{ij} + [\lambda\, e_{kk} - \beta_1 T - \beta_2 C]\, \delta_{ij} \tag{1}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

Using constitutive relations, equation of motion for nonlocal homogenous isotropic thermoelastic solid with diffusion is

$$\mu\, u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \beta_1 T_{,i} - \beta_2 C_{,i} = \rho(1 - \varepsilon^2 \nabla^2)\ddot{u}_i \tag{3}$$

Equations of heat conduction and mass diffusion for nonlocal homogenous isotropic thermoelastic solid with diffusion are given by

$$\rho\, C_E(\dot{T} + \tau_0 \ddot{T}) + \beta_1 T_0(\dot{e}_{kk} + \tau_0 \ddot{e}_{kk}) + a^* T_0(\dot{C} + \tau_0 \ddot{C}) = K T_{,ii} \tag{4}$$

$$D^* \beta_2\, e_{kk,ii} + D^* a^* T_{,ii} + (1 - \varepsilon^2 \nabla^2)(\dot{C} + \tau \ddot{C}) = D^* b^* C_{,ii} \tag{5}$$

where $\mathbf{u} = (u_1,\, u_2,\, u_3)$ is the displacement vector; σ_{ij} are the stress components and e_{ij} are components of strain tensor; e_{kk} is dilatation; σ'_{ij} corresponds to the local thermoelastic solid with diffusion; T is the temperature change measured from the absolute temperature T_0; C_E denotes specific heat at constant strain; K is the thermal conductivity; τ_0 is the relaxation time parameter and τ is the relaxation time of diffusion; C is the concentration; D^* is the thermoelastic diffusion constant; a^* and b^* respectively measures the thermo-diffusion effects and diffusive effects; ρ is mass density; β_1, β_2 are material coefficients with $\beta_1 = (3\lambda + 2\mu)\, \alpha_t$, $\beta_2 = (3\lambda + 2\mu)\, \alpha_c$; λ and μ are Lame's constants; α_t the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusion expansion; ∇^2 denotes the Laplacian operator; $\varepsilon = e_0 a$ is the nonlocal parameter; e_0 corresponds to the material constant; a denotes the characteristic length; δ_{ij} is kronecker delta. In the above equations, superposed dot represents the derivative with respect to time and ',' in the subscript denotes the partial derivatives with respect to x, y, z for i, $j = 1, 2, 3$ respectively.

For two-dimensional problem, we will suppose that all quantities related to the medium are functions of cartesian coordinates x, z ($i.e. \frac{\partial}{\partial y} \equiv 0$) and time t and are independent of y. Displacement vector is considered as

$$\mathbf{u} = (u_1,\, 0,\, u_3) \tag{6}$$

We define the following dimensionless quantities

$$x' = \frac{\omega_1 x}{c_1},\ z' = \frac{\omega_1 z}{c_1},\ u_1' = \frac{\rho\omega_1 c_1}{\beta_1 T_0} u_1,\ u_3' = \frac{\rho\omega_1 c_1}{\beta_1 T_0} u_3,$$

$$t' = \omega_1 t,\ T' = \frac{T}{T_0},\ C' = \frac{\beta_2}{\beta_1 T_0} C,\ \tau_0' = \omega_1\tau_0,\ \tau' = \omega_1\tau \qquad (7)$$

where $\omega_1 = \frac{\rho C_E c_1^2}{K}$, $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$

Now using equation (7) in equations (3), (4), (5) and suppressing the primes, we obtain

$$\alpha_1 \nabla^2 \mathbf{u} + \alpha_2\, grad\, div\, \mathbf{u} - grad\, T - grad\, C = (1-\varepsilon_1^2\nabla^2)\,\ddot{\mathbf{u}} \qquad (8)$$

$$\tau_t^0\,(\dot{T} + \alpha_3\, div\, \dot{\mathbf{u}} + \alpha_4\, \dot{C}) = \nabla^2 T \qquad (9)$$

$$\alpha_5\,\nabla^2\, div\, \mathbf{u} + \alpha_6\,\nabla^2 T - \alpha_7\,\nabla^2 C + (1-\varepsilon_1^2\nabla^2)\,\tau_c^0\,\dot{C} = 0 \qquad (10)$$

where

$$\alpha_1 = \frac{\lambda+\mu}{\lambda+2\mu},\ \alpha_2 = \frac{\mu}{\lambda+2\mu},\ \alpha_3 = \frac{\beta_1^2 T_0}{\rho K\omega_1},\ \alpha_4 = \frac{a^*\beta_1 T_0 c_1^2}{K\omega_1\beta_2},\ \alpha_5 = \frac{D^*\beta_2^2\omega_1}{\rho c_1^4},$$

$$\alpha_6 = \frac{D^* a^*\omega_1\beta_2}{\beta_1 c_1^2},\ \alpha_7 = \frac{D^* b^*\omega_1}{c_1^2}\ \varepsilon_1^2 = \frac{\varepsilon^2\omega_1^2}{c_1^2},\ \tau_t^0 = 1 + \tau_0\,\omega_1\frac{\partial}{\partial t},\ \tau_c^0 = 1 + \tau\,\omega_1\frac{\partial}{\partial t}$$

The displacement vector $\mathbf{u}$ is related to the potential functions $\phi_1(x,z,t)$ and $\phi_2(x,z,t)$ as

$$u_1 = \frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial z}, \quad u_3 = \frac{\partial\phi_1}{\partial z} - \frac{\partial\phi_2}{\partial x} \qquad (11)$$

Using equation (11) in equations (8)-(10), we obtain

$$(\alpha_1+\alpha_2)\nabla^2\phi_1 - T - C = (1-\varepsilon_1^2\nabla^2)\ddot{\phi_1} \qquad (12)$$

$$\alpha_1\nabla^2\phi_2 = (1-\varepsilon_1^2\nabla^2)\ddot{\phi_2} \qquad (13)$$

$$(\frac{\partial}{\partial t} + \tau_0\,\omega_1\frac{\partial^2}{\partial t^2})\,(T + \alpha_4\, C + \alpha_3\,\nabla^2\phi_1) = \nabla^2 T \qquad (14)$$

$$\alpha_5\,\nabla^4\phi_1 + \alpha_6\nabla^2 T - \alpha_7\,\nabla^2 C + (1-\varepsilon_1^2\nabla^2)\,(\frac{\partial}{\partial t} + \tau\,\omega_1\frac{\partial^2}{\partial t^2})\,C = 0 \qquad (15)$$

Equations (12), (14) and (15) show that ϕ_1, T and C are coupled and ϕ_2 remains decoupled.

III. Plane Wave

Notes

We consider a plane wave propagating in a nonlocal homogenous isotropic thermoelastic media with diffusion and assume the solution of the form

$$(\phi_1,\, \phi_2,\, T,\, C) = (\bar{\phi_1},\, \bar{\phi_2},\, \bar{T},\, \bar{C})\ exp\{ik(\mathbf{n}.\mathbf{r} - ct)\} \tag{16}$$

where $\omega = kc$ is the frequency, c is the wave velocity, k is the wave number, $\bar{\phi_1}$, $\bar{\phi_2}$, $\bar{T}$, $\bar{C}$ are undetermined amplitudes that depend on the time and co-ordinates $\mathbf{r} = (x, 0, z)$, $\mathbf{n}$ is the unit vector.

Using equation (16) in equations (12)-(15), we obtain

$$(B_1k^2 + \omega^2)\bar{\phi_1} = \bar{T} + \bar{C} \tag{17}$$

$$(B_2k^2 - \omega^2)\bar{\phi_2} = 0 \tag{18}$$

$$(k^2 - B_3)\bar{T} - \alpha_4 B_3 \bar{C} + \alpha_3 B_3 k^2 \bar{\phi_1} = 0 \tag{19}$$

$$\alpha_5 k^4 \bar{\phi_1} - \alpha_6 k^2 \bar{T} + (B_4 k^2 - B_5)\bar{C} = 0 \tag{20}$$

where

$$B_1 = \varepsilon_1^2\omega^2 - 1,\ \ B_2 = \alpha_1 - \varepsilon_1^2\omega^2,\ \ B_3 = i\omega + \tau_0\,\omega_1\,\omega^2$$

$$B_4 = \alpha_7 - \varepsilon_1^2(i\omega) - \varepsilon_1^2\tau\,\omega_1\,\omega^2,\ \ B_5 = i\omega + \tau\,\omega_1\,\omega^2$$

Solving equations (17), (19) and (20) for $\bar{\phi_1}$, $\bar{T}$, $\bar{C}$ we obtain a cubic equation in k^2 as

$$F_1k^6 + G_1k^4 + H_1k^2 + J_1 = 0 \tag{21}$$

where

$$F_1 = B_4B_1 + \alpha_5,\ \ J_1 = B_3B_5\omega^2$$

$$G_1 = \alpha_3\alpha_6B_3 + B_4\omega^2 - B_1B_3B_4 + \alpha_3B_3B_4 - B_5B_1 + \alpha_4\alpha_5B_3 - \alpha_4\alpha_6B_1B_3 - \alpha_5B_3$$

$$H_1 = -B_4B_3\omega^2 - B_5\omega^2 + B_5B_3B_1 - \alpha_3B_3B_5 - \alpha_4\alpha_6B_3\omega^2$$

Solving equation (21), we obtain six values of k in which three values k_1, k_2, k_3 correspond to positive z-direction and the other three values of k correspond to negative z-direction. Corresponding to k_1, k_2 and k_3 there exist three coupled waves, namely, longitudinal wave(P), thermal wave(T) and mass diffusion wave(MD).

The expressions for the phase velocity, attenuation coefficients, penetration depth and specific loss of above waves are evaluated as

Phase Velocity: The phase velocities v_1, v_2 and v_3 of P-wave, T-wave, and MD-wave, respectively, are given by

$$v_j = \frac{\omega}{|Re(k_j)|} \quad j = 1, 2, 3 \tag{22}$$

Attenuation Coefficients: The attenuation coefficients Q_1, Q_2 and Q_3 of P-wave, T-wave and MD-wave, respectively, can be written as

$$Q_j = Im(k_j) \quad j = 1, 2, 3 \tag{23}$$

Penetration Depth: The penetration depth D_1, D_2 and D_3 of P-wave, T-wave and MD-wave, respectively, is defined as

$$D_j = \frac{1}{|Im(k_j)|} \quad j = 1, 2, 3 \tag{24}$$

Specific Loss: The Specific Loss L_1, L_2 and L_3 of P-wave, T-wave and MD-wave, respectively, are given by

$$L_j = 4\pi|\frac{Re(k_j)}{Im(k_j)}| \quad j = 1, 2, 3 \tag{25}$$

Solving equation(18), we obtain two values of k in which one value k_4 corresponds to positive z-direction representing transverse wave(SV) and other value of k corresponds to negative z-direction. The phase velocity of transverse wave is given by $v_4 = \sqrt{B_2}$

IV. Steady Oscillations

Assume that displacement vector, temperature change and concentration are functions as

$$(\mathbf{u}(x,z,t), T(x,z,t), C(x,z,t)) = Re[(\mathbf{u}^*(x,z,t), T^*(x,z,t), C^*(x,z,t))e^{-i\omega t}] \tag{26}$$

Using equation (26) in equations (8),(9),(10), we obtain following system of equations of steady oscillations

$$[(\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\mathbf{u}^* + \alpha_2\, grad\, div\, \mathbf{u}^* - grad\, T^* - grad\, C^* = 0 \tag{27}$$

$$-\tau_t^{01}[\alpha_3\, div\, \mathbf{u}^* + \alpha_4\, C^*] + (\nabla^2 - \tau_t^{01})T^* = 0 \tag{28}$$

$$\alpha_5\, \nabla^2\, div\, \mathbf{u}^* + \alpha_6\, \nabla^2 T^* + [\tau_c^{01} - (\alpha_7 + \varepsilon_1^2\, \tau_c^{01})\, \nabla^2]\, C^* = 0 \tag{29}$$

Notes

where

$$\tau_t^{01} = -i\omega(1 - i\omega\tau_0), \quad \tau_c^{01} = -i\omega(1 - i\omega\tau)$$

We define matrix differential operator

$$\mathbf{B}(\mathbf{D}_x) = \left[B_{mn}(\mathbf{D}_x)\right]_{4\times 4} \tag{30}$$

where

$$B_{m1}(\mathbf{D}_x) = [(\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\delta_{m1} + \alpha_2 \frac{\partial^2}{\partial x\,\partial x^*}$$

$$B_{m2}(\mathbf{D}_x) = [(\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\delta_{m2} + \alpha_2 \frac{\partial^2}{\partial z\,\partial x^*}$$

$$B_{m3}(\mathbf{D}_x) = B_{m4}(\mathbf{D}_x) = -\frac{\partial}{\partial x^*}, \quad B_{3n}(\mathbf{D}_x) = -\tau_t^{01}\,\alpha_3 \frac{\partial}{\partial x^*}$$

$$B_{4n}(\mathbf{D}_x) = \alpha_5\,\nabla^2 \frac{\partial}{\partial x^*}, \quad B_{33}(\mathbf{D}_x) = \nabla^2 - \tau_t^{01}, \quad B_{34}(\mathbf{D}_x) = \tau_t^{01}\,\alpha_4,$$

$$B_{43}(\mathbf{D}_x) = \alpha_2\,\nabla^2, \quad B_{44}(\mathbf{D}_x) = -\alpha_1\,\nabla^2 + (1 - \varepsilon_1^2\,\nabla^2)\tau_c^{01}; \quad m, n = 1, 2$$

For $m = n = 1, \quad x^* = x$ and for $m = n = 2, \quad x^* = z; \qquad \delta_{mn}$ is kronecker's delta.
The system of equations (27)-(29) can be written as

$$\mathbf{B}(\mathbf{D}_x)\mathbf{V}(\mathbf{X}) = 0 \tag{31}$$

where $\mathbf{V} = (u_1^*,\, u_3^*,\, T^*,\, C^*)$ ia a four component vector function.
Assume that

$$-(\alpha_1 + \alpha_2 - \omega^2\,\varepsilon_1^2)(\alpha_1 - \omega^2\varepsilon_1^2)\,(\alpha_1 + \varepsilon_1^2\tau_c^{01}) \neq 0 \tag{32}$$

If condition (32) is satisfied then $\mathbf{B}$ is an elliptic differential operator (Hörmander [18]).

Definition. The fundamental solution of system of equations (27)-(29) is the matrix $\mathbf{A}(\mathbf{X}) = \left[A_{ij}(\mathbf{X})\right]_{4\times 4}$ satisfying the condition

$$\mathbf{B}(\mathbf{D}_x)\mathbf{A}(\mathbf{X}) = \delta(\mathbf{X})\mathbf{I}(\mathbf{X}) \tag{33}$$

where δ is Dirac delta, $\mathbf{I} = \left[\delta_{ij}\right]_{4\times 4}$ is the unit matrix.
We now construct $\mathbf{A}(\mathbf{X})$ in terms of elementary functions.

Ref

18. Hörmander L., Linear partial differential operators. Berlin: Springerverlag; 1963.

Notes

V. Fundamental Solution of System of Equations of Steady Oscillations

We consider the system of equations

$$[(\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\mathbf{u}^* + \alpha_2\, grad\, div\, \mathbf{u}^* - \tau_t^{01}\,\alpha_3\, grad\, T^* + \alpha_5\,\nabla^2\, grad\, C^* = \mathbf{J'} \tag{34}$$

$$-div\,\mathbf{u}^* + (\nabla^2 - \tau_t^{01})T^* + \alpha_2\nabla^2 C^* = L \tag{35}$$

$$-div\,\mathbf{u}^* - \tau_t^{01}\alpha_4\, T^* + [-\alpha_7\,\nabla^2 + (1 - \varepsilon_1^2\,\nabla^2)\tau_c^{01}]C^* = M \tag{36}$$

where $\mathbf{J'}$ is a vector function on E^3 and L, M are scalar functions on E^3. The system of equations (34)-(36) may be written in the following form

$$\mathbf{B}^{tr}(\mathbf{D}_x)\mathbf{V}(\mathbf{X}) = \mathbf{G}(\mathbf{X}) \tag{37}$$

where $\mathbf{B^{tr}}$ is the transpose of matrix $\mathbf{B}$ and $\mathbf{G} = (\mathbf{J'},\ L,\ M)$
Applying operator div to the equation (34), we get

$$[(1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\, div\,\mathbf{u}^* - \tau_t^{01}\,\alpha_3\,\nabla^2 T^* + \alpha_5\,\nabla^4\, C^* = div\,\mathbf{J'} \tag{38}$$

$$-div\,\mathbf{u}^* + (\nabla^2 - \tau_t^{01})T^* + \alpha_2\nabla^2 C^* = L \tag{39}$$

$$-div\,\mathbf{u}^* - \tau_t^{01}\alpha_4\, T^* + [-\alpha_7\,\nabla^2 + (1 - \varepsilon_1^2\,\nabla^2)\tau_c^{01}]C^* = M \tag{40}$$

equations (38)-(40) may be expressed as

$$D(\nabla^2)\,\mathbf{P} = \mathbf{Q} \tag{41}$$

where $\mathbf{P} = (div\,\mathbf{u}^*,\ T^*,\ C^*)$, $\mathbf{Q} = (div\,\mathbf{J'},\ L,\ M) = (d_1,\ d_2,\ d_3)$ and

$$\mathbf{D}(\nabla^2) = [D_{mn}]_{3\times3}$$

$$= \begin{bmatrix} (1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2 & -\tau_t^{01}\,\alpha_3\,\nabla^2 & \alpha_5\,\nabla^4 \\ -1 & \nabla^2 - \tau_t^{01} & \alpha_2\nabla^2 \\ -1 & -\tau_t^{01}\alpha_4 & -\alpha_7\,\nabla^2 + (1 - \varepsilon_1^2\,\nabla^2)\tau_c^{01} \end{bmatrix}_{3\times3} \tag{42}$$

equations (38)-(40) may be expressed as

$$\Gamma_1(\nabla^2)\,\mathbf{P} = \sigma \tag{43}$$

Notes

where

$$\sigma = (\sigma_1,\, \sigma_2,\, \sigma_3); \quad \sigma_n = e_1^* \sum_{m=1}^{3} D_{mn}^*\, d_m$$

$$\Gamma_1(\nabla^2) = e_1^*\, det\, \mathbf{D}(\nabla^2), \;\; e_1^* = -\frac{1}{(1-\omega^2\varepsilon_1^2)\,(\alpha_7 + \varepsilon_1^2\,\tau_c^{01})} \tag{44}$$

and D_{mn}^* is the cofactor of elements D_{mn} of matrix $\mathbf{D}$
From equations (42) and (44), we have

$$\Gamma_1(\nabla^2) = \prod_{m=1}^{3} (\nabla^2 + \Lambda_m^2)$$

where $\Lambda_m^2,\; m = 1,2,3$ are roots of equation $\Gamma_1\,(-r) = 0$ (with respect to r)
Applying $\Gamma_1(\nabla^2)$ to the equation (27) and using equation (43), we obtain

$$\Gamma_1(\nabla^2)\,[(\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2]\mathbf{u}^* = -\alpha_2\, grad\,\sigma_1 + grad\,\sigma_2 + grad\,\sigma_3 \tag{45}$$

This equation may also be written as

$$\Gamma_1(\nabla^2)\,\Gamma_2(\nabla^2)\,\mathbf{u}^* = \sigma' \tag{46}$$

where

$$\Gamma_2(\nabla^2) = (\alpha_1 - \omega^2\varepsilon_1^2)\nabla^2 + \omega^2 \tag{47}$$

and

$$\sigma' = -\alpha_2\, grad\,\sigma_1 + grad\,\sigma_2 + grad\,\sigma_3 \tag{48}$$

It can be written as

$$\Gamma_2\,(\nabla^2) = (\nabla^2 + \Lambda_4^2) \tag{49}$$

where Λ_4^2 is a root of equation $\Gamma_2\,(-r) = 0$ (with respect to r)
From equations (43) and (46), we can write

$$\mathbf{\Theta}(\nabla^2)\,\mathbf{V}(\mathbf{X}) = \hat{\sigma}(\mathbf{X}) \tag{50}$$

where $\hat{\sigma}(\mathbf{X}) = (\sigma',\, \sigma_2,\, \sigma_3)$ and $\mathbf{\Theta}(\nabla^2) = \left[\Theta_{gh}(\nabla^2)\right]_{4\times 4}$

$$\Theta_{mm}(\nabla^2) = \Gamma_1(\nabla^2)\,\Gamma_2(\nabla^2)$$

$$\Theta_{33} = \Theta_{44} = \Gamma_1(\nabla^2)$$

$$\Theta_{gh}(\nabla^2) = 0$$

$$m = 1,2; \quad g,h = 1,2,3,4; \quad g \neq h$$

Equations (41),(44) and (48) can also be written as

$$\sigma' = c_{11}(\nabla^2)\, grad\, div\, \mathbf{J}' + c_{21}(\nabla^2)\, grad\, L + c_{31}(\nabla^2)\, grad\, M \tag{51}$$

$$\sigma_2 = c_{12}(\nabla^2)\, div\, \mathbf{J}' + c_{22}(\nabla^2)\, L + c_{32}(\nabla^2)\, M \tag{52}$$

$$\sigma_3 = c_{13}(\nabla^2)\, div\, \mathbf{J}' + c_{23}(\nabla^2)\, L + c_{33}(\nabla^2)\, M \tag{53}$$

where

$$c_{11}(\nabla^2) = -\alpha_2\, e_1^*\, D_{11}^* + e_1^*\, D_{12}^* + e_1^*\, D_{13}^*, \quad c_{12} = e_1^*\, D_{12}^*, \quad c_{13} = e_1^*\, D_{13}^*,$$

$$c_{21}(\nabla^2) = -\alpha_2\, e_1^*\, D_{21}^* + e_1^*\, D_{22}^* + e_1^*\, D_{23}^*, \quad c_{22} = e_1^*\, D_{22}^*, \quad c_{23} = e_1^*\, D_{23}^*,$$

$$c_{31}(\nabla^2) = -\alpha_2\, e_1^*\, D_{31}^* + e_1^*\, D_{32}^* + e_1^*\, D_{33}^*, \quad c_{32} = e_1^*\, D_{32}^*, \quad c_{33} = e_1^*\, D_{33}^*$$

From equations (51)-(53), we get

$$\hat{\sigma}(\mathbf{X}) = \mathbf{H}^{tr}\,(\mathbf{D}_x)\,\mathbf{G}(\mathbf{X}) \tag{54}$$

where

$$\mathbf{H} = \left[H_{gh}\right]_{4X4}$$

$$H_{m1}(\mathbf{D}_x) = c_{11}(\nabla^2)\,\frac{\partial^2}{\partial x\,\partial x^*}, \quad H_{m2}(\mathbf{D}_x) = c_{11}(\nabla^2)\,\frac{\partial^2}{\partial z\,\partial x^*}, \quad H_{m3}(\mathbf{D}_x) = c_{21}(\nabla^2)\,\frac{\partial}{\partial x^*},$$

$$H_{m4}(\mathbf{D}_x) = c_{31}(\nabla^2)\,\frac{\partial}{\partial x^*}, \quad H_{3n}(\mathbf{D}_x) = c_{12}(\nabla^2)\,\frac{\partial}{\partial x^*}, \quad H_{4n}(\mathbf{D}_x) = c_{13}(\nabla^2)\,\frac{\partial}{\partial x^*},$$

$$H_{33}(\mathbf{D}_x) = c_{22}(\nabla^2), \quad H_{34}(\mathbf{D}_x) = c_{32}(\nabla^2), \quad H_{43}(\mathbf{D}_x) = c_{43}(\nabla^2),$$

$$H_{44}(\mathbf{D}_x) = c_{44}(\nabla^2); \qquad m,n = 1,2 \tag{55}$$

Notes

For $m = n = 1, \;\; x^* = x$ and for $m = n = 2, \;\; x^* = z$
From equations (37), (46) and (50), we obtain

$$\mathbf{\Theta}\,\mathbf{V} = \mathbf{H}^{tr}\,\mathbf{B}^{tr}\,\mathbf{V}$$

Notes

Above equation can be rewritten as

$$\mathbf{H}^{tr}\,\mathbf{B}^{tr} = \mathbf{\Theta}$$

Therefore, we have

$$\mathbf{B}(\mathbf{D}_x)\,\mathbf{H}(\mathbf{D}_x) = \mathbf{\Theta}(\nabla^2) \tag{56}$$

we assume that

$$\lambda_m^2 \neq \lambda_n^2 \neq 0; \quad m, n = 1, 2, 3, 4; \quad m \neq n$$

We now define

$$\mathbf{W}(\mathbf{X}) = \left[W_{rs}(\mathbf{X})\right]_{4\times 4}$$

$$W_{mm}(\mathbf{X}) = \sum_{n=1}^{4} q_{1n}\xi_n(\mathbf{X}), \;\; W_{33}(\mathbf{X}) = W_{44}(\mathbf{X}) = \sum_{n=1}^{3} q_{2n}\xi_n(\mathbf{X}), \;\; W_{uv}(\mathbf{X}) = 0$$

$m = 1, 2; \quad u, v = 1, 2, 3, 4; \quad u \neq v$

where

$$\xi_n(\mathbf{X}) = -\frac{1}{4\,\Pi\,|\mathbf{X}|}\,\exp\left(i\Lambda_n|\mathbf{X}|\right), \quad n = 1, 2, 3, 4$$

$$q_{1l} = \prod_{m=1,\, m\neq l}^{4} (\Lambda_m^2 - \Lambda_l^2)^{-1}, \quad l = 1, 2, 3, 4$$

$$q_{2u} = \prod_{m=1,\, m\neq u}^{3} (\Lambda_m^2 - \Lambda_u^2)^{-1}, \quad u = 1, 2, 3 \tag{57}$$

Now, we prove the following Lemma:

Lemma: The matrix $\mathbf{W}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}(\nabla^2)$, that is

$$\mathbf{\Theta}(\nabla^2)\,\mathbf{W}(\mathbf{X}) = \delta(\mathbf{X})\,\mathbf{I}(\mathbf{X}) \tag{58}$$

Proof: To prove the Lemma, it is sufcient to show that

$$\Gamma_1(\nabla^2)\,\Gamma_2(\nabla^2)\,W_{11}(\mathbf{X}) = \delta(\mathbf{X})$$

$$\Gamma_1(\nabla^2)\,W_{33}(\mathbf{X}) = \delta(\mathbf{X}) \tag{59}$$

Consider

$$q_{21} + q_{22} + q_{23} = \frac{-g_1 + g_2 - g_3}{g_4}$$

where

$$g_1 = (\Lambda_2^2 - \Lambda_3^2), \quad g_2 = (\Lambda_1^2 - \Lambda_3^2), \quad g_3 = (\Lambda_1^2 - \Lambda_2^2),$$

$$g_4 = (\Lambda_1^2 - \Lambda_2^2)\,(\Lambda_1^2 - \Lambda_3^2)(\Lambda_2^2 - \Lambda_3^2)$$

Solving above relations, we get

$$q_{21} + q_{22} + q_{23} = 0 \tag{60}$$

Similarly, from equation (57) we can also find out

$$q_{22}\,(\Lambda_1^2 - \Lambda_2^2) + q_{23}\,(\Lambda_1^2 - \Lambda_3^2) = 0 \tag{61}$$

$$q_{23}(\Lambda_1^2 - \Lambda_3^2)\,(\Lambda_2^2 - \Lambda_3^2) = 1 \tag{62}$$

Also, we have

$$(\nabla^2 + \Lambda_m^2)\,\xi_n(\mathbf{X}) = \delta(\mathbf{X}) + (\Lambda_m^2 - \Lambda_n^2)\,\xi_n(\mathbf{X}), \qquad m, n = 1, 2, 3 \tag{63}$$

Now consider

$\Gamma_1(\nabla^2)\,W_{33}(\mathbf{X})$

$$= (\nabla^2 + \Lambda_1^2)\,(\nabla^2 + \Lambda_2^2)\,(\nabla^2 + \Lambda_3^2)\sum_{n=1}^{3} q_{2n}\,\xi_n(\mathbf{X})$$

$$= (\nabla^2 + \Lambda_2^2)\,(\nabla^2 + \Lambda_3^2)\sum_{n=1}^{3} q_{2n}\,[\delta(\mathbf{X}) + (\Lambda_1^2 - \Lambda_n^2)\,\xi_n(\mathbf{X})]$$

$$= (\nabla^2 + \Lambda_2^2)\,(\nabla^2 + \Lambda_3^2)\,[\,\delta(\mathbf{X})\sum_{n=1}^{3} q_{2n} + \sum_{n=2}^{3} q_{2n}\,(\Lambda_1^2 - \Lambda_n^2)\,\xi_n(\mathbf{X})]$$

Notes

$$= (\nabla^2 + \Lambda_2^2)\,(\nabla^2 + \Lambda_3^2)\,\sum_{n=2}^{3} q_{2n}\,(\Lambda_1^2 - \Lambda_n^2)\,\xi_n(\mathbf{X})$$

$$= (\nabla^2 + \Lambda_3^2)\sum_{n=2}^{3} q_{2n}\,(\Lambda_1^2 - \Lambda_n^2)\,[\delta(\mathbf{X}) + (\Lambda_2^2 - \Lambda_n^2)\,\xi_n(\mathbf{X})]$$

Notes

$$= (\nabla^2 + \Lambda_3^2)\sum_{n=3}^{3} q_{2n}\,(\Lambda_1^2 - \Lambda_n^2)\,(\Lambda_2^2 - \Lambda_n^2)\,\xi_n(\mathbf{X})$$

$$= (\nabla^2 + \Lambda_3^2)\,q_{23}\,(\Lambda_1^2 - \Lambda_3^2)\,(\Lambda_2^2 - \Lambda_3^2)\,\xi_3(\mathbf{X})$$

$$= (\nabla^2 + \Lambda_3^2)\,\xi_3(\mathbf{X})$$

$$= \delta(\mathbf{X})$$

Similarly, equation $(59)_1$ can be proved
Now, Define matrix

$$\mathbf{A}(\mathbf{X}) = \mathbf{H}(\mathbf{D}_x)\,\mathbf{W}(\mathbf{X}) \tag{64}$$

Using equations (56), (58) and (64), we obtain

$$\mathbf{B}(\mathbf{D}_x)\,\mathbf{A}(\mathbf{X}) = \mathbf{B}(\mathbf{D}_x)\,\mathbf{H}(\mathbf{D}_x)\,\mathbf{W}(\mathbf{X}) = \mathbf{\Theta}(\nabla^2)\,\mathbf{W}(\mathbf{X}) = \delta(\mathbf{X})\,\mathbf{I}(\mathbf{X}) \tag{65}$$

Therefore, $\mathbf{A}(\mathbf{X})$ is solution of equation (33).
Hence, we have proved the following Theorem:

Theorem: The matrix $\mathbf{A}(\mathbf{X})$ defined by the equation (64) is the fundamental solution of system of equations (27)-(29).

VI. Basic Properties of the Matrix A(X)

Property 1. Every column of the matrix $\mathbf{A}(\mathbf{X})$ is the solution of equations (27)-(29) for all points $\mathbf{X} \,\epsilon\, E^3$ except the origin.

Property 2. The matrix $\mathbf{A}(\mathbf{X})$ can be written as

$$\mathbf{A} = \left[A_{rs}\right]_{4\times 4}$$

$$\mathbf{A}_{pq}(\mathbf{X}) = \mathbf{H}_{pq}(\mathbf{D}_x)\,W_{11}(\mathbf{X}),$$

$$\mathbf{A}_{pm}(\mathbf{X}) = \mathbf{H}_{pm}(\mathbf{D}_x)\,W_{33}(\mathbf{X}),$$

$$p = 1, 2, 3, 4; \quad q = 1, 2; \quad m = 3, 4.$$

VII. Special Case

If we neglect nonlocal parameter ($\varepsilon = 0$) in equations (27)-(29), we obtain the system of equations of steady state oscillations for homogenous isotropic generalized thermoelastic solid with diffusion as:

$$[\alpha_1 \nabla^2 + \omega^2]\mathbf{u}^* + \alpha_2\, grad\, div\, \mathbf{u}^* - grad\, T^* - grad\, C^* = 0 \tag{66}$$

$$-\tau_t^{01}[\alpha_3\, div\, \mathbf{u}^* + \alpha_4\, C^*] + (\nabla^2 - \tau_t^{01})T^* = 0 \tag{67}$$

$$\alpha_5\, \nabla^2\, div\, \mathbf{u}^* + \alpha_6\, \nabla^2 T^* + [\tau_c^{01} - \alpha_7\, \nabla^2]\, C^* = 0 \tag{68}$$

The fundamental solution of above system of equations is similar as obtained by Kumar and Kansal [29].

VIII. Numerical Results and Discussion

For numerical calculations, values of relevant parameters for homogenous isotropic generalized thermoelastic solid with diffusion have been swiped from Sharma et al. [17] given in Table 1.

Phase velocity, attenuation coefficient, penetration depth and specific loss are computed numerically by using software MATLAB. Variation of Phase velocity, attenuation coefficient, penetration depth and specific loss with respect to angular frequency of P-wave, T-wave, MD-wave and SV-wave are shown graphically in four types of elastic solids:

Table 1: Numerical values of parametres

Notation	value	Notation	value
λ	$7.76 \times 10^{10}\, Nm^{-2}$	α_t	$1.78 \times 10^{-5}\, K^{-1}$
μ	$3.86 \times 10^{10}\, Nm^{-2}$	α_c	$2.65 \times 10^{-4}\, m^3 Kg^{-1}$
K	$386\, Jm^{-1}S^{-1}K^{-1}$	e_0	0.38
C_E	$383.1\, JKg^{-1}K^{-1}$	a	0.5431×10^{-9}
ρ	$8.954 \times 10^3\, Kgm^{-3}$	T_0	$293\, K$
a^*	$1.2 \times 10^4\, m^2 KS^2$	τ_0	$0.5\, S$
b^*	$0.9 \times 10^6\, Kgm^5 S^2$	τ	$0.5\, S$
D^*	$0.88 \times 10^{-8}\, KgSm^{-3}$		

1. Nonlocal thermoelastic solid with diffusion
2. Local thermoelastic solid with diffusion
3. Nonlocal thermoelastic solid without diffusion
4. Local thermoelastic solid without diffusion

In all graphs, solid line and dashed line represent the impact of local and nonlocal parameter on the variation of phase velocity, attenuation coefficient, penetration depth and specific loss in thermoelastic solid without diffusion respectively whereas dash-dotted and dotted line represent variation in local and nonlocal thermoelastic solid with diffusion respectively.

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Notes

Phase velocity: Figures 1-3 represent the impact of nonlocal and diffusion parameter on the phase velocities v_1, v_2 and v_3 of P-wave, T-wave, and MD-wave respectively with respect to angular frequency ω. From figures 1-2, it is clear that behaviour of phase velocity of P-wave and T-wave with respect to angular frequency is same with difference in magnitude values. The phase velocities v_1 and v_2 decreases monotonically, reaches to minimum value at $\omega = 10^0$ for all types of thermoelastic solids. The values of v_1 and v_2 for local solid are more than that of nonlocal solid. From physical point of view, the stresses produced in nonlocal medium is weak due to impact of nano-structured particles and change in stress causes the change in wave characteristics accordingly. Also phase velocity has lower value in elastic solid with diffusion in comparison to elastic solid without diffusion. Thermoelastic waves exhibit different dispersion characteristics in thermodiffusive solid which in turn influence the Phase velocity, attenuation coefficient, penetration depth and specific loss. Figure 3 shows that phase velocity v_3 of

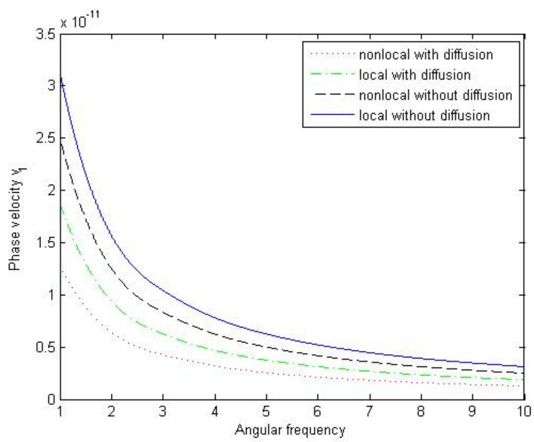

Figure 1: Variation of phase velocity of P-wave

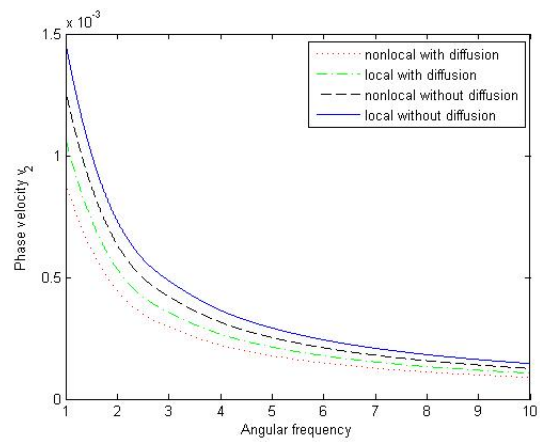

Figure 2: Variation of phase velocity of T-wave

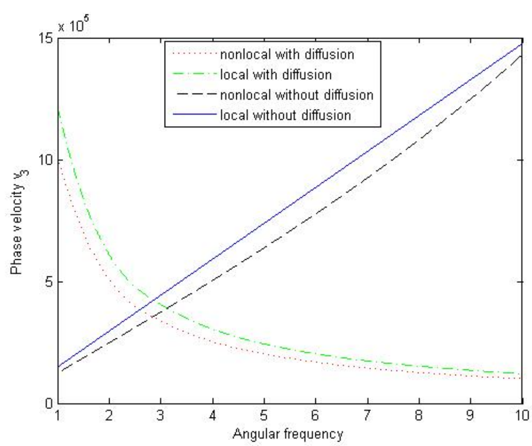

Figure 3: Variation of phase velocity of MD-wave

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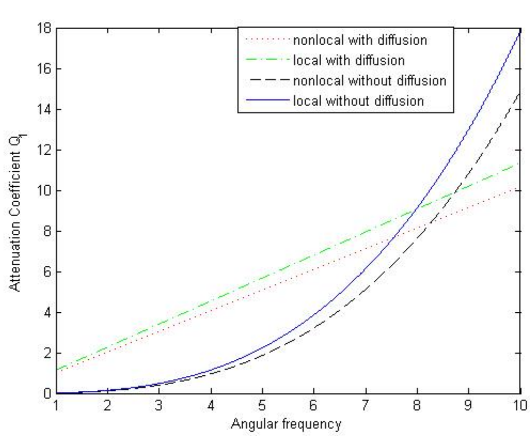

Figure 4: Variation of attenuation coefficient of P-wave

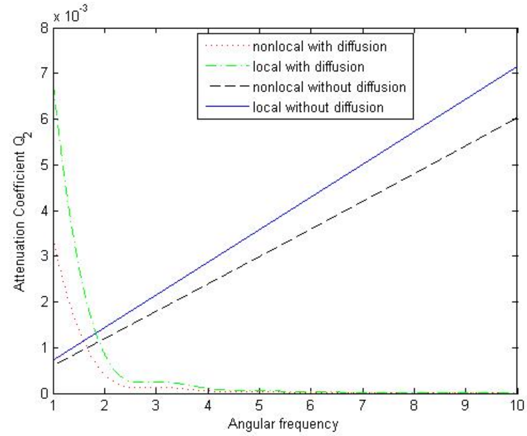

Figure 5: Variation of attenuation coefficient of T-wave

MD-wave decreases monotonically reaches to minimum value at $\omega = 10^0$ for thermoelastic solid with diffusion whereas it increases linearly, reaches to maximum value at $\omega = 10^0$ for elastic solid without diffusion.

Attenuation Coefficients: Figures 4-6 represent the impact of nonlocal and diffusion parameter on the attenuation coefficients Q_1, Q_2 and Q_3 of P-wave, T-wave, and MD-wave respectively with respect to angular frequency ω. It has been observed from the figure 4 that attenuation coefficient Q_1 of P-wave increases parabolically for $1^0 \leq \omega \leq 10^0$ in elastic solid without diffusion. For thermoelastic solid with diffusion, Q_1 increases linearly in local as well as nonlocal solid. Figure 5 shows that attenuation coefficient Q_2 of T-wave increases linearly with increase in ω in thermoelastic solid without diffusion whereas in thermodiffusive solid, it decrease sharply for $1^0 \leq \omega \leq 2.5^0$. From figure 6, it is clear that attenuation coefficient Q_3 of MD-wave increases linearly in all types of solids with difference in magnitude. The value of Q_3 is lower in diffusive solid in comparison to solid without diffusion. Also, figures 4-6 show that attenuation coefficients has smaller value in nonlocal solid in comparison to local elastic solid due to nonlocal parameter.

Penetration Depth: Figures 7-9 represent the effect of diffusion and nonlocal parameter on the penetration depth D_1, D_2 and D_3 of P-wave, T-wave,

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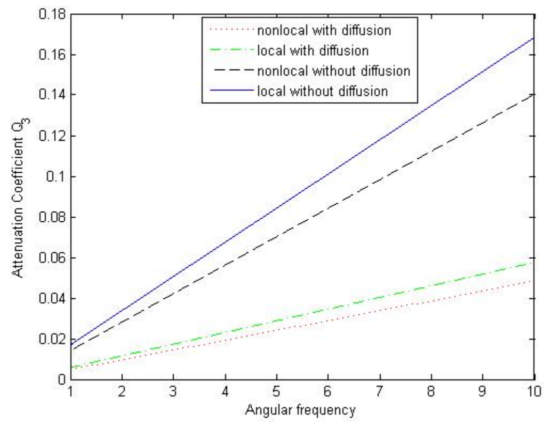

Figure 6: Variation of attenuation coefcient of MD-wave

and MD-wave respectively with respect to angular frequency ω. Figure 7 depicts the variation of penetration depth D_1 of P-wave with respect to angular frequency ω. In thermoelastic solid without diffusion, the penetration depth D_1 decreases sharply for $1^0 \leq \omega \leq 2.5^0$ and then slowly for $\omega \geq 2.5^0$. The value of D_1 decreases monotonically, reaches to minimum value at $\omega = 10^0$ in diffusive elastic solid. It has been observed from the figure 8 that penetration depth D_2 of T-wave increases in diffusive solid and decreases in solid without diffusion with increase in angular frequency ω having greater value in local solid in comparison to nonlocal elastic solid. From figure 9, it is clear that penetration depth D_3 of MD-wave decreases monotonically in all types of solids with difference in magnitude. Also, the value of D_3 is lower in diffusive solid in comparison to solid without diffusion.

Specific Loss: Figures 10-12 show the effect of diffusion and nonlocal parameter on the specific loss L_1, L_2 and L_3 of P-wave, T-wave, and MD-wave respectively with respect to angular frequency ω. From figures 10 and 12, it has been observed that behaviour of specific loss L_1 of P-wave is similar to behaviour of specific loss L_3 of MD-wave with difference in magnitude. Specific loss L_1 and L_3 decreases monotonically, reaches to minimum value at $\omega = 10^0$ in thermoelastic solid without diffusion whereas it increases linearly, reaches to maximum value at $\omega = 10^0$ in thermodiffusive solid. Figure 11 represents the variation of specific loss L_2 of T-wave with respect to an-

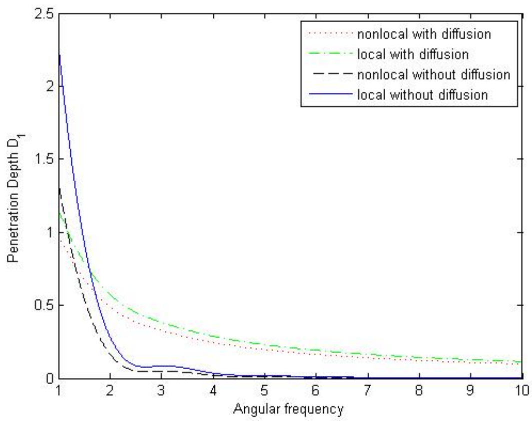

Figure 7: Variation of penetration depth of P-wave

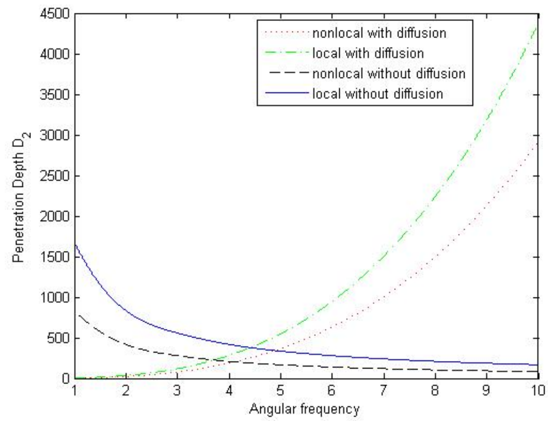

Figure 8: Variation of penetration depth of T-wave

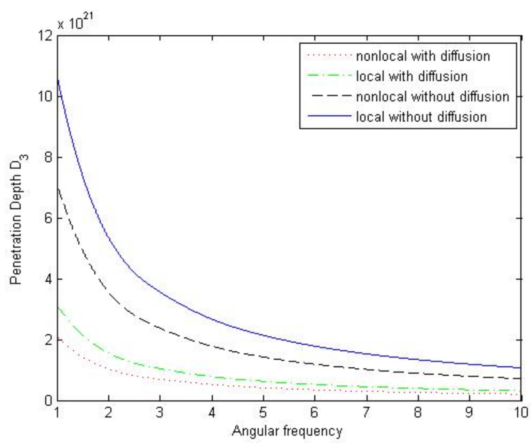

Figure 9: Variation of penetration depth of MD-wave

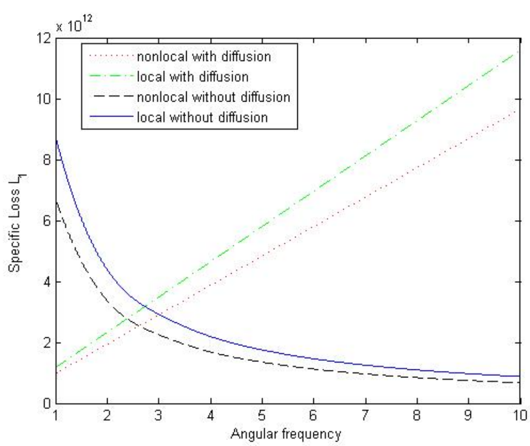

Figure 10: Variation of specific loss of P-wave

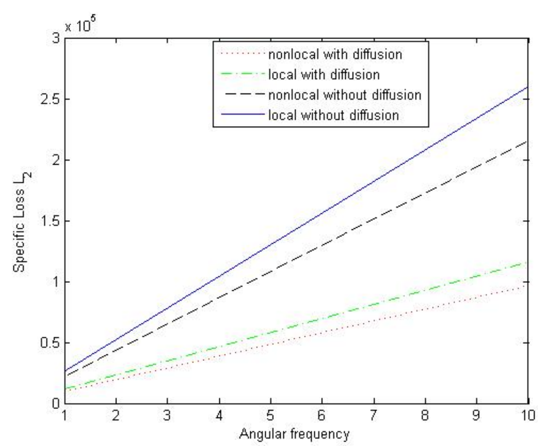

Figure 11: Variation of specific loss of T-wave

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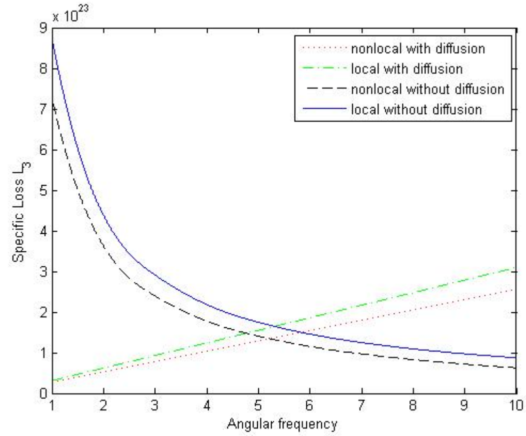

Figure 12: Variation of specific loss of MD-wave

gular frequency ω. In all types of solids, L_2 increases linearly, having lower value in diffusive solid as comparison to solid without diffusion. Also from figures 10-12, it is clear that specific loss has greater value in local solid as comparison to nonlocal elastic solid.

IX. Conclusion

We have examined the effects of nonlocal parameter on the propagation of plane wave in nonlocal homogenous isotropic thermoelastic diffusion.
The major consequences of current problems are:
(1)There exist three coupled waves namely P-wave, T-wave, MD-wave and one transverse wave(SV) propagating with different phase velocities. Furthermore phase velocity, attenuation coefficients, penetration depth and specific loss with respect to angular frequency are studied graphically.
(2) It has been found that characteristics of all the waves are affected by diffusion and nonlocal parameter of the medium.
(3) The fundamental solution of system of differential equations for steady oscillations has been constructed.
(4) The analysis of fundamental solution $\mathbf{M}(\mathbf{X})$ of the system of equations (27)-(29) are helpful to investigate three dimensional problems of nonlocal homogenous isotropic elastic solid with diffusion.
(5) The graphical analysis of present work is very helpful in order to investigate the various fields of aerospace, electronics and geophysics like volcanology, telecommunication etc.

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Solve the 3x+1 Problem by the Multiplication and Division of Binary Numbers

By Jishe Feng
Longdong University

Abstract- The 3x + 1 problem is the following: Suppose we start with a positive integer, and if it is odd then multiply it by 3 and add 1, and if it is even, divide it by 2. Then repeat this process as long as you can. Will you eventually reach the integer 1, no matter what you started with? Collatz conjecture (or 3n + 1 problem) has been explored for about 85 years. In this paper, we convert an integer number from decimal to binary and convert the Collatz function to a binary function, which involves the multiplication and division of two binary numbers. Finally, by iterating the Collatz function, we eventually reach the integer number 1.

Keywords: *3x + 1 problem, binary number, collatz conjecture, sharkovskii ordering, lattice path.*

GJSFR-F Classification: *JEL Code: C2*

SOLVETHE3X1PROBLEMBYTHEMULTIPLICATIONANDDIVISIONOFBINARYNUMBERS

Solve the 3x+1 Problem by the Multiplication and Division of Binary Numbers

Jishe Feng

Abstract- The $3x$+ 1 problem is the following: Suppose we start with a positive integer, and if it is odd then multiply it by 3 and add 1, and if it is even, divide it by 2. Then repeat this process as long as you can. Will you eventually reach the integer 1, no matter what you started with? Collatz conjecture (or $3n$ + 1 problem) has been explored for about 85 years. In this paper, we convert an integer number from decimal to binary and convert the Collatz function to a binary function, which involves the multiplication and division of two binary numbers. Finally, by iterating the Collatz function, we eventually reach the integer number 1.

Keywords: 3x + 1 problem, binary number, collatz conjecture, sharkovskii ordering, lattice path.

I. Introduction

The $3x+1$ problem, also known as the Collatz conjecture, $3x+1$ mapping, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, or Syracuse problem [1], is one of the unsolved problems in mathematics. Paul Erdos (1913-1996) commented on the intractability of the $3x+1$ problem [2], stating that "Mathematics is not ready for those problems yet".

The $2x+1$ problem states that, for any positive integer x, if x is even, divide it by 2; if x is odd, multiply it by 3 and add 1. Repeating this process continuously leads to the conjecture that no matter which number is initially chosen, the result will always reach 1 eventually.

We use the notations as in [4,7], and describe a Collatz function as follows:

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \tag{1}$$

Let N denote the set of positive integers. For $n \in N$, and $k = 0, 1, 2, 3, \cdots$, $T^0(n)$ and $T^{k+1}(n)$ denote n and $T(T^k(n))$, respectively. Concerning the behavior of the iteration of the Collatz function, for any integer n, there must exist an integer r so that

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Ref

1. Jeffrey C. Lagarias. The 3x + 1 Problem and Its Generalizations. American Mathematical Monthly, Vol. 92, No. 1, pp 3-23.(1985).

$$T^r(n) = 1. \tag{2}$$

a) The modified Sarkovskii ordering and integer lattice

We convert the last row of numbers into the first column to get a modified Sarkovskii ordered integer lattice[6] as the following,

$$\begin{array}{lllllllllll}
1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & 19, & \cdots \\
2, & 2\cdot 3, & 2\cdot 5, & 2\cdot 7, & 2\cdot 9, & 2\cdot 11, & 2\cdot 13, & 2\cdot 15, & 2\cdot 17, & 2\cdot 19, & \cdots \\
2^2, & 2^2\cdot 3, & 2^2\cdot 5, & 2^2\cdot 7, & 2^2\cdot 9, & 2^2\cdot 11, & 2^2\cdot 13, & 2^2\cdot 15, & 2^2\cdot 17, & 2^2\cdot 19, & \cdots \\
2^3, & 2^3\cdot 3, & 2^3\cdot 5, & 2^3\cdot 7, & 2^3\cdot 9, & 2^3\cdot 11, & 2^3\cdot 13, & 2^3\cdot 15, & 2^3\cdot 17, & 2^3\cdot 19, & \cdots \\
2^4, & 2^4\cdot 3, & 2^4\cdot 5, & 2^4\cdot 7, & 2^4\cdot 9, & 2^4\cdot 11, & 2^4\cdot 13, & 2^4\cdot 15, & 2^4\cdot 17, & 2^4\cdot 19, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

In the first row, they are odd numbers from left to right, that are $1, 3, 5, 7, 9, 11, 13, \cdots$. From the second row, each number is two times the number in its previous row, and so on.

b) The algebraic formula and Collatz graph

If we draw a line segment with an arrow between two digits in the lattice of integers in the modified Sarkovskii ordering, one being the original value x and the other being its value of the Collatz function $T(x)$, and then connect $T(x)$ to $T^2(x)$, and so on $T^2(x)$ to $T^3(x), \cdots$, we get a graph which can be called a *Collatz graph*.

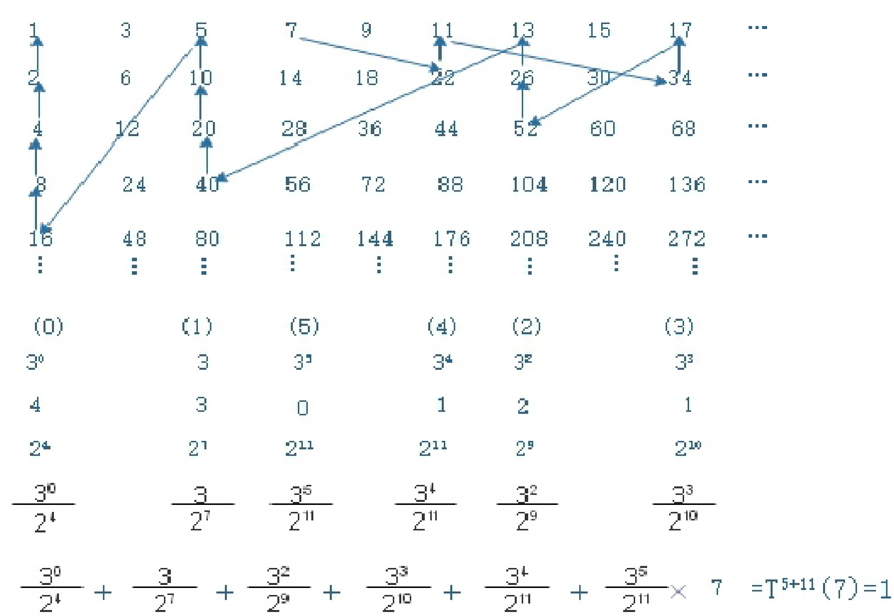

Fig. 1: The Collatz graph of $T^{16}(7) = T(5, 11, 7) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

Ref

6. Jishe FENG, Xiaomeng WANG, Xiaolu GAO, Zhuo PAN. The research and progress of the enumeration of lattice paths. Frontiers of Mathematics in China, 2022, 17(5): 747-766.

Notes

For different integers k and l, if there is a common vertex in their Collatz graphs, their graphs will overlap from that point onwards until they reach the minimum value of 1. Using the Collatz function $T(x)$, we can obtain an algebraic formula of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r} \cdot x$, where r is the number of vertical segments and m is the number of oblique segments in the Collatz graph,

$$T^{m+r}(n) = T(m,r,n) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \cdots + \frac{3^m}{2^r} \cdot x = 1. \tag{3}$$

For example, $n = 7$, $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{16}(7) = T(5,11,7) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{11}} \cdot 7 = 1,$$

and the Collatz graph is Fig. 1.

And $n = 36$, $36 \to 18 \to 9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{21}(36) = T(6,15,36) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{15}} \cdot 36 = 1$$

and the Collatz graph is Fig. 2.

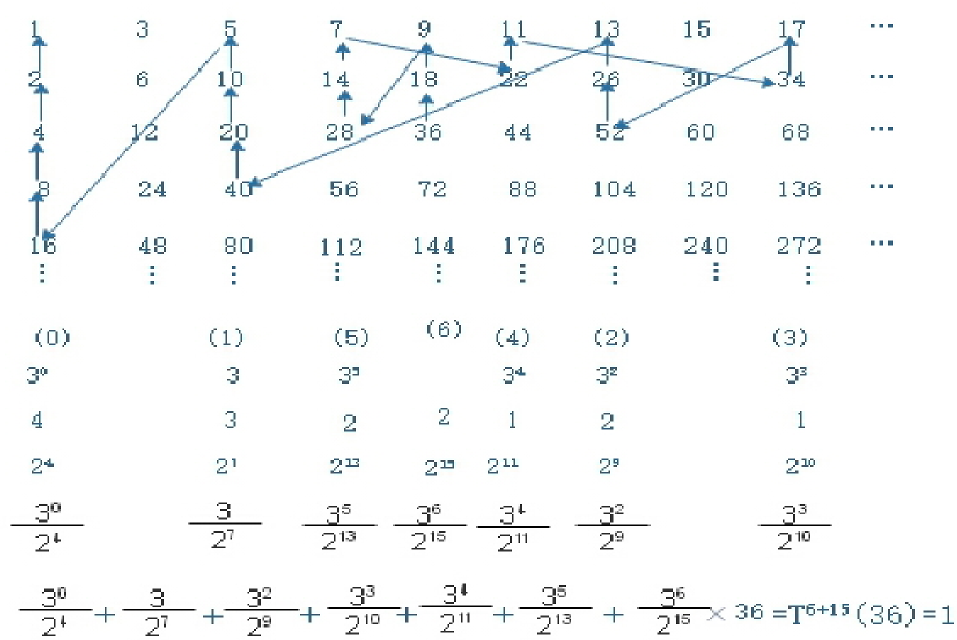

Fig. 2: The Collatz graph of $T^{21}(36) = T(6,15,36) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

We observe that there is a property present,

Proposition 1 *For positive integers i, j, k, l, and $l_k, l_{k-1}, \cdots, l_1$, if $i > j$, then there is a recurrence relation*

$$T^i(n) = \frac{3^k}{2^l}T^j(n) + \frac{3^{k-1}}{2^{l_k}} + \cdots + \frac{3^2}{2^{l_3}} + \frac{3}{2^{l_2}} + \frac{1}{2^{l_1}}$$

where $k + l = i - j$, and $l \geq l_k \geq l_{k-1} \geq \cdots \geq l_1$.

For example, there are

$$T^3(97) = \frac{3}{2^2} \cdot 97 + \frac{1}{2^2} = 73$$

$$T^{18}(97) = \frac{3^7}{2^{11}} \cdot 97 + \frac{3^6}{2^{11}} + \frac{3^5}{2^9} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^2} + \frac{1}{2} = 107$$

We can get the recurrence formula about the Collatz function,

$$\begin{aligned} T^{26}(97) &= \frac{3^3}{2^5} \cdot T^{18}(97) + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} \\ &= \frac{3^{10}}{2^{16}} \cdot 97 + \frac{3^9}{2^{16}} + \frac{3^8}{2^{14}} + \frac{3^7}{2^{12}} + \frac{3^6}{2^{11}} + \frac{3^5}{2^{10}} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} \\ &= 91, \end{aligned}$$

II. Numerical Example

Using the above Collatz graphs of the integer lattice of the modified Sarkovskii ordering, we give the following algebraic formulas,

$$T^{19}(9) = T(6, 13, 9) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{13}} \cdot 9 = 1, \tag{4}$$

$$T^{15}(23) = T(4, 11, 23) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{11}} \cdot 23 = 1, \tag{5}$$

$$T^{17}(15) = T(5, 12, 15) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{12}} + \frac{3^5}{2^{12}} \cdot 15 = 1, \tag{6}$$

$$T^{12}(17) = T(3, 9, 17) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^9} \cdot 17 = 1, \tag{7}$$

$$T^{19}(61) = T(5, 14, 61) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{14}} + \frac{3^5}{2^{14}} \cdot 61 = 1. \tag{8}$$

$$T^{16}(397) = T(5, 11, 397) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{17}} + \frac{3^6}{2^{20}} + \frac{3^7}{2^{20}} \cdot 397 = 1, \tag{9}$$

Notes

Notes

Example 2 For the formula

$$T(6,14,18)=\frac{1}{2^4}+\frac{3}{2^7}+\frac{3^2}{2^9}+\frac{3^3}{2^{10}}+\frac{3^4}{2^{11}}+\frac{3^5}{2^{13}}+\frac{3^6}{2^{14}}\cdot 18=1,$$

we rewrite it as an integer equation,

$$3^6\cdot 18+3^5\cdot 2+3^4\cdot 2^3+3^3\cdot 2^4+3^2\cdot 2^5+3\cdot 2^7+2^{10}=2^{14}.$$

Proof. To calculate the power of 3 and the value of 18 using powers of 2,

$$\begin{aligned}3&=2+1\\3^2&=2^3+1\\3^3&=2^4+2^3+2+1\\3^4&=2^6+2^4+1\\3^5&=2^7+2^6+2^5+2^4+2+1\\3^6&=2^9+2^7+2^6+2^4+2^3+1\\18&=2^4+2\end{aligned}$$

substituting these expressions into the left-hand side of the above equation, one obtains,

$$\begin{aligned}&3^6\cdot 18+3^5\cdot 2+3^4\cdot 2^3+3^3\cdot 2^4+3^2\cdot 2^5+3\cdot 2^7+2^{10}\\&=(2^9+2^7+2^6+2^4+2^3+1)\cdot(2^4+2)+(2^7+2^6+2^5+2^4+2+1)\cdot 2\\&+(2^6+2^4+1)\cdot 2^3+(2^4+2^3+2+1)\cdot 2^4+(2^3+1)\cdot 2^5+(2+1)\cdot 2^7+2^{10},\end{aligned}$$

and get the value 2^{14} which is equal to the right value of the equation.

III. Convert the Integer Number from Decimal to Binary

Be inspired by the above, we use binary to rewrite the Collatz function (1) as the following formulas (2) and (3). We denote a binary number, which is a string of 0s and 1s, as $n=(1\times\cdots\times)_2$, where $\times$ is either 1 or 0, e.g. $3=(11)_2$,

$$T(n)=T((1\times\cdots\times)_2)=\begin{cases}(11)_2\cdot(1\times\cdots\times 1)_2+1, & \text{if } n \text{ is odd number,}\\ \frac{(1\times\cdots\times 10\cdots 00)_2}{(10)_2}, & \text{if } n \text{ is even number.}\end{cases} \tag{10}$$

The result is

$$T(n)=T((1\times\cdots\times)_2)=\begin{cases}(1\times\times\times 10\cdots 0)_2, & \text{if } n \text{ is odd number,}\\ (1\times\cdots\times 10\cdots 0)_2, & \text{if } n \text{ is even number.}\end{cases} \tag{11}$$

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Namely, when n is an odd number, we multiply it with $(11)_2$ and add 1 to the end of the binary number. For example, $T(97) = T(1100001)$ in Fig. 3. When n is an even number, the division is equal to deleting the zero at the end in the binary number. We give the iteration of the Collatz function for $1, 5, 7, 9$, 97 in binary as the following five tables.

We convert the modified Sarkovskii ordered integer lattice[6] from decimal to binary as the follows,

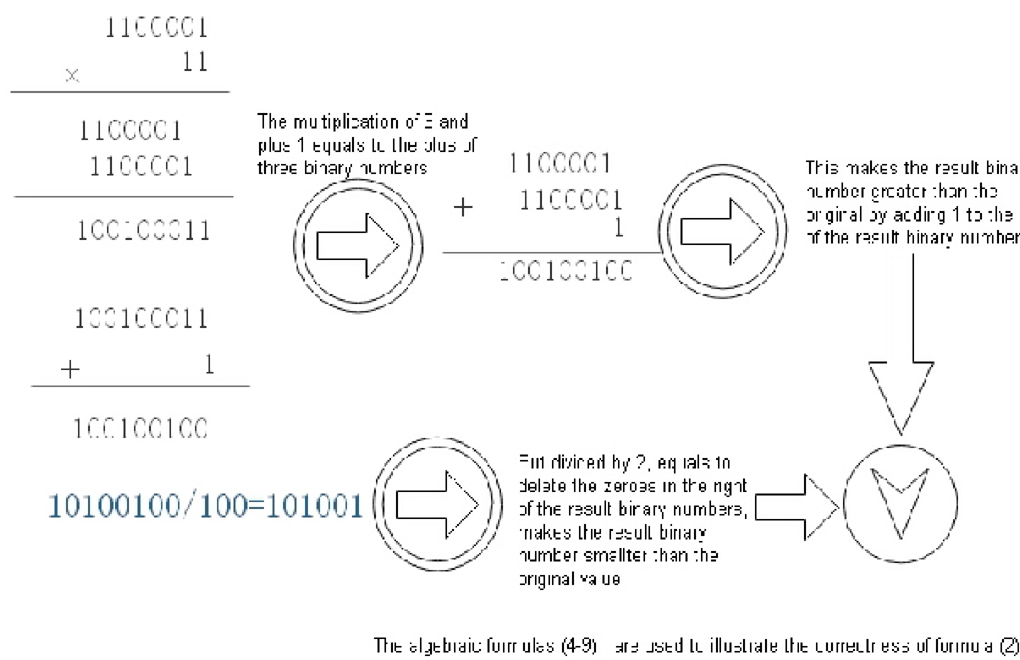

Fig. 3: For the Collatz function $T(97)$ in binary, the first step is the multiplication in left, the second step is division in the right bottom.

1,	11,	101,	111,	1001,	1011,	1101,	1111,	10001,	$\cdots$
10,	110,	1010,	110,	10010,	10110,	11010,	11110,	100010,	$\cdots$
100,	1100,	10100,	1100,	100100,	101100,	110100,	111100,	1000100,	$\cdots$
1000,	11000,	101000,	11000,	1001000,	1011000,	1101000,	1111000,	10001000,	$\cdots$
10000,	110000,	1010000,	110000,	10010000,	10110000,	11010000,	11110000,	100010000,	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Example 3 For positive integer 1, we manipulate the iteration of the Collatz function in both decimal and binary numbers,

ith	*0*	*1*	*2*	*3*
decimal	*1*	*4*	*2*	*1*
binary	*1*	*100*	*10*	*1*

Notes

Example 4 *For positive integer* $5 = (101)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers,*

ith	*0*	*1*	*2*	*3*	*4*	*5*
decimal	*5*	*16*	*8*	*4*	*2*	*1*
binary	*101*	*10000*	*1000*	*100*	*10*	*1*

Example 5 *For* $7 = (111)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*7*	*8*	*11*	*12*	*16*
decimal	*7*	*22*	*11*	*34*	*17*	*52*	*13*	*40*	*5*	*16*	*1*
binary	*111*	*10110*	*1011*	*100010*	*10001*	*110100*	*1101*	*101000*	*101*	*10000*	*1*

Example 6 *For* $9 = (1001)_2$, *we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*14*	*15*	*19*
decimal	*9*	*28*	*14*	*7*	*22*	*11*	*34*	*17*	*5*	*16*	*1*
binary	*1001*	*11100*	*1110*	*111*	*10110*	*1011*	*100010*	*10001*	*101*	*10000*	*1*

Example 7 *For* $97 = (1100001)_2$, *we manipulate the iteration of the Collatz function in binary as the following table,*

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Notes

97	**1100001**	206	11001110	**425**	**110101001**	866	1101100010
292	100100100	**103**	**1100111**	1276	10011111100	**433**	**110110001**
146	10010010	310	100110110	638	1001111110	1300	10100010100
73	**1001001**	**155**	**10011011**	**319**	**100111111**	650	1010001010
220	11011100	466	111010010	958	1110111110	**325**	**101000101**
110	1101110	**233**	**11101001**	**479**	**111011111**	976	1111010000
55	**110111**	700	1010111100	1438	10110011110	488	111101000
166	10100110	350	101011110	719	**1011001111**	244	11110100
83	**1010011**	**175**	**10101111**	2158	100001101110	122	1111010
250	11111010	526	1000001110	**1079**	**10000110111**	**61**	**111101**
125	**1111101**	**263**	**100000111**	3238	110010100110	184	10111000
376	101111000	790	1100010110	**1619**	**11001010011**	92	1011100
188	10111100	**395**	**110001011**	4858	1001011111010	46	101110
94	1011110	1186	10010100010	**2429**	**100101111101**	**23**	**10111**
47	**101111**	**593**	**1001010001**	7288	1110001111000	70	1000110
142	10001110	1780	11011110100	3644	111000111100	**35**	**100011**
71	**1000111**	890	1101111010	1822	11100011110	106	1101010
214	11010110	**445**	**110111101**	**911**	**1110001111**	**53**	**110101**
107	**1101011**	1336	10100111000	2734	101010101110	160	10100000
322	101000010	668	1010011100	**1367**	**10101010111**	80	1010000
161	**10100001**	334	101001110	4102	1000000000110	40	101000
484	111100100	**167**	**10100111**	2051	**100000000011**	20	10100
242	11110010	502	111110110	6154	1100000001010	10	1010
121	**1111001**	**251**	**11111011**	**3077**	**110000000101**	**5**	**101**
364	101101100	754	1011110010	9232	10010000010000	16	10000
182	10110110	**377**	**101111001**	4616	1001000001000	8	1000
91	**1011011**	1132	10001101100	2308	100100000100	4	100
274	100010010	566	1000110110	1154	10010000010	2	10
137	**10001001**	**283**	**100011011**	**577**	**1001000001**	1	1
412	110011100	850	1101010010	1732	11011000100		

Notes

Corollary 8 The Collatz function makes an odd integer number in binary bigger by adding 1 or 2 bits to the left of the sequence of 1s and 0s, and an even integer number in binary smaller by deleting all zeros at the end of the sequence of 1s and 0s. Thus, although in some cases the value of The Collatz function T(x) may be bigger than x in decimal, in general, the iteration of the Collatz function will make an integer number smaller and smaller, eventually reaching the smallest positive integer number 1.

We can rewrite the Collatz conjecture in binary, which makes it an easier problem to solve, thus allowing us to completely solve the Collatz conjecture.

Fact 9 For any positive integer, under the Collatz function, the sequence of integer numbers in binary will eventually reach 1.

Proof. *For an odd binary integer, we multiply it by $(11)_2$ and add 1 in the last bit, the result number must be an even number in binary which at least one zero at the end. We delete all these zeros, namely it is the division. This is the above corollary 8. Thus, we repeat this process as long as we can, because the bits of the sequence in binary of a positive integer number is finite. Eventually, we must in finitely steps reach the smallest positive integer number 1.*

Remark 10 We can say that the 3x+1 problem is a converse proposition of "period three implies chaos" [4], and it is also an example of any one positive integer number having a period of 3 in the Collatz function.

IV. Conclusions

We rewrite the Collatz function in binary, which makes the $3x + 1$ problem easier. The multiplications of $(11)_2$ and divisions of $(10)_2$ make the positive integer number smaller and smaller with the iterations of the Collatz function. In some cases, the value of the Collatz function $T(x)$ may be bigger than x, thus allowing us to completely solve the Collatz conjecture.

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Notes

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An Extension of 'In-Radius Property' of Pythagorean Triangles

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Introduction- It is a well-known fact that the in- radius of a Pythagorean triangle (A right-triangle whose sides form a Pythagorean triple) is always an integer [1]. The purpose of this note is to extent this result in the following sense.

If in any Pythagorean triangle a string of a finite number (say, k) of equal circles, inside the triangle, are so taken that

i. each of the k circles touches a given side (other than the hypotenuse)
ii. each of the (k-2) non-extreme circles also touch the two neighbouring circles.
iii. the extreme two circles touch the nearest other side also.

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ANEXTENSIONOFINRADIUSPROPERTYOFPYTHAGOREANTRIANGLES

Strictly as per the compliance and regulations of:

Ref

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An Extension of 'In-Radius Property' of Pythagorean Triangles

K. B Subramaniam [α] & Aji Thomas [σ]

Introduction

It is a well-known fact that the in- radius of a Pythagorean triangle (A right-triangle whose sides form a Pythagorean triple) is always an integer [1]. The purpose of this note is to extent this result in the following sense.

If in any Pythagorean triangle a string of a finite number (say, k) of equal circles, inside the triangle, are so taken that

i. each of the k circles touches a given side (other than the hypotenuse)
ii. each of the (k-2) non-extreme circles also touch the two neighbouring circles.
iii. the extreme two circles touch the nearest other side also.

We claim that these circles will have a rational radius for all k. We also work out the value of r explicitly.

Before proceeding for the proof, we need to use the following facts

a. A special category of **Pythagorean triples** is that of primitive Pythagorean triples which are merely Pythagorean triples having no common factors.

b. Every Pythagorean triple is of the form $2ab$, a^2-b^2, a^2+b^2, where a and b are positive coprime integers and $a>b$ [2].

Proof:

Let ΔABC be right angled at B.

Without loss of generality, we can assume that the sides of ΔABC form a primitive pythagorean triple. Let $AB=2ab$, $BC=a^2-b^2$ and $AC=a^2+b^2$, where a and b are coprime with $a>b$.

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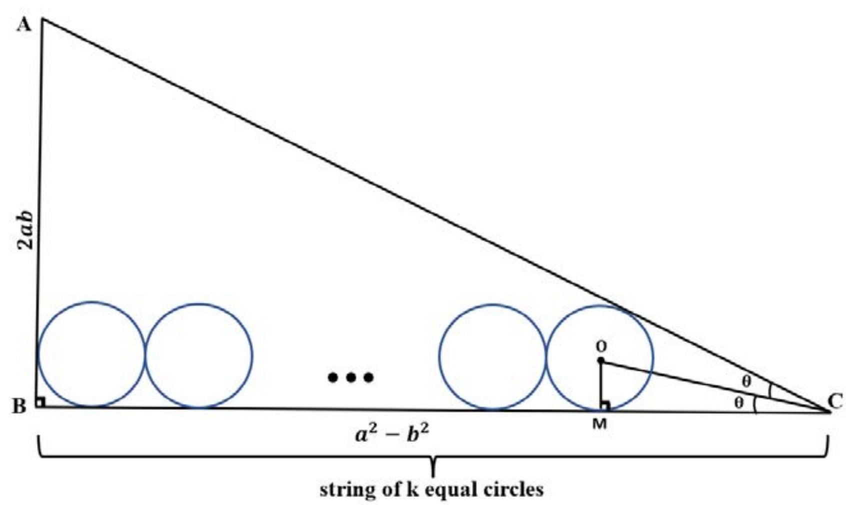
We need to consider two cases depending on whether the strings of circles are taken on BC or on AB. Accordingly, we have to prove our assertion considering both the cases.

Case 1: String of circles lying along BC



It may be noted here that in this case $(2k-1)r < (a^2 - b^2)$ (1)

Let O be the centre of the circle (nearest to AC) and OM $\perp$ BC.

Let r be the radius of each of these circles.

Clearly, OC bisects $\angle$ACB. Let $\angle$ACB = 2θ so that $\angle$OCB = θ.

We have,

$$\tan\theta = \frac{\text{OM}}{\text{MC}} = \frac{r}{a^2 - b^2 - (2k-1)r}$$

Also,

$$\tan 2\theta = \frac{\text{AB}}{\text{BC}} = \frac{2ab}{a^2 - b^2}$$

$$\Rightarrow \frac{2ab}{a^2 - b^2} = \frac{\dfrac{2r}{(a^2 - b^2) - (2k-1)r}}{1 - \dfrac{r^2}{[(a^2 - b^2) - (2k-1)r]^2}}$$ (By the duplication formula for tangent function)

$$\Rightarrow \{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}r^2 - (a^2 - b^2)(4abk + a^2 - b^2 - 2ab)r + ab(a^2 - b^2)^2 = 0$$

Notes

Notes

i.e, $Ar^2 - Br + C = 0$

where,

$$A = 4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)$$

$$B = (a^2 - b^2)(4abk + a^2 - b^2 - 2ab)$$

$$C = ab(a^2 - b^2)^2$$

we have,

$$B^2 - 4AC = (a^2 - b^2)^2(4abk + a^2 - b^2 - 2ab)^2 - 4ab(a^2 - b^2)^2\{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}$$

$$= (a^4 - b^4)^2$$

which clearly asserts that r must be rational

now,

$$r = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{(a^2 - b^2)(4abk + a^2 - b^2 - 2ab) \pm (a^4 - b^4)}{2\{4abk^2 + 2(a^2 - b^2 - 2ab)k - (a^2 - b^2)\}}$$

$$= \frac{(a^2 - b^2)(4abk + a^2 - b^2 - 2ab) \pm (a^4 - b^4)}{2(2ak - b - a)(2bk - b + a)}$$

Here we claim that we have to discard the plus sign.

Because if we take plus sign then r becomes

$$\frac{(a^2 - b^2)\{(4abk + a^2 - b^2 - 2ab) + (a^2 + b^2)\}}{2(2ak - b - a)(2bk - b + a)}$$

$$= \frac{a(a^2 - b^2)(2bk - b + a)}{(2ak - b - a)(2bk - b + a)}$$

$$= \frac{a(a^2 - b^2)}{(2ak - b - a)}$$

By the condition (1), we have

$$\frac{a^2 - b^2}{r} > 2k - 1$$

$$\Rightarrow \frac{2ak-b-a}{a} > 2k-1$$

$$\Rightarrow -b > 0$$

Which is absurd. This justifies our claim
If we take minus sign r becomes

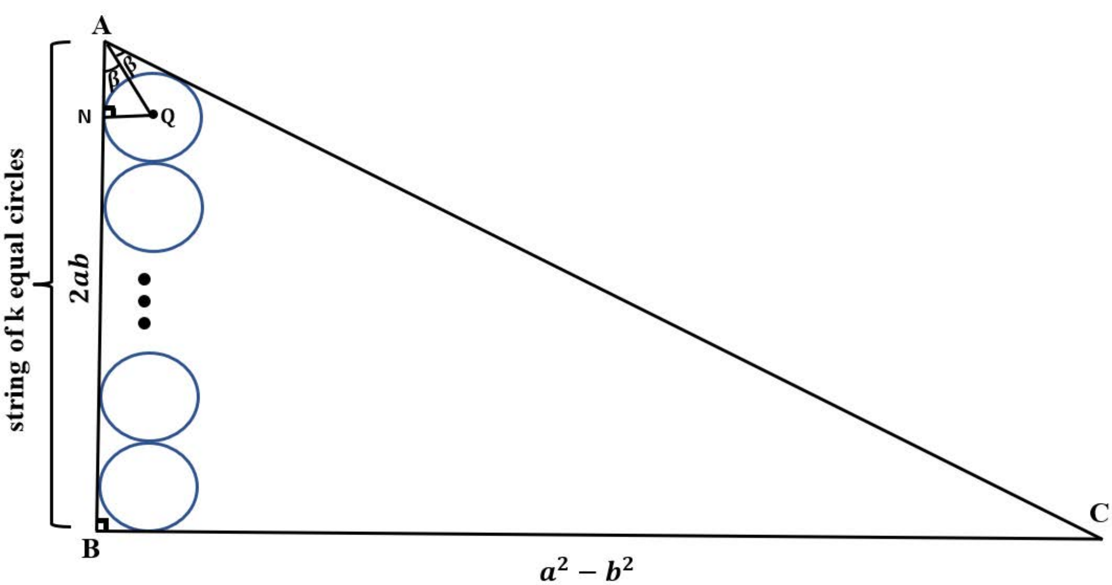
$$\frac{(a^2-b^2)\{(4abk+a^2-b^2-2ab)-(a^2+b^2)\}}{2(2ak-b-a)(2bk-b+a)}$$

$$=\frac{b(a^2-b^2)(2ak-b-a)}{(2bk-b+a)(2ak-b-a)}$$

$$=\frac{b(a^2-b^2)}{(2k-1)b+a}$$

Clearly, this option satisfies the condition (1)

Case 2: String of circles lying along BC



It may be noted here that **(2k - 1)r < 2ab** (2)

Let Q be the centre of the circle (nearest to AC) and QN ⊥ AB.

Let r be the radius of each of these circles.

Notes

Clearly, QA bisects $\angle$BAC. Let $\angle$BAC = 2β so that $\angle$QAB = β.

We have,

$$\tan\beta = \frac{QN}{NA} = \frac{r}{2ab-(2k-1)r}$$

Also,

$$\tan 2\beta = \frac{BC}{AB} = \frac{a^2-b^2}{2ab}$$

$$\Rightarrow \frac{a^2-b^2}{2ab} = \frac{\dfrac{2r}{2ab-(2k-1)r}}{1-\dfrac{r^2}{\{2ab-(2k-1)r\}^2}}$$ (By the duplication formula for tangent function)

$$\Rightarrow \{(a^2-b^2)(k^2-k)+ab(2k-1)\}r^2-(ab)\{(a^2-b^2)(2k-1)+2(ab)\}r+(a^2-b^2)(ab)^2=0$$

i.e, $Ar^2-Br+C=0$, where

$$A=(a^2-b^2)(k^2-k)+ab(2k-1)$$

$$B=(ab)\{(a^2-b^2)(2k-1)+2(ab)\}$$

$$C=(a^2-b^2)(ab)^2$$

$$B^2-4AC=(ab)^2\{(a^2-b^2)(2k-1)+2(ab)\}^2-4\{(a^2-b^2)(k^2-k)+ab(2k-1)\}(a^2-b^2)(ab)^2$$

$$=(ab)^2(a^2+b^2)^2$$

which clearly asserts that r must be rational

now,

$$r=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

$$=\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\}\pm ab(a^2+b^2)}{2\{(a^2-b^2)(k^2-k)+ab(2k-1)\}}$$

Notes

$$= \frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} \pm ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

Here we claim that we have to discard the plus sign.

Because if we take plus sign then r becomes

$$\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} + ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{2(ab)(a+b)(ka-kb+b)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{(ab)(a+b)}{(ka+kb-a)}$$

By the condition (2), we have

$$\frac{(2k-1)r}{2} < ab$$

$$\Rightarrow \frac{r(ka+kb-a)}{(a+b)} > \frac{(2k-1)r}{2}$$

$$\Rightarrow k < \frac{b}{a+b}$$

Which is absurd (as k is an integer). This justifies our claim

If we take minus sign r becomes

$$\frac{(ab)\{(a^2-b^2)(2k-1)+2(ab)\} - ab(a^2+b^2)}{2(ka-kb+b)(ka+kb-a)}$$

$$= \frac{ab(a-b)(ka+kb-a)}{(ka-kb+b)(ka+kb-a)}$$

$$= \frac{ab(a-b)}{k(a-b)+b}$$

Clearly, this option satisfies the condition (2)

Notes

Remark

Interestingly, the values of r in in both the cases are free from k in the numerator. ◻

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Quasi-Cyclic Codes Over Finite Chain $\boldsymbol{m}\Theta$ Pseudo Field F($\boldsymbol{p}^{\boldsymbol{k}}$Z, 1)

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Abstract- The $m\Theta$ sets present an enrichment from the logical viewpoint compared with the classical sets. The subset of the $m\Theta$ invariants of a $m\Theta$ set is a classical set, which leads to the canonical construction of the structures of modal Θ-valent pseudo field. In this note the purpose is to define on a finite chain $m\Theta$ pseudo field, $\mathbb{F}(p^k\mathbb{Z},1)$, the structures of Quasi- Cyclic codes of length r.

Keywords: $\boldsymbol{m}\Theta$ *set,* $\boldsymbol{m}\Theta$ *pseudo field, chain* $\boldsymbol{m}\Theta$ *pseudo field, quasi-cyclic* $\boldsymbol{m}\Theta$ *codes, linear* $\boldsymbol{m}\Theta$ *codes.*

GJSFR-F Classification: *DDC Code: 663.1 LCC Code: TP505*

QUASICYCLICCODESOVERFINITECHAINPSEUDOFIELDFZ1

Strictly as per the compliance and regulations of:

Ref

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Quasi-Cyclic Codes Over Finite Chain $m\Theta$ Pseudo Field $\mathbb{F}(p^k\mathbb{Z}, 1)$

Pemha Binyam Gabriel Cedric

Abstract- The $m\Theta$ sets present an enrichment from the logical viewpoint compared with the classical sets. The subset of the $m\Theta$ invariants of a $m\Theta$ set is a classical set, which leads to the canonical construction of the structures of modal Θ- valent pseudo field. In this note the purpose is to define on a finite chain $m\Theta$ pseudo field, $\mathbb{F}(p^k\mathbb{Z}, 1)$, the structures of Quasi-Cyclic codes of length r.

Keywords: $m\Theta$ set, $m\Theta$ pseudo field, chain $m\Theta$ pseudo field, quasi-cyclic $m\Theta$ codes, linear $m\Theta$ codes.

I. Introduction

Cyclic codes are among the most useful and well-studied code families for various reasons, such as effective encoding and decoding. A cyclic code can be viewed as an ideal in a certain quotient ring obtained from a polynomial ring with coefficients from a finite field [1, 2]. Quasi-Cyclic codes are a generalization of cyclic codes [6, 8]. Algebraically, Quasi-Cyclic codes are modules rather than ideals [10, 13].

A $m\Theta$ approach of the notion of sets has allowed to bring out the new classes of sets: $m\Theta$ sets. The notion of modal Θ-valent set ($m\Theta$ set) noted $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$, p prime, is defined by F. Ayissi Eteme in [12, 16, 7]. Research on modal algebra has evolved and led to the theory of $m\Theta$ codes [11, 15, 17].

The theory of error-correcting $m\Theta$ codes over finite fields has experienced tremendous growth since its inception [5]. Progress has been attained in the direction of determining the structural properties of $m\Theta$ codes over large families of $m\Theta$ fields. This paper is a contribution along those lines as we focus on codes over finite $m\Theta$ pseudo fields with a linear lattice of $m\Theta$ ideals (the so-called chain $m\Theta$ pseudo fields).

The purpose of this paper is to obtain structure theorems for Quasi-Cyclic codes in more general setting. The structures of Quasi-Cyclic codes of length r over finite chain $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$ are established when r is not divisible by the characteristic of the residue $m\Theta$ pseudo field $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$. Some cases where r is divisible by the characteristic of the residue $m\Theta$ pseudo field $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$ are also considered.

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After presenting preliminary concepts and results on $m\Theta$ set in Section 2. Section 3 presents a canonical construction of the structures of modal Θ-valent field and modal Θ-valent field. Section 4 is devoted to the notion of modal Θ-valent extension of a finite field. Section 5 define the intrinsic polynomial representation of the $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, r)$. Section 6 presents the $m\Theta$ Quasi-Cyclic codes. Lastly, section 7 presents the structure of Quasi-Cyclic code over finite chain $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$.

II. Preliminaries

a) *The modal Θ-valent set structure and the algebra of* $\left(\mathbb{F}_{p\mathbb{Z}},\, F_\alpha\right)$

$m\Theta$ sets are considered to be non-classical sets which are compatible with a non-classical logic called the chrysippian $m\Theta$ logic.

Definition 0.1. *[14] Let E be a non-empty set, I be a chain whose first and last elements are 0 and 1 respectively, $(F_\alpha)_{\alpha\in I_*}$ where $I_* = I\setminus\{0\}$ be a family of applications form E to E.*
A $m\Theta$ set is the pair $\left(E,\, (F_\alpha)_{\alpha\in I_*}\right)$ simply denoted by $(E,\, F_\alpha)$ satisfying the following four axioms :

- $\underset{\alpha}{\cap} F_\alpha(E) = \underset{\alpha\in I_*}{\cap}\{F_\alpha(x) : x\in E\} \neq \emptyset$;
- $\forall\alpha,\beta\in I_*,\ if\ \alpha\neq\beta\ then\ F_\alpha\neq F_\beta$;
- $\forall\alpha,\beta\in I_*,\ F_\alpha\circ F_\beta = F_\beta$;
- $\forall x,y\in E,\ if\ \forall\alpha\in I_*, F_\alpha(x) = F_\alpha(y)\ then\ x = y$.

Theorem 0.1. *[16] (The theorem of $m\Theta$ determination)*
Let (E, F_α) be a $m\Theta$ set.

$$\forall x,\, y\in E,\ x =_\Theta y\ if\, and\, only\, if\ \forall\alpha\in I_*,\ F_\alpha(x) = F_\alpha(y).$$

Proof 0.1. *[16]*

Definition 0.2. *[5] Let $\mathrm{C}(E, F_\alpha) = \underset{\alpha\in I_*}{\cap} F_\alpha(E)$. We call $\mathrm{C}(E, F_\alpha)$ the set of $m\Theta$ invariant elements of the $m\Theta$ set (E, F_α).*

Proposition 0.1. *[16] Let (E, F_α) be a $m\Theta$ set. The following properties are equivalent:*

1. $x\in\underset{\alpha\in I_*}{\cap} F_\alpha(E)$;
2. $\forall\alpha\in I_*,\ F_\alpha(x) = x$;
3. $\forall\alpha,\beta\in I_*,\ F_\alpha(x) = F_\beta(x)$;
4. $\exists\mu\in I_*,\ x = F_\mu(x)$.

Ref

14. F. Ayissi Eteme, *Anneau chrysippien Θ-valent*, CRAS, Paris 298, série 1, 1984, pp.1 - 4.

Proof 0.2. [16]

Definition 0.3. [12]

Let (E, F_α) and (E', F'_α) be two $m\Theta$ sets. Let X be a non-empty set. We shall call

1. *(E', F'_α) a modal Θ-valent subset of (E, F_α) if the structure of $m\Theta$ set (E', F'_α) is the restriction to E' of the structure of the $m\Theta$ set (E, F_α), this means:*
   - $E' \subseteq E$;
   - $\forall \alpha : \alpha \in I_*,\ F'_\alpha = F_{\alpha|_{E'}}$.
2. *X a modal Θ-valent subset of (E, F_α) if:*
   - $X \subseteq E$;
   - *$(X, F_{\alpha|X})$ is a $m\Theta s$ which is a modal Θ-valent subset of (E, F_α).*

In all what follows we shall write $F_\alpha x$ for $F_\alpha (x)$, $F_\alpha E$ for $F_\alpha (E)$, etc $\cdots$

Let $p \in \mathbb{N}$, a prime number. Let us recall that if $a \in \mathbb{F}_{p\mathbb{Z}}$.

$$\mathbb{F}_{p\mathbb{Z}} = \mathbb{F}_p \cup \{x_{p\mathbb{Z}} : \neg (x \equiv 0\,(mod p))\}\,; \quad \mathbb{F}_p = \{0,\,1,\,2,\,\cdots,p-1\}.$$

We define the $m\Theta$ support of a denoted $s\,(a)$ as follows:

$$s\,(a) = \begin{cases} a & if\, a \in \mathbb{F}_p; \\ x & if\, a = x_{p\mathbb{Z}}\, with \,\rceil (x \equiv 0\,(mod\,p))\,. \end{cases}$$

Thus $s\,(a) \in \mathbb{F}_p$.

Definition 0.4. [14] *Let $\perp$ be a binary operation on $\mathbb{F}_p$. So, $\forall a, b \in \mathbb{F}_p$, $a \perp b \in \mathbb{F}_p$. Let $x,\, y \in \mathbb{F}_{p\mathbb{Z}}$. We define a binary operation $\perp^*$on $\mathbb{F}_{p\mathbb{Z}}$ as follows :*

$$x \perp^* y = \begin{cases} s(x) \perp s(y) & if\, \begin{cases} x, y \in \mathbb{F}_p \\ (s(x) \perp s(y)) \equiv 0\,(mod\,p) & otherwise \end{cases} \\ (s(x) \perp s(y))_{p\mathbb{Z}} & otherwise. \end{cases}$$

$\perp^*$ as defined above on $\mathbb{F}_{p\mathbb{Z}}$ will be called a $m\Theta$ law on $\mathbb{F}_{p\mathbb{Z}}$ for $x, y \in \mathbb{F}_{p\mathbb{Z}}$. Thus we can define $x + y \in \mathbb{F}_{p\mathbb{Z}}$ and $x \times y \in \mathbb{F}_{p\mathbb{Z}}$ for every $x, y \in \mathbb{F}_{p\mathbb{Z}}$, where $+$ and $\times$ are $m\Theta$ addition and $m\Theta$ multiplication respectively.

Theorem 0.2. [12] *$(\mathbb{F}_{p\mathbb{Z}}, F_\alpha, +, \times)$ is a $m\Theta$ ring of unity 1 and of $m\Theta$ unity $\frac{1}{p\mathbb{Z}}$.*

Proof 0.3. [12]

Remark 0.1. *Since p is prime, $(\mathbb{F}_{p\mathbb{Z}},\, F_\alpha)$ is a $m\Theta$ field.*

Definition 0.5. [4] *x is a divisor of zero in $(\mathbb{F}_{p\mathbb{Z}},\, F_\alpha)$ if it exists $y \in \mathbb{F}_{p\mathbb{Z}}$ such that $x \times y = 0$*

Example 0.1. [4]

$p = 2$, we have $\mathbb{F}_{2\mathbb{Z}} = \{0,\, 1,\, 1_{2\mathbb{Z}},\, 3_{2\mathbb{Z}}\}$

The table of $m\Theta$ determination and tables laws of $\mathbb{F}_{2\mathbb{Z}}$.

Ref

16. F. Ayissi Eteme, *Logique et Algèbre de structure mathé matiques modales Θ-valentes chrysippiennes*, Edition Hermann, Paris, 2009.

$\mathbb{F}_{2\mathbb{Z}}$	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
F_1	0	1	1	*0*
F_2	0	1	0	*1*

$+^\Theta$	*0*	*1*	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
0	*0*	*1*	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
1	*1*	*0*	*0*	*0*
$1_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$	*0*	*0*	*0*
$3_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$	*0*	*0*	*0*

$\times^\Theta$	*0*	*1*	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
0	*0*	*0*	*0*	*0*
1	*0*	*1*	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
$1_{2\mathbb{Z}}$	*0*	$1_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
$3_{2\mathbb{Z}}$	*0*	$3_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$

Observation:

$\mathbb{F}_{2\mathbb{Z}}$ has no divisor of zero, is a $m\Theta$ ring from four elements, that's a $m\Theta$ field of four elements.

III. CANONICAL CONSTRUCTION OF MODAL Θ-VALENT FIELDS ($m\,\Theta f$) AND MODAL Θ-VALENT PSEUDO FIELDS($m\Theta pf$)

Let p be a prime number, $k \neq 0$ a positive integer, $q = p^k$ and $\mathbb{F}_q$ a finite field with q elements. Two $m\Theta f$ K_1 and K_2 of same characteristic p and of same cardinal p^{2k} are $m\Theta$ isomorphic.

a) *Canonical construction of modal Θ-valent fields ($m\Theta f$)[9]*

Consider that $k = 1$, so $q = p$. $\mathbb{F}_p = \frac{\mathbb{Z}}{p\mathbb{Z}}$ is the prime field of characteristic p and of p elements. The modal Θ-valent quotient ring ($m\Theta qr$) $\mathbb{F}_{p\mathbb{Z}}$ as the modal Θ-valent quotient $\frac{\mathbb{Z}_{p\mathbb{Z}}}{p\mathbb{Z}_{p\mathbb{Z}}}$.

Let $\mathbb{F}^*_{p\mathbb{Z}} = \mathbb{F}_{p\mathbb{Z}} - \{0\}$. $\forall x \in \mathbb{F}^*_{p\mathbb{Z}}$, $\exists x' \in \mathbb{F}^*_{p\mathbb{Z}}$ / $x \cdot x' = \frac{1_{p\mathbb{Z}}}{p\mathbb{Z}_{p\mathbb{Z}}}$.

$\mathbb{F}_{p\mathbb{Z}}$ has p^2 elements but has no proper sub $m\Theta$ ring verifying the preceding property for $\mathbb{F}^*_{p\mathbb{Z}}$.

For which reason, $\mathbb{F}_{p\mathbb{Z}}$ is the prime $m\Theta f$ with p^2 elements. $\mathbb{F}_p$ is the prime sub field of the $m\Theta$ invariants of $\mathbb{F}_{p\mathbb{Z}}$. Let f be a polynomial with coefficients in $\mathbb{F}_p$. Clearly, it is all the same that:

1. $f_p(x)$ irreducible over $\mathbb{F}_p$.
2. $f_{p\mathbb{Z}}(x)$ irreducible over $\mathbb{F}_{p\mathbb{Z}}$.

Observations:

Let $\mathbb{F}(p\mathbb{Z}, r) = \frac{\mathbb{F}_{p\mathbb{Z}}[X]}{(f(X))}$ be the $m\Theta r$ modulo $f(x)$, ($m\Theta r(f)$). $f(x) \in \mathbb{F}_p[X]$. $deg(f) = r$, $r \in \mathbb{N}^*$, f irreducible over $\mathbb{F}_p$.

$$\mathbb{F}_{p\mathbb{Z}}[X] \longrightarrow \mathbb{F}(p\mathbb{Z}, r) \,:\, g \longmapsto r_g;\; g = q_g \cdot f(x) + r_g,\; 0 \leq dg(r_g) < dg(f).$$

$(\mathbb{F}_{p\mathbb{Z}})^r \longrightarrow \mathbb{F}(p\mathbb{Z}, r) \,:\, (a_0, \cdots, a_{r-1}) \longmapsto \sum_{i=0}^{r-1} a_i x^i$ is a bijection and therefore becomes a $m\Theta r$ isomorphism for the $m\Theta$ laws modulo $f(x)$. Since f is irreducible over $\mathbb{F}_{p\mathbb{Z}}$, $\mathbb{F}(p\mathbb{Z}, r)$ is a $m\Theta f$.

9. F. Ayissi Eteme, *clrm Θ introducing pure and applied mathematics*, Lambert academic publishing saarbrüken, Germany, 2015.

Theorem 0.3. *1.* $\mathbb{F}(p\mathbb{Z}, r)$ *is a* $m\Theta f$ *of cardinal* p^{2r}*;*

2. $\mathbb{F}_{p\mathbb{Z}}$ *is its prime sub* $m\Theta f$ *of cardinal* p^2*;*

3. $\mathbb{F}_{p\mathbb{Z}}$ *and* $\mathbb{F}(p\mathbb{Z}, r)$ *are booth of characteristic* p *since* $\forall i$:

$$i = 0, \cdots, p-1; \quad \underbrace{1 + 1 + \cdots + 1}_{i\,times} + \underbrace{1_{p\mathbb{Z}} + \cdots + 1_{p\mathbb{Z}}}_{(p-i)\,times} = 0$$

Proof 0.4. [9]

According to a previous notation,

$$\mathbb{F}(p\mathbb{Z}, 1) = \mathbb{F}_{p\mathbb{Z}},\ \mathbb{F}(p, 1) = \frac{\mathbb{Z}}{p\mathbb{Z}},\ \mathbb{F}(p, r) = \mathbb{GF}(p, r).$$

b) Canonical construction of modal Θ-valent pseudo fields ($m\Theta pf$)

Consider that $k \neq 1$, so $q = p^k$. Let then $\mathbb{F}(p^k\mathbb{Z}, 1)$ denote the quotient $m\Theta r$ $\mathbb{F}_{p^k\mathbb{Z}} = \frac{\mathbb{Z}_{p\mathbb{Z}}}{p^k\mathbb{Z}_{p\mathbb{Z}}}$ and let

$$O(p^k, 1) = O_{p^k} = \{\frac{a}{p^k\mathbb{Z}_{p\mathbb{Z}}} : a \in \mathbb{Z}_{p\mathbb{Z}},\ s(a)/p^k\} = \{\frac{a}{p^k\mathbb{Z}} : a \in \mathbb{Z},\ a/p^k\}.$$

Let $\mathbb{F}^*(p^k\mathbb{Z}, 1) = \mathbb{F}(p^k\mathbb{Z}, 1) - O(p^k, 1)$; $k \in \mathbb{N}^*$. Then $\forall x : x \in \mathbb{F}^*(p^k\mathbb{Z}, 1)$, $\exists x' : x' \in \mathbb{F}^*(p^k\mathbb{Z}, 1) : x \cdot x' = \frac{1_{p\mathbb{Z}}}{p^k\mathbb{Z}_{p\mathbb{Z}}}$.

So we call $\mathbb{F}_{p^k\mathbb{Z}}$ a $m\Theta$ pseudo field ($m\Theta pf$). $\mathbb{F}_{p^k\mathbb{Z}}$ has p^{k+1} elements and is of characteristic p^k. It has no proper sub $m\Theta pf$ with the same as the preceding properties for $\mathbb{F}^*(p^k\mathbb{Z}, 1)$. Finally, $\mathbb{F}(p^k\mathbb{Z}, 1)$ is the prime $m\Theta pf$ with p^{k+1} elements.

Let now $f \in \mathbb{Z}_{p^k}[X] : dg(f) = r$ and f irreducible over $\mathbb{Z}_{p^k} = \frac{\mathbb{Z}}{p^k\mathbb{Z}}$. Let $\mathbb{F}(p^k\mathbb{Z}, r) = \frac{\mathbb{F}(p^k\mathbb{Z}, 1)[X]}{(f(X))}$ $m\Theta r$ modulo $f(x)$. $\mathbb{F}(p^k\mathbb{Z}, r)$ is a $m\Theta pf$.

$(\mathbb{F}(p^k\mathbb{Z}, 1))^r \longrightarrow \mathbb{F}(p^k\mathbb{Z}, r) : (a_0, \cdots, a_{r-1}) \longmapsto \sum_{i=0}^{r-1} a_i x^i$ is a bijection and therefore a $m\Theta$ ring modulo $f(X)$ isomorphism. Since $card\mathbb{F}(p^k\mathbb{Z}, 1) = p^{k+1}$, $card\mathbb{F}(p^k\mathbb{Z}, r) = p^{(k+1)r}$.

Theorem 0.4. [9] $\forall k \in \mathbb{N} - \{0\}$,

1. $\mathbb{F}(p^k\mathbb{Z}, r)$ *is a* $m\Theta pf$ *of cardinal* $p^{(k+1)r}$.

2. $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is its prime sub* $m\Theta pf$ *of* p^{k+1} *elements.*

3. $\mathbb{F}(p^k\mathbb{Z}, 1)$ *and* $\mathbb{F}(p^k\mathbb{Z}, r)$ *are booth of characteristic* p^k.

Proof 0.5. [9]

$\mathbb{F}(p^k, r) = G\mathbb{F}(p^k, r)$ is the sub pseudo field of the $m\Theta$ invariants of the $m\Theta pf$ $\mathbb{F}(p^k\mathbb{Z}, r)$. $\mathbb{F}_{p^k} = \frac{\mathbb{Z}}{p^k\mathbb{Z}}$ is the prime sub pseudo field of the $m\Theta$ invariants of $\mathbb{F}_{p^k\mathbb{Z}}$; the prime sub $m\Theta pf$ with p^{k+1} elements.

Ref

9. F. Ayissi Eteme, *chrm Θ introducing pure and applied mathematics*, Lambert academic publishing saarbrüken, Germany, 2015.

Theorem 0.5. [9]

1. *Any $m\Theta f$ K of characteristic p prime and then of cardinal p^{2r}, $r \in \mathbb{N}^*$ is $m\Theta$ isomorphic to the $m\Theta f$ $\mathbb{F}(p\mathbb{Z}, r)$;*
2. *Any $m\Theta f$ K' of characteristic p^k, p prime and then of cardinal $p^{(k+1)r}$, $r \in \mathbb{N}^*$ is $m\Theta$ isomorphic to the $m\Theta f$ $\mathbb{F}(p^k\mathbb{Z}, r)$.*

Proof 0.6. [9]

IV. MODAL Θ-VALENT EXTENSION OF A FINITE FIELD

Note that K is a finite field of cardinal p^n, $p, n \in \mathbb{N}^*$ and then of characteristic p prime. $\beta \in K$, of minimal polynomial $m_\beta(x) \in \mathbb{F}_p[x]$, $r = deg_{\mathbb{F}_p}(m_\beta(x)) \in \mathbb{N}^*$, $m_\beta(x)$ is irreducible over $\mathbb{F}_p$.

Observations: Let $I_\beta = \langle m_\beta(x) \rangle_{\mathbb{F}_p[x]}$ the principal ideal of $\mathbb{F}_p[x]$ generated by $m_\beta(x)$. Since $\mathbb{F}_p \subset \mathbb{F}_{p\mathbb{Z}}$, $\mathbb{F}_p[x] \subset \mathbb{F}_{p\mathbb{Z}}[x]$.
Let $a \in {\mathbb{F}^*}_{p\mathbb{Z}}$: $\exists \mu$, $F_\mu a \neq 0$, then $F_\mu a \in {\mathbb{F}^*}_p$, thus $m_\beta(F_\mu a) \neq 0$ and since $F_\mu m_\beta(a) = m_\beta(F_\mu a)$, $F_\mu m_\beta(a) \neq 0$. Then $m_\beta(a) \neq 0$.

Therefore, $m_\beta(x)$ is also irreducible over $\mathbb{F}_{p\mathbb{Z}}$. It is known that $\frac{\mathbb{F}_{p\mathbb{Z}}[x]}{\langle m_\beta(x) \rangle}$ is a $m\Theta$ field with p^{2r} elements and then of characteristic p. $\frac{\mathbb{F}_p[x]}{m_\beta(x)}$ is its subfield of the Θ-invariants who has p^r elements and characteristic p.

Let $I_{\beta\ p\mathbb{Z}} = \langle m_\beta(x) \rangle_{\mathbb{F}_{p\mathbb{Z}}[x]}$ the principal $m\Theta$ ideal of $\mathbb{F}_{p\mathbb{Z}}[x]$ generated by $m_\beta(x)$. $\forall \alpha$, $F_\alpha I_{\beta\ p\mathbb{Z}} = I_\beta$ therefore $I_{\beta\ p\mathbb{Z}}$ is a $m\Theta$ maximal ideal of $\mathbb{F}_{p\mathbb{Z}}[x]$. Then define $\Phi_{\beta\ p\mathbb{Z}} : \mathbb{F}_{p\mathbb{Z}}[x] \longrightarrow \mathbb{F}(p\mathbb{Z}, n)$ as follows; if $f(x) = \sum_{i=0}^{q} a_i x^i \in \mathbb{F}_{p\mathbb{Z}}[x]$,

$$\Phi_{\beta\ p\mathbb{Z}}(f(x)) = f(\beta) = \sum_{i=0}^{q} a_i \beta^i \in \mathbb{F}(p\mathbb{Z}, n).$$

By definition $\Phi_{\beta\ p\mathbb{Z}}$ is a $m\Theta$ ring morphism since then $\Phi_{\beta\ p\mathbb{Z}}(\mathbb{F}_{p\mathbb{Z}}[x]) = \{f(\beta) \mid f(x) \in \mathbb{F}_{p\mathbb{Z}}[x]\}$ is a sub $m\Theta$ field of $\mathbb{F}(p\mathbb{Z}, n)$. Therefore the following diagram $m\Theta$ commutes

$$\begin{tikzcd}
\mathbb{F}_{p\mathbb{Z}}[x] \arrow[r, "\Phi_{\beta\ p\mathbb{Z}}"] \arrow[d, "\varphi_{\Phi_{\beta\ p\mathbb{Z}}}"'] & \varphi_{\Phi_{\beta\ p\mathbb{Z}}}(\mathbb{F}_{p\mathbb{Z}}[x]) \arrow[r, "i_{p\mathbb{Z}}"] & \mathbb{F}(p\mathbb{Z}, n) \\
\frac{\mathbb{F}_{p\mathbb{Z}}[x]}{I_\beta} \arrow[ur, "\tilde{\Phi}_{\beta\ p\mathbb{Z}} = \frac{\Phi_{\beta\ p\mathbb{Z}}}{\langle m_\beta(x) \rangle}"'] & &
\end{tikzcd}$$

- $\frac{\mathbb{F}_{p\mathbb{Z}}[x]}{I_\beta} = \frac{\mathbb{F}_{p\mathbb{Z}}[x]}{\langle m_\beta(x) \rangle}$ is a $m\Theta$ field of cardinal p^{2r} and then of characteristic p.

Ref

9. F. Ayissi Eteme, *chrm Θ introducing pure and applied mathematics,* Lambert academic publishing saarbrüken, Germany, 2015.

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- Through the $m\Theta$ ring isomorphism $\widetilde{\Phi}_{\beta\ p\mathbb{Z}}$, $\widetilde{\Phi}_{\beta\ p\mathbb{Z}}(\mathbb{F}_{p\mathbb{Z}}[x])$ becomes a $m\Theta$ subfield of $\mathbb{F}(p\mathbb{Z}, n)$ with the $m\Theta$ field structure of p^{2r} elements exported from $\frac{\mathbb{F}_{p\mathbb{Z}}[x]}{\langle m_\beta(x)\rangle}$ by $\widetilde{\Phi}_{\beta\ p\mathbb{Z}}$.

Notation 0.1.

$$\mathbb{F}_{p\mathbb{Z}}(\beta) = \Phi_{p\mathbb{Z}}(\beta) = \widetilde{\Phi}_{\beta\ p\mathbb{Z}}(\mathbb{F}_{p\mathbb{Z}}[x]) = \{f(\beta)\,|\,f(x) \in \mathbb{F}_{p\mathbb{Z}[x]}\}$$

Theorem 0.6. *1. $\mathbb{F}_{p\mathbb{Z}}[\beta]$ has p^{2r} elements and characteristic p.*

2. $\mathbb{F}_{p\mathbb{Z}}$ is the prime $m\Theta$ subfield of $\mathbb{F}_{p\mathbb{Z}}[\beta]$.

3. Any sub $m\Theta$ field of $\mathbb{F}(p\mathbb{Z}, n)$ containing $\mathbb{F}_{p\mathbb{Z}}$ and β contains $\mathbb{F}_{p\mathbb{Z}}[\beta]$.

4. $\forall a\,;\, a \in \mathbb{F}_{p\mathbb{Z}}[\beta]$, $\exists a_i$, $i = 0, 1, \cdots, r-1\,/\,a_i \in \mathbb{F}_{p\mathbb{Z}}\ :\ a = \sum_{i=0}^{r-1} a_i\beta^i$.

Definition 0.6. *Henceforth we call $\mathbb{F}_{p\mathbb{Z}}[\beta]$ the $m\Theta$ extension of $\mathbb{F}_p$ and $\mathbb{F}_{p\mathbb{Z}}$ to β.*

Definition 0.7. *We call a $m\Theta$ primitive element of $\mathbb{F}(p\mathbb{Z}, n)$ any generator if there exists one, noted α, of $\mathbb{F}(p\mathbb{Z}, n) - \mathbb{F}(p, n)$. This meaning that $\forall a$: $a \in \mathbb{F}(p\mathbb{Z}, n) - \mathbb{F}(p, n), \exists m \in \mathbb{N} :\ 0 \leq m \leq \omega(\mathbb{F}^*(p\mathbb{Z}, n);\ a = \alpha^m$.*

Example 0.2. *$2_{3\mathbb{Z}}$ and $5_{3\mathbb{Z}}$ are two $m3$ generators of $\mathbb{F}_{3\mathbb{Z}}$.*

Proposition 0.2. *If $\alpha \in \mathbb{F}(p\mathbb{Z}, n)$ is a $m\Theta$ primitive element then $\mathbb{F}(p\mathbb{Z}, n) = \mathbb{F}_{p\mathbb{Z}}(\alpha)$.*

Proof 0.7. *Suppose $u \in \mathbb{F}(p\mathbb{Z}, n) - \mathbb{F}(p, n)$ and α is a $m\Theta$ primitive element: $\exists m$, $m \in \mathbb{N}\ :\ 0 \leq m \leq \omega(\mathbb{F}^*(p\mathbb{Z}, r)$, $u = \alpha^m$. Let $f(x) = x^m \in \mathbb{F}_{p\mathbb{Z}}[x]$, $\Phi_{\beta\,p\mathbb{Z}}(f(x)) = f(\alpha) = x^m$.*

Therefore $u = \alpha^m = f(\alpha) \in \mathbb{F}_{p\mathbb{Z}}(\alpha)$. Thus $\mathbb{F}(p\mathbb{Z}, n) = \mathbb{F}_{p\mathbb{Z}}(\alpha)$.

V. The Intrinsic Polynomial Representation of the $m\Theta$ Pseudo Field $\mathbb{F}(p^k\mathbb{Z}, r)$

Let $k \in \mathbb{N}$, r, $p \in \mathbb{N}^*$, p prime $2 \leq p$. It is plain in [7] that:

$$\prod_{a\in\mathbb{F}^*(p^k\mathbb{Z},1)} (x-a) \;=\; \prod_{x\in\tilde{\mathbb{F}^*}_{p^k}} (x-a) \times \prod_{a\in\mathbb{F}^*(p^k\mathbb{Z},1)-\mathbb{F}^*_{p^k}} (x-a)$$

$$= (x^{\varphi(p^k)}-1)(x^{\varphi(p^{k+1})}-1_{p\mathbb{Z}}).$$

$\mathbb{F}(p^k, 1) = \frac{\mathbb{Z}}{p^k\mathbb{Z}}$.

7. F. Ayissi Eteme, *Logique et algèbre de structures mathematiques modales Θ-valentes chrysipiennes*, Hermann, Paris, 2009.

Proposition 0.3. *Let* $< x^{\varphi(p^k)} - 1 >$ *and* $< x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} >$ *be the ideals of* $\mathbb{F}(p^k, 1)[x]$ *respectively generated by* $x^{\varphi(p^k)} - 1$ *and* $x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}}$*, then*

1. $< x^{\varphi(p^k)} - 1 >$ *is a maximal* $m\Theta$ *ideal of* $\mathbb{F}(p^k, 1)[x]$*;*

2. $< x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} > \subsetneq < x^{\varphi(p^k)} - 1 >$.

Proof 0.8. 1. $< x^{\varphi(p^k)} - 1 >$ *is a* $m\Theta$ *ideal since generated by the* $m\Theta\Theta$ *invariant polynomial* $x^{\varphi(p^k)} - 1$*; this* Θ *ideal is a maximal since* $< x^{\varphi(p^k)} - 1 >_{\mathbb{F}_{p^k}[x]}$ *is maximal in* $\mathbb{F}_{p^k}[x]$ *and* $\forall \alpha \in I_*, F_\alpha < x^{\varphi(p^k)} - 1 >=< x^{\varphi(p^k)} - 1 >_{\mathbb{F}_{p^k}[x]}$*. This is sufficient to claim that* $\frac{\mathbb{F}(p^k,1)[x]}{<x^{\varphi(p^k)}-1>}$ *is a* $m\Theta$ *pseudo field, and as such* $m\Theta$ *isomorphic to the* $m\Theta$ *pseudo field* $\mathbb{F}(p^k, \varphi(p^k))$.

2. $x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} \in < x^{\varphi(p^k)} - 1 >$*. Since* $\varphi(p^{k+1}) = p\varphi(p^k)$, $x^{\varphi(p^{k+1})} = x^{p\varphi(p^k)}$*. Henceforth,*

$$\begin{aligned} x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} &= x^{p\varphi(p^k)} - 1_{p\mathbb{Z}} \\ &= (x^{\varphi(p^k)})^p - 1_{p\mathbb{Z}}^p \\ &= (x^{\varphi(p^k)} - 1_{p\mathbb{Z}})^p \\ &= (x^{\varphi(p^k)} - 1)(x^{\varphi(p^k)} - 1_{p\mathbb{Z}})^{p-1} \end{aligned}$$

This last expression shows that $x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} \in < x^{\varphi(p^k)} - 1 >$*. Trivially,* $x^{\varphi(p^k)} - 1 \notin < x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} >$ *. Therefore* $< x^{\varphi(p^{k+1})} - 1_{p\mathbb{Z}} > \subsetneq < x^{\varphi(p^k)} - 1 >$, $\forall k \in \mathbb{N}^*$*. Thus* $< x^{p(p-1)} - 1_{p\mathbb{Z}} > \subsetneq < x^{p-1} - 1 >$.

Definition 0.8. *The* $m\Theta$ *pseudo field* $\mathbb{F}(p^k\mathbb{Z}, \varphi(p^k)) = \frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{<x^{\varphi(p^k)}-1>}$ *is what we call the intrinsic polynomial representation of the* $m\Theta$ *pseudo field* $\mathbb{F}(p^k\mathbb{Z}, r)$.

Corollary 0.1. $\mathbb{F}(p\mathbb{Z}, \varphi(p)) = \frac{\mathbb{F}(p\mathbb{Z},1)[x]}{<x^{(p-1)}-1>}$ *is the intrinsic polynomial representation of* $\mathbb{F}(p\mathbb{Z}, r)$ *with* $r = \varphi(p) = p - 1$, $k = 1$.

Proposition 0.4. *For a finite commutative* $m\Theta$ *pseudo field* $\mathbb{F}(p^k\mathbb{Z}, 1)$ *the following conditions are equivalent:*

1. $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is a local* $m\Theta$ *pseudo field and the maximal* $m\Theta$ *ideal* M *of* $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is principal;*

2. $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is a local principal* $m\Theta$ *ideal pseudo field;*

3. $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is a chain* $m\Theta$ *pseudo field.*

Notes

Notes

Proof 0.9. *$i) \Longrightarrow ii)$. Let I be an $m\Theta$ ideal of $\mathbb{F}(p^k\mathbb{Z}, 1)$. If $I = \mathbb{F}(p^k\mathbb{Z}, 1)$ then I is generated by the identity 1. If $I \subsetneq \mathbb{F}(p^k\mathbb{Z}, 1)$, then $I \subseteq M$. By $i)$, M is generated by an element, say $M = \langle a \rangle$. Therefore, $I = \langle a^i \rangle$, for some integer k. Hence, $\mathbb{F}(p^k\mathbb{Z}, 1)$ is a local principal $m\Theta$ ideal pseudo field.*

$ii) \Longrightarrow iii)$. Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a local principal $m\Theta$ ideal pseudo field with the maximal ideal $M = \langle a \rangle$, and A, B be proper ideals of $\mathbb{F}(p^k\mathbb{Z}, 1)$. Then A, $B \subseteq M$, whence there exist integers l, m such that $A = \langle a^l \rangle$, $B = \langle a^m \rangle$ (l, $m \leq$ the nilpotency of a). Hence, either $A \subseteq B$, or $B \subseteq A$. Thus, $\mathbb{F}(p^k\mathbb{Z}, 1)$ is a chain $m\Theta$ pseudo field.

$iii) \Longrightarrow i)$. Assume $\mathbb{F}(p^k\mathbb{Z}, 1)$ is a finite commutative chain $m\Theta$ pseudo field, then clearly $\mathbb{F}(p^k\mathbb{Z}, 1)$ is local. To show the maximal $m\Theta$ ideal M of $\mathbb{F}(p^k\mathbb{Z}, 1)$ is principal, suppose to the contrary that M is generated by more than one element, say b, c are in the generator set of M and $b \notin c\mathbb{F}(p^k\mathbb{Z}, 1)$, $c \notin b\mathbb{F}(p^k\mathbb{Z}, 1)$. Then $\langle b \rangle \nsubseteq \langle c \rangle$ and $\langle c \rangle \nsubseteq \langle b \rangle$, a contradiction with the assumption that $\mathbb{F}(p^k\mathbb{Z}, 1)$ is a chain $m\Theta$ pseudo field. Thus, M is principal, proving $i)$.

Let a be a fixed generator of the maximal ideal M. Then a is nilpotent and we denote its nilpotency index by t. The ideals of $\mathbb{F}(p^k\mathbb{Z}, 1)$ for a chain

$$\mathbb{F}(p^k\mathbb{Z}, 1) = \langle a^0 \rangle \supsetneq \langle a^1 \rangle \supsetneq \cdots \supsetneq \langle a^{t-1} \rangle \supsetneq \langle a^t \rangle = \langle 0 \rangle.$$

Let $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)} = \frac{\mathbb{F}(p^k\mathbb{Z}, 1)}{M}$. By $^{-} : \mathbb{F}(p^k\mathbb{Z}, 1)[x] \longrightarrow \overline{\mathbb{F}(p^k\mathbb{Z}, 1)}[x]$, we denote the natural $m\Theta$ pseudo field homomorphism that maps $\rho \longmapsto \rho + M$ and the variable x to x.

Proposition 0.5. *Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a finite commutative chain $m\Theta$ pseudo field, with maximal ideal $M = \langle a \rangle$, and let t be a nilpotency a. Then we get the following statements.*

1. *For some prime p and positive integers k, l ($k \geq l$), $|\mathbb{F}(p^k\mathbb{Z}, 1)| = p^{k+1}$, $|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}| = p^{l+1}$, and the characteristic of $\mathbb{F}(p^k\mathbb{Z}, 1)$ and $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$ are powers of p.*
2. *For $i = 0, \cdots, t$, $|\langle a^i \rangle| = |\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|^{t-i}$. In particular, $|\mathbb{F}(p^k\mathbb{Z}, 1)| = |\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|^t$, so, $k = lt$.*

Two $m\Theta$ polynomials f_1, $f_2 \in \mathbb{F}(p^k\mathbb{Z}, 1)[x]$ are called $m\Theta$ coprime if $\langle f_1 \rangle + \langle f_2 \rangle = \mathbb{F}(p^k\mathbb{Z}, 1)[x]$. A $m\Theta$ polynomial $f \in \mathbb{F}(p^k\mathbb{Z}, 1)[x]$ is called basic $m\Theta$ irreducible if $\overline{f}$ is $m\Theta$ irreducible in $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}[x]$. A $m\Theta$ polynomial $f \in \mathbb{F}(p^k\mathbb{Z}, 1)[x]$ is called regular if it is not a zero divisor.

VI. $m\Theta$ Quasi-Cyclic Codes

For a finite $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$, consider the set $\mathbb{F}^r(p^k\mathbb{Z}, 1)$ of n-tuples of elements from $\mathbb{F}(p^k\mathbb{Z}, 1)$ as a module over $\mathbb{F}(p^k\mathbb{Z}, 1)$ in the usual way. A subset $C \subseteq \mathbb{F}^r(p^k\mathbb{Z}, 1)$ is called a linear $m\Theta$ code of length r over $\mathbb{F}(p^k\mathbb{Z}, 1)$ if C is an $\mathbb{F}(p^k\mathbb{Z}, 1)$-submodule of $\mathbb{F}^r(p^k\mathbb{Z}, 1)$. C is called $m\Theta$ cyclic if, for every $m\Theta$ codeword $x = (x_0, x_1, \cdots, x_{r-1}) \in C$, its cyclic shift $(x_{n-1}, x_0, x_1, \cdots, x_{n-2})$ is also in C. An n-tuple $c = (c_0, c_1, \cdots, c_{r-1}) \in \mathbb{F}^r(p^k\mathbb{Z}, 1)$ is identified with the $m\Theta$ polynomial $c_0 + c_1x + \cdots + c_{r-1}x^{r-1}$ in $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{<x^r-1>}$, which is called the $m\Theta$ polynomial representation of $c = (c_0, c_1, \cdots, c_{r-1})$.

It is well known that a code C of length r over $\mathbb{F}(p^k\mathbb{Z}, 1)$ is $m\Theta$ cyclic if and only if the $m\Theta$ set of polynomial representations of its $m\Theta$ codewords is an $m\Theta$ ideal of $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{<x^r-1>}$.

Given $x = (x_0, x_1, \cdots, x_{r-1})$, $y = (y_0, y_1, \cdots, y_{r-1}) \in \mathbb{F}^r(p^k\mathbb{Z}, 1)$, their scalar product is

$$x \cdot y = x_0y_0 + x_1y_1 + \cdots + x_{r-1}y_{r-1}.$$

(evaluated in $\mathbb{F}(p^k\mathbb{Z}, 1)$). Two $m\Theta$ words x, y are called orthogonal if $\forall\alpha \in I_*,\ F_\alpha(x) \cdot F_\alpha(y) = 0$. For a linear $m\Theta$ code C over $\mathbb{F}(p^k\mathbb{Z}, 1)$, its dual code $C^\perp$ is the set of $m\Theta$ words over $\mathbb{F}(p^k\mathbb{Z}, 1)$ that are orthogonal to all $m\Theta$ codewords of C;

$$C^\perp = \{x \in \mathbb{F}(p^k\mathbb{Z}, 1) | \forall\alpha \in I_*,\ F_\alpha(x) \cdot F_\alpha(y) = 0,\ \forall y \in C\}.$$

A $m\Theta$ code C is called self-dual if $C = C^\perp$. For a finite $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$ with maximal ideal $\langle a\rangle$ and the nilpotency t of a is even, the code $\langle a^{\frac{t}{2}}\rangle$ is self-dual and is called the trivial self-dual code.

Proposition 0.6. Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a finite commutative $m\Theta$ pseudo field and

$$a(x) = a_0 + a_1x + \cdots + a_{r-1}x^{r-1};$$

$$b(x) = b_0 + b_1x + \cdots + b_{r-1}x^{r-1} \in \mathbb{F}(p^k\mathbb{Z}, 1)[x].$$

Then $a(x)b(x) = 0$ in $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{<x^r-1>}$ if and only if $(a_0, a_1, \cdots, a_{r-1})$ is $m\Theta$ orthogonal to $(b_{r-1}, b_{r-2}, \cdots, b_0)$ and all its cyclic shifts.

Proof 0.10. Let ζ denote the cyclic shift for $m\Theta$ codewords of length r, i.e., for each $(x_0, x_1, \cdots, x_{r-1}) \in \mathbb{F}^r(p^k\mathbb{Z}, 1)$.

$$\zeta(x_0, x_1, \cdots, x_{r-1}) = (x_{r-1}, x_0, \cdots, x_{r-2}).$$

Notes

Thus, $\zeta^i(b_{r-1}, b_{r-2}, \cdots, b_0), \ i = 1, 2, \cdots, r$ *are all cyclic shifts of* $(b_{r-1}, b_{r-2}, \cdots, b_0)$*.*

Let $c(x) = c_0 + c_1x + \cdots + c_{r-1}x^{r-1} = a(x)b(x) \in \frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{<x^r-1>}$*. Then for* $k = 0, 1, \cdots, r-1$,

$$\begin{aligned} c_k &= \sum_{i+j=k \text{ or } i+j=r-k} a_ib_j \\ &= (a_0, a_1, \cdots, a_{r-1}) \cdot (b_k, b_{k-1}, \cdots, b_{k+1}) \\ &= (a_0, a_1, \cdots, a_{r-1}) \cdot \zeta^{k+1}(b_{r-1}, b_{r-2}, \cdots, b_0). \end{aligned}$$

Therefore, $c(x) = 0$ *if and only if* $c_k = 0$ *for* $k = 0, 1, \cdots, r-1$ *if and only if*

$$(a_0, a_1, \cdots, a_{r-1}) \cdot \zeta^{k+1}(b_{r-1}, b_{r-2}, \cdots, b_0) = 0,$$

for $k = 0, 1, \cdots, r-1$ *if and only if* $(a_0, a_1, \cdots, a_{r-1})$ *is orthogonal to* $(b_{r-1}, b_{r-2}, \cdots, b_0)$ *and all its cyclic shifts, as desired.*

Definition 0.9. *(quasi-cyclic* $m\Theta$ *code)*
A linear $m\Theta$ *code* C *of length* $r = lk$ *over a finite* $m\Theta$ *pseudo field* $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is called a quasi-cyclic* $m\Theta$ *code of index* k *if for every* $m\Theta$ *codeword* $c \in C$ *there exists a number* k *such that the* $m\Theta$ *codeword obtained by* k *cyclic shifts is also a* $m\Theta$ *codeword in* C*. That is,*

$$c = (c_0, c_1, \cdots, c_{r-1}) \in C \Longrightarrow c' = \zeta^k(c) = (c_{r-k}, \cdots, c_0, \cdots, c_{r-k-1}) \in C.$$

In the definition k is defined as the smallest number of cyclic shifts where the $m\Theta$ code is invariant. Quasi-cyclic $m\Theta$ codes are a generalization of cyclic $m\Theta$ codes.

VII. STRUCTURE OF QUASI-CYCLIC CODE OVER FINITE CHAIN $m\Theta$ PSEUDO FIELD $\mathbb{F}(p^k\mathbb{Z}, 1)$

Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a finite chain $m\Theta$ pseudo field with the maximal $m\Theta$ ideal $\langle a \rangle$, and t be the nilpotency of a. There exist a prime p and an integer l such that $|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}| = p^l$, $|\mathbb{F}(p^k\mathbb{Z}, 1)| = p^{lt}$, the characteristic of $\mathbb{F}(p^k\mathbb{Z}, 1)$ and $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$ are powers of p. In this section, we assume r to be a positive integer which is not divisible by p; that implies r is not divisible by the characteristic of the residue $m\Theta$ pseudo field $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$, so that $x^r - 1$ is square free in $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}[x]$. Therefore, $x^r - 1$ has a unique decomposition as a product of basic irreducible pairwise coprime $m\Theta$ polynomials in $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$.

Notes

Lemma 0.1. *Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a finite chain $m\Theta$ pseudo field with the maximal $m\Theta$ ideal $\langle a\rangle$, and t be the nilpotency of a. If f is a regular basic irreducible $m\Theta$ polynomial of the $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$, then $\frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{\langle f\rangle}$ is also a chain $m\Theta$ pseudo field with precisely the following ideals:*

$$\langle 0\rangle,\ \langle 1\rangle,\ \langle 1+\langle f\rangle\rangle,\ \langle a+\langle f\rangle\rangle,\ \cdots,\ \langle a^{t-1}+\langle f\rangle\rangle.$$

Proof 0.11. *First we show that for distinct values of $i,\ j\in\{0,\ 1,\ \cdots,\ t-1\}$, $\langle a^i+\langle f\rangle\rangle\neq\langle a^j+\langle f\rangle\rangle$. Suppose $\langle a^i+\langle f\rangle\rangle=\langle a^j+\langle f\rangle\rangle$, for $0\leq i<j\leq t-1$. Then, there exists $g(x)\in\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ with $deg(g)<deg(f)$ such that $a^i+\langle f\rangle=a^j+\langle f\rangle$. That means $a^jg(x)-a^i\in\langle f\rangle$. As*

$$deg(a^jg(x)-a^i)\leq deg(g)<deg(f)$$

it follows that $a^jg(x)-a^i=0$. Multiplying by a^{t-j} gives $a^{t+i-j}=0$, which is a contradiction to our hypothesis that a has nilpotency t and $0<t+i-j<t$. Let I be a nonzero ideal of $\frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{\langle f\rangle}$ and $h+\langle f\rangle$ a nonzero element of I. By assumption, f is a basic irreducible $m\Theta$ polynomial in $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$, hence, $\bar{f}$ is irreducible in $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}[x]$. Therefore, $gcd(\bar{h},\ \bar{f})=1$, or $\bar{f}$. If $gcd(\bar{h},\ \bar{f})=1$, that is, $\bar{h},\ \bar{f}$ are coprime in $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}[x]$, then $h,\ f$ are coprime in $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$. So there exist $u,\ v\in\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ such that $uh+vf=1$. That implies

$$(u+\langle f\rangle)(h+\langle f\rangle)=1+\langle f\rangle$$

whence $h+\langle f\rangle$ is invertible in $\frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{\langle f\rangle}$. Therefore,

$$I=\frac{\mathbb{F}(p^k\mathbb{Z},\ 1)[x]}{\langle f\rangle}=\langle 1+\langle f\rangle\rangle.$$

For the case $gcd(\bar{h},\ \bar{f})=\bar{f}$, for all $h+\langle f\rangle\in I$, which means $\bar{f}$ divides $\bar{h}$, hence, there exist $w,\ z\in\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ such that $h=wf+az$. Whence

$$h+\langle f\rangle\in\langle a+\langle f\rangle\rangle,\ \ for\ all\ h+\langle f\rangle\in I$$

implying $I\subseteq\langle a+\langle f\rangle\rangle$. Let k be the greatest integer $<t$ such that $I\subseteq\langle a^k+\langle f\rangle\rangle$. Then, as $I\nsubseteq\langle a^{k+1}\langle f\rangle\rangle$, there is a (nonzero) element $h'+\langle f\rangle\in I$ such that $h'+\langle f\rangle\notin\langle a^{k+1}+\langle f\rangle\rangle$. Since $h'+\langle f\rangle\in I\subseteq\langle a^k+\langle f\rangle\rangle$, there exist $w',\ z'\in\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ such that $h'=w'f+a^kz'$. Now $gcd(\bar{z}',\ \bar{f})=1$, or $\bar{f}$. Suppose $gcd(\bar{z}',\ \bar{f})=\bar{f}$, then $\bar{f}$ divides $\bar{z}'$ and so there exist $w'',\ z''\in\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ such that $z'=w''f+az''$. Hence,

$$\begin{aligned} h' &= w'f+a^kz'=w'f+a^k(w''f+az'') \\ &= (w'+a^kw'')f+a^{k+1}z''. \end{aligned}$$

Notes

It follows that $h' + \langle f\rangle \in \langle a^{k+1} + \langle f\rangle\rangle$*, a contradiction. Thus,* $gcd(\bar{z}', \bar{f}) = 1$*. The same argument as above yields that* $z' + \langle f\rangle$ *invertible in* $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle f\rangle}$*, which means that there exists* $z_0 \in \mathbb{F}(p^k\mathbb{Z}, 1)[x]$ *such that*

$$(z' + \langle f\rangle)(z_0 + \langle f\rangle) = 1 + \langle f\rangle.$$

Notes

Therefore

$$\begin{aligned} a^k + \langle f\rangle &= (z_0 + \langle f\rangle)(a^k z' + \langle f\rangle) \\ &= (z_0 + \langle f\rangle)(h' + \langle f\rangle) \in I. \end{aligned}$$

Consequently, $I = \langle a^k + \langle f\rangle\rangle$.

Customarily, for a $m\Theta$ polynomial f of degree k, its reciprocal $m\Theta$ polynomial $x^k f(x^{-1})$ will be denoted by f^*. Thus, for example, if $f(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + a_k x^k$, then

$$\begin{aligned} f^*(x) &= x^k(a_0 + a_1 x^{-1} + \cdots + a_{k-1}x^{-(k-1)} + a_k x^{-k}) \\ &= a_k + a_{k-1}x + \cdots + a_1 x^{k-1} + a_0 x^k. \end{aligned}$$

Moreover, if $f(x)$ is a factor of $x^r - 1$, we denote $\hat{f}(x) = \frac{x^r-1}{f(x)}$.

Theorem 0.7. Assume $\mathbb{F}(p^k\mathbb{Z}, 1)$ *is a finite chain* $m\Theta$ *pseudo field with maximal* $m\Theta$ *ideal* $\langle a\rangle$*, and that* t *is the nilpotency of* a*. Let* $x^r - 1 = f_1 f_2 \cdots f_l$ *be a representation of* $x^r - 1$ *as a product of basic irreducible pairwise-coprime polynomials in* $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$*. Then any ideal in* $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle f\rangle}$ *is a sum of* $m\Theta$ *ideals of the form* $\langle a^j \hat{f}_i + \langle x^r - 1\rangle\rangle$*, where* $0 \le j \le t$*,* $1 \le i \le r$*.*

Proof 0.12. By the Chinese Reminder theorem, we have

$$\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle x^r - 1\rangle} = \frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\cap_{i=1}^{l}\langle f_i\rangle} \cong \bigoplus \sum_{i=1}^{l} \frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle f_i\rangle}.$$

Thus, any $m\Theta$ *ideal* I *of* $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle x^r-1\rangle}$ *is of the form* $\bigoplus\sum_{i=1}^{l} I_i$*, where* I_i *is an* $m\Theta$ *ideal of* $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle f_i\rangle}$*. According to the previous lemma, for* $1 \le i \le r$*,* $I_i = 0$ *or* $I_i = \langle a_k + \langle f_i\rangle\rangle$*, for some* $k \in \{0, \cdots, t-1\}$*. Then* I_i *correspond to* $\langle a^k \hat{f}_i + \langle x^r - 1\rangle\rangle$ *in* $\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle x^r-1\rangle}$*. Consequently,* I *is a sum of ideals of the form* $\langle a^j \hat{f}_i + \langle x^r - 1\rangle\rangle$*.*

Corollary 0.2. Let $\mathbb{F}(p^k\mathbb{Z}, 1)$ be a finite $m\Theta$ pseudo field with maximal $m\Theta$ ideal $\langle a\rangle$, and t be the nilpotency of a. The numbers of quasi-cyclic $m\Theta$ codes over $\mathbb{F}(p^k\mathbb{Z}, 1)$ of length r is $(t+1)^l$, where l is the number of factors in the unique factorization of x^r-1 into a product of monic basic irreducible pairwise coprime $m\Theta$ polynomials.

From now on, in order to simplify notation, we will just write $l_0+l_1x+\cdots+l_{r-1}x^{r-1}$ for the corresponding coset $l_0+l_1x+\cdots+l_{r-1}x^{r-1}+\langle x^r-1\rangle$ in $\frac{\mathbb{F}(p^k\mathbb{Z},1)[x]}{\langle x^r-1\rangle}$

Theorem 0.8. Let C be a quasi-cyclic $m\Theta$ codes of length r over a finite $m\Theta$ pseudo field with maximal $m\Theta$ ideal $\langle a\rangle$, and t be the nilpotency of a. Then there exists a unique family of pairwise coprime monic $m\Theta$ polynomials $F_0, F_1, \cdots, F_t$ in $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ such that $F_0F_1\cdots F_t = x^r-1$ and $C=\langle\widehat{F_1}, a\widehat{F_2}, \cdots, a^{t-1}\widehat{F_t}\rangle$. Moreover

$$|C| = (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{\sum_{i=0}^{t-1}(t-i)deg(F_{i+1})}.$$

Proof 0.13. Let $x^r-1=f_1\cdots f_l$ be the unique factorization of x^r-1 into a product of monic basic irreducible pairwise coprime $m\Theta$ polynomials. C is a direct sum of ideals of the form $\langle a^j\widehat{f_i}\rangle$, where $0\le j\le t$, $1\le i\le l$. After reordering if necessary, we can assume that

$$\begin{aligned} C &= \langle \hat{f}_{k_1+1}\rangle \oplus\cdots\oplus\langle\hat{f}_{k_1+k_2}\rangle\oplus\langle a\hat{f}_{k_1+k_2+1}\rangle\oplus\cdots a\hat{f}_{k_1+k_2+k_3}\rangle\oplus \\ &\quad \cdots\oplus\langle a^{t-1}\hat{f}_{k_1+\cdots+k_t+1}\rangle\oplus\cdots\oplus\langle a^{t-1}\hat{f}_r\rangle \end{aligned}$$

where $k_1,\cdots,k_t\ge 0$ and $k_1+\cdots+k_t+1\le r$. Let $k_0=0$, and k_{t+1} be a nonnegative integer such that $k_1+\cdots+k_t+1\le r$. For $i=0,\cdots,t$, define

$$F_i = f_{k_0+\cdots+k_i+1}\cdots f_{k_0+\cdots+k_i+1}.$$

Then by our construction, it is clear that $F_0,\cdots,F_t$ are pairwise coprime, $F_0\cdots F_t=f_1\cdots f_r=x^r-1$, and

$$C=\langle\widehat{F_1}\rangle\oplus\langle a\widehat{F_2}\rangle\oplus\cdots\oplus\langle a^{t-1}\widehat{F_t}\rangle.$$

To prove the uniqueness, assume $G_0G_1\cdots G_t=x^r-1$ and $C=\langle\widehat{G_1}, a\widehat{G_2}, \cdots, a^{t-1}\widehat{G_t}\rangle$. Then

$$\frac{\mathbb{F}(p^k\mathbb{Z}, 1)[x]}{\langle x^r-1\rangle} = \langle\widehat{G_0}\rangle\oplus\langle\widehat{G_1}\rangle\oplus\cdots\oplus\langle\widehat{G_s}\rangle$$

Notes

thus, $C = \langle\widehat{G_1}\rangle \oplus \langle a\widehat{G_2}\rangle \oplus \cdots \oplus \langle a^{t-1}\widehat{G_s}\rangle$. *Now there exist nonnegative integers* $l_0 = 0, l_1, \cdots, l_{t+1}$ *with* $l_0+l_1+\cdots+l_{t+1} = l$, *and a permutation* $\{f'_1, \cdots, f'_r\}$ *of* $\{f_1, \cdots, f_r\}$ *such that, for* $i = 0, 1, \cdots, t$

$$G_i = f'_{l_0+\cdots+l_i+1} \cdots f'_{l_0+\cdots+l_i+1}.$$

Hence,

$$C = \langle \hat{f'}_{l_1+1}\rangle \oplus \cdots \oplus \langle \hat{f'}_{l_1+l_2}\rangle \oplus \langle a\hat{f'}_{l_1+l_2+1}\rangle \oplus \cdots a\hat{f'}_{l_1+l_2+l_3}\rangle \oplus$$

$$\cdots \oplus \langle a^{t-1}\hat{f'}_{l_1+\cdots+l_t+1}\rangle \oplus \cdots \oplus \langle a^{t-1}\hat{f'}_r\rangle$$

Now for $i = 0, \cdots, t$, *it follows that* $l_i = k_i$, *and, furthermore,*$\{f'_{l_0+\cdots+l_{i+1}}, \cdots, f'_{l_0+\cdots+l_{t+1}}\}$ *is a permutation of* $\{f_{k_0+\cdots+k_{i+1}}, \cdots, f_{k_0+\cdots+k_{t+1}}\}$. *Therefore,* $G_i = F_i$, *for* $i = 0, \cdots, t$.

To calculate the order $|C|$, *note that*

$$C = \langle\widehat{F_1}\rangle \oplus \langle a\widehat{F_2}\rangle \oplus \cdots \oplus \langle a^{t-1}\widehat{F_t}\rangle$$

and for $i = 0, 1, \cdots, t-1$

$$\begin{aligned} |\langle a^i\widehat{F_{i+1}}\rangle| &= \left(\frac{|\mathbb{F}(p^k\mathbb{Z}, 1)|}{|\langle a^{t-i}\rangle|}\right)^{(n-deg\widehat{F_{i+1}})} = \left(\frac{|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|^t}{|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|^i}\right)^{degF_{t+1}} \\ &= (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{(t-i)degF_{t+1}}. \end{aligned}$$

Hence,

$$\begin{aligned} |C| &= |\langle\widehat{F_1}\rangle| \cdot |\langle a\widehat{F_2}\rangle| \cdot \cdots \cdot |\langle a^{t-1}\widehat{F_t}\rangle| \\ &= (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{tdegF_1} \cdot (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{(t-1)degF_2} \cdots (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{degF_t} \\ &= (|\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}|)^{\sum_{i=0}^{t-1}(t-i)deg(F_{i+1})}. \end{aligned}$$

Theorem 0.9. *Let* C *be a quasi-cyclic code of length* r *over a finite chain* $m\Theta$ *pseudo field* $\mathbb{F}(p^k\mathbb{Z}, 1)$, *which has maximal* $m\Theta$ *ideal* $\langle a\rangle$ *and* t *is the nilpotency of* a. *Then there exist polynomials* $g_0, g_1, \cdots, g_{t-1}$ *in* $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ *such that* $C = \langle g_0, ag_1, \cdots, a^{t-1}g_{t-1}\rangle$ *and* $g_{t-1}|g_{t-2}|\cdots|g_1|g_0|(x^r - 1)$.

Proof 0.14. *According to previous theorem, there exists a family of pairwise coprime monic* $m\Theta$ *polynomials* $F_0, F_1, \cdots, F_t$ *in* $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$ *such that* $F_0F_1\cdots F_t = x^r - 1$ *and* $C = \langle\widehat{F_1}, a\widehat{F_2}, \cdots, a^{t-1}\widehat{F_t}\rangle$. *Define*

$$g_i = \begin{cases} F_0F_1\cdots F_t, & \text{if } 0 \le i \le t-2 \\ F_0, & \text{if } i = t-1. \end{cases}$$

Notes

Then clearly $g_{t-1}|g_{t-2}|\cdots|g_1|g_0|(x^r-1)$. *Moreover, for* $0 \le i \le t-1$, *we have*

$$a^i\hat{F}_{i+1} = a^iF_0\cdots F_iF_{i+2}\cdots F_t = a^ig_iF_1\cdots F_i.$$

Therefore, $C \subseteq \langle g_0,\, ag_1,\, \cdots,\, a^{t-1}g_{t-1}\rangle$. *On the other hand,* $g_0 = F_0F_1\cdots F_t \in C$. *Since* F_1, F_2 *are coprime* $m\Theta$ *polynomials in* $\mathbb{F}(p^k\mathbb{Z}, 1)[x]$, *there exist polynomials* $u, v \in \mathbb{F}(p^k\mathbb{Z}, 1)[x]$ *such that* $uF_1 + vF_2 = 1$. *It follows that*

$$\begin{aligned} g_1 &= F_0F_3\cdots F_t = (uF_1+vF_2)F_0F_3\cdots F_t \\ &= uF_0F_1F_3\cdots F_t + cF_0F_2F_3\cdots F_t = u\widehat{F_2} + vg_0 \end{aligned}$$

whence $ag_1 = au\widehat{F_2} + avg_0 \in C$. *Continuing this process, we obtain* $a^ig_i \in C$ *for* $0 \le i \le t-1$, *which implies*

$$\langle g_0,\, ag_1,\, \cdots,\, a^{t-1}g_{t-1}\rangle \subseteq C.$$

Consequently, $C = \langle g_0,\, ag_1,\, \cdots,\, a^{t-1}g_{t-1}\rangle$.

VIII. Conclusion

This note studies the Quasi-Cyclic codes over a finite chain $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$, which leads to the modal structure of the notion Quasi-Cyclic codes over a finite chain pseudo field [3]. It appears that the Structures of Quasi-Cyclic codes of length r over a finite chain $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$ are established when r is not divisible by the characteristic of the residue $m\Theta$ pseudo field $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$. Some cases where r divisible by the characteristic of the residue $m\Theta$ field $\overline{\mathbb{F}(p^k\mathbb{Z}, 1)}$ are also considered.
At the end of this study, some interesting problems remain to be solved:

1. We would like to construct the $m\Theta$ structure of cyclic dual codes and negacyclic codes over finite chain $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$.
2. We would like to define a necessary and sufficient condition for the existence of self-dual cyclic $m\Theta$ codes over a $m\Theta$ pseudo field $\mathbb{F}(p^k\mathbb{Z}, 1)$.

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Notes

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Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

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Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

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Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

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Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

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References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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