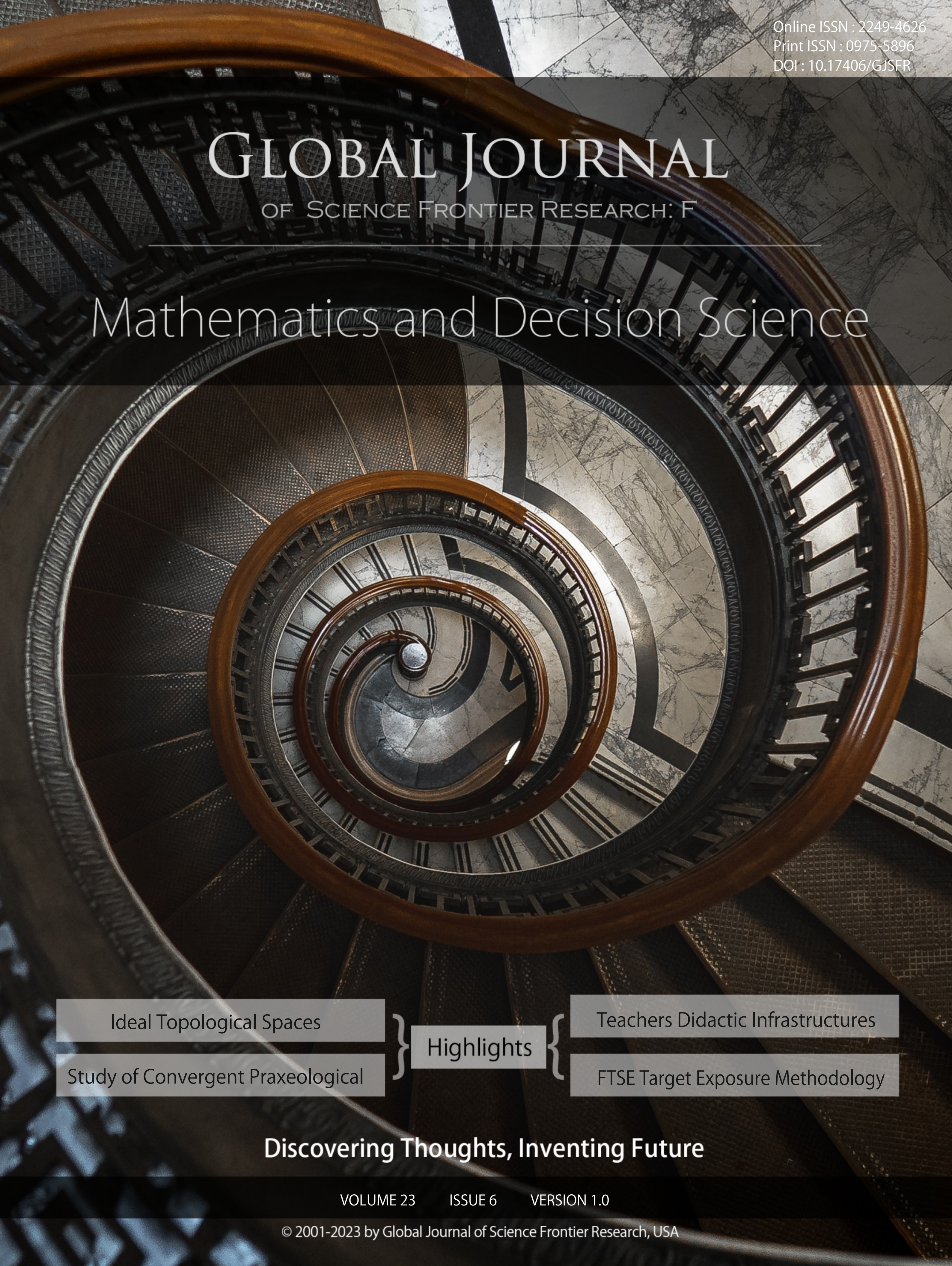

Online ISSN : 2249-4626
Print ISSN : 0975-5896
DOI : 10.17406/GJSFR

Global Journal
of Science Frontier Research: F

Mathematics and Decision Science

Highlights

- Ideal Topological Spaces
- Study of Convergent Praxeological
- Teachers Didactic Infrastructures
- FTSE Target Exposure Methodology

Discovering Thoughts, Inventing Future

VOLUME 23 ISSUE 6 VERSION 1.0

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

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Volume 23 Issue 6 (Ver. 1.0)

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Contents of the Issue

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

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By Julien Riposo & Yang Wang

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GJSFR-F Classification: FOR: 0101

SOLUTIONUNIQUENESSANDCONTINUITYOFTHEFTSETARGETEXPOSUREMETHODOLOGY

Global Journal of Science Frontier Research (F) Volume XXIII Issue VI Version I Year 2023

Solution Uniqueness and Continuity of the FTSE Target Exposure Methodology

Julien Riposo [α] & Yang Wang [σ]

Abstract- The Target Exposure methodology [FTSE] derives a portfolio allocation of assets, each being exposed to multiple factors. We show that, given a set of model parameters and active exposures of the assets to the factors, there exists at most one allocation of the assets. The means to prove this result are (i) mathematical induction on the number of factors, and (ii) a statistical argument averaging the overall exposures of each asset to the considered factors. The model has been set to a system of non-linear exponential functions, and the goal is to prove the existence of at most one solution of this system, as well as its continuity. The theoretical result derived in this paper provides additional insight into the well-adopted Target Exposure methodology and furthers the understanding of this portfolio construction framework that, in many cases, is favored for its weighting transparency.

Keywords: FTSE target exposure; non-linear equations; general topology.

I. Introduction

Since the middle of the twentieth century, the quantitative landscape for modelling the performance of financial asset allocation has been pictured. Concretely, an investor would like to buy or sell some shares of a portfolio constituted of equities, commodities, cryptos, or derivatives, and wishes to allocate efficiently and with risk control.

Markowitz is considered to be the first to have introduced a quantitative theory for allocating assets in an optimized manner, for a given targetted portfolio return [Markowitz]. For such portfolios, performance is measured in terms of portfolio return, while a common metric for risk is its standard deviation. Different metrics are used to measure performance of the portfolio [Sharpe, Riposo]. From this framework, many other quantitative approaches have been developed (see for example [Grinold, Cartea, Brugière]), all aiming at bringing profit to a risky investor. Generally speaking, a paradigm consists in expressing the portfolio return R (vector of real numbers) as follows:

$$R = r_0 + \beta(R_{\mathrm{M}} - r_0) + \epsilon, \tag{1}$$

where r_0 is the return of a risk-free asset (for instance a bond) considered in the portfolio economy, R_{M} is the market return, β is the *exposure* of the market to the investment portfolio, and ϵ is all the information not considered in the first and second terms of this equation. This equation can be proven through the Capital Asset Pricing Model (CAPM) (one of the main building-block articles for the CAPM is [French], and it was introduced by J. Treynor, W.F. Sharpe, J. Lintner, and J. Mossin, independently). In particular, the quantity $R_{\mathrm{M}} - r_0$ is the *risk premium* [Capinski].

However, the Markowitz framework may not be the best one to explain the risk taken by the investor to elaborate her portfolio, as risks are not spcifically identified. In order to address risk dependencies, *factor* models have been introduced (for instance [Brugière, Connor]). If the returns are R, a general factor model writes as:

$$R = A + BF + \mathcal{E}. \tag{2}$$

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Ref

5. [Capinski] Capinski, M.J. and Kopp, E. '*Portfolio Theory and Risk Management*'. Cambridge University Press. 2014.

where A is a constant vector, B is the matrix (number of assets × number of factors) of *factor exposures*, F is the vector (dimension is the number of factors) of *factor characteristics*, and $\mathcal{E}$ is the vector gathering all the information which is not in the first two terms of the above equation. Generally speaking, the factor exposures represent the covariance of the returns with respect to the factor characteristics, and essentially represent how strongly dependent are the returns on the underlying factors (see [Klepfish] for a conciliating estimation of risk factor exposures).

The factor characteristics are of two types. (i) *Endogeneous*: they are statistically derived from the observed returns [FrenchR]; and (ii) *Exogeneous*: they are explanatory variables added to the model, for instance Growth, Price-to-Earning, Price-to-Book, or other fundamental metrics. In the sequence of this paper, we shall focus on the second type of factor characteristics. As an example of such characteristic portfolios, the Fama and French Three Factor Model [Fama] still has some strong supporters today.

However, in the recent decades, most researchers have shown the existence of factor risk premium associated with specific factors as Value, Momentum, Size, Low Volatility, and Quality, see e.g. [Fitzgibbons, Bender, Ghayur]. Many discussions are redefining some factors as actually reflecting one of these five core factors, see e.g. [FrenchR2]. We will not enter the financial context of these five factors, the interested reader can focus on [Zaher].

Earlier studies in the style factors have seen researchers use a long-short approach aiming to capture the pure premium. Much inspired by Markowitz's, optimised portfolios have then been widely used in constructing factor portfolios. Optimization has the flexibility of introducing extra construction considerations simply as constraints without compromising ex-ante objective, providing the problem remains feasible and solvable, e.g. convex [Wilhelm]. Another construction method that has been introduced since the early days of portfolio construction is *Tilting*. Assets under management mostly using Tilting reached $1.45 trillion in December 2021 [Morningstar]. The Target Exposure framework, first introduced to the passive investment community by FTSE Russell indices, is an extension of traditional tilting and which has many practical use cases, e.g. [Wang]. The motivation was to join the transparency and intuition of tilting with the ability to exert explicit control of the portfolio ex-ante outcome that is comparable to optimisation. In short, and as shown in Section 2, the Target Exposure methodology aims at solving a system of exponential functions, giving at final a portfolio allocation (see Equation (6)). In this paper, we are not going to compare different approaches, as this is out of scope, and this will be interesting areas for further studies.

We are focusing on the roots (or solutions) of a non-linear system of equations, the detailed form of which will be discussed in Section 2. In essence, the system of interest writes as:

$$\begin{cases} \mathcal{F}_1(\alpha_1,\dots,\alpha_F) & = & 0 \\ \mathcal{F}_2(\alpha_1,\dots,\alpha_F) & = & 0 \\ & \vdots & \\ \mathcal{F}_F(\alpha_1,\dots,\alpha_F) & = & 0 \end{cases}$$

where the $\mathcal{F}_f$'s, $f \in \{1,2,\dots,F\}$, are sums of product of exponential functions, see Equations (6). We want to check rather if there exists common root(s) for all the $\mathcal{F}_f$, $f \in \{1,2,\dots,F\}$. Many methods have already been developped to solve such non-linear systems, by means of Taylor's polynomial [Burden], quadrature formula [Darvishi, Babajee], or homotopy perturbation method [Golbabai]. A gradient decent method could also be applied to solve such system [Hao]. Using more contemporary approaches, Machine Learning regression technics are applied to estimate solutions of parametrized non-linear system [Freno], or quantum methods allow enhancing the diversity of the solutions, and avoid local minima [Rizk-Allah], in the spirit of simulated annealing. In the more specific context of our study, the functions $\mathcal{F}_f$, $f \in \{1,2,\dots,F\}$, are sum of product of exponentials, see Equation (6). When there is only one factor (i.e. only one unknown variable), we obtain 'generalized Dirichlet polynomials' (although they are not polynomials), which write as:

$$\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i \, e^{b_i \alpha},$$

where the a's and b's are real numbers. In [Jameson], the number of roots of such polynomials are found by the means of the Descartes' rule of signs. By re-ordering this sum such that $b_1 > \cdots > b_N$ (supposing they are all distinct), the number of roots is linked to the number of sign changes in the thus obtained ordered family $\{a_1,\dots,a_N\}$. Our situation is more complicated, since each term is a product of exponentials, each exponent

Ref

12. [FrenchR] French, R. '*Latent Factors in Equity Returns: How Many Are There and What Are They?*'. Journal of Portfolio Management, jpm.2021.1.296

being one unknown. In addition, we will see that we do not need Descartes' rule of signs in our case. To some extend, our problem is a generalization of the one enhanced by the 'generalized Dirichlet polynomials'. While we are not focusing on estimating the solution(s), we are interested in proving that there is at most one solution.

Notes

The rest of this paper is depicted as follows. Section 2 sets the Target Exposure Problem, as the general system of exponential functions. Section 3 solves the Target Exposure Problem when considering only *one* factor. Finally, Section 4 solves by mathematical induction on the number of factors, the general problem under the statistical approximation of the *mean*, assuming that there are factors for which linear combinations of Z-scores do not depend on the considered assets. Section 5 discusses this hypothesis and illustrates the main result, and finally, Section 6 shows a numerical illustration of the main finding of this paper, while Section 7 concludes the paper.

II. The Target Exposure Methodology

We consider a set of $N \in \mathbb{N}^*$ assets of an index which we aim at deriving the weights from the Target Exposure methodology, with $F \in \mathbb{N}^*$ factors. We univoquely assign each asset (resp. factor) to an integer $i \in \{1, \ldots, N\}$ (resp. $f \in \{1, \ldots, F\}$) without any particular order.

The Target Exposure methodology consists in deriving the weight for asset $i \in \{1, \ldots, N\}$ as follows:

$$W_i(\alpha) = \frac{\mathcal{M}_i \prod_{f=1}^{F} s_{i,f}^{\alpha_f}}{\sum_{k=1}^{N} \mathcal{M}_k \prod_{f=1}^{F} s_{k,f}^{\alpha_f}}, \tag{3}$$

where $\mathcal{M}_i$ is the benchmark weight for asset i, typically weight by free float adjusted market capitalisation (but it can more generally be a benchmark of any type, and have no particular assumptions except they are > 0); $s_{i,f} = S(Z_{i,f})$, where S is an increasing positive function (e.g. exponential) applied to the Z-score $Z_{i,f} \in \mathbb{R}$, being the rescaled exposure of asset i exposed to factor f; α_f is the *strength* for factor f, which is an unknown real number; and α is the vector of all the strengths, thus of dimension F.

$W_i(\alpha)$ is the weight for instrument i, and is a function of the tilt strength vector $\alpha \in \mathbb{R}^F$. In practice, the non-linear system mentioned above arises when investors have a set of expected factor exposures as portfolio objective, and try to find a set of strengths that leads to portfolio weights yielding the desired portfolio factor exposures.

More specifically, the *active exposure* $A_f \in \mathbb{R}$ to factor f is defined as the portfolio exposure in exccess of the benchmark exposures and is given by:

$$A_f = \sum_{i=1}^{N} Z_{i,f} \left(W_i(\alpha) - \mathcal{M}_i\right), \quad \forall f \in \{1, \ldots, F\}. \tag{4}$$

Equation (4) is an equation whose unknown is α, vector of F elements, and there are F such equations. We thus have F unknown variables for F equations.

We set the *target index exposure* as:

$$\bar{A}_f = A_f + \sum_{i=1}^{N} \mathcal{M}_i Z_{i,f} = \sum_{i=1}^{N} W_i(\alpha) Z_{i,f}.$$

In practice, investors express their factor exposures expectations via active exposures. Hence $A_f \in \mathbb{R}$ and

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consequently $\bar{A}_f \in \mathbb{R}$ are fixed as parameters of the model. Replacing (3) into this equation leads to:

$$\bar{A}_f = \sum_{i=1}^{N} Z_{i,f} \frac{\mathcal{M}_i \prod_{f=1}^{F} s_{i,f}^{\alpha_f}}{\sum_{k=1}^{N} \mathcal{M}_k \prod_{f=1}^{F} s_{k,f}^{\alpha_f}}.$$

Hence, by multiplying by the denominator of the right-hand-side:

$$\sum_{i=1}^{N} \left(\bar{A}_f - Z_{i,f}\right) \mathcal{M}_i \prod_{h=1}^{F} s_{i,h}^{\alpha_h} = 0.$$

We now set:

$$\begin{cases} a_{i,f} = & \left(\bar{A}_f - Z_{i,f}\right) \mathcal{M}_i \\ b_{i,f} = & \ln s_{i,f} \end{cases} \tag{5}$$

The previous equation becomes:

$$\sum_{i=1}^{N} a_{i,f} \, \mathrm{e}^{\sum_{h=1}^{F} b_{i,h}\alpha_h} = 0, \quad f \in \{1, \dots, F\}. \tag{6}$$

Note that $b_{i,f} = Z_{i,f}$ if and only if S is the exponential function.

We end up with F equations, each being a weighted sum of exponential functions. Thus, this is a system of F equations with F unknowns, which we call a *Target Exposure Problem*. Each equation is a sum of N terms, and to each of these terms we have the variable α_h involved, for all $h \in \{1, \dots, F\}$. In addition, it is assumed that all the a's and b's are non-zero numbers and all distinct. We also assume that the matrices $(a_{i,f})_{1\le i\le N, 1\le f\le F}$ and $(b_{i,f})_{1\le i\le N, 1\le f\le F}$ have their rows and columns linearly independent (otherwise we remove the redundant rows and columns). It is worth pointing out that we systematically assume that these families are connected through Equations (5) in the rest of this paper.

From now on, it is useful to write, for all $f \in \{1, \dots, F\}$, the following function:

$$\mathcal{F}_f(\alpha) = \sum_{i=1}^{N} a_{i,f} \, \mathrm{e}^{\sum_{h=1}^{F} b_{i,h}\alpha_h},$$

so that the goal is to find the number of solution(s) for Equation (6), given by:

$$\mathcal{F}_f(\alpha) = 0, \quad \forall f \in \{1, \dots, F\}.$$

We note that the functions $\mathcal{F}_f$ are all continuous, and C^∞-differentiable on $\mathbb{R}^F$.

III. Unique and Continuous Solution with One Factor

We consider the function given by:

$$\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i \, \mathrm{e}^{b_i \alpha},$$

Notes

where the family of numbers $(a_i)_{i\in\{1,...,N\}}$ and $(b_i)_{i\in\{1,...,N\}}$ are non-zero, mutually distinct and as defined by Equations (5) (with $f = 1$ omitted).

The following theorem can be proven by using the Descartes' rule of signs applied to 'generalized Dirichlet polynomials'. We however prove it without using the rule.

Proposition 1. The function $\mathcal{F}$ has exactly one root on $\mathbb{R}$. In addition, the function $\mathcal{F}$ has exactly one root on $\mathbb{R}_+$ if and only if the following condition is satisfied:

Notes

$$\bar{A}_1 \geq \frac{\sum_{i=1}^{N} Z_i \mathcal{M}_i}{\sum_{i=1}^{N} \mathcal{M}_i}.$$

The inequality condition states that the target index exposure should be higher than the market capitalisation weighted average of the Z-scores, so that the strength surely is positive.

Proof

Without loss of generality, we reorder the terms and set:

$$b_1 > \cdots > b_N.$$

Bearing in mind the constraints given by Equations (5), we then have:

$$\exists! k \in \{1,\ldots,N-1\} \quad \forall (l,j) \in \{1,\ldots,k\} \times \{k+1,\ldots,N\} \quad a_l < 0 \text{ and } a_j > 0. \tag{7}$$

We introduce the number b such that $b_k > b > b_{k+1}$, and

$$\tilde{\mathcal{F}}(\alpha) = \mathrm{e}^{-b\alpha}\,\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i\,\mathrm{e}^{(b_i-b)\alpha}.$$

The functions $\tilde{\mathcal{F}}$ and $\mathcal{F}$ have the same roots. In addition, the function $\tilde{\mathcal{F}}$ is of class C^1 (even C^∞), and therefore its derivative is given by:

$$\tilde{\mathcal{F}}'(\alpha) = \sum_{i=1}^{N} a_i\,(b_i - b)\,\mathrm{e}^{(b_i-b)\alpha}.$$

Now, bearing Equation (7) in mind, we note that $a_i(b_i - b) < 0$, for all $i \in \{1,\ldots,N\}$. Indeed, if $i \leq k$, then $a_i < 0$ and $b_i - b > b_i - b_k \geq 0$; and if $i > k$, then $a_i > 0$ and $b_i - b < b_i - b_{k+1} \leq 0$. Thus, we have $\tilde{\mathcal{F}}'(\alpha) < 0$ for any $\alpha \in \mathbb{R}$, hence the function $\tilde{\mathcal{F}}$ is strictly decreasing on $\mathbb{R}$. In addition, we note that:

$$\tilde{\mathcal{F}}(\alpha) \underset{\alpha\to+\infty}{\sim} a_1\,\mathrm{e}^{(b_1-b)\alpha} < 0.$$

We additionally have:

$$\tilde{\mathcal{F}}(\alpha) \underset{\alpha\to-\infty}{\sim} a_N\,\mathrm{e}^{(b_N-b)\alpha} > 0.$$

Henceforth, we conclude that $\mathcal{F}$ has a unique root on $\mathbb{R}$.

Finally, we have:

$$\tilde{\mathcal{F}}(0) = \mathcal{F}(0) = \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} \left(\bar{A}_1 - Z_i\right) \mathcal{M}_i = \bar{A}_1 \sum_{i=1}^{N} \mathcal{M}_i - \sum_{i=1}^{N} Z_i \mathcal{M}_i$$

$$= \left(\sum_{i=1}^{N} \mathcal{M}_i\right) \left(\bar{A}_1 - \frac{\sum_{i=1}^{N} Z_i \mathcal{M}_i}{\sum_{i=1}^{N} \mathcal{M}_i}\right).$$

□

This concludes the proof.

Proposition 2. The root of $\mathcal{F}$ is a continuous function of the a_i's and b_i's.

Proof
In light of the proof of Proposition 1, we assume that $\mathcal{F}$ is strictly decreasing (otherwise focus on $\tilde{\mathcal{F}}$ instead of $\mathcal{F}$).

If the root $\bar{\alpha}$ is on $\mathbb{R}_+$, then we have:

$$\mathcal{F}(\bar{\alpha}) = 0 \Leftrightarrow \sum_{i=1}^{N} a_i \, e^{b_i \bar{\alpha}} = 0 \Leftrightarrow a_1 \, e^{b_1 \bar{\alpha}} = -\left(\sum_{i=2}^{N} a_i \, e^{b_i \bar{\alpha}}\right) \Rightarrow |a_1| \, e^{b_1 \bar{\alpha}} \le \left(\sum_{i=2}^{N} |a_i|\right) e^{b_2 \bar{\alpha}},$$

hence, since $\sum_{i=2}^{N} |a_i| \le \sum_{i=1}^{N} |a_i|$, we have:

$$\bar{\alpha} \le \frac{1}{b_1 - b_2} \ln\left(\frac{\sum_{i=1}^{N} |a_i|}{|a_1|}\right).$$

If the root $\bar{\alpha}$ is on $\mathbb{R}_-$, then, we have:

$$\mathcal{F}(\bar{\alpha}) = 0 \Leftrightarrow a_N \, e^{b_N \bar{\alpha}} = -\left(\sum_{i=1}^{N-1} a_i \, e^{b_i \bar{\alpha}}\right) \Rightarrow |a_N| \, e^{b_N \bar{\alpha}} \le \left(\sum_{i=1}^{N-1} |a_i|\right) e^{b_{N-1} \bar{\alpha}},$$

hence:

$$\bar{\alpha}\,(b_N - b_{N-1}) = (-\bar{\alpha})\,(b_{N-1} - b_N) \le \ln\left(\frac{\sum_{i=1}^{N} |a_i|}{|a_N|}\right) \Leftrightarrow -\bar{\alpha} \le \frac{1}{b_{N-1} - b_N} \ln\left(\frac{\sum_{i=1}^{N} |a_i|}{|a_N|}\right).$$

In any case, we have:

$$|\bar{\alpha}| \le \max\left(\frac{1}{b_1 - b_2} \ln\left(\frac{\sum_{i=1}^{N} |a_i|}{|a_1|}\right), \frac{1}{b_{N-1} - b_N} \ln\left(\frac{\sum_{i=1}^{N} |a_i|}{|a_N|}\right)\right) \overset{\text{def}}{=} K.$$

Let $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ be two sequences of numbers converging to a_i and b_i, respectively, and for each $i \in \{1, \ldots, N\}$, then set:

$$K_n = \max\left(\frac{1}{b_{1,n} - b_{2,n}} \ln\left(\frac{\sum_{i=1}^{N} |a_{i,n}|}{|a_{1,n}|}\right), \frac{1}{b_{N-1,n} - b_{N,n}} \ln\left(\frac{\sum_{i=1}^{N} |a_{i,n}|}{|a_{N,n}|}\right)\right),$$

which converges to K.

We set:

$$\mathcal{F}_n(\alpha) = \sum_{i=1}^{N} a_{i,n}\, \mathrm{e}^{b_{i,n}\alpha},$$

and we call $\bar{\alpha}_n$ the root of the function $\mathcal{F}_n$, function of class C^∞.

Notes

The sequence $(K_n)_{n\in\mathbb{N}}$ is converging to K, so that it is bounded, so there exists K_{Max} such that:

$$\forall n \in \mathbb{N} \quad |\bar{\alpha}_n| \leq K_{\mathrm{Max}}.$$

The sequence of functions $(\mathcal{F}_n^{(k)})_{n\in\mathbb{N}}$ converges to $\mathcal{F}^{(k)}$ (k^{th} derivative), for any $k \in \mathbb{N}$. Since $\mathcal{F}_n$ and $\mathcal{F}$ are functions of class C^∞, we can use the Hadamard lemma: there exist functions $\mathcal{G}_n$ and $\mathcal{G}$ (they have no real root and are both of class C^∞) such that:

$$\left\{ \begin{array}{rcl} \mathcal{F}_n(\alpha) & = & (\alpha - \bar{\alpha}_n)\, \mathcal{G}_n(\alpha) \\ \mathcal{F}(\alpha) & = & (\alpha - \bar{\alpha})\, \mathcal{G}(\alpha) \end{array} \right. \tag{8}$$

In fact, the Hadamard lemma allows writing:

$$\left\{ \begin{array}{rcl} \mathcal{G}_n(\alpha) & = & \displaystyle\int_0^1 \mathcal{F}_n'(\bar{\alpha}_n + t(\alpha - \bar{\alpha}_n))\, \mathrm{d}t \\ \mathcal{G}(\alpha) & = & \displaystyle\int_0^1 \mathcal{F}'(\bar{\alpha} + t(\alpha - \bar{\alpha}))\, \mathrm{d}t \end{array} \right.$$

Clearly, we have:

$$|\mathcal{F}_n'(\alpha)| \leq \sum_{i=1}^{N} |a_{i,n}\, b_{i,n}|\, \mathrm{e}^{b_{i,n}\alpha},$$

so that $\mathcal{F}_n'$ is dominated by an integrable function on the compact $[-K_{\mathrm{Max}}, K_{\mathrm{Max}}]$. Therefore, from the Dominated Convergence Theorem, we deduce that the sequence of functions $(\mathcal{G}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{G}$.

We set $\epsilon > 0$ and we want to prove that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, the root $\bar{\alpha}_n$ of $\mathcal{F}_n$ is contained in $]\bar{\alpha} - \epsilon, \bar{\alpha} + \epsilon[$, which will prove continuity.

By a *reductio ad absurdum*, we assume that for any n_0, there exists $n \geq n_0$ such that $\bar{\alpha}_n \notin]\bar{\alpha} - \epsilon, \bar{\alpha} + \epsilon[$. This means that we can extract of sub-sequence $(\mathcal{F}_{\psi(n)})_{n\in\mathbb{N}}$ (ψ is an increasing function) such that:

$$|\bar{\alpha} - \bar{\alpha}_{\psi(n)}| > \epsilon.$$

The sequence $(\bar{\alpha}_{\psi(n)})_{n\in\mathbb{N}}$ takes its values on the compact set $[-K_{\mathrm{Max}}, K_{\mathrm{Max}}]$, and we can take another sub-sequence $(\bar{\alpha}_{\phi\circ\psi(n)})_{n\in\mathbb{N}}$ (ϕ is an increasing function) such that the sequence $(\bar{\alpha}_{\phi\circ\psi(n)})_{n\in\mathbb{N}}$ converges to $\beta \in [-K_{\mathrm{Max}}, K_{\mathrm{Max}}]$ (Bolzano-Weierstrass property). In particular, we have:

$$|\bar{\alpha} - \bar{\alpha}_{\phi\circ\psi(n)}| > \epsilon$$

and

$$|\bar{\alpha} - \beta| > \epsilon.$$

This proves that $\bar{\alpha} \neq \beta$.

We consider the sequence $(\mathcal{F}_{\phi\circ\psi(n)})_{n\in\mathbb{N}}$ given by $\mathcal{F}_{\phi\circ\psi(n)}(\alpha) = (\alpha - \bar{\alpha}_{\phi\circ\psi(n)})\,\mathcal{G}_{\phi\circ\psi(n)}(\alpha)$. This sequence converges to $\mathcal{F}(\alpha) = (\alpha - \beta)\,\mathcal{G}(\alpha)$. But then $\bar{\alpha} = \beta$, which is absurd. This concludes the proof. □

Notes

IV. Unique and Continuous Solution with Multiple Factors

This section is the core of the paper. We come back to the most general case, given by Equation (6), with $F > 1$. We first introduce one definition and two lemmas.

Definition 1 (Mean approximation). *Let $F \in \mathbb{N}^* \setminus \{1\}$. Suppose $f \in \{1, \ldots, F\}$, and $(c_1, \ldots, c_{f-1}, c_{f+1}, \ldots, c_F)$ a point on a compact set of $\mathbb{R}^{F-1}$. We say that the family of numbers $(b_{i,f})_{i\in\{1,\ldots,N\}}$ satisfies the f-*Mean approximation *if:*

$$\exists M \in \mathbb{R} \quad \forall i \in \{1, \ldots, N\} \quad \exists \epsilon_i \in \mathbb{R} \quad \sum_{\substack{h=1\\h\neq f}}^{F} b_{i,h}\, c_h = M + \epsilon_i.$$

Note that M does not depend on i, which is the main advantage of this notion, as we are going to see, but obviously depends on the vector $(c_1, \ldots, c_{f-1}, c_{f+1}, \ldots, c_F)$. However, the second term ϵ_i is supposed to be small, that is $\epsilon_i = o(1)$ (usual Landau's little-o notation). In practice, this means that the $b_{i,h}$'s have the same magnitude order, for any instrument, *perhaps* except for some factor f. We refer the reader to Section 5 for a deeper discussion on this approximation.

Lemma 1. We consider the function $\mathcal{F}$ given by:

$$\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i\, e^{b_i\alpha + \epsilon_i},$$

where the a_i's and b_i's are the numbers as defined in Section 2, and $\epsilon_i = o(1)$. Then the function $\mathcal{F}$ has at most one root on any compact set of $\mathbb{R}$, which is the same as the function

$$\alpha \mapsto \sum_{i=1}^{N} a_i\, e^{b_i\alpha}.$$

Proof

We follow the same steps as the ones for Proposition 1. At some point, we have

$$\tilde{\mathcal{F}}(\alpha) = e^{-b\alpha}\,\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i\, e^{(b_i - b)\alpha + \epsilon_i}.$$

Hence, we have:

$$\tilde{\mathcal{F}}'(\alpha) = \sum_{i=1}^{N} a_i\,(b_i - b)\, e^{(b_i-b)\alpha}\, e^{\epsilon_i} = \sum_{i=1}^{N} a_i\,(b_i - b)\, e^{(b_i-b)\alpha}\,(1 + \epsilon_i)\,.$$

Notes

This means that inside the sum, the second term $a_i\,(b_i - b)\,\mathrm{e}^{(b_i-b)\alpha}\,\epsilon_i$ is negligeable in comparison with $a_i\,(b_i - b)\,\mathrm{e}^{(b_i-b)\alpha}$, so that we still have $\tilde{\mathcal{F}}'(\alpha) < 0$ for any α on a compact set of $\mathbb{R}$. We conclude. □

The fact that we restrict on a compact set of $\mathbb{R}$, i.e. on an interval of the form $[c, d]$ $(c < d)$ is essential: we have $\sum_{i=1}^{N} a_i = 0$ if and only if 0 is the root of $\mathcal{F}$ and any compact set of $\mathbb{R}$ containing 0 can be chosen. If $\sum_{i=1}^{N} a_i \neq 0$, then assume without any loss of generality that the root is positive. Suppose $c > 0$. The exponent $(b_i - b)\alpha$ is in $[(b_i - b)c, (b_i - b)d]$ if $b_i > b$ or in $[(b_i - b)d, (b_i - b)c]$ if $b_i < b$; thus, since these two last segments do not contain 0, ϵ_i can indeed be sufficiently small in comparison with $|b_i - b|\alpha$. If now $c \leq 0$, then there exists $c' > 0$ such that the root is contained in the compact $[c', d]$, and we come back to the previous case: ϵ_i can finally be chosen as small as we want.

Lemma 2. Let $F \geq 1$. Any compact set $\mathcal{K}$ of $\mathbb{R}^F$ is included into a Cartesian product of closed intervals.

Any compact set of $\mathbb{R}^F$ is not necessarily a Cartesian product of compact sets (think of a disk), which is why Lemma 2 is going to be useful.

Proof

We endow $\mathbb{R}^F$ with the usual Euclidean metric, and with its canonical basis, so that any element $x \in \mathbb{R}^F$ can be written as $x = (x_1, \ldots, x_F)$. The compact set $\mathcal{K}$ can be parametrized with specific coordinates. In particular, we can write

$$\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_F,$$

where $\mathcal{K}_i$ is a part of $\mathbb{R}$. Since $\mathcal{K}$ is compact, it is bounded, and $\mathcal{K}_i$ is bounded as well, for all $i \in \{1, \ldots, F\}$. Therefore, $\mathcal{K}_i$ admits a minimum (resp. maximum) number $c_i \in \mathbb{R}$ (resp. $d_i \in \mathbb{R}$) in the sense that $c_i \leq x_i$ (resp. $d_i \geq x_i$), for any $x_i \in \mathcal{K}_i$. Therefore, we have $\mathcal{K}_i \subset [c_i, d_i]$, which is compact on $\mathbb{R}$. Hence $\mathcal{K} \subset [c_1, d_1] \times \cdots \times [c_F, d_F]$, concluding the proof. □

Remark 1. The previous lemma implies that any compact set $\mathcal{K}$ of $\mathbb{R}^F$ verifies

$$\mathcal{K} \subset \Gamma \times \mathcal{C},$$

where Γ is a compact set of $\mathbb{R}$ and $\mathcal{C}$ is a compact set of $\mathbb{R}^{F-1}$.

Equipped with the Mean approximation and the lemmas, the main result of the paper is the following.

Theorem 1 (Solution Uniqueness and Continuity). *Under the f-Mean approximation (see Definition 1) for some $f \in \{1, \ldots, F\}$, $F \in \mathbb{N}^*$, the system given by Equations (6) with constraints given by Equations (5) has at most one root on any compact set of $\mathbb{R}^F$. In addition, the root, when it exists, is a continuous function of the a's and b's.*

Proof

We can prove the result by induction on F.

Initiation

This case has been treated when $F = 1$. This is Proposition 1 and Proposition 2. In addition, the Mean approximation is trivially verified: the sum is zero, and $M = \epsilon_i = 0$ for any $i \in \{1, \ldots, N\}$.

Heredity

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We assume the result to be true for any system of $F-1 \geq 1$ equations and unknowns.

One the one hand, let $f \in \{1,\dots,F\}$ be the factor corresponding to the Mean approximation, and we focus on the f^{th} function $\mathcal{F}_f$. We note that

$$\mathcal{F}_f(\alpha_1,\dots,\alpha_f,\dots,\alpha_F) = \sum_{i=1}^{N} \left(a_{i,f}\, \mathrm{e}^{\sum_{\substack{l=1 \\ l\neq f}}^{F} b_{i,l}\alpha_l} \right) \mathrm{e}^{b_{i,f}\alpha_f}.$$

Without loss of generality, we reorder the terms so that

$$b_{1,f} > \cdots > b_{N,f},$$

which implies that

$$\exists! k \in \{1,\dots,N-1\} \quad \forall (l,j) \in \{1,\dots,k\} \times \{k+1,\dots,N\} \quad a_{l,f} < 0 \text{ and } a_{j,f} > 0. \tag{9}$$

First, we fix the variables $(\alpha_1,\dots,\alpha_{f-1},\alpha_{f+1},\dots,\alpha_F) \in \mathbb{R}^{F-1}$, say

$$(\alpha_1,\dots,\alpha_{f-1},\alpha_{f+1},\dots,\alpha_F) = (c_1,\dots,c_{f-1},c_{f+1},\dots,c_F),$$

and apply Proposition 1. Thus, there exists exactly one solution $\bar{\alpha}_f(c_1,\dots,c_{f-1},c_{f+1},\dots,c_F) \in \mathbb{R}$ such that

$$\begin{aligned} \mathcal{F}_f(c_1,\dots,c_{f-1},\bar{\alpha}_f(c_1,\dots,c_{f-1},c_{f+1},\dots,c_F),c_{f+1},\dots,c_F) = 0, \\ \forall (c_1,\dots,c_{f-1},c_{f+1},\dots,c_F) \in \mathbb{R}^{F-1}. \end{aligned} \tag{10}$$

On the other hand, we consider the $F-1$ equations, by omitting the f^{th} one:

$$\mathcal{F}_h(\alpha_1,\dots,\alpha_{f-1},\alpha_f,\alpha_{f+1},\dots,\alpha_F) = \sum_{i=1}^{N} a_{i,h}\, \mathrm{e}^{\sum_{l=1}^{F} b_{i,l}\alpha_l}, \quad h \in \{1,\dots,f-1,f+1,\dots,F\}.$$

We have

$$\mathcal{F}_h(\alpha_1,\dots,\alpha_{f-1},\alpha_f,\alpha_{f+1},\dots,\alpha_F) = \sum_{i=1}^{N} \left(a_{i,h}\, \mathrm{e}^{b_{i,f}\alpha_f} \right) \mathrm{e}^{\sum_{\substack{l=1 \\ l\neq f}}^{F} b_{i,l}\alpha_l}.$$

Thus, we indeed fix $\alpha_f = c_f \in \mathbb{R}$, and, for any $h \in \{1,\dots,f-1,f+1,\dots,F\}$, let $\mathcal{H}_h^f$ be the function given by:

$$\mathcal{H}_h^f(\alpha_1,\dots,\alpha_{f-1},\alpha_{f+1},\dots,\alpha_F) = \mathcal{F}_h(\alpha_1,\dots,\alpha_{f-1},c_f,\alpha_{f+1},\dots,\alpha_F).$$

The induction hypothesis allows to assert that the system formed by the $F-1$ equalities

$$\mathcal{H}_h^f(\alpha_1,\dots,\alpha_{f-1},\alpha_{f+1},\dots,\alpha_F) = 0, \quad \forall h \in \{1,\dots,f-1,f+1,\dots,F\},$$

admits at most one solution

$$\alpha^*(c_f) \stackrel{\text{def}}{=} \left(\alpha_1^*(c_f),\dots,\alpha_{f-1}^*(c_f),\alpha_{f+1}^*(c_f),\dots,\alpha_F^*(c_f)\right) \in \mathcal{C}(c_f) \subset \mathbb{R}^{F-1} \tag{11}$$

where $\mathcal{C}(c_f)$ is a compact set of $\mathbb{R}^{F-1}$ depending on c_f (the matrices of the a's and b's have their rows and columns linearly independent). Here, we note that the α_h^*'s are *functions* of c_f, hence the above system can be

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seen as a parametrized system of equations.

Thus, we have

$$\mathcal{F}_h\left(\alpha_1^*(c_f),\ldots,\alpha_{f-1}^*(c_f),c_f,\alpha_{f+1}^*(c_f),\ldots,\alpha_F^*(c_f)\right)=0,\quad \forall c_f\in\mathbb{R}. \tag{12}$$

Notes

In light of Equation (10), the task is now to prove the eventual existence of $\alpha_f \in \mathbb{R}$ (i.e. one particular fixed value c_f) such that

$$\bar{\mathcal{F}}_f(\alpha_f)\stackrel{\text{def}}{=}\mathcal{F}_f\left(\alpha_1^*(\alpha_f),\ldots,\alpha_{f-1}^*(\alpha_f),\alpha_f,\alpha_{f+1}^*(\alpha_f),\ldots,\alpha_F^*(\alpha_f)\right)=0. \tag{13}$$

Indeed, by fixing $\alpha_f \in \mathbb{R}$, in order to obtain Equation (13), just choose

$$(c_1,\ldots,c_{f-1},c_{f+1},\ldots,c_F)=\left(\alpha_1^*(\alpha_f),\ldots,\alpha_{f-1}^*(\alpha_f),\alpha_{f+1}^*(\alpha_f),\ldots,\alpha_F^*(\alpha_f)\right)$$

to insert into Equation (10), and then we must prove the existence of an α_f such that

$$\bar{\alpha}_f\left(\alpha_1^*(\alpha_f),\ldots,\alpha_{f-1}^*(\alpha_f),\alpha_{f+1}^*(\alpha_f),\ldots,\alpha_F^*(\alpha_f)\right)=\alpha_f,$$

so that we will have the same function arguments as the ones in Equation (13), concluding the uniqueness of the root. Thus, in the rest of this proof, we will focus on the function $\bar{\mathcal{F}}_f$.

We have

$$\bar{\mathcal{F}}_f(\alpha_f)=\sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f)}\right)\mathrm{e}^{b_{i,f}\alpha_f}. \tag{14}$$

By induction, the function α_h^* is continuous on $\mathbb{R}$, for all $h\in\{1,\ldots,f-1,f+1,\ldots,F\}$. Thus, the function α^* (see Equation (11)) is continuous on $\mathbb{R}$, and $(\alpha^*)^{-1}(\mathcal{C}(\alpha_f))$ is a compact set of $\mathbb{R}$, for any $\alpha_f\in\mathbb{R}$. Therefore, α_f can always be defined on a compact set of $\mathbb{R}$ without loss of generality.

Let $\alpha_f\in\Gamma$ where Γ is an arbitrary compact set of $\mathbb{R}$, and let $\mathcal{C}$ be an arbitrary compact set of $\mathbb{R}^{F-1}$. If $\Gamma\cap(\alpha^*)^{-1}(\mathcal{C})=\emptyset$, then there is no $\alpha_f\in\Gamma$ such that $\alpha_f\in(\alpha^*)^{-1}(\mathcal{C})\Leftrightarrow\alpha^*(\alpha_f)\in\mathcal{C}$, and there is no common root of the functions $\mathcal{F}_h$'s and $\mathcal{F}_f$ on the compact set $\Gamma\times\mathcal{C}$ of $\mathbb{R}^F$. Thus, we now assume that $\Gamma\cap(\alpha^*)^{-1}(\mathcal{C})\stackrel{\text{def}}{=}\Gamma'\neq\emptyset$.

The function α_h^* is continuous on Γ', and so as

$$\alpha\mapsto a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha)},$$

for any $i\in\{1,\ldots,N\}$.

Then this last function is bounded on Γ' and reaches its bounds (using Weierstrass' Extreme Value Theorem, since Γ' is compact), that is

$$\exists\alpha_f^{\mathrm{L}},\alpha_f^{\mathrm{U}}\in\Gamma'\quad a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{L}})}\leq a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f)}\leq a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{U}})},\quad\forall\alpha_f\in\Gamma'.$$

Therefore, by multiplying by $\mathrm{e}^{b_{i,f}\alpha_f}>0$ and summing over i, we have

$$\sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{L}})}\right)\mathrm{e}^{b_{i,f}\alpha_f}\leq\bar{\mathcal{F}}_f(\alpha_f)\leq\sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\l\neq f}}^{F}b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{U}})}\right)\mathrm{e}^{b_{i,f}\alpha_f},\quad\forall\alpha_f\in\Gamma'.$$

We now use the Mean approximation, see Definition 1:

$$\begin{cases} \exists M \in \mathbb{R} \quad \forall i \in \{1,\dots,N\} \quad \exists \epsilon_i \in \mathbb{R} \quad \sum_{\substack{l=1\\ l\neq f}}^{F} b_{i,l}\,\alpha_l^*\left(\alpha_f^{\mathrm{U}}\right) = & M + \epsilon_i. \\ \exists m \in \mathbb{R} \quad \forall i \in \{1,\dots,N\} \quad \exists \eta_i \in \mathbb{R} \quad \sum_{\substack{l=1\\ l\neq f}}^{F} b_{i,l}\,\alpha_l^*\left(\alpha_f^{\mathrm{L}}\right) = & m + \eta_i. \end{cases}$$

Regarding the upper bound, we have

$$\sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\ l\neq f}}^{F} b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{U}})}\right)\mathrm{e}^{b_{i,f}\alpha_f} = \sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{M+\epsilon_i}\right)\mathrm{e}^{b_{i,f}\alpha_f} = \mathrm{e}^{M}\sum_{i=1}^{N} a_{i,f}\,\mathrm{e}^{b_{i,f}\alpha_f+\epsilon_i},$$

and regarding the lower bound, we have as well

$$\sum_{i=1}^{N}\left(a_{i,f}\,\mathrm{e}^{\sum_{\substack{l=1\\ l\neq f}}^{F} b_{i,l}\alpha_l^*(\alpha_f^{\mathrm{L}})}\right)\mathrm{e}^{b_{i,f}\alpha_f} = \mathrm{e}^{m}\sum_{i=1}^{N} a_{i,f}\,\mathrm{e}^{b_{i,f}\alpha_f+\eta_i},$$

hence

$$\mathrm{e}^{m}\sum_{i=1}^{N} a_{i,f}\,\mathrm{e}^{b_{i,f}\alpha_f+\eta_i} \leq \bar{\mathcal{F}}_f\left(\alpha_f\right) \leq \mathrm{e}^{M}\sum_{i=1}^{N} a_{i,f}\,\mathrm{e}^{b_{i,f}\alpha_f+\epsilon_i}. \tag{15}$$

Therefore, according to Lemma 1, the lower and upper bounds have at most one unique identical root α_f on Γ', hence, the function $\bar{\mathcal{F}}_f$ has the same unique root, if it exists. Then there is at most one $\alpha_f \in \Gamma$ such that $\alpha_f \in (\alpha^*)^{-1}(\mathcal{C}) \Leftrightarrow \alpha^*(\alpha_f) \in \mathcal{C}$, and there exists at most one common root of the functions $\mathcal{F}_h$'s and $\mathcal{F}_f$ on the compact set $\Gamma \times \mathcal{C}$ of $\mathbb{R}^F$.

As a synthesis, there exists at most a root α_f of $\bar{\mathcal{F}}_f$ on the compact set Γ' of $\mathbb{R}$ such that, for all $h \in \{1,\dots,f-1,f+1,\dots,F\}$, we have

$$\begin{cases} \mathcal{F}_h\left(\alpha_1^*(\alpha_f),\dots,\alpha_{f-1}^*(\alpha_f),\alpha_f,\alpha_{f+1}^*(\alpha_f),\dots,\alpha_F^*(\alpha_f)\right) & = & 0 \\ \mathcal{F}_f\left(\alpha_1^*(\alpha_f),\dots,\alpha_{f-1}^*(\alpha_f),\alpha_f,\alpha_{f+1}^*(\alpha_f),\dots,\alpha_F^*(\alpha_f)\right) & = & 0 \end{cases}$$

If it exists, the point $\left(\alpha_1^*(\alpha_f),\dots,\alpha_{f-1}^*(\alpha_f),\alpha_f,\alpha_{f+1}^*(\alpha_f),\dots,\alpha_F^*(\alpha_f)\right)$ is the root of this system on the compact set $\Gamma \times \mathcal{C}$ of $\mathbb{R}^F$.

If we now consider *any* compact set $\mathcal{K}$ of $\mathbb{R}^F$, Lemma 2 allows to write that $\mathcal{K} \subset \Gamma \times \mathcal{C}$, where Γ is a compact set of $\mathbb{R}$ and $\mathcal{C}$ is a compact set of $\mathbb{R}^{F-1}$. The previous study allows to assess that there exists at most one root of the system on $\Gamma \times \mathcal{C}$, therefore, there exists at most one root on the compact $\mathcal{K}$.

Finally, in light of Inequations (15) and bearing Proposition 2 in mind, the root is a continuous function of the a's and b's.

Conclusion

We have proven that, using the Mean approximation, the system given by Equations (6) admits at most one solution on any compact set of $\mathbb{R}^F$, for any $F \in \mathbb{N}^*$, and this solution is a continuous function of the a's and b's.

Notes

V. Discussion

In this section, we discuss the approximation explicited in Definition 1. We would like to elaborate on the practical meaning of this assumption for particular values of F. Then we heurestically illustrate the need for the compactness in the context of researching the solution of the Target Exposure Problem. We end this section by an attempt of the generalization of the proof, without the need of the Mean approximation.

Notes

Mean approximation when $F = 1$
As there is just one factor $f = F = 1$, the sum is empty in Definition 1, and is equal to 0. Thus $M = \epsilon_i = 0$. The Mean approximation is trivial.

Mean approximation when $F = 2$
We have two factors f and $h \neq f$. Thus Definition 1 gives

$$\exists M \in \mathbb{R} \quad \forall i \in \{1, \dots, N\} \quad b_{i,h} c_h = M + \epsilon_i, \quad h \neq f.$$

When $c_h = 0$, this is reduced to the single factor form. We note that $b_{i,h}$ is commonly a style factor Z-score. In practice, they are generally observed as a close to normal distribution in a large universe. Z-scores have mean 0, standard deviation 1. Additionally, it is common practice to construct the Z-scores with values constrained in $[-3, 3]$. Provided that $c_h \neq 0$, when c_h is reasonably small, typically $0 < c_h < 1$, we can set $M = 0$ and ϵ_i is a random value in the range $[-3c_h, 3c_h]$.

The case $c_h < 1$ is a typical observation in broad multi-factor passive investment solutions, limited by the requirement of diversification, liquidity and capacity. While a five-factor system, namely Value, Quality, Size, Momentum and Low Volatility, is a common investment consideration, as the number of factors grows, the Mean approximation approaches a statistical approximation (see Figure 2).

Mean Approximation when $F \gg 1$
We assume here that the b's are all random variables which are independent and identically distributed. This means that the instruments all have the same dependency to the overall pool of considered factors. We can think of the Mean approximation as the law of large numbers. In fact, adding one more term, which is related to factor f, wouldn't change the argument made here. Thus, we have

$$\lim_{F \to +\infty} \frac{1}{F} \sum_{h=1}^{F} b_{i,h} c_h = \mathbb{E}(b_{i,h} c_h).$$

Here, we see that M is playing the role of $\mathbb{E}(b_{i,h} c_h)$, while ϵ_i clearly is evolving as $1/\sqrt{F}$. In essence, the sum converges in law to a normal random variable. According to the Lindeberg-Lévy Central Limit Theorem (CLT), we have:

$$\frac{1}{F} \sum_{h=1}^{F} b_{i,h} c_h \underset{\text{in law}}{\sim} \mathcal{N}\left(M, \frac{\mathbb{V}\text{ar}(b_{i,h} c_h)}{F}\right).$$

In this sense, the Mean approximation used here is a statistical approximation, overall approximating the exposures by their average over all individual assets, and the accuracy is dependent on the variability in the distribution, i.e. the dispersion around the average value, or the deviation of the exposures to an overall average over the pool of instruments. To some extent, the Mean approximation in the case of large number of factors and the law of large numbers are the two faces of the same coin.

Although a statistical approximation of the style of CLT makes sense when the sample is large enough, so that the observable can be approximated by its average, the Mean approximation given by Definition 1 is a useful tool to turn the general case $F > 1$ into the already well-established one $F = 1$, but is perhaps not necessary to prove uniqueness of solution. Thus, proving uniqueness of the solution of Equations (6) without any statistical approximation remains an open question.

We now would like to illustrate the meaning of compactness in the proof of Theorem 1. Compactness is behaving as a 'measure instrument' looking for the solution (as a microscope chasing a particular bacteria). The solution might not exist on a particular compact area of $\mathbb{R}^F$, but 'moving' the set of observations (as focusing the microscope length on somewhere else in the sample) in another area of $\mathbb{R}^F$ might lead to the discovery of the solution. Increasing the size of the considered compact set (as changing length focus of a microscope) increases the chance of finding the solution. In addition, as finite union of (not necessarily intersecting) compacts is compact, finding the solution with distinct compact sets increases the chance of finding the solution.

We end this discussion section by an attempt for a generalisation of the proof without using the Mean approximation. We come back to the proof of Theorem 1, before the stage of applying the Mean approximation.

Notes

Let $\alpha \in \Gamma'$, and we set

$$\mathcal{E}_i(\alpha) = \sum_{\substack{h=1 \\ h \neq f}}^{F} b_{i,h} \alpha_h^*(\alpha).$$

We conceptualise the *constraint linear approximation* of the functions $\mathcal{E}_i$'s, as follows: we claim that

$$\forall i \in \{1, \dots, N\} \quad \exists (\lambda_i, \mu_i) \in \mathbb{R}^2 \quad \forall \alpha \in \Gamma' \quad \mathcal{E}_i(\alpha) = \lambda_i + \mu_i \alpha + \epsilon_i(\alpha), \tag{16}$$

where $b_{1,f} + \mu_1 > \cdots > b_{N,f} + \mu_N$, $|\epsilon_i(\alpha)| \leq \theta_i$, and $\theta_i > 0$ is the *precision*.

Indeed, fixing $i \in \{1, \dots, N\}$, the function $\mathcal{E}_i$ is continuous on the compact set Γ'. From the Weierstrass Approximation Theorem, for any $\theta_i > 0$, we deduce that there exists a polynomial $P_i \in \mathbb{R}[X]$ such that

$$\forall \alpha \in \Gamma' \quad |\mathcal{E}_i(\alpha) - P_i(\alpha)| \leq \theta_i,$$

We choose the minimal θ_i such that P_i is a polynomial of order 1 and we impose the consecutive inequalities $b_{1,f} + \mu_1 > \cdots > b_{N,f} + \mu_N$. Equation (16) together with the constraints follow, and the claim is proven.

From this claim, we write

$$\bar{\mathcal{F}}_f(\alpha) = \sum_{i=1}^{N} a_{i,f}\, \mathrm{e}^{b_{i,f}\alpha + \mathcal{E}_i(\alpha)} = \sum_{i=1}^{N} a_{i,f} \mathrm{e}^{\lambda_i}\, \mathrm{e}^{(b_{i,f}+\mu_i)\alpha}\, \mathrm{e}^{\epsilon_i(\alpha)}.$$

Since we have

$$b_{1,f} + \mu_1 > \cdots > b_{N,f} + \mu_N,$$

then Equation (9) still applies by replacing $a_{i,f}$ with $a_{i,f}\mathrm{e}^{\lambda_i}$ and $b_{i,f}$ with $b_{i,f} + \mu_i$ (the variable k doesn't change). We set

$$\bar{\mathcal{F}}_f^0(\alpha) = \sum_{i=1}^{N} a_{i,f}\, \mathrm{ch}(\theta_i)\, \mathrm{e}^{\lambda_i}\, \mathrm{e}^{(b_{i,f}+\mu_i)\alpha},$$

and

$$F_f(\alpha) = -\sum_{i=1}^{k} a_{i,f}\, \mathrm{sh}(\theta_i)\, \mathrm{e}^{\lambda_i}\, \mathrm{e}^{(b_{i,f}+\mu_i)\alpha} + \sum_{i=k+1}^{N} a_{i,f}\, \mathrm{sh}(\theta_i)\, \mathrm{e}^{\lambda_i}\, \mathrm{e}^{(b_{i,f}+\mu_i)\alpha} > 0,$$

where ch and sh are the hyperbolic consine and hyperbolic sine, respectively. The function $\bar{\mathcal{F}}_f^0$ verifies Proposition 1, from which we deduce that $\bar{\mathcal{F}}_f^0$ has at most one root α_f on the compact set Γ'.

Furthermore, we have $-\theta_i \leq \epsilon_i(\alpha) \leq \theta_i$, that is $\mathrm{e}^{-\theta_i} \leq \mathrm{e}^{\epsilon_i(\alpha)} \leq \mathrm{e}^{\theta_i}$, hence

Notes

$$\begin{cases} i \leq k \quad \Rightarrow \quad a_{i,f}\mathrm{e}^{\theta_i} \leq a_{i,f}\mathrm{e}^{\epsilon_i(\alpha)} \leq a_{i,f}\mathrm{e}^{-\theta_i} \quad (a_{i,f} < 0) \\ i > k \quad \Rightarrow \quad a_{i,f}\mathrm{e}^{-\theta_i} \leq a_{i,f}\mathrm{e}^{\epsilon_i(\alpha)} \leq a_{i,f}\mathrm{e}^{\theta_i} \quad (a_{i,f} > 0) \end{cases}$$

By multiplying by $\mathrm{e}^{\lambda_i}\,\mathrm{e}^{(b_{i,f}+\mu_i)\alpha} > 0$, summing over i, we deduce that

$$\forall \alpha \in \Gamma' \quad \left|\bar{\mathcal{F}}_f(\alpha) - \bar{\mathcal{F}}_f^0(\alpha)\right| \leq F_f(\alpha). \tag{17}$$

Inequation (17) means that we can approximate $\bar{\mathcal{F}}_f$ by $\bar{\mathcal{F}}_f^0$ with precision F_f. If θ_i is sufficiently small, we have

$$\bar{\mathcal{F}}_f(\alpha) = \bar{\mathcal{F}}_f^0(\alpha) + \mathrm{o}(\theta_i),$$

so that $\bar{\mathcal{F}}_f$ indeed has at most one root on Γ'. However, θ_i has no practical reasons to be small. We thus could extend the degree of the polynomial P_i but we couldn't prove that the function $\bar{\mathcal{F}}_f^0$ thus obtained has at most one root. From this approach, it seems that the only possibility is to have a polynomial P_i of degree 1. In addition, the fact that θ_i is small has no particular connection with the Mean approximation. The end of Section 6 (Figures 2 and 3) shows some illustration with FTSE All-World Index data.

However, the order-3 polynomial approximation is excellent (see end of Section 6). Equation (16) should thus become

$$\forall i \in \{1, \dots, N\} \quad \exists \left(\lambda_i, \mu_i, \mu_i^{(2)}, \mu_i^{(3)}\right) \in \mathbb{R}^4 \quad \forall \alpha \in \Gamma' \quad \mathcal{E}_i(\alpha) = \lambda_i + \mu_i\alpha + \mu_i^{(2)}\alpha^2 + \mu_i^{(3)}\alpha^3 + \epsilon_i(\alpha),$$

and all the previous mathematics should be re-performed through the function

$$\mathcal{F}(\alpha) = \sum_{i=1}^{N} a_i\,\mathrm{e}^{b_i\alpha + c_i\alpha^2 + d_i\alpha^3},$$

with the c_i's and d_i's appropriately defined. See Figures 3 and 4 for a comprehensive numerical illustration with FTSE All-World Index data.

VI. Numerical Illustrations

In this section, we provide two numerical illustrations for the uniqueness of the solution of system of Equations (6). Finding numerically the roots for such system is a complex task in itself, since in order to numerically find the solution, an objective function (here the set of the $\mathcal{F}_f$'s functions) is required, hence the numerical problem must diverge from the mathematical problem. In practice, the objective function is minimised iteratively, and at each iteration, the system comes more and more to a minimum. Reaching this minimum needs, at iteration 1, a starting point, named *initial enhancement*. We can illustrate the main result of this paper by an appropriate choice of values for the initial enhancement, corresponding to a trial and error approach, so that we can judge of the stability of the discovered minimum point.

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That said, we need to fix the values for the a's, b's and the $\bar{A}_f$'s. In our experiment, we choose the following values $N = 10$, $F = 3$, $S = \exp$ (so that the b's are the Z-scores), and:

$$(Z_{i,f})_{1\leq i\leq N,1\leq f\leq F} = (b_{i,f})_{1\leq i\leq N,1\leq f\leq F} = \begin{pmatrix} 0.2849081 & 0.9052735 & 0.6710882 \\ -1.2086324 & 1.3848464 & 0.7704434 \\ 0.5253426 & 0.1693195 & 0.4171714 \\ 1.1848723 & -0.1850475 & 0.5078646 \\ 2.8027457 & 1.2794519 & 0.6511082 \\ 1.1879513 & 1.8178505 & 0.2750094 \\ 2.1976343 & 0.7550815 & 1.7427517 \\ 1.4541885 & 1.0356254 & 0.5881843 \\ 0.5923255 & 1.2455047 & 0.9488955 \\ 1.4507721 & 1.9529544 & -0.4500539 \end{pmatrix};$$

$$(\bar{A}_f)_{1\leq f\leq F} = \left(\frac{1}{N}\sum_{i=1}^{N} Z_{i,1},\ (1+1/100)\,\frac{1}{N}\sum_{i=1}^{N} Z_{i,2},\ (1+2/100)\,\frac{1}{N}\sum_{i=1}^{N} Z_{i,3}\right);$$

$$(a_{i,f})_{1\leq i\leq N,1\leq f\leq F} = (\bar{A}_f)_{1\leq f\leq F} - (b_{i,f})_{1\leq i\leq N,1\leq f\leq F}.$$

For the Z-scores, we took realisations of a Gaussian random variable of mean 1 and standard deviation 0.9. The Newton-Raphson iterative method has been used to reach for a minimum, and the objective function $\mathcal{O}$ has been set to be

$$\mathcal{O}(\alpha_1, \alpha_2, \alpha_3) = \sum_{f=1}^{3} |\mathcal{F}_f(\alpha_1, \alpha_2, \alpha_3)|.$$

More specifically, we have generated 1000 initial enhancements, each given by the realisation of a Gaussian random vector of mean 1 and standard deviation 0.5. We have also done the exercise with another series of 1000 simulations, but with mean 10 and standard deviation 5, and another 1000 with mean 100 and standard deviation 50. The roots found are systematically the same for all the 3000 simulations:

$$(\alpha_1,\ \alpha_2,\ \alpha_3) = (-0.002802974,\ 0.04459991,\ 0.06373927).$$

At this point, the value of the $\mathcal{F}_f$'s are of order 10−15, which is of the precision of the machine. Finally, the estimation precision is of the order 10−15, also reaching the machine precision. The number of iterations varies, depending on the initial enhancement, but never reaches 1000.

Within this set of parameters, we start to meet numerical issues if the initial enhancement have too negative values. For instance, if the initial value are $(-1, -1, -1)$, the roots found are

$$(\alpha_1,\ \alpha_2,\ \alpha_3) = (-10.17499,\ -11.11835,\ -12.90939),$$

but the magnitude order for the values of the functions $\mathcal{F}_f$'s at this point are of order 10^{-6} (same for the estimated precision), which shows this point is a local minimum for the objective function, but clearly not the global one.

This shows the limitation of the numerical method, and further numerical studies would perhaps be needed to further illustrate the main result of this paper.

Notes

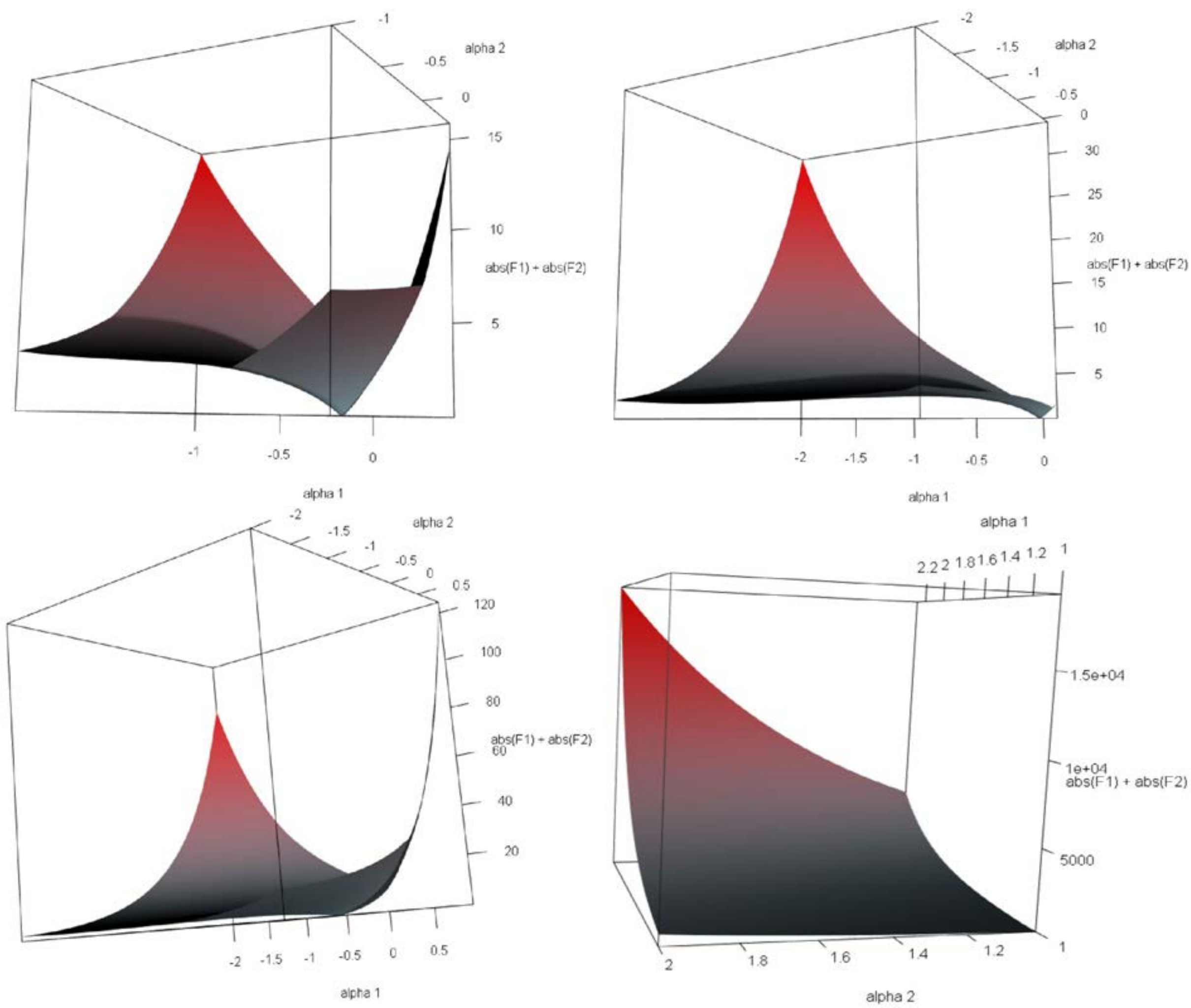

Figure 1: Plot of the objective function $|\mathcal{F}_1(\alpha_1, \alpha_2)| + |\mathcal{F}_2(\alpha_1, \alpha_2)|$ versus α_1 and α_2 around the unique root $(-0.0002063940,\, 0.02590912)$.

We conclude this part with an additional graphical illustration. We choose $F = 2$ and $N = 10$, and the Z-scores are the first two columns of the above Z-scores matrix. The found that the root is given by

$$(\alpha_1,\, \alpha_2) = (-0.0002063940,\, 0.02590912),$$

with values for the $\mathcal{F}_f$'s of order 10^{-14} (same for the estimated precision). Figure 2 plots the 3D-surface of the objective function $|\mathcal{F}_1(\alpha_1, \alpha_2)| + |\mathcal{F}_2(\alpha_1, \alpha_2)|$ versus α_1 and α_2. As shown, when we leave the neighbourhood of the root, the objective function significantly increases from 0. We also note that some convexity appears in the negative values of α_1 and α_2 (see the bottom of the surface in the top right figure), showing that another local minimum, which is not the global one, could be reached using the Newton-Raphson method once the initial enhancement is sufficiently close to it; but it is unlikely to be found in this area.

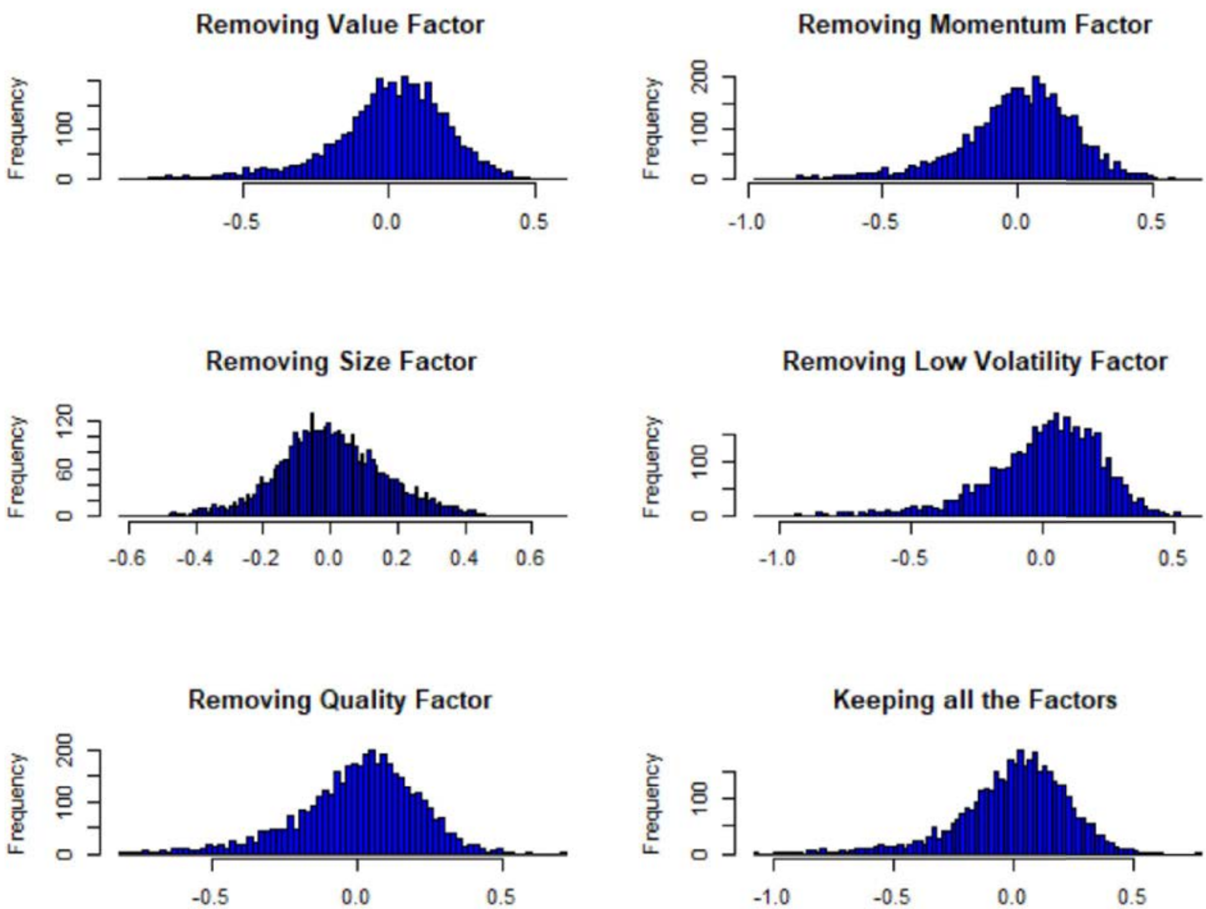

Figure 2: Histograms of the sum of Z-scores weighted by the root numbers (five factors), from the FTSE All-World Index data (see Equation (18)). The numbers are comprised between -1 and 1, and centered around 0 with a standard deviation of approx. 0.5. The 'symmetry' suggests that the Mean approximation is a statistical argument. The presence of extreme values also suggests that the Mean approximation is not a necessary condition to have uniqueness, but sufficient only.

To illustrate the numerical example using practical data, we use FTSE All-World index as the opportunity set. It includes listed companies from both developed and emerging markets, which are classified as large and mid size companies by market cap (for details, see [GEIS]). This index represents a portfolio with around $N \approx 4100$ equity instruments, and $F = 5$ factors, Value, Momentum, Size, Low Volatily, and Quality.

The proof of Theorem 1 is using the Mean approximation, see Definition 1. Testing this approximation is important. We note that the strengths *a priori* are dependent variables. For the given set of parameters imposed by the data, the strengths $(\alpha_{\hbar}^*)_{\hbar \in \{1,\ldots,5\}}$ designate the unique solution of the system found at rebalance date 21/03/2022. Fixing factor $f \in \{1, 2, \ldots, 5\}$, we know that the numbers $(\alpha_{\hbar}^*)_{\hbar \in \{1,\ldots,5\}\setminus\{f\}}$ are functions of α_f^*. Bearing Equation (14) in mind, we see that

$$\bar{\mathcal{F}}_f(\alpha_f^* + \alpha) = \sum_{i=1}^{N} \left(a_{i,f}\, e^{\sum_{\substack{l=1 \\ l \neq f}}^{F} Z_{i,l}\alpha_l^*(\alpha_f^* + \alpha)} \right) e^{Z_{i,f}\alpha_f^* + Z_{i,f}\alpha},$$

where α is a perturbation of the solution strength α_f^*. Thus, a proxy to have access to the variation of the numbers $(\alpha_{\hbar}^*)_{\hbar \in \{1,\ldots,5\}\setminus\{f\}}$ with respect to α_f^* is to replace $Z_{i,f}\alpha$ with $\alpha\alpha_f^*$: $Z_{i,f}\alpha$ is itself a pertubation, so as $\alpha\alpha_f^*$. In fact, the proxy consists in identifying the perturbation of α for the Z-score with the variation of α_f^*.

Ref
16. [GEIS] https://www.ftserussell.com/products/indices/geisac

Notes

Bearing this in mind, it is interesting to focus on the quantity $\mathcal{H}_f$, function of the perturbation α of the Z-score $Z_{i,f}$'s and given by

$$\mathcal{H}_f(\alpha) = \sum_{\substack{h=1 \\ h \neq f}}^{F} Z_{i,h} \alpha_h^*(\alpha), \tag{18}$$

where we excluded a given factor f in the sum, and where $\alpha_h^*(\alpha)$ is the strength found after the Z-scores are perturbed by α. In Figure 2, we plotted the histogram of the sum $\mathcal{H}_f$ (one realization for one value of α, chosen as realized Gaussian random variables of parameters $(0, 1)$ (size of sample: 1000)), and we see that most of the values are concentrated around 0, but there are more extreme ones (even if < 1). The root uniqueness may not need the Mean approximation.

We now focus on Equation (16). First, Figure 3 shows the $\mathcal{H}_f$'s (see Equation (18)) with respect to α. For the chosen instrument, the linear approximation is quite correct as witnessed by the R^2 values, see Figure 3. However, Figure 4 shows the same plots for another instrument, and the linear approximation is not convincing. We calculated the averaged R^2 for all the factors and for all the instruments, at all the rebalance dates between 2015 and 2022 (8 samples). It is given by 0.748, with a standard deviation of 0.300.

As specified in Section 5, the order-3 polynomial is an excellent approximation for all the factors, instruments, and considered years. We also calculated the averaged R^2, which is 0.998, with a standard deviation of 0.015.

VII. Conclusion

In this paper, we have explained the Target Exposure methodology, abundantly used by the index industry and passive investment community. The Target Exposure derives an allocation of considered assets, allowing an investor to build portfolios that are exposed to various factor risks. It provides the investors a constrution tool that gives transparency and intuition inherited from traditional tiling. Weighting transparency is a growing consideration of the passive investment community, especially when it includes sustainable investment goals such as ESG or carbon emission intensity. The discussion on the uniqueness of this allocation for a given set of exposures targets furthers our understanding of the target exposure framework.

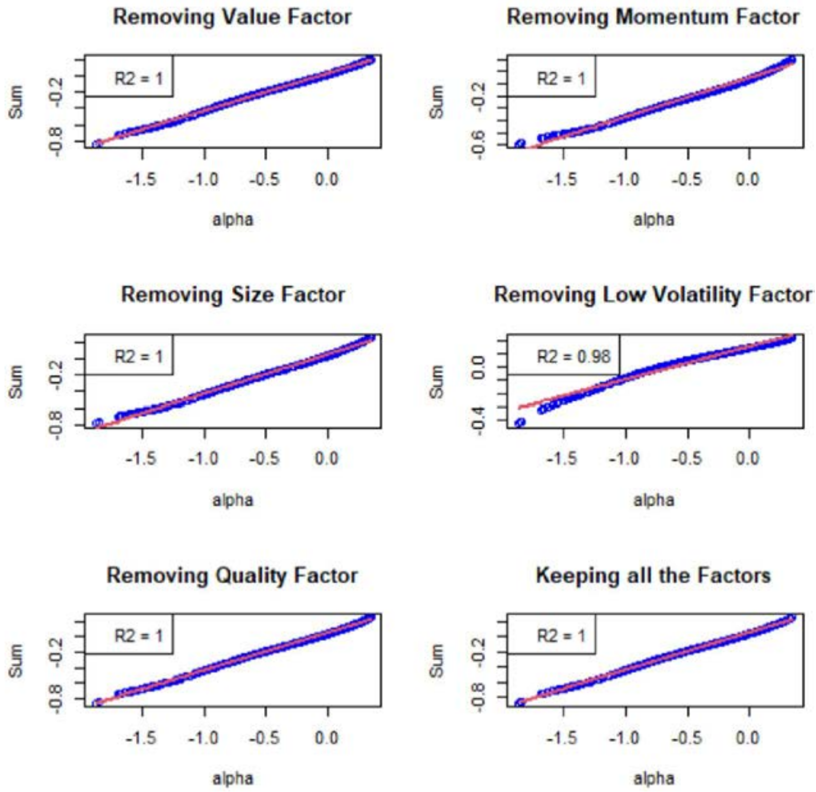

Figure 3: Scatter plot of the sums as defined by the $\mathcal{H}_f$'s (see Equation (18)) vs α defined around 0, for a chosen instrument for which the linear approximation (see Equation (16)) is good.

We have first reduced the Target Exposure model to a system of non-linear exponential functions. We have fully established the uniqueness of the solution in case we've considered only one factor. Then, as the most difficult part, we have proven the uniqueness of the solution for any number of factors, under a useful statistical approximation, the *Mean* approximation.

Consisting in averaging the exposures of each asset to their factors overall, this approximation allowed completing the mathematical induction approach on the number of factors: we were able to show that there exists at most one allocation for a given considered universe of parameters, reduced to given compact sets of multi-dimensioned real vector space. The fact that the result is true for any compact set is not restrictive at all: if the compact set is too small, the unique solution will unlikely be contained in it. Thus, it is suggested to move the compact set in the universe of parameters, or increase its size, until finding the solution. We see that compact sets here are the mathematical justification for playing with the set of parameters, in the research of the unique possible allocation.

Tilting is one of the most common for use modern portfolio construction methodologies. Target Exposure aims to extend the tilting capability to incorporate explicit ex-ante outcomes while keeping the transparent weighting formulation. The extension to Target Exposures introduces a system of non-linear equations. The discussion of the root of this system of equations, particularly its uniqueness, has presented us with an interesting and challenging task. This paper has laid the groundwork for potential deeper dive into this system. While multiple solutions of such a system would further lead to possible discussions on different portfolios yielding identical investment objectives, our study so far has shown that such a scenario is not likely. The research proposed in this paper shows that the system underpinned by the Target Exposure problem has at most one real and continuous solution.

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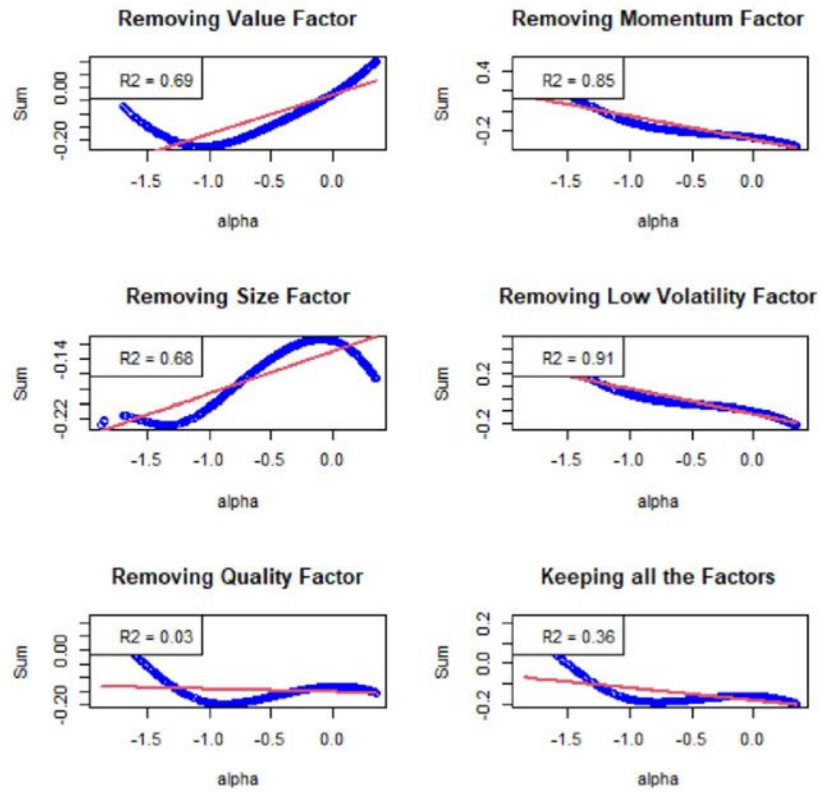

Figure 4: Same as Figure 3 but for another instrument, for which the linear approximation (see Equation (16)) is not a good approximation. However, for all the instruments, the order-3 polynomial approximation is a very good approximation.

Acknowledgments

We are very thankful for the discussions we had with the Index Research & Design Team at London Stock Exchange Group.

Conflict of Interest

The authors declare no conflict of interest. The views expressed in this paper are those of the authors and do not necessarily reflect the views and policies of LSEG - FTSE.

Data Availability Statement

Data available on request due to privacy/ethical restrictions.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Comparison of Mathematical Models for Thermodynamic Prediction of High Voltage Underground Cables

By Amórtegui G. Francisco Javier, Escobar P. Andrés Felipe & León G. Luis Miguel
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GJSFR-F Classification: LCC: QA1-939

Comparison of Mathematical Models for Thermodynamic Prediction of High Voltage Underground Cables

Comparación de Modelos Matemáticos Para la Predicción Termodinámica de Cables Subterráneos de Alta Tensión

Amórtegui G. Francisco Javier [α], Escobar P. Andrés Felipe [σ] & León G. Luis Miguel [ρ]

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I. Introduction

En la actualidad gran parte de los sistemas de transmisión y distribución son concebidos como instalaciones subterráneas debido a que estas permiten un mejoramiento en el uso del espacio público, disminución de costos de mantenimiento y aumento de la confiabilidad y disponibilidad del sistema de energía. Sin embargo, es necesario tener en cuenta otros factores que afectan la instalación, como lo son el aumento de las pérdidas de potencia reactiva debido al incremento de las componentes capacitivas e inductivas de la instalación, así como la disminución de la capacidad de corriente en los conductores, que en eventos de sobrecarga implican aumentos sustanciales de la temperatura, disminuyendo la vida útil de este activo y en caso de falla en la instalación, altos costos de reparación.

El aumento de la temperatura en los conductores implica el degradamiento de la sección aislante del cable, que en general es de tipo XLPE. Este material envejece progresivamente al estar sometido a ciclos de temperatura que sobrepasan los niveles especificados por el fabricante determinados a partir de ensayos de certificación [1]; por tal motivo, es de gran importancia conocer y estudiar el comportamiento térmico del cable ante diferentes condiciones de operación. Una de las formas de analizar el

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1. R. Xia, S. Meng, and Y. Wang, "Long term reliability tests of 500 kV XLPE cable system," *POWERCON 2014 - 2014 Int. Conf. Power Syst. Technol. Towar. Green, Effic. Smart Power Syst. Proc.*, no. Powercon, pp. 1521–1524, 2014.

comportamiento térmico es mediante el uso de modelos numéricos capaces de predecir las temperaturas a las que estará expuesto cada región del cable y así tener un control más riguroso sobre las condiciones de operación de este, permitiendo extender la vida útil del cable [2], [3], [4], [5].

La complejidad de los modelos matemáticos puede comprender desde la solución numérica de ecuaciones diferenciales ordinarias (EDOs) que describen el fenómeno de forma unidimensional y concentrada, hasta la solución discretizada de ecuaciones diferenciales parciales (EDPs) distribuidas en todo un campo espacial, algunos modelos térmicos de los cables han sido comparados entre si previamente [6]. En este caso, se aborda un problema en el que es necesario vincular dos tipos de comportamientos físicos, por un lado, ecuaciones que describan la distribución de densidad de corriente eléctrica en el conductor y por el otro, ecuaciones que, a partir de los resultados obtenidos en el modelo anterior, detallen el comportamiento dinámico de los flujos de calor existentes en todo el cable.

En este documento se exponen dos modelos termoeléctricos utilizando análisis numéricos de diferente complejidad. Los resultados obtenidos con cada uno son comparados con mediciones experimentales para verificar su capacidad de predicción ante fenómenos reales. Finalmente se presentan las ventajas y hallazgos más importantes de cada modelo a partir de la comparación realizada.

II. Método Basado En Análisis De Elementos Finitos

El método de elementos finitos consiste en la generación de un dominio de cálculo, en el cual mediante una discretización espacial resuelve una serie de ecuaciones diferenciales parciales que le dan solución a un problema en específico. Para modelar el comportamiento termodinámico de un cable subterráneo de alta tensión es conveniente recrear de la manera más precisa la geometría de la sección transversal de este, de igual forma es conveniente conocer con la mayor precisión cada una de las propiedades electromagnéticas y termodinámicas de los materiales que componen al cable.

El cálculo implementando este modelo se desarrolla de forma secuencial iterativa, siguiendo los pasos a continuación: a partir de las condiciones iniciales y las condiciones de frontera se calcula la densidad de corriente e implícitamente la densidad de potencia inicial disipada en cada uno de los elementos diferenciales que componen al dominio computacional; consiguientemente, a partir de la densidad de potencia disipada se calculan los flujos de energía a través del dominio y con estos la temperatura alcanzada en cada elemento diferencial en el primer intervalo de tiempo discreto; finalmente, se ajustan las propiedades físicas de los elementos diferenciales a la temperatura encontrada para cada uno y se vuelve a iniciar el cálculo desde la potencia disipada. El proceso secuencial descrito se repite en cada intervalo de tiempo hasta alcanzar la totalidad de tiempo descrito por el estudio, permitiendo conocer no sólo la temperatura a la que llegara el cable si no también su respuesta transitoria. La construcción y consideraciones del modelo se desarrollan a continuación:

Ref

6. L. Yang *et al.*, "Comparison of conductor-Temperature calculations based on different radial-position-Temperature detections for high-voltage power cable," *Energies*, vol. 11, no. 1, 2018.

a) Parámetros básicos

Para el estudio del fenómeno termodinámico se tomó un cable subterráneo de media tensión AWG 2/0 de 25 kV, con las siguientes dimensiones, propiedades eléctricas y térmicas (tomadas de trabajos previamente realizados [6], [7]) y condiciones ambientales de prueba controladas.

Tabla 1: Características geométricas del cable

COPPER XLPE 25kV AWG2/0					
Conductor		**A is lamiento**		**Chaqueta**	
Œrea	Diámetro	Espesor	Diámetro	Espesor	Diámetro
mm^2	mm	mm	mm	mm	mm
67.4	9.57	6.6 ffi 5%	23.4	1.78	29.1

Tabla 2: Características eléctricas de los materiales

	Conductividad	**Permitividad R.**	**Permeabilidad R.**
	MS/m		
Cobre 20flC	53.3	1	1
Cobre 100flC	47.9	1	1
XLPE	1E-24	2.5	1
PE	1E-24	2.25	1
SCP	2.25E-6	2.25	1

Tabla 3: Características térmicas de los materiales

	Densidad	**Conductividad**	**Capacidad T.**
	Kg/m^3	**W/(m·K)**	**J/(kg·K)**
Cobre	8960	401	385
XLPE	930	0.32	2302
PE	935	0.38	2302
SCP	1055	10	2405

Tabla 4: Condiciones ambientales

T. Ambiente	**Humedad R.**	**Viento**
flC		m/s
19.7	0.48	0

Ref

7. K. Y. Lee, J. S. Yang, Y. S. Choi, and D. H. Park, "Specific heat and thermal conductivity measurement of XLPE insulator and semiconducting materials," *Proc. IEEE Int. Conf. Prop. Appl. Dielectr. Mater.*, pp. 805–809, 2007.

El dominio computacional se recreó de la forma más parecida a la geometría real del cable y se discretizo espacialmente con 52.438 elementos de estructura triangular.

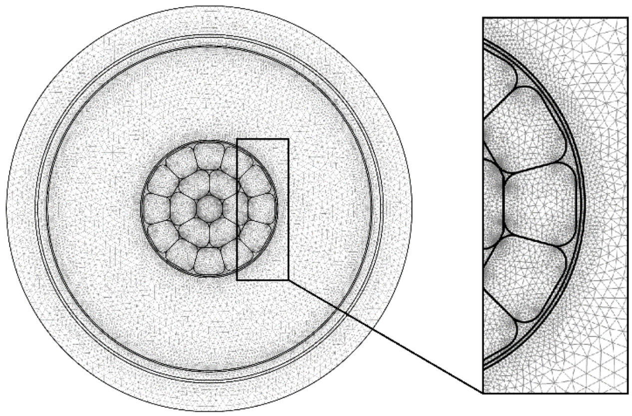

Figura 1: Discretización espacial del dominio computacional

b) Ecuaciones gobernantes del modelo

El estudio por elementos finitos se divide en dos físicas que se comunican entre ellas al finalizar cada intervalo discreto de tiempo. En un primer paso se resuelven las ecuaciones que gobiernan el fenómeno electromagnético y en un segundo paso se resuelven las ecuaciones que definen el fenómeno termodinámico. A nivel electromagnético, el modelo soluciona la ley de Maxwell Ampere en forma bidimensional mediante el uso de un vector magnético potencial $\vec{A}$ ortogonal al plano en el que se trabaja, lo que permite que el sistema se calcule bajo sólo una variable dependiente. Para ello es necesario definir las siguientes ecuaciones que relacionan al vector $\vec{A}$ en el dominio de la frecuencia.

$$\nabla \times \vec{H} = \vec{J} \tag{1}$$

$$\vec{B} = \nabla \times \vec{A} \tag{2}$$

$$\vec{J} = \sigma\vec{E} + j\omega\vec{D} + \vec{J_e} \tag{3}$$

$$\vec{E} = -j\omega\vec{A} \tag{4}$$

Haciendo una serie de combinaciones matemáticas se puede encontrar la ecuación diferencial parcial que define todo el sistema a nivel electromagnético únicamente en función del vector $\vec{A}$.

$$-\omega^2\varepsilon\vec{A} + j\omega\sigma\vec{A} + \nabla \times \left(\mu^{-1}\nabla \times \vec{A}\right) = 0 \tag{5}$$

Esta ecuación se aplica en cada uno de los nodos de los elementos diferenciales del dominio computacional, dando como resultado una ecuación matricial que al solucionarse da el valor de $\vec{A}$ en cada uno de los elementos diferenciales del sistema. Finalmente, a partir del vector $\vec{A}$, se puede encontrar el valor de potencia disipada a nivel diferencial mediante la siguiente ecuación.

Notes

$$Q_{EM} = \frac{1}{2} Re(\vec{J} \cdot \vec{E}^{*} + \vec{B} \cdot \vec{H}^{*}) \tag{6}$$

El fenómeno termodinámico toma el campo escalar de potencia electromagnética disipada y lo integra en la ecuación de calor donde la potencia disipada es igual a la suma de la potencia almacenada en forma de calor en estado transitorio, la potencia transferida por conducción y la potencia transferida por convección. Por lo tanto, la ecuación de calor toma la siguiente forma.

$$\rho C_T \frac{\partial T}{\partial t} + \nabla \cdot (k \nabla T) + \rho C_T \vec{v} \cdot \nabla T = Q_{EM} \tag{7}$$

Bajo esta ecuación el modelo por elementos finitos toma una complejidad elevada por lo que se considera una buena aproximación cambiar el vector convectivo por un coeficiente de convección h.

$$\rho C_T \frac{\partial T}{\partial t} + \nabla \cdot (k \nabla T) + h \cdot \Delta T = Q_{EM} \tag{8}$$

Para este caso, el campo escalar de temperatura T unifica la ecuación. Al resolver la ecuación matricial con una sola incógnita se encuentra el valor de T en cada elemento diferencial del dominio. Como último paso de la primera iteración, cada material definido se ajusta a la nueva temperatura de operación y así se repite el número de iteraciones necesarias hasta llegar al estado estable.

c) Resultados y consideraciones

La fuerza electromotriz inducida en el cable produce un flujo de corriente eléctrica en su interior con una uniformidad del 99.62% con respecto al centro del núcleo, la cual decae de forma linealizada en función al aumento de temperatura.

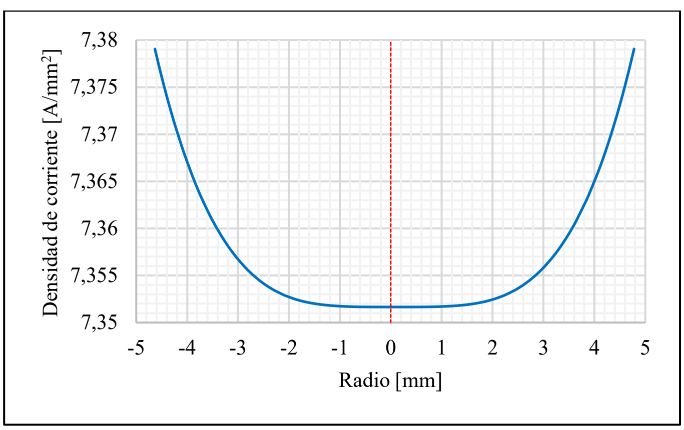

Figura 2: Distribución radial de la densidad de corriente en el cable

Debido a que el dominio y la fuente de calor presentan simetría cilíndrica, del flujo de calor es totalmente radial, a diferencia del núcleo del cable donde se presentan pequeñas alteraciones en la dirección del flujo.

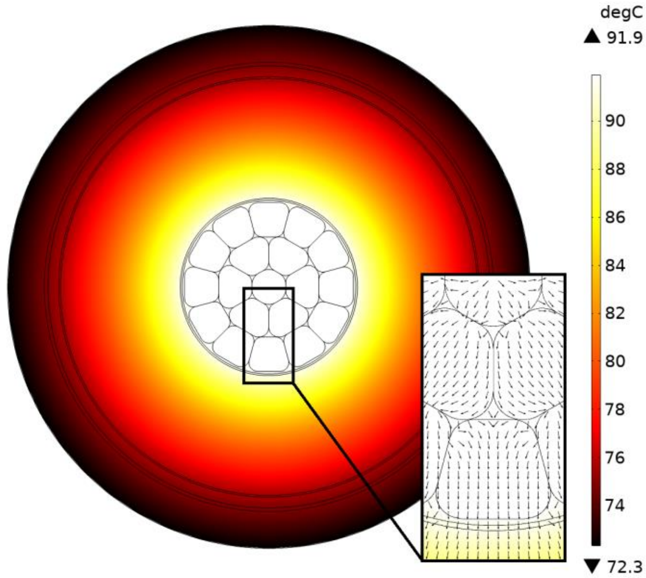

Figura 3: Temperatura y flujo de calor en el conducto en estado estable

III. Método Basado En Análisis De Parámetros Concentrados

El desarrollo del modelo, al igual que el anterior, se encuentra dividido en dos etapas, termodinámica y eléctrica, ambas ligadas entre sí a partir de ecuaciones diferenciales ordinarias. A continuación, se presentan las ecuaciones que dominan el comportamiento del cable y sus simplificaciones, así como también se presenta el modelo construido en diagrama de bloques para solucionar las ecuaciones.

a) Ecuaciones gobernantes del modelo

Se utilizó el modelo π de una línea de transmisión para simular la respuesta eléctrica del cable, de forma que la corriente de calentamiento inducida en el cable circula únicamente a través de las componentes resistiva e inductiva. El transformador de inducción de corriente se modela como una fuente de corriente constante, suposición válida cuando la componente inductiva del cable es más representativa que la resistiva, pues esta última varía en función de la temperatura, comportamiento que puede aproximarse de forma lineal como:

$$R(T) = R_{Ti}(1+\propto (T - T_i)) = \alpha_1 T + \beta_1 \quad (9)$$

El cambio de esta resistencia, que también se incluye en el modelo anterior, afecta en mayor medida y de forma directa el modelo térmico, pues a pesar de que la corriente se mantenga constante gracias al gran componente inductivo, el incremento resistivo implica una mayor disipación de potencia en el cable y, por ende, mayor temperatura de calentamiento. El circuito equivalente final es presentado en la Figura 4, en esta se aprecian las entradas y salidas del sistema, en específico, la entrada que controla la amplitud de la fuente controlada de corriente, la entrada del valor de la resistencia del circuito en función de la temperatura y como salida se tiene la corriente

Notes

que circula por el circuito. Esta corriente (en valor RMS) sirve de entrada al modelo térmico del cable.

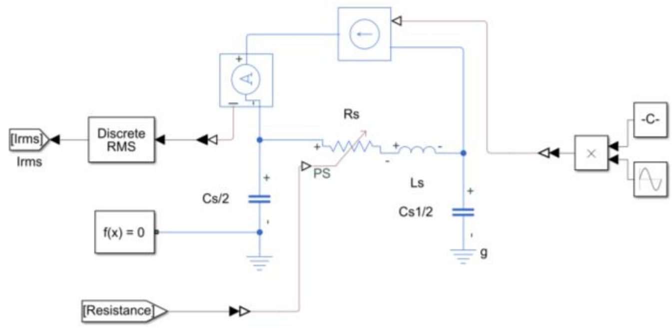

Figura 4: Circuito equivalente del cable a ensayar utilizando parámetros concentrados

Dentro del modelamiento térmico se tienen varias aproximaciones: la primera de ellas referida a la cantidad de materiales que componen el cable, pues este sólo contempla dos materiales, el conductor de cobre y el aislante XLPE que lo rodea. La segunda de las aproximaciones refiere al análisis de únicamente un metro de cable, en el cual se asume que la temperatura es homogénea. La tercera de las aproximaciones se refiere al conductor por el cual circula la corriente, este material al ser cobre presenta conductividad térmica muy alta, por lo cual es posible aproximar que la temperatura en toda el área del conductor es la misma y dentro de él no existe un gradiente de temperaturas. En la Figura 5 se muestra el corte transversal que se analiza.

El desarrollo del modelo termodinámico consiste en analizar dos elementos concéntricos, siendo el elemento central (conductor) fuente de calor del que lo envuelve (aislante). El valor específico de la fuente de calor está dado por la ley de Joule, de acuerdo con la corriente que circula por el cable. El comportamiento térmico del cable se modela a través la ecuación diferencial parcial que describe la transferencia de calor entre el núcleo y el aislante (7).

Al ser este un sistema coaxial en donde a través de toda la longitud se tiene la misma temperatura y donde la fuente se encuentra en todo el centro del sistema, el cambio será únicamente percibido de forma radial y por supuesto, temporal (dos variables r, t). Se desarrollan así dos ecuaciones que describen la transferencia de calor por conducción dentro del material y por conducción y convección dentro y fuera del material:

Para $0 < r < r_1$

$$\frac{1}{r}\frac{1}{\partial r}\left(k_1(T_1)r\frac{\partial T_1}{\partial r}\right) + \frac{J^2}{\sigma(T_1)} = \rho_1 C_{T1}(T_1)\frac{\partial T_1}{\partial t} \qquad (10)$$

Para $r_1 < r < r_2$

$$\frac{1}{r}\frac{1}{\partial r}\left(k_2(T_2)r\frac{\partial T_2}{\partial r}\right) = \rho_2 C_{T2}(T_2)\frac{\partial T_2}{\partial t} \qquad (11)$$

En donde los subíndices 1 y 2 hacen referencia al material cobre y XLPE, respectivamente.

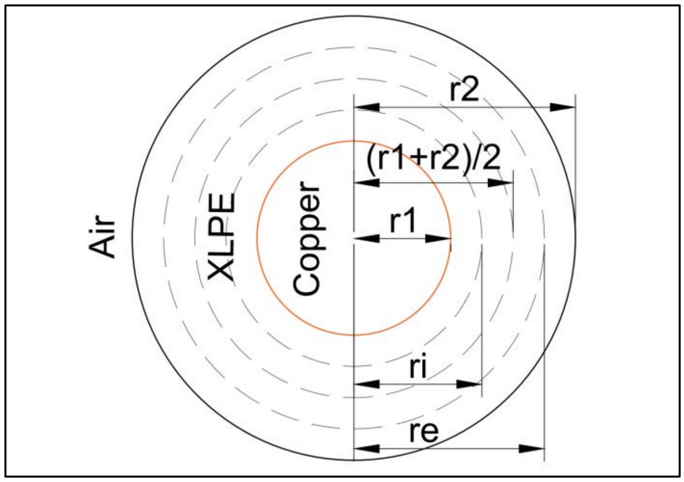

Figura 5: Acotaciones geométricas del modelo térmico desarrollado para el cable

Las ecuaciones anteriores ecuaciones quedan ligadas a una condición inicial:

$$t = 0; \quad 0 \leq r \leq r_2; \quad T_1(r_1) = T_2(r_2) = T_\infty$$

Y las siguientes condiciones de frontera:

$$t \geq 0; \quad r = 0 \rightarrow \frac{\partial T_1}{\partial r} = 0$$

$$t \geq 0; \quad r = r_1 \rightarrow T_1(r_1) = T_2(r_2); \quad k_1(T_1)\frac{\partial T_1}{\partial r} = k_2(T_2)\frac{\partial T_2}{\partial r}$$

$$t \geq 0; \quad r = r_2 \rightarrow -k_2(T_2)\frac{\partial T_2}{\partial r} = h_g(T_2, T_\infty)[T_2(r_2) - T_\infty]$$

Siendo h_g el coeficiente de transferencia de calor global que tiene en cuenta las contribuciones tanto de convección como de radiación y que depende altamente de la temperatura. Ahora, empleando la metodología de modelado en componentes concentrados descrita en [8], y considerando las aproximaciones descritas al inicio de la sección, las ecuaciones (10) y (11) pueden simplificarse como sigue:

Conductor – aislante:

$$\rho_1 C_{T1}(T_1)A_1\frac{\partial T_1}{\partial t} = -\frac{T_1 - T_2}{R_i} + I^2 R_{el}(T_1) \tag{12}$$

Aislante – exterior: 1

$$\rho_2 C_{T2}(T_2)A_2\frac{\partial T_2}{\partial t} = \frac{T_1 - T_2}{R_i} - \frac{T_2 - T_\infty}{R_e} \tag{13}$$

Donde el término δcA representa la capacitancia térmica (C) del material por unidad de longitud (recordando que se analiza apenas un metro del material) y los

Ref

8. F. Scarpa and M. De Rosa, "Transient heat conduction in wires with heat sources; lumped and distributed solution techniques," *Heat Transf. Res.*, vol. 47, no. 8, pp. 753–765, 2016.

términos RTi y RTe, son las resistencias térmicas radiales del interior y hacia el exterior, respectivamente, descritas por: 1

$$R_i = \frac{s_i}{k_{is}\bar{A}_i}; \quad R_e = \frac{s_e}{k_{is}\bar{A}_e} + \frac{1}{h_{global}2\pi r_2} \tag{10}$$

Con los términos $\bar{A}$ siendo las áreas longitudinales evaluadas en los puntos medios ri y re, de igual forma, los términos s son las longitudes desde el conductor hasta el punto medio y desde el punto medio hasta el exterior. Adicionalmente, debido a que este modelo no toma en cuenta la variación de coeficiente de convección (h_{global}), el cual, de acuerdo con la temperatura ambiente, humedad y disposición geométrica típicamente puede variar entre 5 y 25 $[W/m^2K]$ [9]. Para el desarrollo del modelo se ajustó esta variable entre un valor de 9 y 10. Finalmente, todo el modelo térmico puede ser resumido en las ecuaciones (15) y (16) con las cuales es posible representar el circuito térmico modelado con parámetros concentrados y que es representado en la Figura 6.

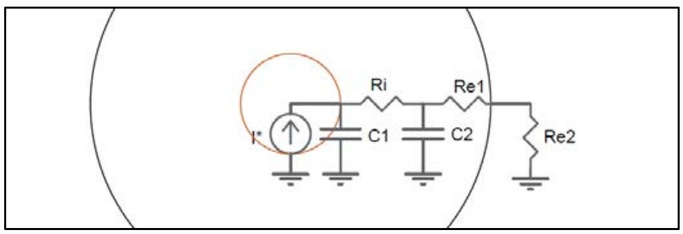

Figura 6: Circuito térmico equivalente del cable luego de emplear la teoría de termodinámica y modelado en parámetros concentrados

$$C_1 \frac{\partial V_1}{\partial t} = -\frac{V_1 - V_2}{R_i} + I^* \tag{15}$$

$$C_2 \frac{\partial V_2}{\partial t} = \frac{V_1 - V_2}{R_i} - \frac{V_2 - V_\infty}{R_e} \tag{16}$$

Con las siguientes condiciones iniciales:

$$t = 0; \quad V_1 = V_2 = V_\infty$$

b) Elaboración del diagrama de bloques

luego de modelar ambos fenómenos que ocurren en el cable se procede a interconecta ambos modelos entre sí. Las ecuaciones diferenciales del modelo térmico son escritas en forma de diagrama de bloques como se muestra en la Figura 7; al final, la salida de este modelo es la temperatura en los dos puntos de análisis, que son la superficie del conductor en contacto con el aislante y el punto medio del aislante, donde se ubican las capacitancias térmicas de la Figura 6. Es válido mencionar que propiedades como el calor específico y la conductividad térmica del aislante también varían con la temperatura [7]; sin embargo, en este modelo no se tiene en consideración estos fenómenos y por tanto se utilizan valores constantes definidos a temperatura ambiente.

Ref

9. S. W. Churchill and H. H. S. Chu, "Correlating equations for laminar and turbulent free convection from a horizontal cylinder," *Int. J. Heat Mass Transf.*, vol. 18, no. 9, pp. 1049–1053, 1975.

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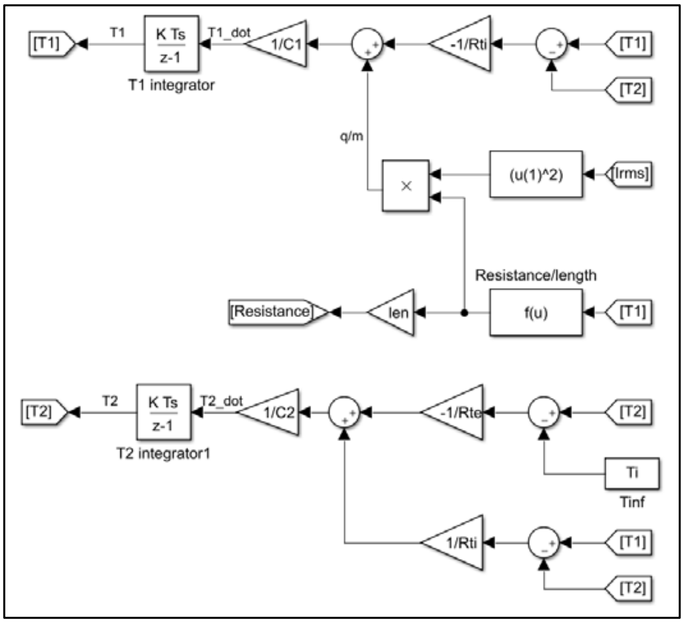

Figura 7: Modelo termodinámico del cable representado a través de diagrama de bloques

IV. Ensayo Experimental y Comparación

Para la verificación de los resultados obtenidos con ambos modelos se realizó un montaje experimental en el cual, mediante un transformador de corriente se indujeron 380 A a un conductor XLPE AWG 2/0 de 25 kV por un periodo de 6 horas, en un ambiente controlado a $20 \pm 1°C$ dando como resultado la siguiente curva de calentamiento del núcleo del cable.

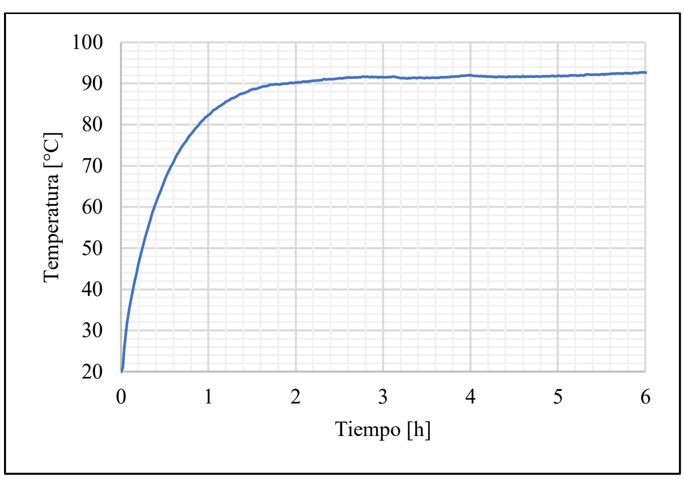

Figura 8: Datos experimentales que ilustran el incremento de la temperatura en el núcleo conductor del cable

La medición de temperatura fue realizada perforando las capas dieléctricas del cable e introduciendo un termopar de contacto tipo K en la zona superficial del núcleo; los datos fueron registrados en intervalos de 20 segundos.

Los resultados presentados por cada modelo y el experimental se resumen en la Figura 9, la comparación del error de la respuesta tomando como base el resultado experimental se muestra en la Figura 10, mientras que los recursos computacionales empleados por cada uno se resumen en la Tabla 5.

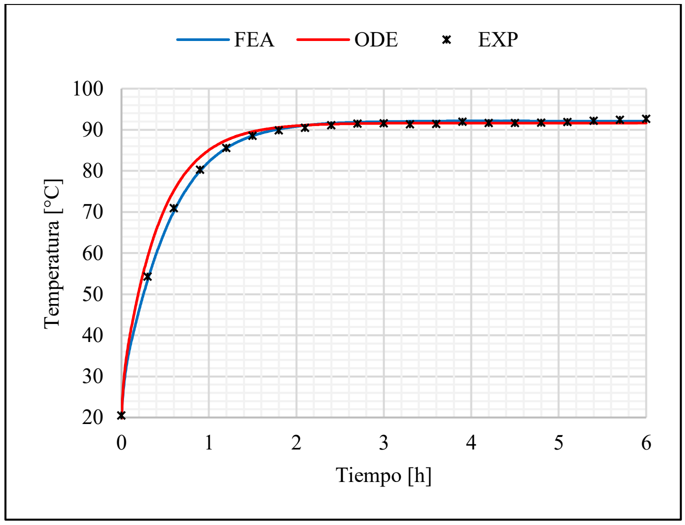

Figura 9: Comparación de los resultados de los modelos computacionales y el ensayo experimental

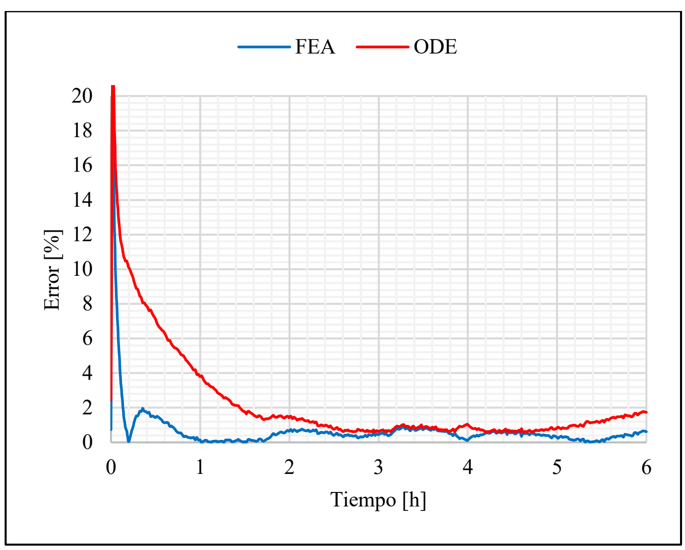

Figura 10: Error relativo de temperatura con respecto al ensayo experimental

En promedio, en el modelo de elementos finitos, el error absoluto en la predicción de temperatura en estado transitorio es de 0.42°C y para el caso de estado estable se obtuvo un promedio de error absoluto de 0.32°C. Por otra parte, el modelo de parámetros concentrados presentó un error promedio de 2.08°C en la predicción de estado transitorio y un error promedio de 0.27°C en la de estado estable.

Al comparar ambas respuestas es notorio que el análisis por elementos finitos presenta una mejor exactitud en la predicción del comportamiento transitorio; una de las posibles razones de este resultado es debida a que el coeficiente de convección integrado en el modelo responde a los cambios en la temperatura de la superficie del cable, mientras que el modelo de parámetros concentrados no toma en cuenta estas variaciones. Los errores de las respuestas de estado estable de ambos modelos se encuentran por debajo del 1.16%. El error presente es debido a que ambos modelos toman en cuenta un valor constante de temperatura ambiente promedio y omiten las pequeñas variaciones que se presentan en el tiempo.

En cuanto a eficiencia computacional, el modelo de parámetros concentrados presenta en promedio, un tiempo de solución 7 veces menor en comparación con el modelo de elementos finitos, haciendo uso de tan solo un 55.6% de memoria con respecto al otro.

Tabla 5: Recursos computacionales empleados por cada método

	Elementos finitos	Parámetros concentrados
Tiempo [s]	256	37
Memoria [GB]	3.54	1.97

V. Conclusiones

Los modelos matemáticos descritos predicen de forma apropiada el comportamiento térmico de estado estable de una muestra de cable aislado en XLPE. Estos pueden parametrizarse para simular cualquier muestra de cable y de esta forma calcular las corrientes necesarias para alcanzar un valor de temperatura deseado.

El modelo de análisis de elementos finitos presenta una mayor exactitud en comparación con el modelo de parámetros concentrados en estado transitorio, mientras que ambos presentan exactitudes similares en la predicción de temperatura de estado estable.

Queda a selección del investigador el uso de uno u otro modelo, esta decisión dependerá principalmente de la capacidad computacional con la que se cuente y el nivel de exactitud que necesite para analizar el estado transitorio del comportamiento térmico.

Ambos modelos pueden emplearse como herramientas de predicción de calentamiento ante entradas de corriente; sin embargo, también pueden ser empleados como planta (el modelo de parámetros concentrados pues presenta una mayor facilidad de extensión) para el diseño de controladores de corriente en los cables teniendo como entrada su temperatura. Esta última aplicación puede resultar puntualmente útil para la ejecución de pruebas de certificación o pre-cualificación de muestras de cables que necesiten ser evaluadas.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Exploring b_I^{**}- Hyperconnectedness and b_I^* - Separation in Ideal Topological Spaces

By Donna Ruth Talo-Banga & Michael P. Baldado Jr.
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Abstract- We came up with the concept b^*-open set which has stricter condition with respect to the notion b-open sets, introduced by Andrijevic [3] as a generalization of Levine's [11] generalized closed sets. Anchoring on this concept, we defined b_I^{**}-hyperconnected sets and b^* -separated sets.

Topology is seen in many areas of science, for example, it is used to model the space-time notion of the universe. It is sometimes investigated in non-conventional ways, for example Donaldson [7] utilized mathematical concepts used by physicists to solve topological problems. These problems includes new topological sets like b^*-open set.

Keywords and phrases: b^*-open sets, b_I^*-open sets, ideals, b_I^{**}-hyperconnected sets, b_I^* -separated sets.

GJSFR-F Classification: *MSC Code: 54A05*

Strictly as per the compliance and regulations of:

Global Journal of Science Frontier Research (F) Volume XXIII Issue VI Version I Year 2023

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Exploring b_I^{**}-Hyperconnectedness and b_I^*-Separation in Ideal Topological Spaces

Donna Ruth Talo-Banga [α] & Michael P. Baldado Jr. [σ]

Abstract- We came up with the concept $\boldsymbol{b}^*$-open set which has stricter condition with respect to the notion *b*-open sets, introduced by Andrijevic [3] as a generalization of Levine's [11] generalized closed sets. Anchoring on this concept, we defined $\boldsymbol{b_I^{**}}$-hyperconnected sets and $\boldsymbol{b}^*$ -separated sets.

Topology is seen in many areas of science, for example, it is used to model the space-time notion of the universe. It is sometimes investigated in non-conventional ways, for example Donaldson [7] utilized mathematical concepts used by physicists to solve topological problems.

These problems includes new topological sets like $\boldsymbol{b}^*$-open set.

A subset *B* of a topological space *W* is called a $\boldsymbol{b}^*$*-open relative to an ideal I* (or b_I^* -open), if there is an open set *P* with $P \subseteq \mathrm{Int}(B)$, and a closed set *S* with $\mathrm{Cl}(B) \subseteq S$ such that $(\mathrm{Int}(S) \cup \mathrm{Cl}(\mathrm{Int}(B))) \backslash B \in I$, and $B \backslash (\mathrm{Int}(\mathrm{Cl}(B)) \cup \mathrm{Cl}(P)) \in I$.

In this study, we gave some of the important properties of $\boldsymbol{b_I^{**}}$-hyperconnected sets and $\boldsymbol{b}^*$-separated sets.

Keywords and phrases: b^*-open sets, b_I^* -open sets, ideals, $\boldsymbol{b_I^{**}}$ -hyperconnected sets, b_I^* -separated sets.

I. Introduction

Topology is seen in many areas of science [14]. It is applied in biochemistry [5] and information systems [19]. Topology as a mathematical system is fundamentally comprised of open sets together with the operations union and intersection. Over time, open sets were generalized to different varieties. To name some, we have, Stone [20] introduced regular open set. Levine [10] introduced semi-open sets. Njasted [16] introduced α-open sets. Mashhour et al. [13] introduced pre-open sets. Abd El-Monsef et al. [1] introduced β-open set.

In the year 1970, Levine [11] introduced generalized closed sets, and achoring on this notion, Andrijevic [3] presented yet another generalization of open sets called b-open sets. This study uses the notion of b-open sets to come up with a new concept called b^*-open sets.

The concept ideal topological spaces was first seen in [9]. Vaidyanathaswamy [23] investigated this concept in point set topology. Tripathy and Shravan [17, 18], Tripathy and Acharjee [21], Tripathy and Ray [22], Catalan et al. [6] also made investigations on ideal topological spaces.

Several concepts in topology were generalized using this structure. One of which is the concept b^*-open sets. Using the notion of b^*-open sets, we introduced the concepts b^*-compact sets, compatible b_I^*-compact sets, countably b_I^*-compact sets, b_I^*-connected sets, in ideal generalized topological spaces.

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Let W be a non-empty set. An ideal I on a set W is a non-empty collection of subsets of W which satisfies:

1. $B \in I$ and $D \subseteq B$ implies $D \in I$.
2. $B \in I$ and $D \in I$ implies $B \cup D \in I$.

Let W be a topological space and B be a subset of W. We say that B is *b^*-open set* if $B = \mathrm{cl}(\mathrm{int}(B)) \cup \mathrm{int}(\mathrm{cl}(B))$. For example, consider $W = \{a, b, c\}$ and the topology $\varsigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, W\}$ on W. Then the b^*-open subsets are $\varnothing$, $\{a, b\}$, $\{c\}$ and W.

Let W be a topological space and B be a subset of W. The set B is called *b^*-open relative to an ideal* I (or *b_I^*-open*), if there is an open set P with $P \subseteq \mathrm{Int}(B)$, and a closed set S with $\mathrm{Cl}(B) \subseteq S$ such that

1. $(\mathrm{Int}(S) \cup \mathrm{Cl}(\mathrm{Int}(B)))\backslash B \in I$, and
2. $B\backslash(\mathrm{Int}(\mathrm{Cl}(B)) \cup \mathrm{Cl}(P)) \in I$.

In addition, we say that a set B is a b_I^*-closed set if B^C is b_I^*-open.

Consider the ideal space $(\{q, r, s\}, \{\varnothing, \{q\}, \{r\}, \{q, r\}, \{q, r, s\}\}, \{\varnothing, \{r\}\})$. Then $B = \{r, s\}$ is a b^*-open with respect to the ideal $I = \{\varnothing, \{r\}\}$. To see this, we let P be the open set $\{r\}$ and S be the closed set $\{r, s\}$. Then $\mathrm{Int}(S) \cup \mathrm{cl}(\mathrm{int}(\{r, s\}))\backslash\{r, s\} = \mathrm{int}(\{r, s\}) \cup \mathrm{cl}(\{r\})\backslash\{r, s\} = \{r\} \cup \{r, s\}\backslash\{r, s\} = \{r, s\}\backslash\{r, s\} = \varnothing \in I$. Also, $\mathrm{Int}(\mathrm{cl}(\{r, s\}) \cup \mathrm{cl}(P)\backslash\{r, s\} = \mathrm{int}(\{r, s\}) \cup \mathrm{cl}(\{r\})\backslash\{r, s\} = \{r\} \cup \{r, s\}\backslash\{r, s\} = \{r, s\}\backslash\{r, s\} = \varnothing \in I$. This shows that $B = \{r, s\}$ is a b_I^*-open.

The succeeding sections presents the rudimentary properties of b_I^{**}-hyperconnected spaces and b_I^*-separated spaces.

II. Results

This section presents the results of this study.

a) Preliminary Result: The following Lemmas were established in [4]. They will used in the proofs of some of the succeeding statements. In particular, Lemma 2.1 is used in Theorem 2.13 and Remark 2.15, while Lemma 2.2 is used in Lemma 2.19.

Lemma 2.1. *[4] Let (X, τ, I) be an ideal topological space. Then every b^*-open set is a b_I^*-open set.*

Lemma 2.2. *[4] Let (X, τ, I) be an ideal topological space with $I = \{\varnothing\}$. Then A is a b^*-open set if and only if A is a b_I^*-open set.*

b) b_I^{**}-Hyperconnected Ideal Topological Spaces: The concept $*$-hyperconnectedness was introduced by Ekici et al. [8], and the concept $I*$-hyperconnectedness was introduced by Abd El-Monsef et al. [12]. These insights motivated us to create the concept called b_I^{**}-hyperconnectedness. One may see [15] to gain more insights on these ideas.

Definition 2.3. *Let (X, τ) be a topological space and I be an ideal on X. A function $()^*(I, \tau) : P(X) \to P(X)$ given by $A^*(I, \tau) = \{x \in X : A \cup U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ is called a local of A with respect to τ and I.*

Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, and $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ (note that τ is a topology on X and I is an ideal on X). Then, $\varnothing^* = \varnothing$, $\{a\}^* = \{c\}$, $\{b\}^* = \{c\}$, $\{c\}^* = X$, $\{a, b\}^* = \{c\}$, $\{a, c\}^* = X$, $\{b, c\}^* = X$ and $X^* = X$.

Ref

4. M. P. Baldado. b^*_J sets and b^*_J-compact ideal spaces. European Journal of Pure & Applied Mathematics, 16(3), 2023.

Definition 2.4. *Let (X,τ) be a topological space and I be an ideal on X. The Kuratowski closure operator $Cl()^*(I,\tau): P(X) \to P(X)$ for the topology $\tau^*(I,\tau)$ is given by $Cl(A)^*(I,\tau) = A \cup A^*$.*

Consider the ideal space in the previous example. We have, $Cl(\varnothing)^* = \varnothing \cup \varnothing^* = \varnothing \cup \varnothing = \varnothing$, $Cl(\{a\})^* = \{a\} \cup \{a\}^* = \{a\} \cup \{c\} = \{a,c\}$, $Cl(\{b\})^* = \{b\} \cup \{b\}^* = \{b\} \cup \{c\} = \{b,c\}$, $Cl(\{c\})^* = \{c\} \cup \{c\}^* = \{c\} \cup X = X$, $Cl(\{a,b\})^* = \{a,b\} \cup \{a,b\}^* = \{a,b\} \cup \{c\} = X$, $Cl(\{a,c\})^* = \{a,c\} \cup \{a,c\}^* = \{a,c\} \cup X = X$, $Cl(\{b,c\})^* = \{b,c\} \cup \{b,c\}^* = \{b,c\} \cup X = X$, and $Cl(X)^* = X \cup X^* = X \cup X = X$.

Definition 2.5. *Let (X,τ) be a topological space and I be an ideal on X. The Kuratowski interior operator $Int()^*(I,\tau): P(X) \to P(X)$ for the topology $\tau^*(I,\tau)$ is given by $Int(A)^*(I,\tau) = X - Cl(X-A)^*$.*

Definition 2.6. *An ideal space (X,τ,I) is called $*$-hyperconnected [8] if $cl(A)^* = X$ for all non-empty open set $A \subseteq X$.*

Definition 2.7. *An ideal space (X,τ,I) is called I^*-hyperconnected [2] if $X - cl(A)^* \in I$ for all non-empty open set $A \subseteq X$.*

Definition 2.8. *An ideal topological space (X,τ,I) is said to be b_I^{**}-hyperconnected space if $X - cl(A)^* \in I$ for every non-empty b_I^*-open subset A of X.*

The next theorem says that the family of all b_I^{**}-hyperconnected space contains all $I*$-hyperconnected space.

Theorem 2.9. *Let (X,τ,I) be an ideal topological space. If X is $I*$-hyperconnected, then it is b_I^{**}-hyperconnected also.*

Proof. Let X be $I*$-hyperconnected, and A be a non-empty open set. Because X is I_*-hyperconnected, we have $X - cl(A)^* \in I$ for all non-empty open set $A \subseteq X$. And, because an open set is also a b_I^*-open set, we have $X - cl(A)^* \in I$ for all non-empty b_I^*-open set $A \subseteq X$. Hence, X is b_I^{**}-hyperconnected.

The next lemma is clear.

Lemma 2.10. *Let (X,τ) be a topological space. Then the intersection of any family of ideals on X is an ideal on X.*

Theorem 2.11 is taken from [2]. It says that when I is the minimal ideal, then the notions $*$-hyperconnected and I^*-hyperconnected are equivalent.

Theorem 2.11 *[2] Let (X,τ) be a clopen ideal topological space with $I = \{\varnothing\}$. Then, X is $*$-hyperconnected if and only if it is I^*-hyperconnected.*

The next remark is clear.

Remark 2.12. *If (X,τ) is a clopen topological space (a space in which every open set is also closed), then A is open if and only if A is b^*-open.*

Theorem 2.13 says that in a clopen space, with respect to the minimal ideal I, the notions b_I^{**}-hyperconnected and I^*-hyperconnected are equivalent.

Theorem 2.13 *Let (X,τ,I) be a clopen ideal topological space with $I = \{\varnothing\}$. Then, X is I^*-hyperconnected if and only if it is b_I^{**}-hyperconnected.*

Ref

2. M. E. Abd El-Monsef, A. A. Nasef, A. E. Radwan, F. A. Ibrahem, and R. B. Esmaeel. Some properties of semi-open sets with respect to an ideal.

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Proof. Suppose that X is I^*-hyperconnected. Let A is a non-empty element of τ. Then $X - cl^*(A) \in I$. By Remark 2.12 and Lemma 2.2, every open set is precisely b_I^*-open. Thus, $X - cl^*(A) \in I$ for all b_I^*-open set $A\ (\neq \varnothing)$. Therefore, X is b_I^{**}-hyperconnected also. Conversely, suppose that X is b_I^{**}-hyperconnected. Let A be a non-empty b_I^*-open set. Then $X - cl^*(A) \in I$. By Remark 2.12 and Lemma 2.2, b_I^*-open set is precisely open. Thus, $X - cl(A)^* \in I$ for all open set $A\ (\neq \varnothing)$. Therefore, X is I^*-hyperconnected also. □

Corollary 2.14 says that in a clopen ideal topological space, relative to the minimal ideal I, the notions b_I^{**}-hyperconnected, I^*-hyperconnected, and $*$-hyperconnected are equivalent.

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Corollary 2.14. Let (X, τ, I) be a clopen ideal topological space with $I = \{\varnothing\}$. Then the following statements are equivalent.

i. X is $*$-hyperconnected.

ii. X is I^*-hyperconnected.

iii. X is b_I^{**}-hyperconnected.

Theorem 2.15 may be an important property.

Remark 2.15. If an ideal topological space $(X, \tau, \{\varnothing\})$ is a $b_I^*{*}$-hyperconnected space, then $X - cl^*(A) \in I$ for every non-empty b^*-open subset A of X.

To see this, let A is a non-empty b-open set. Then by Lemma 2.2 A is b_I^*-open. Since X is b_I^{**}-hyperconnected, $X - cl^*(A) \in I$.

Theorem 2.16 is a characterization of b_I^{**}-hyperconnected space.

Theorem 2.16. Let (X, τ, I) be an ideal topological space. Then the following statements are equivalent.

i. X is a b_I^{**}-hyperconnected space.

ii. $int(A)^* \in I$ for all b_I^*-closed proper subset A of X.

Proof. $(i) \Rightarrow (ii)$ Let B be b_I^*-closed. Then $X - B$ is b_I^*-open. Since $B \neq X$, $X - B \neq \varnothing$. Hence, by assumption we have $\text{int}(B)^* = X - \text{cl}(X - B)^* \in I$.
$(ii) \Rightarrow (i)$ Let $A\ (\neq X)$ be a non-empty b_I^*-open set. Then $X - A$ is a non-empty b_I^*-open set. Hence, by assumption, we have $X - \text{cl}(A)^* = X - \text{cl}(X - (X - A))^* = \text{int}(X - A)^* \in I$. Thus, X is b_I^{**}-hyperconnected. □

c) b_I^*-Separated Ideal Topological Spaces: In this section, we present the concepts b_I^*-separated sets and b_I^*-connected sets. We also present some of their important properties.

Definition 2.17. Let (X, τ, I) be an ideal topological space and A be a subset of X. The b^*-closure of A, denoted by $cl_{b^*}(A)$, is the smallest b^*-closed set that contains A. The b_I^*-closure of A, denoted by $cl_{b_I^*}(A)$, is the smallest b_I^*-closed set that contains A.

Next, we define b_I^*-separated sets, b_I^*-connected sets, and b_I^*-connected spaces.

Definition 2.18. Let (X, τ, I) be an ideal topological space. A pair of subsets, say A and B, of X is said to be b_I^*-separated if $cl_{b_I^*}(A) \cap B = \varnothing = A \cap cl_{b_I^*}(B)$. A subset A of X is said to be b_I^*-connected if it cannot be expressed as a union of two b_I^*-separated sets. The topological space X is said to be b_I^*-connected if it is b_I^*-connected as a subset.

Lemma 2.19 says that every b_I^*-connected space is connected. Recall, a space is connected if it cannot be written as a union of two non-empty open sets.

Lemma 2.19. Let (X, τ) be a τ_ς-space (a topological space in which every element is b^*-open also) and I be an ideal in X. If X is b_I^*-connected, then it is connected.

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Proof. Suppose that to the contrary X is not connected. Let A and B be non-empty disjoint elements of τ with $X = A \cup B$. By Lemma 2.1, A and B are b_I^*-open sets also. Because $A = B^C$ and $B = A^C$, A and B are also b_I^*-closed. And so, $A = \text{cl}_{b_I^*}(A)$ and $B = \text{cl}_{b_I^*}(B)$. Thus, $\text{cl}_{b_I^*}(A) \cap B = A \cap B = \varnothing$ and $A \cap \text{cl}_{b_I^*}(B) = A \cap B = \varnothing$. This implies that X is b_I^*-separated, that is X is not b_I^*-connected, a contradiction. □

Remark 2.20. Let (X, τ) be a topology and I be an ideal in X. If $Y \subseteq X$, then $I_Y = \{Y \cap A : A \in I\}$ is an ideal in the relative topology (Y, τ_Y).

To see this, for the first property, let $B \in I_Y$ and $A \subseteq B$. Then $A \subseteq B \subseteq Y$. Now, if $A \in I_Y$, then there exist $C \in I$ such that $Y \cap C = A$. Note that $A \subseteq B \subseteq C$. Hence, $A, B \in I$. Thus, $A = Y \cap A \in I_Y$. Next, for the second, let $D, E \in I_Y$. Then $D \subseteq Y$ and $E \subseteq Y$. if $D \in I_Y$, then there exist $F \in I$ such that $Y \cap F = D$. Similarly, if $E \in I_Y$, then there exist $G \in I$ such that $Y \cap G = E$. Since I is an ideal, $F \cup G \in I$. Now, because $D \cup E \subseteq F \cup G$, $D \cup E \in I$. Thus, $D \cup E = (D \cup E) \cap Y \in I_Y$.

The next statement, Theorem 2.24, presents a way to construct b_I^*-open sets in a subspace.

Theorem 2.21. Let (X, τ, I) be an ideal topological space, and $Y \subseteq X$. If A is a b_I^*-open subset of X, then $A \cap Y$ is a $b_{I_Y}^*$-open set in Y.

Proof. Let A be a b_I^*-open set in X (τ, I). Then there exists an open set O with $O \subseteq \text{int}(A)$, and a closed set F with $\text{cl}(A) \subseteq F$ such that $\text{int}(F) \cup \text{cl}(\text{int}(A)) \backslash A \in I$, and $A \backslash \text{int}(\text{cl}(A)) \cup \text{cl}(O) \in I$. Let $O' = O \cap Y$, and $F' = F \cap Y$. Then O' is open in Y, and F' is closed in Y. Also, $\text{int}(F) \cup \text{cl}(\text{int}(A)) \backslash A \supseteq (\text{int}(F) \cup \text{cl}(\text{int}(A))) \cap Y \backslash (A \cap Y) \supseteq (\text{int}(F)) \cap Y \cup \text{cl}(\text{int}(A))) \cap Y \backslash (A \cap Y) \supseteq \text{int}(F \cap Y) \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y) \supseteq \text{int}(F') \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y)$, and $A \backslash \text{int}(\text{cl}(A)) \cup \text{cl}(O) \supseteq (A \cap Y) \backslash (\text{int}(\text{cl}(A)) \cup \text{cl}(O)) \cap Y \supseteq (A \cap Y) \backslash \text{int}(\text{cl}(A)) \cap Y \cup \text{cl}(O) \cap Y \supseteq (A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O \cap Y) \supseteq (A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O')$. Hence, by heredity $\text{int}(F') \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y) \in I$, and $(A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O') \in I$. This shows that $A \cap Y$ is a $b_{I_Y}^*$-open set in Y. □

Corollary 2.22. Let (X, τ, I) be an ideal topological space and $Y \subseteq X$. If A is a b_I^*-closed set in X (τ, I), then $A \cap Y$ is a $b_{I_Y}^*$-closed set in Y.

Proof. If A is b_I^*-closed, then A^C is b_I^*-open. By Theorem 2.21, $A^C \cap Y$ is $b_{I_Y}^*$-open. Hence, $A \cap Y = (A^C \cap Y)^C$ is $b_{I_Y}^*$-closed in Y. □

Remark 2.23. Let (X, τ, I) be an ideal topological space and $Y \subseteq X$. Then $I_Y = \{A \cap Y : A \in I\}$ is a subset of I.

The next statement, Theorem 2.24, say something about the closure of a set in the subspace.

Theorem 2.24. Let (X, τ, I) be an ideal topological space, and Y be an open subset of X. If $A \subseteq X$, then $cl_{b_{I_Y}^*}(A \cap Y) \subseteq cl_{b_I^*}(A) \cap Y$.

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Proof. Since $\mathrm{cl}_{b_I^*}(A)$ is a b_I^*-closed set in X, by Corollary 2.22 $\mathrm{cl}_{b_I^*}(A)\cap Y$ is a $b_{I_Y}^*$-closed set in Y. Hence, $\mathrm{cl}_{b_{I_Y}^*}(A\cap Y)\subseteq \mathrm{cl}_{b_{I_Y}^*}(\mathrm{cl}_{b_I^*}(A)\cap Y)=\mathrm{cl}_{b_I^*}(A)\cap Y$. □

The next statement, Theorem 2.25, says that if two sets, say A and B, are separated in the mother space, then $A\cap Y$ and $B\cap Y$ are also separated in the subspace.

Theorem 2.25. Let (X,τ,I) be an ideal topological space, and Y be a subset of X. If A and B are b_I^*-separated in X, then $A\cap Y$ and $B\cap Y$ are $b_{I_Y}^*$-separated in Y.

Proof. If A and B are b_I^*-separated in X, then $\mathrm{cl}_{b_I^*}A\cap B=\varnothing=A\cap \mathrm{cl}_{b_I^*}B$. Thus, by Theorem 2.24 $\varnothing=\varnothing\cap Y=(\mathrm{cl}_{b_I^*}A\cap B)\cap Y=((\mathrm{cl}_{b_I^*}A)\cap Y)\cap(B\cap Y)\supseteq \mathrm{cl}_{b_{I_Y}^*}(A\cap Y)\cap(B\cap Y)$ and $\varnothing=\varnothing\cap Y=(A\cap \mathrm{cl}_{b_I^*}B)\cap Y=(A\cap Y)\cap((\mathrm{cl}_{b_I^*}B)\cap Y)\supseteq(A\cap Y)\cap \mathrm{cl}_{b_{I_Y}^*}(B\cap Y)$. Thus, $A\cap Y$ and $B\cap Y$ are $b_{I_Y}^*$-separated. □

The next statement, Remark 2.26, says that the non-empty components of a space that makes it b_I^*-separated are b_I^*-open.

Remark 2.26. Let (X,τ,I) be a b_I^*-separated ideal topological space. If $X=A\cup B$ with $A\neq\varnothing$, $B\neq\varnothing$ such that $\mathrm{cl}_{b_I^*}A\cap B=\varnothing=A\cap \mathrm{cl}_{b_I^*}B$, then A and B are b_I^*-open.

To see this, we have $A^C=\mathrm{cl}_{b_I^*}(B)$ and $B^C=\mathrm{cl}_{b_I^*}(A)$. Hence, A^C and B^C are b_I^*-closed. Thus, A and B are b_I^*-open.

Recall, a pair of subsets, say A and B, of X is said to be b_I^*-separated if $\mathrm{cl}_{b_I^*}(A)\cap B=\varnothing=A\cap \mathrm{cl}_{b_I^*}(B)$. A subset A of X is said to be b_I^*-connected if it cannot be expressed as a union of two b_I^*-separated sets. A topological space X is said to be b_I^*-connected if it is b_I^*-connected as a subset.

The next statement, Theorem 2.27, says that two b_I^*-separated set cannot contain portions of a connected set.

Theorem 2.27. Let (X,τ,I) be a b_I^*-separated ideal topological space, and A be a b_I^*-connected set. If $A\subseteq H\cup G$ with H and G are b_I^*-separated sets, then either $A\subseteq H$ or $A\subseteq G$.

Proof. Suppose that to the contrary, $A=(A\cap H)\cup(A\cap G)$ with $A\cap H\neq\varnothing$ and $A\cap G\neq\varnothing$. Since H and G are b_I^*-separated sets, $\mathrm{cl}_{b_I^*}(A\cap H)\cap(A\cap G)\subseteq \mathrm{cl}_{b_I^*}H\cap G=\varnothing$ and $(A\cap H)\cap \mathrm{cl}_{b_I^*}(A\cap G)\subseteq H\cap \mathrm{cl}_{b_I^*}G=\varnothing$. Thus, $\mathrm{cl}_{b_I^*}(A\cap H)\cap(A\cap G)=\varnothing=(A\cap H)\cap \mathrm{cl}_{b^*}(A\cap G)$. Therefore, A can be expressed as a union of two b_I^*-separated sets $A\cap H$ and $A\cap G$. This is a contradiction. □

The next statement, Theorem 2.28, says that subsets of each of two b_I^*-separated sets are also separated.

Theorem 2.28. Let (X,τ,I) be an ideal topological space, and, A and B be b_I^*-separated sets. If $C\subseteq A$ $(C\neq\varnothing)$ and $D\subseteq B$ $(D\neq\varnothing)$, then C and D are also b_I^*-separated.

Proof. Suppose that A and B are b_I^*-separated. Then $\mathrm{cl}_{b_I^*}A\cap B=\varnothing=A\cap \mathrm{cl}_{b_I^*}B$. Thus, $\mathrm{cl}_{b_I^*}C\cap D\subseteq \mathrm{cl}_{b_I^*}A\cap B=\varnothing$ and $C\cap \mathrm{cl}_{b_I^*}D=A\cap \mathrm{cl}_{b_I^*}B=\varnothing$. Hence, $\mathrm{cl}_{b_I^*}C\cap D=\varnothing=C\cap \mathrm{cl}_{b_I^*}D$. Therefore, C and D is b_I^*-separated. □

Notes

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Study of Convergent Praxeological Needs for Teachers' Didactic Infrastructures

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Abstract- The aim of this study is to reflect on teacher training, based essentially on the Didactic Anthropological Theory and on research aimed at studying the praxeological needs of teachers. It is a qualitative study of a theoretical-bibliographical nature, as it produces reflections based on our theoretical framework and research that deals with teacher training in the light of the Didactic Anthropological Theory. Our reflections led us to consider teacher training, didactic infrastructures for teacher training and the praxeological needs of the teacher. In addition, we took as an example one of the episodes from the experimental phase of Lobo's research (2019), which aimed to study the knowledge of plane analytic geometry that can be acquired by students (teacher trainees) from a Mathematics degree course in Bahia (Brazil), participating in supervised curricular internship classes, who were involved in a training process based on a Study and Research Pathway. The aim is to reflect on the complexity of building teaching praxeologies.

Keywords: *teacher training, didactic infrastructures, teacher praxeological needs.*

GJSFR-F Classification: *FOR Code: 130202*

STUDYOFCONVERGENTPRAXEOLOGICALNEEDSFORTEACHERSDIDACTICINFRASTRUCTURES

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Global Journal of Science Frontier Research (F) Volume XXIII Issue VI Version I Year 2023

Study of Convergent Praxeological Needs for Teachers' Didactic Infrastructures

Estudo De Necessidades Praxeológicas Convergentes Para Infraestruturas Didáticas De Professores

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Resumo- O estudo que propomos tem por objetivo tecer reflexões sobre a formação de professores, apoiando-nos essencialmente na Teoria Antropológica do Didático e em pesquisas voltadas ao estudo de necessidades praxeológicas do professor. É uma pesquisa qualitativa de cunho teórico-bibliográfica, pois produz reflexões tecidas a partir de nosso referencial teórico e de pesquisa que tratam da formação de professores à luz da Teoria Antropológica do Didático. Nossas reflexões nos levaram a ponderações sobre a formação de professores, as infraestruturas didáticas para a formação de professores e as necessidades praxeológicas do professor. Além disso, tomamos como exemplo um dos episódios da fase experimental da pesquisa de Lobo (2019) que teve por objetivo estudar os conhecimentos em geometria analítica plana que podem ser adquiridos por estudantes (professores-estagiários) de um curso de Licenciaturas em Matemática na Bahia (Brasil), participantes de turmas de estágio curricular supervisionado, que foram envolvidos em um processo de formação apoiando-se em um Percurso de Estudo e Pesquisa. O intuito é tecer reflexões sobre a complexidade de construção de praxeologias docentes.

Palavras-chave: formação de professores, infraestruturas didáticas, necessidade praxeológicas do professor.

Abstract- The aim of this study is to reflect on teacher training, based essentially on the Didactic Anthropological Theory and on research aimed at studying the praxeological needs of teachers. It is a qualitative study of a theoretical-bibliographical nature, as it produces reflections based on our theoretical framework and research that deals with teacher training in the light of the Didactic Anthropological Theory. Our reflections led us to consider teacher training, didactic infrastructures for teacher training and the praxeological needs of the teacher. In addition, we took as an example one of the episodes from the experimental phase of Lobo's research (2019), which aimed to study the knowledge of plane analytic geometry that can be acquired by students (teacher trainees) from a Mathematics degree course in Bahia (Brazil), participating in supervised curricular internship classes, who were involved in a training process based on a Study and Research Pathway. The aim is to reflect on the complexity of building teaching praxeologies.

Keywords: teacher training, didactic infrastructures, teacher praxeological needs.

I. Introduction

Teacher training is one of the crucial problems faced by training institutions and researchers in mathematics didactics. Generally, it is concerned with specific aspects, and some official documents confirm the need for research in this area, due to the gap between what we want teaching to be and how it is carried out (André, 2001). In this direction, the Referentials for teacher training state that:

> [...] despite the commitment of many and the progress of the experiments already carried out, there is a huge gap - and not just in Brazil - between the knowledge and actions of the majority of teachers in practice and the new conceptions of

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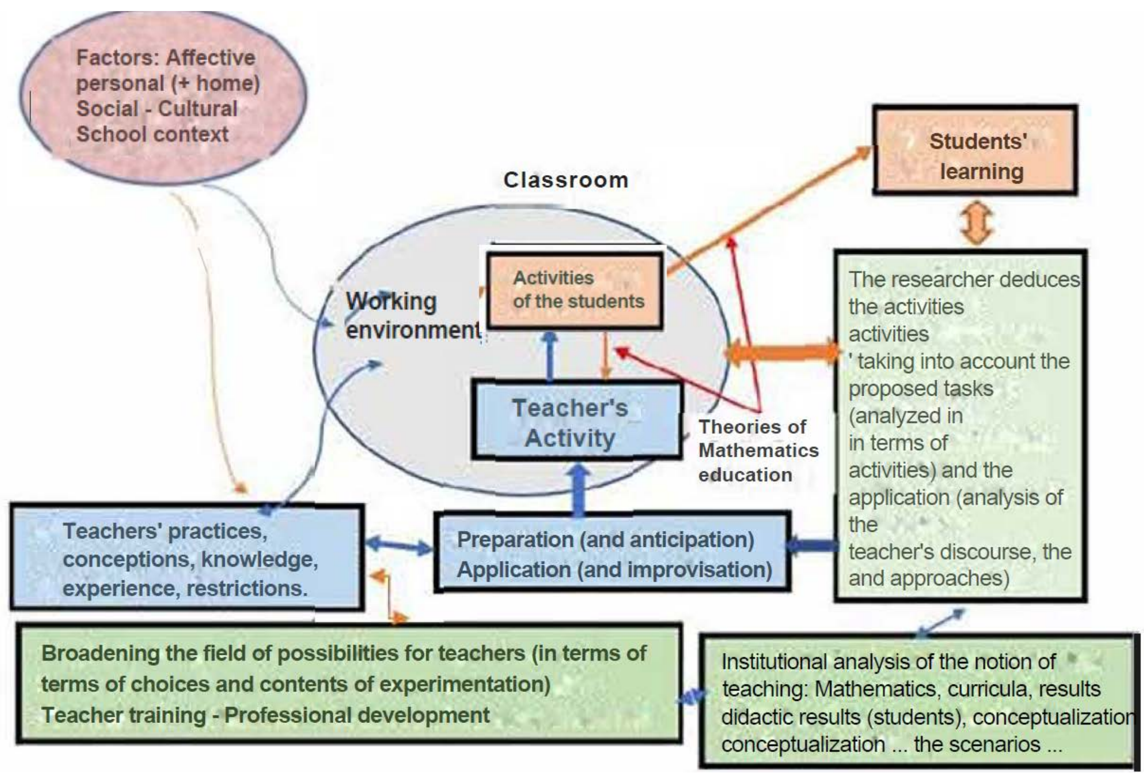
teacher work that these movements have been producing. It is therefore a question not just of doing training better, but of doing it differently. These changes require, among other things, that teachers reconstruct their practices and, to do this, it is necessary to "build bridges" between the reality of their work and what is targeted. (Brasil, 1999, p. 16).

As this statement alludes to practicing teachers, it refers to a special type of training, continuing education, which is now considered essential for classroom teachers, both to update their knowledge and techniques in the specific area they teach, and to develop skills and attitudes. It also suggests that the concept of teacher training should be questioned, because it can be conceived in different ways, considering the objectives, content and methods.

In order to analyze and interpret the findings from the teacher's practices and their student's learning activities, an appropriate theoretical framework and methodology must be used. Figure 1 presents a structure that maps out the paths to be taken in order to read the reality of the mathematics classroom.



Source: Adapted from Abboud-Blanchard, Robert, Rogalski &Vandebrouck (2017, p.11).

Figure 1: Classroom reading grid

The scheme (Figure 1) allows us to study and understand students' mathematical learning in the context of the teaching they receive at school, but also elements of the teacher's practice. It is therefore necessary to study students' (mathematical) activities in the classroom, what they do (or don't do), what they say (or don't say), what they write (or don't write), even if we can only collect traces of them, because what they think remains unobservable (Abboud-Blanchard et al., 2017). This requires choosing an appropriate theoretical framework for studying students in situations, distinguishing between tasks and activities, and focusing the study on learning.

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Global Journal of Science Frontier Research (F) Volume XXIII Issue VI Version I Year 2023

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We understand that knowing, as a personal construction, does not only occur cognitively; it is necessary for the subject to identify with what they learn so that they can give their own meaning to the relationship they build with knowing. Learning takes on an active meaning for the individual and is linked to the moment and situation in which the learning takes place. In this way, the object of analysis - when studying learning processes - should be the relationships in which subjects engage with knowing. In other words, to question this set of relationships with knowing, is to be interested in the process in which the subject is integrated into their environment.

As for theories of Mathematics Education, it is important to use them to ensure a certain coherence in the decomposition of the reality involved. Situation theory, for example, can help design situations that are potentially favorable to learning and whose implementation needs to be tested. The didactic variables available to the teacher allow him/her to influence the possible activities of the students. The didactic contract serves to specify the expectations, explicit or not, of the teacher and students towards each other, and highlighting it can help us understand what can distort or reinforce the games in which students are involved (Abboud-Blanchar et al., 2017).

The Anthropological Theory of the Didactic (the theoretical-methodological framework of our study) allows, for example, reference mathematical analyses based on the identification of the praxeologies at play, ranging from the types of tasks and techniques to the technologies in use and the theories in which they are embedded. In addition, the phenomena identified are part of various levels of determination, from the classroom to society (aspects that we will delve into in this text).

In this article, we reflect on the praxeological needs of math teachers from the perspective of the Anthropological Theory of the Didactic (ATD) developed by Chevallard (1999) and collaborators.

Olarría and Sierra (2011, pp. 466-467) state that, in the context of ATD, the profession of math teacher,

> must be equipped with its own didactic-mathematical resources, which constitute the necessary *infrastructure*[1] to deal with the difficulties, problems and challenges that continually arise in teaching and which, due to their complexity, cannot - and should not - be dealt with by the teacher alone. For this reason, the problem of teacher training should be seen as an aspect of one of the "great problems" of didactics: *the links between the development of didactic science, the development of the educational system and the training* of its agents. This is why the shortcomings of the education system should not be attributed solely to the responsibility of the individual teacher, nor to their level of training.

[2] The notion of *infrastructure* (or *substructure*) is, in the ATD, a general concept: it refers to the underlying base needed to develop any determined, superstructural activity. It should be clear, for example, that the "superstructural" activity that consists in watching TV at home requires an enormous infrastructural base. In a school system Σ, the infrastructure allows the appropriate actors of Σ to engage in the superstructural activities of creating and managing the schools σ that the system Σ will consist of. In each of these schools σ there are also infrastructural means to create and manage classes ϲ, for example by solving problems of time and place of operation. In each class, there are similarly infrastructural devices that allow the superstructural activities that make up the class to be carried out. In a mathematics class, there is a gradually built infrastructure allowing the mathematical (superstructural) activities to be carried out by the students. To be able to write that we have $141217/3215763 = 0.04391... \approx 4.39\%$, for example, we need to have available the division operation and the system $\mathbb{D}^{\geq}$ of nonnegative decimal numbers, without forgetting a sufficient calculation time by hand or a calculator, together with the notions of "almost equality" and percentage and their respective symbols (≈ and %). It should be noted that, in many cases, at least within the *paradigm of visiting works*, the time taken to build the mathematical infrastructure leaves relatively little room for the (superstructural) mathematical activities that this infrastructure is supposed to make possible. Things go differently within the *paradigm of questioning the world*, insofar as the mathematical infrastructure is built according to the needs of the superstructural mathematical activities that one wishes to develop. In this perspective, it should be noted that the infrastructure made available by the Internetand digital information technology offers a quite favorable frameworktothe pedagogies ofinquiry. (Chevallard & Bosch, 2022). In: http://www.dicionariodidatica.ufba.br/infraestrutura-e-superestrutura/

Therefore, in this text we present some of the constitutive elements of the teacher's praxeological equipment (Chevallard & Cirade, 2010), with a focus on a qualitative study of a theoretical-bibliographical nature, since it produces reflections based on our theoretical framework and research that deals with teacher training.

In terms of the literature review, we relied mainly on Chevallard (1999). Cirade (2019), Wozniak (2020), among other authors. We focused our reflections on the following aspects: mathematical praxeologies, infrastructures for teacher training, and the praxeological needs of the teacher.

In the next section, we present some constructs from ATD that are fundamental to the construction of this text.

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II. Anthropological theory of the Didactic

In the context of this theory, didactics is defined as the science of the conditions and restrictions of the social dissemination of praxeologies: the didactics of mathematics is therefore the science of the conditions and restrictions of the social dissemination of mathematical praxeologies (Chevallard & Cirade, 2010). These authors differentiate between *conditions* and *restrictions*: a restriction is a condition considered, from a given institutional position, at a given time, to be non-modifiable. A condition is a constraint considered to be modifiable.

What didactics studied, first and foremost, were the conditions and restrictions created by what Chevallard and Cirade (2010) call the *didactic*, i.e. all the personal or institutional "facts and gestures" inspired by a didactic intention. This intention is intended to ensure that a person or institution complies with a given praxeological content. Didactics has focused mainly on the study of the *didactic* created in the classroom by the teacher. Against this limitation of the field of study, the authors assert that the theory of didactic transposition[2] highlights the conditions not created by the teacher, which are often constraints for him or her, and, more broadly, the conditions created at other levels of what is known as the scale of levels of didactic codetermination, shown in Figure 2.

The Anthropological Theory of the Didactic (ATD) studies the conditions of possibility and functioning of Didactic Systems, understood as subject-institution-knowing relations (in reference to the didactic system treated by Brousseau (1986), student-teacher-knowing).

Chevallard (1999) asserts that ATD studies man in relation to mathematical knowing, and more specifically, in relation to mathematical situations, and places mathematical activity and, consequently, the study of mathematics within the set of human activities and social institutions.

In ATD, the notions of (types of) task, (type of) technique, technology and theory make it possible to model social practices in general, and particularly mathematical activity, on the basis of three postulates:

1. Every institutional practice can be analyzed, from different points of view and in different ways, in a system of relatively well-defined tasks.
2. The fulfillment of every task derives from the development of a technique

The word technique is used here as a "way of doing" a task, but not necessarily as a structured and methodical or algorithmic procedure.

[2] Didactic transposition is "the set of transformations that a 'wise' knowing undergoes in order to be taught". In this definition, we can distinguish "knowing" from "taught knowledge". Knowing does not exist "in a vacuum" within a social void: all knowing appears, at a given moment, in a given society, as anchored in one or more institutions. (Chevallard, 1989, apud Almouloud, 2022, p. 140)

The institutional relationship that is established between an institution I (student, teacher, ...) and an object O depends on the positions they occupy in that institution, and the set of tasks that these people must fulfill using certain techniques. According to Chevallard (1992, p. 86),

> An object exists from the moment that a person X or an institution I recognizes it as existing (for them). More precisely, we can say that the object O exists for X (respectively for I) if there is an object, which I will denote by R(X, O) (respectively $R_I(O)$), which I will call X's personal relationship with O (respectively I's institutional relationship with O).

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Tasks are identified by an action verb, which alone would characterize a task genre, for example: calculate, decompose, solve, add, which do not define the content under study. On the other hand, "solving a fractional equation" or "decomposing a rational fraction into simple elements" characterize types of tasks in which certain tasks are found, such as "solving the equation x +2=0" or "decomposing the fraction 7/9 into simpler fractions" (Silva, 2005).

For a given task, there is usually one technique or a limited number of techniques recognized in the institution that problematized that task, although there may be alternative techniques in other institutions. Most institutional tasks become routine when they no longer present any problems. This means that in order to produce techniques, you need to have an effectively problematic task that stimulates the development of at least one technique to answer the questions posed by the task. The techniques produced in this way are then organized so that they function regularly in the institution.

These two postulates result in a "practical-technical" block made up of a type of task and a technique that can be identified in everyday language as "know-how to do". (Chevallard, 2002a, p. 3)

The third postulate to be stated refers to the ecology of tasks:

3. The ecology of tasks, in other words, the conditions and constraints that allow them to be produced and used in institutions. Thus,

[...] the ecology of tasks and techniques are the conditions and restrictions that allow them to be produced and used in institutions and it is assumed that, in order to exist in an institution, a technique must be understandable, legible and justified [...] this ecological need implies the existence of a descriptive and justificatory discourse of tasks and techniques that we call the technology of technique. This postulate also implies that every technology needs a justification which we call the theory of technique and which constitutes its ultimate foundation (Bosch; Chevallard, 1999, pp. 85-86).

It is assumed that in order to exist in an institution, a technique must at least be comprehensible, legible and justified, which would be a minimum condition to allow its control and guarantee the effectiveness of the chosen tasks. These ecological conditions and restrictions imply the existence of a discourse that describes and justifies the tasks and techniques that Bosch & Chevallard (1999) call the technology of technique. Every technology also needs a justification, which they call the theory of technology.

A complex of techniques, technologies and theories organized around a type of task forms a praxeological organization (or praxeology) (Bosch & Chevallard, 1999). A set of techniques, technologies and theories The word praxeology is formed by two Greek terms, praxis and logos, which mean practice and reason respectively. It refers to the fact that in an institution, a human practice is always accompanied by a more or less developed discourse of a logos that justifies it, accompanies it and gives it reason.

The praxeology associated with a knowing is the combination of two blocks: knowing-how to do (technical/practical) and knowing (technological/theoretical), whose ecology refers to the conditions of its construction and life in the educational institutions that produce, use or transpose it. The conditions for the "survival" of knowing and doing are considered here in analogy to an ecological study: what is the habitat? What is the niche? What role does this knowing or know-how play in the "food chain"? These answers help us understand the mathematical organization determined by a praxeology.

Chevallard (1999) observes that the praxeologies (or organizations) associated with mathematical knowing are of two kinds: mathematical and didactic. Mathematical organizations refer to the mathematical reality that can be constructed to be developed in a classroom and didactic organizations refer to the way in which this construction is carried out; thus, there is a relationship between the two types of organization that Chevallard (2002a) defines as the phenomenon of codetermination between mathematical and didactic organizations.

In a process of knowing/knowledge formation, praxeologies age because their theoretical and technological components lose their credibility. Constantly, new praxeologies emerge in a given institution I that can be produced or reproduced if they exist in any institution I'. The passage from the praxeology of institution I to that of institution I' is called Transposition by Chevallard (2002b), more specifically, Didactic Transposition, when the target institution is an educational institution (school, class, etc.).

In the next section, we present some reflections from research in mathematics didactics on teaching practices, the conditions and constraints related to these practices, and teacher training.

III. Teaching Infrastructures for Teacher Training

Cirade (2019, p. 341) identifies some difficulties encountered by trainers in the exercise of their profession. The author observes that "a difficulty having been recognized by a person or an institution ξ, can be transmuted, for a person or an institution ξ^*, into the form of a question to be answered and that "the recognition that a difficulty affects the exercise of the profession, its transmutation into a question Q, the construction of an answer R and the control of the validity and value of this answer are by definition the responsibility of the *noosphere of the profession*" (Chevallard, 2013, p. 88). From this perspective, Chevallard (2011, p. 12) states that.

> Given an activity project Π0 in which such an institution or person U_0 plans to get involved, what is, for this institution or person, the praxeological equipment $\{\wp\}$ that can be considered indispensable or simply useful in the conception and realization of this project?

In the case of U_0 being a teacher training institution and Π_0 being the teacher training project, the question for Cirade (2019) is to study the praxeologies that are useful or indispensable for the realization of this project, since both the U_0 institution and the Π0 project are decisive in this study. In this didactic institution U_0, it is a question of establishing teaching praxeologies around a question Q, with students X and study directors Y, in order to constitute a study *milieu* M and face it in order to produce an answer $R^{♥}$ (optimal answer). This system is modeled by the Herbartian scheme, presented here in its semi-developed form:

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$$[S(X\,;\,Y\,;\,Q) \curvearrowright M] \hookrightarrow R^{\heartsuit}.$$

In the process of studying Q, various resources can be mobilized: the resources that make up the "didactic milieu" or milieu for the study (of Q), M is the set of resources useful for studying the question Q, producing the answer $R^{\heartsuit}$ and validating it. The upward curving arrow (➡) indicates that it is the didactic system $S\,(X;\,Y;\,Q)$ that constitutes, that "manufactures" this milieu. Therefore,

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> M is not created in advance; it is created in parallel with the search for answers. The construction of *millieu* M involves activating gestures in five moments: observing, analyzing, evaluating, developing, disseminating and defending objects, works, resources, information, etc. that can be incorporated, in whole or in part, into the *millieu* and be an indispensable part of the construction of answer $R^{\heartsuit}$. (Chevallard, 2009a, p. 20)

The *milieu* M can be represented as:

$$M=\{P_1^{\diamond}, P_2^{\diamond}, P_3^{\diamond},\cdots,P_\nu^{\diamond},\Theta_{\nu+1},\cdots\Theta_{\mu},\ O_{\mu+1},\cdots O_{\pi}\}.$$

Thus, the Herbartian scheme developed would be represented as follows:

$$[S\ (X;\ Y;\ Q) \rightarrow -R_1^{\diamond}, R_2^{\diamond}, R_3^{\diamond}, \cdots, R_n^{\diamond}, Q_{n+1}, \cdots Q_m, O_{m+1}, \cdots\ O_p{}''] \rightarrow R^{\heartsuit}]$$

Chevallard states that

> The elements of $R_i^{\diamond}$ for i=1,⋯,n are the "stamped" answers, "validated" by institutions, for example, the class book, a website, the teacher's course, a lecture note, etc. The elements of Q_j for j=n+1,⋯,m are questions derived from Q, i.e. questions formulated by trying to answer Q. The elements of $O_l$for l=m+1,⋯,p are works, theories, experimental set-ups, praxeologies that are believed to be useful for deconstructing the answer $R^{\heartsuit}$. (Chevallard, 2009b, pp. 21-22, our translation)

From this perspective, adapting it to our study context, we reformulated Cirade's[3] (2019) question as follows: "How can we establish a certain organization of mathematical knowledge in a class at a given level of schooling?" The process of studying this question and its derivative questions allows us to build a milieu of teaching praxeologies, which must be analyzed and evaluated in order to develop teaching products (Cirade, 2019).

The author notes that, for the didactic systems studied in teacher training, a $R_i^{\Diamond}$ answer could be a lesson report, an extract from a textbook, a teacher's website, etc. It will therefore be necessary to observe, analyze and evaluate the corpus, which can be of a diverse nature; and we can see that, in addition to teaching praxeologies stricto sensu, we are required to integrate corpus study praxeologies into the milieu M: this is an important issue at the training level.

It is always complex to justify to students preparing for the teaching profession the need to carry out mathematical praxeological analysis, as well as didactic praxeological analysis and the relationship between the two (Cirade, 2019). This question concerns the raison d'être of didactic analysis and how to highlight it. From this perspective, the author notes that it is necessary to carry out a praxeological

[3] How can we establish a certain organization of mathematical knowledge in a middle or high school class? (Cirade, 2019, p. 342, our translation)

analysis of the work in terms of the direction of study, which will include identifying the didactic moments of study[4] and identifying the types of tasks to be studied.

The reasons for a didactic analysis, which should be highlighted in (initial) teacher training, lead to the question of didactic infrastructures in training, in the dual sense of the study (the teaching praxeologies) and the direction of the study (the training praxeologies). For a Π_0 project, with the aim of encouraging students to engage in work to design mathematical and didactic praxeologies in a scientifically based way, Cirade (2019) identified some conditions under which this project should develop:

- The introduction of the professional gesture in training, with regard to the analysis of corpus data from teaching practice, leads to the introduction of corpus study praxeologies in the M milieu that enable this analysis.
- The question "What mathematical concept should be taught?" can be studied scientifically using the notion of praxeology, which allows us to understand what is proposed in the prescribed/suggested curriculum, what is found in textbooks, etc. The question concerns, in a non-independent way, the four components of praxeology (type of tasks, technique, technology and theory) considered.

Praxeological analysis makes it possible to change the students' relationship with the objects under consideration (the mathematics/statistics to be taught).

Cirade (2019) states that there are many conditions that need to be taken into account when studying a project like Π0, such as the primacy given to technology, seen as a producer of techniques, rather than justifying an emerging technique, or the unavailability in the profession of teaching materials for designing collections of study and research activities. The conditions raised above do not exhaust the work to be done to identify the conditions linked to the praxeologies of the study of the corpus, and it would be important to continue exploring other levels of didactic codetermination.

This scale of levels of didactic codetermination (Figure 2) distinguishes, from bottom to top (see Figure 2), the level of the discipline to which the intended praxeological content belongs (mathematics, French grammar, biology, etc.), then the level of pedagogy, then that of the school, as well as that of society and, finally, that of civilization. Contrary to a tradition that saw the pedagogical level (home to the conditions and constraints considered non-specific to a given praxeological content) as the alpha and omega of the ecology of school didactics, didacticians have studied the conditions and constraints at the level of the discipline, sometimes forgetting then the constraints at a higher level, without which many phenomena affecting the dissemination of the discipline cannot be explained (Chevallard & Cirade, 2010).

Each level in Figure 2 imposes, at some point in the life of the educational system, a set of constraints and support points. At the higher levels (Civilization, Society, School and Pedagogy), there are more generic types of constraints, in which society, through educational institutions, organizes the study of different subjects. The lower levels correspond to the conditions and restrictions directly linked to the different components of a discipline, according to the way it is structured in the educational institution in question.

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[4] These are the moments discernible in study processes. ln the study of a how question relating to a type of tasks T, ATD recognizes six such "study moments": the moment of the first encounter with T; the moment of exploration of T and emergence of a technique ,; the moment to build the techΩnological and theoretical block [θ / Θ]; the moment to work on the praxeology proΩduced, [T / τ / θ / Θ] and particularly on the technique ,; the moment to institutionalise it; the moment to evaluate the praxeology produced and one's relation to it. (See also (Yves Chevallard, with Marianna Bosch, 2020, p. 26)

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Civilization	Level- 3
Society	Level-2
School	Level-1
Pedagogy	Level 0
Discipline of Studies	Level 1
Domain of Study	Level 2
Education Department	Level 3
Topic of Study	Level 4
Subject of study	Level 5

Source: Chevallard (2002b, p.50)

Figure 2: Levels of didactic codetermination

To give just one example, at the deepest level of the scale, one can go beyond the theme or subject to explore conditions related to sectors, domains and the discipline itself (mathematics, for example)

Other conditions are very important: the denial of the need for scientifically-based praxeological equipment is, in our opinion, one of the most important restrictions within the scope of such a project, because in the teaching profession, the situation described by Chevallard (1997, p. 23) still seems relevant to us:

> the absence of a language that is sufficiently rich and widely shared to allow an objective (and not simply personal) analysis of even the most common professional situations should be noted, resulting in a weak collective and individual capacity to communicate, to debate, to even think about the objects of an activity that easily gets stuck in the repetition of gesture and technical solipsism.

This weighs heavily on teacher training and gives rise to a number of difficulties that trainers encounter, such as justifying to students the need to rely on scientifically based tools. But this is not the only source of these difficulties, which are obviously due to the complexity of the issue at stake in the study of training, i.e. the teaching of praxeologies, but also to the poor development of training praxeologies and therefore the virtual absence of didactic infrastructures for teacher training.

Cirade (2006) distinguishes at least three types of teaching praxeologies directly related to mathematics teacher training: mathematical praxeologies for teaching (mathematical knowledge to be taught), mathematical praxeologies for teaching (mathematical knowledge necessary for teaching, which cannot be reduced to praxeologies for teaching) and didactic praxeologies (necessary for designing, managing, analyzing and evaluating the way teaching is carried out). From this perspective, Olarría and Sierra (2011. p. 467) reinforce this idea when they state that:

> Among the praxeologies necessary for teaching mathematics are multiple mathematical organizations (i.e. types of mathematical tasks, techniques and technological-theoretical discourses) that are institutionally new, i.e. absent from both high school and university, where future teachers received their previous

mathematical training. These praxeologies contain, but far exceed, the set of mathematical knowledge that must be taught, but they are not reducible to the "wise" mathematical organizations that future teachers learn in college.

Cirade's work (2006) shows the enormous problematicity of the mathematics taught in elementary school and how the mathematical resources that could allow this problematicity to be addressed are still very far removed from the mathematical culture of both teachers and many members of the mathematically literate community (Olarría& Sierra, 2011).

Chevallard and Cirade (2010) emphasize that, in order to teach mathematics, there is, among the relevant knowledge, the tool for teaching mathematics, which is mathematics itself. It is therefore important to take into account the praxeologies for the profession, i.e. all the praxeologies with which the profession can benefit from equipping itself. The authors further reinforce this perspective when they state that

> Of course, this category contains the subcategory of praxeologies to be taught, but it is far from being reduced to it: mathematically speaking, it includes the indispensable knowledge to identify the praxeologies to be taught. The (vague and evolving) set of mathematical praxeologies to be taught can then be included in another subcategory, that of praxeologies for teaching, which includes, along with the didactic praxeologies related to this or that mathematical praxeology to be taught, the mathematical praxeologies directly useful for designing and constructing these didactic praxeologies (the elaboration of which also presupposes praxeologies for the profession that are not, strictly speaking, praxeologies for teaching). (Chevallard &Cirade, 2010, p. 3)

We emphasize with these authors that the profession must equip itself with useful praxeologies in order to contribute to the construction of a validated response to identify the mathematical praxeologies that should be taught. Also, to avoid the phenomenon of "monumentalism[5]",

> which often permeates the didactic-mathematical training of future teachers, it is necessary to make this knowledge appear with meaning, that is, as answers to crucial questions for the teacher in training. (Higueras& García, 2011, p. 460)

It is important, therefore, to build training devices that have the potential to enable trainee teachers to construct personal answers as a result of a set of answers to questions generated by a generating question stemming from a teacher's didactic problem. It's not just a matter of setting trainee teachers the task of generating new school situations, but of generating mathematical and didactic organizations using "mathematical-didactic knowledge and previously constructed answers in a meaningful, controlled and intentional way". (Higueras & García, 2011, p. 460)

In the next section, we discuss some of the praxeologies necessary for the teacher's teaching practice.

IV. The Teacher's Praxeological Needs

We reflect on and question the praxeological needs of the teacher, based on Wozniak (2020). As the director of study, the teacher allows students to build a relationship with knowing in accordance with what the school institution wishes to

[5] Chevallard (2006) defines the phenomenon of monumentalism as the result of the loss of the raison d'être of the mathematical praxeologies studied at school. Similarly, we believe that the training of mathematics-didactic teachers often suffers from the same phenomenon: the loss of the raison d'être that gives meaning to the mathematical-didactic knowledge that trainee teachers study in faculties of Education Sciences (Higueras & García, 2011, pp. 433-434).

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establish. To do this, once the praxeologies to be taught have been clearly identified, the teacher designs the mathematical organizations to be used in lessons and chooses the didactic organizations adapted to their project.

Wozniak (2020) reflects on a methodological issue to discover the teacher's praxeological needs for teaching mathematics, which Chevallard and Cirade (2010, p. 44) call normal "praxeological equipment". For the author, identifying these needs makes it possible to anticipate what the teachers' difficulties might be in exercising their profession. This identification is made according to a praxeological model of reference. The question addressed by the author starts from what Chevallard (2011, p. 98) calls the "primordial problem":

> Given an activity project in which such an institution or person plans to get involved, what, for this institution or person, is the praxeological equipment that can be considered indispensable or simply useful in the conception and realization of this project?

The author notes that teachers' primary needs concern knowing of the mathematical skills to be taught. Cirade (2006) shows how a didactic device involving a question forum, created by Yves Chevallard in the training of future teachers, is particularly effective for:

- (a) revealing that teachers' questions are actually questions from a profession;
- (b) working collectively on these questions;
- (c) bringing to light certain praxeological needs, whether explicitly expressed or emerging from the study of the questions themselves. (Wozniak, 2020, pp. 788-789)

As we saw in the previous section, Cirade (2006) identifies what this specific knowings is made of in relation to the knowings taught to secondary school mathematics teachers. She distinguishes between the mathematics to be taught, the mathematics for the teacher which is "the mathematics that the teacher may see fit to mobilize to equip his or her thinking and action" (p. 185) and the mathematics for teaching which begins when teachers "begin to question the reasons for the existence of such and such a notion, such and such a theory, such and such a theorem" (p. 133). Chevallard and Cirade (2010, apud Worzniak, 2020, p.790) structure praxeologies for the profession as follows:

> Of course, this category contains the subcategory of praxeologies to be taught; but it is far from being reduced to it: at the mathematical level, it thus includes the knowledge indispensable for identifying the praxeologies to be taught. The (vague and evolving) set of mathematical praxeologies to be taught can then be included in another subcategory, that of praxeologies for teaching, which includes, along with the didactic praxeologies related to such and such a mathematical praxeology to be taught, the mathematical praxeologies that are directly useful for the design and construction of these didactic praxeologies (the elaboration of which also implies praxeologies for the profession that are not, strictly speaking, praxeologies for teaching). We can therefore write the following: praxeologies for the profession $\supset$ praxeologies for teaching $\supset$ praxeologies to be taught

For Wozniak (2020), studying issues related to teachers' praxeological needs means approaching the question of conditions and constraints not from a top-down perspective - how didactic determinants at higher levels of didactic codetermination

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(Figure 2) impact on the didactic system - but from a bottom-up perspective that comes from the didactic system: what do teachers need to teach? The question is therefore how to identify these needs, which can be related to both mathematical and didactic organization.

To answer this question, we rely on three determinants identified by Wozniak (2020), namely: the importance of ecological analyses, naturalistic observations and the fate of didactic engineering, important aspects that we discuss in the next three sections.

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V. Ecological Analysis

Ecological analyses (Artaud, 1997) and the study of didactic transposition phenomena make it possible to discover certain praxeological needs of teachers. Thus, for example, Chevallard and Wozniak (2011) studied why textbooks for the third grade of secondary school did not introduce probabilities according to a frequentist approach, even though this aspect was present in the school programs of the time.

The epistemological study carried out by these authors based on the book *Introduction à la théorie des Probabilité*, by B.V. Gnedenko and A. Khintchine, showed how the problem of the frequentist approach makes it possible to establish the rules for calculating probability. Chevallard and Wozniak (2011) show how the frequentist model makes it possible to establish the rules for calculating probability, which they consider to be an essential methodological aspect, since it is a way of understanding why didactic phenomena are what they are, in this case, the lack of teaching of a particular object of knowing. They also consider that a historical review of the construction of the notion of probability has made it possible to illustrate how the classic definition of "number of favorable cases/number of possible cases" can lead to the dissociation of the calculation of probabilities from its statistical basis in school culture

Wozniak (2020) states that the teaching of probability thus becomes the teaching of syntax without semantics: for a student, the probability of an event is nothing more than what is obtained by applying the rules of probability calculation.

By analyzing the respective roles of estimation and prediction in the probabilistic modeling of statistical variability, he diagnosed a need for mathematical and didactic knowledge for teaching probability from a frequentist approach in the ninth grade of the French system.

VI. Naturalistic Observations

Wozniak (2020) states that observations in which the teacher is free to act offer the opportunity to compare what is done with what could be done, in order to determine what should be done to enable students to build appropriate relationships between knowings and the institution of reference.

We agree with the author when she observes that at the methodological level, praxeological needs are revealed by the distance between the practices observed and the praxeological model of reference in terms of mathematical organization and didactic organization. This model depends on the focus of knowing and the institutional relationship with this object that prevails in the reference institution. This distance is measured especially through the technological discourse that reveals to the class the knowledge used, describes it, explains it, justifies it, questions it and, finally, validates what has been built together. To do this, words, notations and ostensibles are needed so that the class can tell itself what knowledge it has built collectively and refer to it.

Notes

In Wozniak (2012), the author proposed a classification of praxeologies according to the role of technological discourse. He states that in silent praxeology, the role of discourse is only visible from its praxis component through the technique used, while the logos component is inaudible or silent. A weak praxeology allows the logos component to be glimpsed through the ostensibles associated with the technique used, while the technological discourse is implicit or limited to the description of the technique. Finally, a strong praxeology dialectically implements the two components praxis and logos to act, think and validate the action.

It also ensures that if the use of silent or weak praxeologies is an indication of praxeological needs, it is still necessary to validate what has been identified as a need of the profession and not just of the teachers observed. This is done by considering what individual practices reveal about the constraints of the didactic system.

VII. The Fate of Didactic Engineering

From the perspective of the Didactics of Mathematics, it is known that a Didactical Engineering is designed to seek answers to a research question which, most of the time, is not necessarily a concern of teachers, or at least not in the same terms. Wozniak (2020) states that this partly explains why it is not enough to propose problem situations from Didactical Engineering for teachers to adopt and implement them as designed.

For teachers to "benefit" from the products of didactic engineering, they need to understand their raison d'être and be able to "read" the experience as an answer to a question. From the point of view of the dialectic of media and *milieux*[6], Didactic Engineering is a media for the teacher-experimenter, and a tool that allows them to interrogate experimentation in order to constitute it as a milieu for the development of their praxeological equipment (Wozniak, 2020).

Research Engineering is a potential resource. This situation is similar to that of a naturalistic observation, in which engineering plays the role of one resource among others, whether naturalistic or organized around Didactic Research Engineering, and in which deviation from the praxeological model of reference is always an indicator.

Perrin-Glorian (2011) has shown the complexity of the reception of Didactical Engineering by a school institution in relation to the types of questions it answers, whether for research, training or the design of teaching situations in the classroom. It seems that teachers' ability to use the proposed didactic tools depends on constraints that go beyond those that prevail only in the classroom when they teach.

Teachers' praxeological needs are symptoms of the conditions and constraints of their situation, and the (re)knowledge of useful mathematical and didactic praxeologies is not only the problem of the teacher, but also of the profession as a whole.

Wozniak (2020) has identified several complementary ways of identifying teachers' praxeological needs. The author notes that these praxeological needs help to establish a set of facts that validate the elements brought to light. Each of the paths considered is based on the ecological analysis triptych of what is - what could be - what should be, which is a set of conditions and constraints, and is based on comparison with a praxeological reference model. In order to carry out such and such a project, it is

[6] The dialectic of media and milieux considers that any statement made (by a media) is, a priori, a conjecture that has not yet been proven, and that the search for proof is cohesive with the determination of milieux for the question under investigation. Note that experiments in the sense of the experimental sciences, reasoning and calculations are essential types of media that the investigator tries to "make speak" (Sineae Kim (2015, apud Artaud, 2019, p.249) This dialectic thus implies, in particular, carrying out tasks of the type "verifying a statement" and "controlling a result". These types of tasks are part of the study of a praxeology and are not specific to mathematics, but we will consider them in the context of the study of mathematics below. For example, we can see types of tasks such as verifying a geometric statement, verifying the result of a numerical calculation, verifying the behavior of a sequence or a function, etc (Arthaud, 2019, pp. 249-250).

necessary to implement such and such a praxeology or praxeologies, and it is the distance from this initially established praxeological complex that allows gaps and needs to be identified.

In the next section, based on an example, we reflect on the complexity of the process of constructing teacher praxeologies.

VIII. Complexity of the Process of Building Teaching Praxeologies: An Example

Notes

We take as an example one of the episodes from the experimental phase of the research by Freitas (2019), whose aim was to study the knowledge of plane analytic geometry that can be acquired by students (teacher trainees) from a Mathematics degree course in Bahia (Brazil), participating in supervised curricular internship classes, who were involved in a training process based on a Study and Research Pathway. The aim was also to analyze the benefits obtained in this training process for projecting this knowledge into secondary education. The theoretical-methodological device was based mainly on DBT and the constructs of a Study and Research Pathway for Teacher Training (SRP-TT).

Based on the paradigm of questioning the world, Chevallard (2001, 2009a, 2009b, 2011, 2013) presents a didactic device called the Study and Research Pathway (SRP), which extends the didactic system to integrate the Herbartian scheme[7]: S (X; Y; Q) ➥ $R^{♥}$. The scheme indicates that students X investigate a question Q under the direction of Y, with the aim of giving an answer R to Q. In other words, a question Q is explored and an answer R must be produced. This process is indicated by the arrow ➥. The symbol ♥ as the exponent of R represents the institutional relativity of knowing, i.e. that the answer R is produced under certain conditions and restrictions specific to that institution. (Chevallard, 2009b, apud Almouloud et al. 2021).

Almouloud & al. (2021) state that the SRP is characterized by a generative question Q, the answer to which is not immediate; hence the need to formulate other questions derived from Q. Remember that the construction of the didactic *milieu* M is simultaneous with the construction of answers to the questions derived from Q. It is hypothesized that students expand their possibilities for action by formulating questions, looking for resources and sources of information, constructing answers, evaluating them and defending them critically to other students (Chevallard, 2012 apud Almouloud et al. 2021).

With regard to the Study and Research Pathway for Teacher Training (SRP-TT), its aim is to familiarize teachers in initial or continuing training with the SRP as a useful didactic device for their professional development.

> In order to prepare for an effective transition from the monumentalist paradigm to the paradigm of questioning the world, teacher training itself needs didactic devices that are not based solely on the monumentalist paradigm and, for this reason, it is necessary to resort in some way to devices with a SRP-type structure (Study of questions, media, *milieux*) (Ruiz-Olarría, 2015, p. 136).

The SRP-TT is also developed from a generating question Q0-TT which must be formulated to search for contexts linked to teacher training, which must focus on a

[7] Called the Herbartian scheme by the German pedagogue, Johann Friedrich Herbart (1776-1841), who is considered the father of scientific pedagogy.

teaching object to be worked on considering the school level in which these teachers are inserted.

Returning to Fritos' work, we focused our analysis on the episode concerning the students' work on constructing Study and Research Paths to answer the following question: What should be taught about Plane Analytic Geometry to high school students, and how should it be taught?

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It was hoped that these students had expanded their praxeological equipment with regard to GAP, more specifically, with regard to the point and line in the plane studied in previous episodes, so that they could propose didactic organizations, or a teaching plan, that would answer the following questions (Figure 3):

Part 1: Construct a schema or conceptual map, globally representing the evolution of the study carried out, in terms of the mathematical objects studied.
Part 2: Based on the reflections made in the didactic modeling workshop, develop a didactic organization of study for your 3rd year high school students, based on the following question: how to teach the analytic **geometry of the point and the line?**

Source: Freitas (2019, p. 295)

Figure 3: Didactic integration activity

The author hoped that the students would analyze all the material they had built and propose a study for their potential high school students. The activity was carried out in groups, with construction, study and research taking place, followed by sharing.

Freitas (2019) found that the students were more committed to developing the mathematical content than in the first moment of planning, at the beginning of the study on the alignment of three points, as can be seen in the following extract:

H1 - We prepared the three-point alignment activity, five questions for them to identify if they understood how this... if it is aligned and this question here which is more... which also involves the first degree exponential function. Now we have to finish? The activity from the other days. Because here, look, on the first day we only chose one question, which is this one, in addition to the examples we're going to give, which is going to be more about construction. The second is also about construction. And then, on the third day, there's this one, and then, to understand when we take... as we did here, the general equation of the line. With... with the general equation of the line there are those cases when "B" is zero and when it's parallel to "X". Now what did we think? To put this into Geogebra, so we could draw a line. And then, from the... quotient, they see the... happen in Geogebra. Not on the board, as we're seeing.

H2 - Oh, yes, yes.

H1 - Because there are times when even the drawing on the board is a bit complicated for them to visualize. It may not be the best... ((overlapping voices))

H2 - Visualize it, right?

H1 - Now... now here I'm in doubt. If it's to continue... because, if it's to continue the content, you see the reduced equation, you see the angle, in this case the slope of the line, and the angle. Then there's the reduced line and the parametric line, but since we didn't... in the conceptual map we only went as far as the general equation, we're also only going as far as the general equation in the plan.

H2- (Who writes) [00:01:44] Geogebra?

H1 - Geogebra, every time we write it, it looks like this.

(Audio transcription, Group A production),

Source (Freitas, 2019, p. 346)

This transcript shows the students planning tasks to propose in the context of the didactic organization and the lesson plan, which they classify as exercises, construction tasks and visualization tasks using Geogebra. They wondered how far they would go with the content. They decided that they would comply with the proposal of the map built in the previous session. This moment

The students' planning was very significant, although it was not possible to identify possible praxeologies for teaching GAP in the audios. The trainees' work, despite the difficulties in expressing themselves in formal Portuguese, reveals the activity of the practice of being a teacher, planner, organizer from a didactic and mathematical point of view, considering, among other things, the pedagogical aspects. (Freitas, 2019, p. 347)

In the last phase of the experiment, the aim was to finalize the study by socializing and debating the collective construction, with a view to finding the answer to the study's guiding question, Q_0. In addition, Freitas (2019) proposed collectively evaluating the device as a proposal for training (future) teachers.

In order to answer question Q_0, one of the groups of students realized that they had organized a teaching plan that contained the following items: the educational institution; the workload: 10 hours of study; the class: third year high school; the contents: coordinate system, point and line, the objectives, the methodology, the resources and the assessment. Regarding the objectives, the group presented the following:

- Understand the procedures used to identify the coordinates of the point;
- Geometrically represent given points on the Cartesian plane;
- Understand collinearity;
- Recall basic concepts about points and lines;
- Explore some of the tools in the Geogebra software;

  Recognize a line, a semi-straight line, a line segment and the different positions relative to the line;
- Define the distance between two points on the Cartesian plane (production excites Group A). (Freitas, 2019, pp.347-348)

Regarding the didactic organization of the work, the excerpt below reveals how this organization was thought up by the students.

Introduce the content by presenting the orthogonal Cartesian system and its axes, then specify the generalization of the ordered pair.

Then ask the class to represent some points geometrically. After understanding what each of the coordinates of the ordered pair represents, ask the class to graph two points using Geogebra software, and to find strategies for calculating the distance between two given points.

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Next, still using Geogebra, we will construct a right-angled triangle with vertices A, B and C, recalling the Pythagorean Theorem and specifying that the calculation can be done using it.

We will use example 1, attached, and ask the class to solve the following exercise, using the knowledge acquired so far.

(UFU-MG) The points A(2, y), B(1,-4) and C(3, -1) are given. What must the value of y be for triangle ABC to be right-angled at B?

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Starting the study of the line with the alignment of three points: ask the class to think of two points in pairs and record them individually. Then represent them graphically. Ask the students what they understand about collinearity, in order to construct the formal definition together with the class, and present ways of verifying whether it exists or not. By calculating the determinant of the matrix. Propose solving the attached activity, using the knowledge acquired during the discussions and explanation of the content.

From the knowledge of the alignment of three points, with the help of the triangle constructed in one of the previous lessons, we will deduce the general equation of the line.

The particular cases of the line will be studied, using Geogebra software to show geometrically what happens in each case, addressing aspects relating to the slope of the line.

Note: The procedures may be altered or added to.

Resources: blackboard, computer, eraser, paintbrush, textbook, exercise list; assessment will be continuous throughout the students' teaching and learning process.

Analyzing participation and solving the proposed exercises.

(Written production Group A) (Freitas, 2019, p. 348)

The author summarized the most significant points of the group's oral presentation:

- *Develop exploratory activities using the Geogebra interface to represent points on the Cartesian plane;*
- *Identify the mathematical objects of study in the curriculum, which a priori the students should have studied in previous years and the need to articulate them in order to teach GA, for example matrices and determinants;*
- *Use the Geogebra interface to enable visualization and exploration of the content and propose "exercises" (tasks) with demonstration;*
- *Incorporating a questioning and participatory approach to the students in the process of constructing knowledge, getting the student to think and discuss the content until they reach a generalization;*
- *They discussed didactic time and cognitive time, i.e. each class has a different amount of time to progress with the content. The students also stated that they should know the "level of the class", in order to approach previous content, which is important for the continuity of the work;*
- *In the activities using the software interface, the students listed orthogonal coordinates, the representation of points and the study of the distance between two points,*

- *They emphasized the importance of monitoring the construction of the activities using the software, with a view to the teaching objectives for each proposed activity;*
- *In the case of the study of distance, they brought up the possibility of replicating the demonstration with the right triangle, carried out during the study;*
- *They proposed working in pairs to study the alignment of three points;*
- *They proposed working on the deduction of the general equation of the line and then, from the particular cases, the slope and the angular coefficient;*
- *They identified that the proposal was still incomplete, as it could be expanded and improved.*
- *Progressively proposing the study of segmented and reduced equations, while at the same time continuously assessing, always with the student's participation in the construction of knowledge. (Freitas, 2019, pp. 348-349)*

From the students' performance in this task, there are indications that those who participated in all the episodes would have understood the need for detailed planning, well thought out and supported by didactic reflections and the mathematical praxeologies being studied. (Freitas, 2019)

The author also points out, with regard to the mathematical organization, that the students proposed a set of tasks, such as, for example, tasks on the alignment of three points and the distance between two points, the completion of which was planned based on the use of the software and the others as fixation exercises.

The tasks were organized as a set of task types, using T_{Ai} for group A tasks and T_{Bi}, for group B tasks.

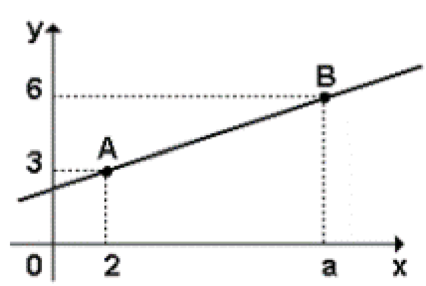
T_{A1}*: determine the distance between two points (-1,1) and (3,2);*

T_{A2}*: check that points A (0, 4), B (-6, 2) and C (8, 10) are aligned, algebraically.*

T_{A3}*: Determine the value of y so that points P (1, 3), Q (3, 4) and R (y, 2) are the vertices of any triangle.*

T_{A4}*: Points A (-1, 2), B (3, 1) and C (a, b) are collinear. For C to lie on the abscissa axis, what must be the values of a and b?*

T_{A5} *(UFSM) The figure shows the graph of a function of the 1st degree that passes through points A and B, where a≠2. The point where the line AB intersects the x-axis has an abscissa equal to: (a) 1-a; (b) a-2;(c)* $\frac{3a-12}{a-2}$*;(d) 4-a; (e) 12 - 3a.(Figure 3)*



Source: Freitas (2019, p.350)

Figure 4: Figure of task T_{A5}

T_{A6}*: determine the value of k so that the equation kx - y - 3k + 6 = 0 represents the line passing through the point (5,0).*

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T_{A8}: EBSERH - AOCP 2016). If there is a line whose equation is y - 2x -10 = 0, is it correct to say that this line passes through which of the following two points? (a) A (5, 0) and B (-20, 35) ; (b) C (12, 21) and D (0, 20) ; (c) E (14, -15) and F (-7, 7): (d) G (5, 30) and H (0.5, 4); (e) A (0, 10) and B (-13, 16). (Written production of group A) (Freitas, 2019, p. 350))

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It can be seen that all the types of tasks proposed by the students are described in the dominant model of the textbooks analyzed, but the techniques envisioned by the students are part of the alternative model proposed by Freitas (2019) in the training. Regarding the technological-theoretical discourse, they relied on what had been developed during the training, such as the right triangle and Pythagoras' Theorem. T_{A5}(UFSM) formulation shows a conceptual problem related to confusion between the graph of a function and its figural representation.

Regarding the alignment of three points, the students suggest as a technological-theoretical discourse, the condition of collinearity of points to deduce the alignment condition and the general equation of the line. According to Freitas (2019, p. 351)

> The trainee teachers' productions revealed that technological knowledge appeared as something "natural", or naturalized in the teaching process, even if they didn't have a deeper understanding of it. Apart from visualization, other potentialities of the dynamic environment were not explored.

Group B's presentation identifies the following contents:

- Introduce analytical geometry so that students understand the content and can apply and identify it in everyday life;
- Identify the relationships between plane geometric figures and the Cartesian plane;
- Understand how studies of the optical system can help in the study of points and lines (Written production. Group B) (Freitas, 2019, p. 351)

And it describes the likely stages of its didactic sequence, with the didactic and pedagogical procedures for each lesson:

Lesson 1: We will start the lesson interactively to find out the students' previous knowledge of the content on the study of points. We will ask questions such as: what is a point? Can planet earth be considered a point in relation to the universe?

Then we can define what a point is in mathematics. Next, we'll work with a cardboard image of a neighborhood common to the students (e.g. the school district) and ask them to identify buildings that are between the intersection of two streets. The main objective is to remember the Cartesian plane.

Lesson 2: We'll start with a ruler and a Xerox or graph paper. We will ask the students to measure the distance between certain points on the Cartesian plane, starting with the distance between points that are on horizontal and vertical lines, which will allow us to associate it with the modulus of a number. After that, we will ask the students to measure the distance between points or measures of segments that are on non-horizontal and vertical lines, so we will show the use of the Pythagorean Theorem to solve this problem, in addition, we will ask the students questions about the midpoint and equidistance between points.

Lesson 3: We will share exercises from the previous lesson and discuss the interdisciplinary nature of mathematics in different areas in relation to the Cartesian system. We will use examples from geography involving the content of longitude and latitude, how a ship is located in the middle of the ocean and an airplane in the sky.

Next, we will propose a research activity on what the teacher has discussed. The class will be divided into groups of 5 students and the teacher will assign topics to them. It is expected that the school will have a computer room. If not, it will be agreed in the previous lesson that research will be carried out for the students to bring in. The lesson will be to maintain these presentations.

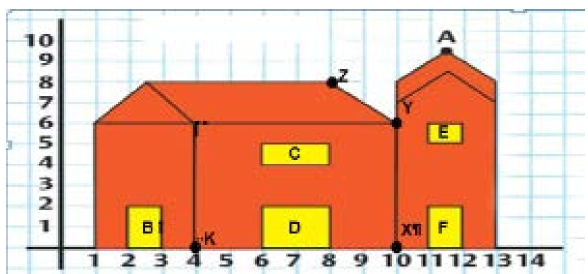
Lesson 4: The group presentations will take place.

Lesson 5: The lesson will be expository and we will talk about the alignment of 3 points. There will be an activity before the lecture and another afterwards to establish the method of determining whether the three points are aligned.

Lesson 6: We will continue with the exercises from the previous lesson, applying the following methods: matrices by Sarrus, cofactors, Chió Rule.

Lesson 7: Assessment (Written production, Group B) (Freitas, 2019, pp. 351-352)

With regard to mathematical organization, group B presented a set of task types, based on Figure 4.



Source: Freitas (2019, p. 352)

Figure 5: Illustration of group B tasks

The types of tasks that group B developed from Figure 4 are explained below:

T_{B1}: *knowing that the width of door A can be represented by the distance between the points (2, 0) and (3, 0). What is the width of the door?*

T_{B2}: *as we can see on the Cartesian plane, the height of door D is defined by the following points, (6, 2) and (8, 2). What is the height of the door?*

T_{B3}: *find the midpoint of the width of door D, knowing that the start point is (6,0) and the end point is (8,0).*

T_{B4}: *Determine points X and Y, then calculate the distance between points X and Y and find the midpoint of points X and Y.*

T_{B5} *Find the point Z, and using the point Y found in the previous question calculate the distance between the point Y and Z and find its midpoint.*

T_{B6}: *Find point K and calculate the distance between K and Y and then calculate the midpoint of points K and Y. (Written production, Group B) (*Freitas, 2019, p. 353)

Tasks T_{B4} , T_{B5} $T_{B6}$are tasks of the same type: identifying the coordinates of points and calculating distance.

Freitas points out that the two groups realized that their proposals were not yet ready, given the various aspects that had been pointed out and which had not been taken into account by the two groups. These joint reflections helped the students to realize that they had to redesign their proposal.

The ”finalization” of the intervention proposal for high school students and the emerging need to (re)plan it, stand out in the context of the research as a *didactic moment of evaluation* of the experimental device, with the presentation of the answer to

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the initial question Q_0 (*how to teach the analytic geometry of the point and the line?*) of the device.

At the end of this training, Freitas (2019, p. 354) observes that the

> [...] the study leaves open the desire to improve the knowings acquired by the subjects and to see it put into practice, or even to improve and experiment with the teaching plan organized by the trainees. However, the limitations of the research make this second phase of experiments immediately unfeasible, which we leave for future studies.

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An important aspect highlighted by the author is the "volume of each piece of knowing" necessary for teaching practice. At each work session, certain specific mathematical and didactic praxeologies, referring to the teaching situations themselves, were mobilized or (re)signified by the subjects. The author observes that this movement, typical of working with the SRP, guided by the generative questions of each phase of the experiment, generated a framework of teaching, mathematical and didactic praxeologies with the support of ATD, some of which were structured (developed) by the subjects during the different sessions.

Freitas (2019) points out that from a macro perspective, the device would have enabled subjects to (re)signify certain teaching praxeologies. In addition, the mathematical and didactic organizations proposed by the trainee teachers referred to potential students and were not actually applied to (real) high school students. It would be relevant to observe how the trainees, who were the subjects of Freitas' research (2019), would develop the teaching sequences that they had developed in the schools that were part of the supervised internship.

The conclusions drawn from Freitas' research (2019) are in line with the results of Artaud, Cirade and Michel Jullien (2011), who observed the positive and negative points of the implementation of aSRP by student-teachers and their attempts to design a SRP, loaded with a complex of conditions and restrictions that favor, allow or, on the contrary, hinder the dissemination of the notion of SRP. Among the favorable conditions, the authors mention

> the technological elements that will justify the need for SRP, in particular the improvement of motivation and the amalgamation of the mathematical organizations produced. The amalgamation of mathematical organizations will, however, be limited, from the point of view of the praxeologies implemented, by the thematic vision of mathematical organizations that we have seen some people propose and whose prevalence in the profession we know: in fact, certain existing structures in the secondary mathematics teaching system in France, such as the division into chapters or a chronogenesis reading of the content of the syllabus, drive this thematic vision. (Artaud et al., 2011, pp. 792-793)

The authors also point out that this amalgam is faced with the problem of understanding the moment of institutionalization and its articulation with other moments in the study of regional or even local mathematical organization.

In the episode by Freitas (2019) that we analyzed, we noticed, as in the research by Artaud et al. (2011. pp. 792-793) that

> synthesis occurs very early on, from two points of view. On the one hand, it interrupts the dynamics of the study and therefore gives shape to diffracted mathematical organizations. On the other hand, as the technological-theoretical moment has not really taken place, due to the lack of - adequate - articulation with the exploratory moment, the statements recorded in the synthesis are not

related to the practices that require them and therefore have only a very partial status of a technological element.

As we saw in the previous section, the issues related to teachers' praxeological needs are related to conditions and restrictions that have high impacts on the didactic system and on the search for an answer to the question: what do teachers need to teach?

IX. Conclusions

In this text, we reflect on teacher training based on the Anthropological Theory of the Didactic and on research aimed at studying teachers' praxeological needs. Our literature review led us to reflect on the didactic infrastructures for teacher training and the praxeological needs of teachers. In addition, we took as an example one of the episodes from the experimental phase of Freitas' research (2019), which aimed to study the knowledge of plane analytic geometry that can be acquired by students (teacher trainees) from a Mathematics degree course in Bahia (Brazil), participating in supervised curricular internship classes, who were involved in a training process based on a Study and Research Pathway.

With regard to didactic infrastructures for teacher training (Cirade, 2020), we infer that the praxeological equipment of the profession should include a set of mathematical knowledge that allows teachers to question their projects for teaching mathematical concepts, reformulate it or even, in certain cases, discard it; in short, make a didactic decision that has the potential to advance their students in the appropriation of these concepts.

Finally, Wozniak (2020) identifies several ways of finding the teacher's praxeological needs. These paths are complementary and help to establish a set of facts which, when constituted as a whole, validate the elements brought to light. Each of the paths considered is based on the ecological analysis triptych of the following questions: What is it? What could it be? What should it be? These questions involve a set of conditions and restrictions, and are based on comparison with a praxeological reference model. In order to carry out such a project, it is necessary to implement the praxeologies deemed relevant to the realization of this project, and it is the distance from this initially established praxeological complex that allows us to identify gaps and needs (Wozniak, 2020).

With regard to the complexity of the processes of constructing teaching praxeologies, the analysis of the findings related to our example shows that the device enabled the subjects to (re)signify certain teaching knowledge and that there would have been a change in the knowledge that was part of the research subjects' praxeological equipment. However, the design of the students' SRPs reveals the need to expand the praxeological equipment of these students with regard to vectors in the plane and in space, based on the consolidation of knowledge of synthetic geometry and GAP, in order to constitute a fundamental technological-theoretical block in the construction of Linear Algebra knowledge (Freitas, 2019).

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Quasi-P-Normal and N-Power Class Q Composite Multiplication Operatos on the Complex Hilbert Space

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Abstract- In this paper, the condition under which composite multiplication operators on $L^2(\mu)$-space become Quasi-P-Normal operators and n-Power class Q operator have been obtained in terms of radon-nikodym derivative f_0.

Keywords: composite multiplication operator, conditional expectation, Quasi-p-normal, multiplication operator, class Q operator.

GJSFR-F Classification: MSC 2010: 47B33, 47B20, 46C05.

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Quasi-P-Normal and N-Power Class Q Composite Multiplication Operatos on the Complex Hilbert Space

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Abstract- In this paper, the condition under which composite multiplication operators on $L^2(\mu)$ -space become Quasi-P-Normal operators and n-Power class Q operator have been obtained in terms of radon-nikodym derivative f_0.

Keywords: composite multiplication operator, conditional expectation, Quasi-p-normal, multiplication operator, class Q operator.

I. Introduction

Let (X,Σ,μ) be a σ-finite measure space. Then a mapping T from X into X is said to be a measurable transformation if $T^{-1}(E)\in\Sigma$ for every $E\in\sum$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(E))=0$ whenever $\mu(E)=0$. If T is non-singular then the measure μT^{-1} defined as $\mu T^{-1}(E)=\mu(T^{-1}(E))$ for every E in Σ, is an absolutely continuous measure on Σ with respect to μ.Since μ is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function f_0 in $L^1(\mu)$ such that $\mu T^{-1}(E)=\int_E f_0\, d\mu$ for every $E\in\Sigma$. The function f_0 is called the Radon-Nikodym derivative of μT^{-1} with respect to μ.

Every non- singular measurable transformation T from X into itself induces a linear transformation C_T on $L^p(\mu)$ defined as $C_T f = f\circ T$ for every f in $L^p(\mu)$. In case C_T is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by T. We restrict our study of the composition operators on $L^2(\mu)$ which has Hilbert space structure. If u is an essentially bounded complex-valued measurable function on X, then the mapping M_u on $L^2(\mu)$ defined by $M_u f = u\cdot f$, is a continuous operator with range in $L^2(\mu)$. The operator M_u is known as the multiplication operator induced by u.

A composite multiplication operator is linear transformation acting on a set of complex valued Σ measurable functions f of the form

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$$M_{u,T}(f) = C_T M_u(f) = u \circ T \ f \circ T$$

Where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

In the study considered is the using conditional expectation of composite multiplication operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \le p \le \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g f \, d\mu = \int_A g E(f) \, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [1-4].

a) *Normal operator*

Let H be a Complex Hilbert Space. An operator T on H is called normal operator if $T^*T = TT^*$

b) *Quasi-normal operator*

Let H be a Complex Hilbert Space. An operator T on H is called Quasi-normal operator if $T T^* T = T^* T T$, ie, T^*T commute with T

c) *Quasi p-normal operator [13]*

Let H be a Complex Hilbert Space. An operator T on H is called Quasi-normal operator if $T^*T\left(T + T^*\right) = \left(T + T^*\right)T^*T$

d) *2-Power -normal operator*

Let H be a Complex Hilbert Space. An operator T on H is called 2 power-normal operator if $T^2 T^* = T^* T^2$

e) *Class Q-operator [14]*

Let H be a Complex Hilbert Space. An operator T on H is called Quasi-normal operator if $T^{*2} T^2 = \left(T^* T\right)^2$

II. Related Work in the Field

The study of weighted composition operators on L^2 spaces was initiated by R. K. Singh and D. C. Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in $L^p(\sum), (1 \le p < \infty)$ spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X,E)$ has been studied in [7]. Recently S. Senthil, P. Thangaraju, Nithya M, Surya devi B and D. C. Kumar, have proved several theorems on n-normal , n-quasi-normal, k-paranormal, and (n,k) paranormal of composite multiplication operators on L^2 spaces [8-12]. In this

Ref

5. Singh, RK & Kumar, DC, Weighted composition operators, Ph.D. thesis, Univ. of Jammu (1985).

paper we investigate composite multiplication operators on $L^2(\mu)$-space become Quasi-P-Normal operators and n-Power class Q operator have been obtained in terms of radon-nikodym derivative f_0 .

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III. Characterization on Composite Multiplication of Quasi P Normal Operators on L^2-Space

a) Proposition

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$.Then for $u \geq 0$

(i) $M^*_{u,T} M_{u,T} f = u^2 f_0 f$

(ii) $M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

(iii) $M^n_{u,T}(f) = (C_T M_u)^n (f) = u_n \ (f \circ T^n)$, $u_n = u \circ T . u \circ T^2 . u \circ T^3 ....u \circ T^n$

(iv) $M^*_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$

(v) $M^{*^n}_{u,T} f = u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}$

where $E(u f_0) \circ T^{-(n-1)} = E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} .... E(u f_0) \circ T^{-(n-1)}$

Theorem 3.1

Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then the following statements are equivalent

(i) $M_{u,T}$ is Quasi p-normal operator

$$u \circ T \ u^2 \circ T \ h \circ T \ f \circ T + h u E\left(h u^2 f\right) \circ T^{-1} = h u^2 u \circ T \ f \circ T + h^2 u^3 E(f)$$

Proof:

For $f \in L^2(\mu)$, $M_{u,T}$ is Quasi P-normal operator if

$\left(M_{u,T} + M^*_{u,T}\right)\left(M^*_{u,T} M_{u,T}\right) f = \left(M^*_{u,T} M_{u,T}\right)\left(M_{u,T} + M^*_{u,T}\right) f$ and we have,

$$\left(M_{u,T} + M^*_{u,T}\right)\left(M^*_{u,T} M_{u,T}\right) f = M_{u,T}\left(M^*_{u,T} M_{u,T}\right) f + M^*_{u,T}\left(M^*_{u,T} M_{u,T}\right) f$$

$$= M_{u,T} M^*_{u,T} \left(u \circ T \ f \circ T\right) + M^*_{u,T} M^*_{u,T} \left(u \circ T \ f \circ T\right)$$

$$= M_{u,T}\left[h u E\left(uf \circ T\right) \circ T^{-1}\right] + M^*_{u,T}\left[h u E\left(uf \circ T\right) \circ T^{-1}\right]$$

$$= M_{u,T}\left[h u^2 f\right] + M^*_{u,T}\left[h u^2 f\right]$$

$$= u \circ T \left(h u^2 f\right) \circ T + h u E\left(h u^2 f\right) \circ T^{-1}$$

$$= u \circ T \ u^2 \circ T \ h \circ T \ f \circ T + h u E\left(h u^2 f\right) \circ T^{-1}$$

Consider

$$\left(M^{*}{}_{u,T}\, M_{u,T}\right)\left(M_{u,T} + M^{*}{}_{u,T}\right) f = \left(M^{*}{}_{u,T}\, M_{u,T}\right) M_{u,T}\, f + \left(M^{*}{}_{u,T}\, M_{u,T}\right) M^{*}{}_{u,T}\, f$$

$$= \left(M^{*}{}_{u,T}\, M_{u,T}\right)\left(u \circ T\ f \circ T\right) + \left(M^{*}{}_{u,T}\, M_{u,T}\right)\left(h\, u\, E(f) \circ T^{-1}\right)$$

$$= M^{*}{}_{u,T}\ u \circ T\ \left(u \circ T\ f \circ T\right) \circ T + M^{*}{}_{u,T}\ u \circ T\ \left(h\, u\, E(f) \circ T^{-1}\right) \circ T$$

$$= h\, u\, E\left[u \circ T\ u \circ T^{2}\ f \circ T^{2}\right] \circ T^{-1} + h\, u\, E\left[u \circ T\ h \circ T\ u \circ T\ E(f)\right] \circ T^{-1}$$

$$= h\, u^{2}\ u \circ T\ f \circ T + h^{2}\ u^{3}\ E(f)$$

Suppose, $M_{u,T}$ is Quasi P-normal operator. Then

$$\left(M_{u,T} + M^{*}{}_{u,T}\right)\left(M^{*}{}_{u,T} M_{u,T}\right) f = \left(M^{*}{}_{u,T} M_{u,T}\right)\left(M_{u,T} + M^{*}{}_{u,T}\right) f$$

$$\Leftrightarrow u \circ T\ u^{2} \circ T\ h \circ T\ f \circ T + h\, u\, E\ \ h\, u^{2} \left(f \circ T\right)^{-1} = h\, u^{2}\ u \circ T\ f \circ T + h^{2}\ u^{3}\ E(f)$$

Theorem 3.2

Let the $M_{u,T}$ be a composite multiplication operator on $L^{2}(\mu)$. Then the following statements are equivalent

(i) $M^{*}{}_{u,T}$ is Quasi p-normal operator

$$h^{2}\, u^{3}\, E(f) \circ T^{-1} + h \circ T^{2}\ u \circ T\ u^{2} \circ T^{2}\ E(f) \circ T$$

(ii) $= h \circ T\ u^{2} \circ T\ E(h)\ E(u) E(f) \circ T^{-1} + h \circ T\ u \circ T\ u^{2} \circ T\ f \circ T$

Proof:

For $f \in L^{2}(\mu)$, $M^{*}{}_{u,T}$ is Quasi P-normal operator if

$$\left(M^{*}{}_{u,T} + M_{u,T}\right)\left(M_{u,T}\, M^{*}{}_{u,T}\right) f = \left(M_{u,T}\, M^{*}{}_{u,T}\right)\left(M^{*}{}_{u,T} + M_{u,T}\right) f$$

and then we have

$$\left(M^{*}{}_{u,T} + M_{u,T}\right)\left(M_{u,T}\, M^{*}{}_{u,T}\right) f = M^{*}{}_{u,T}\left(M_{u,T}\, M^{*}{}_{u,T}\right) f + M_{u,T}\left(M_{u,T}\, M^{*}{}_{u,T}\right) f$$

$$= M^{*}{}_{u,T} M_{u,T}\left[h\, u\, E(f) \circ T^{-1}\right] + M_{u,T}\, M_{u,T}\left[h\, u\, E(f) \circ T^{-1}\right]$$

$$= M^{*}{}_{u,T}\ u \circ T\left[h\, u\, E(f) \circ T^{-1}\right] \circ T + M_{u,T}\ u \circ T\left[h\, u\, E(f) \circ T^{-1}\right] \circ T$$

$$= M^{*}{}_{u,T}\left[u \circ T\ h \circ T\, u \circ T\ E(f)\right] + M_{u,T}\left[u \circ T\ h \circ T\, u \circ T\ E(f)\right]$$

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$$= h\,u\,E\left(u\circ T\ h\circ T\,u\circ T\ E(f)\right)\circ T^{-1} + u\circ T\left[u\circ T\ h\circ T\,u\circ T\ E(f)\right]\circ T$$

$$= h^2\,u^3\,E(f)\circ T^{-1} + h\circ T^2\,u\circ T\ u^2\circ T^2\,E(f)\circ T$$

Notes

Consider

$$\left(M_{u,T}\,M^*_{u,T}\right)\left(M^*_{u,T}+M_{u,T}\right)f = \left(M_{u,T}\,M^*_{u,T}\right)M^*_{u,T}\ f + \left(M_{u,T}\,M^*_{u,T}\right)M_{u,T}\ f$$

$$= \left(M_{u,T}\,M^*_{u,T}\right)h\,u\,E(f)\circ T^{-1} + \left(M_{u,T}\,M^*_{u,T}\right)(u\circ T\ f\circ T)$$

$$= M_{u,T}\ h\,u\,E\left(h\,u\,E(f)\circ T^{-1}\right)\circ T^{-1} + M_{u,T}\ h\,u\,E\left(u\circ T\ f\circ T\right)\circ T^{-1}$$

$$= M_{u,T}\ h\,u\,E(h)\circ T^{-1}E(u)\circ T^{-1}E(f)\circ T^{-2} + M_{u,T}\ h\,u^2\,f$$

$$= u\circ T\left(h\,u\,E(h)\circ T^{-1}\,E(u)\circ T^{-1}\,E(f)\circ T^{-2}\right)\circ T + u\circ T\,(h\,u^2 f)\circ T$$

$$= h\circ T\ u^2\circ T\ E(h)\ E(u)\,E(f)\circ T^{-1} + h\circ T\ u\circ T\ u^2\circ T\ f\circ T$$

Suppose $M^*_{u,T}$ is Quasi p-normal operator. Then

$$\left(M^*_{u,T}+M_{u,T}\right)\left(M_{u,T}\,M^*_{u,T}\right)f = \left(M_{u,T}\,M^*_{u,T}\right)\left(M^*_{u,T}+M_{u,T}\right)f$$

$$\Leftrightarrow h^2\,u^3\,E(f)\circ T^{-1} + h\circ T^2\,u\circ T\ u^2\circ T^2\,E(f)\circ T$$

$$= h\circ T\ u^2\circ T\ E(h)\ E(u)\,E(f)\circ T^{-1} + h\circ T\ u\circ T\ u^2\circ T\ f\circ T$$

IV. Characterizations on N Power Class Q Composite Multiplication Operatos on L^2-Space

Theorem 4.1

Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}$ is n power class Q composite multiplication operator if and only if

$$h\,u\,E(h)\circ T^{-1}\,E(u)\circ T^{-1}\,E(u_{2n})\circ T^{-2}\,f\circ T^{2n-2}$$
$$= h\,u\,E(u_n)\circ T^{-1}\,h\circ T^{n-1}\,u\circ T^{n-1}\,E(u_n)\circ T^{n-2}\,f\circ T^{2n-2}$$

Proof:

Now Consider,

$$M^{*2}{}_{u,T}\,M^{2n}{}_{u,T}\ f = M^{*2}{}_{u,T}\left[u_{2n} f\circ T^{2n}\right]$$

where $u_{2n} = u\circ T^2\ u\circ T^4 \ldots\ldots\ldots\ldots\ldots\ u\circ T^{2n}$

$$= M^*_{u,T} \left(h\,u\,E\left(u_{2n}\, f \circ T^{2n}\right) \circ T^{-1}\right)$$

$$= M^*_{u,T}\, h\,u\,E(u_{2n}) \circ T^{-1}\, f \circ T^{2n-1}$$

$$= h\,u\,E\left(h\,u\,E(u_{2n}) \circ T^{-1}\, f \circ T^{2n-1}\right) \circ T^{-1}$$

$$= h\,u\,E(h) \circ T^{-1}\, E(u) \circ T^{-1}\, E(u_{2n}) \circ T^{-2}\, f \circ T^{2n-2}$$

Next we consider,

$$\left(M^*_{u,T} M^n_{u,T}\right)^2 f = \left(M^*_{u,T} M^n_{u,T}\right)\left(M^*_{u,T} M^n_{u,T}\right) f$$

$$= \left(M^*_{u,T} M^n_{u,T}\right) M^*_{u,T}\left(u_n\, f \circ T^n\right)$$

where $u_n = u \circ T\;\, u \circ T^2 \ldots\ldots\ldots\ldots\ldots\; u \circ T^n$

$$= \left(M^*_{u,T} M^n_{u,T}\right) h\,u\,E\left(u_n\, f \circ T^n\right) \circ T^{-1}$$

$$= \left(M^*_{u,T} M^n_{u,T}\right) h\,u\,E(u_n) \circ T^{-1}\, f \circ T^{n-1}$$

$$= M^*_{u,T}\;\, u_n \left(h\,u\,E(u_n) \circ T^{-1}\, f \circ T^{n-1}\right) \circ T^n$$

$$= M^*_{u,T}\;\, u_n\, h \circ T^n\, u \circ T^n\, E(u_n) \circ T^{n-1}\, f \circ T^{2n-1}$$

$$= h\,u\,E\left(u_n\, h \circ T^n\, u \circ T^n\, E(u_n) \circ T^{n-1}\, f \circ T^{2n-1}\right) \circ T^{-1}$$

$$= h\,u\,E(u_n) \circ T^{-1}\, h \circ T^{n-1}\, u \circ T^{n-1}\, E(u_n) \circ T^{n-2}\, f \circ T^{2n-2}$$

Given $M_{u,T}$ is n power class Q composite multiplication operator

$$\Leftrightarrow M^{*2}_{u,T}\, M^{2n}_{u,T}\; f = \left(M^*_{u,T} M^n_{u,T}\right)^2 f$$

$$\Leftrightarrow h\,u\,E(h) \circ T^{-1}\, E(u) \circ T^{-1}\, E(u_{2n}) \circ T^{-2}\, f \circ T^{2n-2}$$

$$= h\,u\,E(u_n) \circ T^{-1}\, h \circ T^{n-1}\, u \circ T^{n-1}\, E(u_n) \circ T^{n-2}\, f \circ T^{2n-2}$$

Theorem 4.2

Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M^*_{u,T}$ is n power class Q composite multiplication operator if and only if

$$u \circ T\; u^2 \circ T^2\, h \circ T^2\, E(u\,h) \circ T^{-(2n-3)}\, E(f) \circ T^{-(2n-2)}$$

$$= u^2 \circ T\; h \circ T\; E(u\,h) \circ T^{-(n-2)}\, h \circ T^{-(n-2)}\, E(u\,h) \circ T^{-(2n-3)}\, E(f) \circ T^{-(2n-2)}$$

Notes

Notes

Proof:
Now if we consider

$$M^2{}_{u,T}\, M^{*2n}{}_{u,T}\ f = M^2{}_{u,T}\left(h\,u\,E(h\,u)\circ T^{-(2n-1)}\ E(f)\circ T^{-2n}\right)$$

$$= M_{u,T}\left(u\circ T\left(h\,u\,E(h\,u)\circ T^{-(2n-1)}\ E(f)\circ T^{-2n}\right)\circ T\right)$$

$$= M_{u,T}\left(h\circ T\ \ u^2\circ T\ E(h\,u)\circ T^{-(2n-2)}\,E(f)\circ T^{-(2n-1)}\right)$$

$$= u\circ T\ \left(h\circ T\ \ u^2\circ T\ E(h\,u)\circ T^{-(2n-2)}\,E(f)\circ T^{-(2n-1)}\right)\circ T$$

$$= u\circ T\ u^2\circ T^2\ h\circ T^2\ E(u\,h)\circ T^{-(2n-3)}\,E(f)\circ T^{-(2n-2)}$$

and we consider

$$\left(M_{u,T}\, M^{*n}{}_{u,T}\right)^2 f = \left(M_{u,T}\, M^{*n}{}_{u,T}\right)\left(M_{u,T}\, M^{*n}{}_{u,T}\right) f$$

$$= \left(M_{u,T}\, M^{*n}{}_{u,T}\right) M_{u,T}\ \ u\,h\,E(u\,h)\circ T^{-(n-1)}\ E(f)\circ T^{-n}$$

$$= \left(M_{u,T}\, M^{*n}{}_{u,T}\right) u\circ T\left(u\,h\,E(u\,h)\circ T^{-(n-1)}\ E(f)\circ T^{-n}\right)\circ T$$

$$= M_{u,T}\ M^{*n}{}_{u,T}\left(u^2\circ T\ \ h\circ T\ E(u\,h)\circ T^{-(n-2)}\ \ E(f)\circ T^{-(n-1)}\right)$$

$$= M_{u,T}\ u\,h\,E(u\,h)\circ T^{-(n-1)}\ E\left(u^2\circ T\ \ h\circ T\ E(u\,h)\circ T^{-(n-2)}\ \ E(f)\circ T^{-(n-1)}\right)\circ T^{-n}$$

$$= M_{u,T}\ \left(u\,h\,E(u\,h)\circ T^{-(n-1)}\ u^2\circ T^{-(n-1)}\ h\circ T^{-(n-1)}\,E(u\,h)\circ T^{-(2n-2)}\ E(f)\circ T^{-(2n-1)}\right)$$

$$= u\circ T\left(\ u\,h\,E(u\,h)\circ T^{-(n-1)}\ u^2\circ T^{-(n-1)}\ h\circ T^{-(n-1)}\,E(u\,h)\circ T^{-(2n-2)}\ E(f)\circ T^{-(2n-1)}\right)\circ T$$

$$= u^2\circ T\ \ h\circ T\ E(u\,h)\circ T^{-(n-2)}\ h\circ T^{-(n-2)}\ E(u\,h)\circ T^{-(2n-3)}\ E(f)\circ T^{-(2n-2)}$$

Since $M_{u,T}$ is a Composite multiplication operator, by definition

$$\Leftrightarrow\ M^2{}_{u,T}\, M^{*2n}{}_{u,T}\ f = \left(M_{u,T}\, M^{*n}{}_{u,T}\right)^2 f$$

$$\Leftrightarrow\ u\circ T\ u^2\circ T^2\ h\circ T^2\ E(u\,h)\circ T^{-(2n-3)}\ E(f)\circ T^{-(2n-2)}$$
$$= u^2\circ T\ \ h\circ T\ E(u\,h)\circ T^{-(n-2)}\ h\circ T^{-(n-2)}\ E(u\,h)\circ T^{-(2n-3)}\ E(f)\circ T^{-(2n-2)}$$

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 6 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Behavior Dynamics of the Lotka-Volterra Mapping Composition, with Transitive Tournaments Describing Models of Sexually Transmitted Diseases

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Abstract- Mathematical ecology as a science began to take shape at the beginning of the XX century. Its emergence was facilitated by the works of outstanding mathematicians like Vito Volterra and his contemporaries L. Lotka and V. A. Kostitsin. Further development of mathematical ecology is associated with the names of G. F. Gause, A. N. Kolmogorov, Yu. Odum, Yu. M. Svirezhev, R. A. Poluektov, etc. Mathematical methods have penetrated most deeply into the study of the dynamics of the number of biological populations, which occupy a central place in the problems of ecology and population genetics. The paper considers the composition of two Lotka-Volterra mappings operating in a two-dimensional simplex with transitive tournaments, with two inversely directed edges, since the composition can be used to simulate sexually transmitted diseases. All fixed points are found for the composition and their characters are studied, as well as the dynamics of the asymptotic behavior of the trajectory, i.e. the phase portrait, is shown for each component of the composition.

GJSFR-F Classification: LCC: QH541.15.E26

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Abstract- Mathematical ecology as a science began to take shape at the beginning of the XX century. Its emergence was facilitated by the works of outstanding mathematicians like Vito Volterra and his contemporaries L. Lotka and V. A. Kostitsin. Further development of mathematical ecology is associated with the names of G. F. Gause, A. N. Kolmogorov, Yu. Odum, Yu. M. Svirezhev, R. A. Poluektov, etc. Mathematical methods have penetrated most deeply into the study of the dynamics of the number of biological populations, which occupy a central place in the problems of ecology and population genetics. The paper considers the composition of two Lotka-Volterra mappings operating in a two-dimensional simplex with transitive tournaments, with two inversely directed edges, since the composition can be used to simulate sexually transmitted diseases. All fixed points are found for the composition and their characters are studied, as well as the dynamics of the asymptotic behavior of the trajectory, i.e. the phase portrait, is shown for each component of the composition.

I. Introduction

Biologists, epidemiologists, economists, and mathematicians have been trying to work together for a long time. In the twenties of the last century, articles by S. N. Berstein appeared, the first of which was "On the application of mathematics to biology "("Science in Ukraine", 1922, issue 1, p. 14). His next work in 1924 is called "The solution of a mathematical problem related to the theory of heredity". In the future , the genetic theme can be traced in the work of A. S. Serebrovsky (Reports of the USSR Academy of Sciences, 1934, 2, 33) with the unusual-sounding title "On the properties of Mendelistic equalities"and the article by V. I. Glivenko "Mendeleev algebra"(Reports of the USSR Academy of Sciences, 1936,13, 371). This topic was further developed in Europe, for example, Etherington, 1939-1941; Reiersol, 1962; Holgate, 1967-1968. The fate of the remarkable work of A. N. Kolmogorov, I. G. Petrovsky, N. S. is instructive. Piskunov "Study of the diffusion equation connected with an increase in the amount of matter and its application to a biological problem "(Bulletin of Moscow State University, 1937, 1, issue 6, 1–26). It investigated the displacement of an unstable genotype by a stable one. The work has had many imitations in other areas of mathematical natural science. It continued to be studied and quoted even at a time when even mentioning the gene was impossible, which was greatly facilitated by the successful title of the work.

It is characteristic of mathematics that the formation of new directions often arises on the basis of new tasks. In particular, all classical analysis arose on the basis of problems of physics, mechanics and geometry. However, later it turned out that the scope of mathematical analysis is much wider. Quite a lot of problems related to chemistry or technology, to various branches of engineering and natural science, including problems of biology and even economics, are solved by methods of mathematical analysis. However, to solve a number of problems in the field of biology, epidemiology of economics, technology, having a cybernetic nature, i.e. related to information flows and management, the methods of classical analysis are not applicable. Tasks of this nature stimulated the development of new branches of mathematics, such as information theory, the doctrine of control systems, as well as automata

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theory, game theory, various sections of mathematical programming, etc. A common feature of these new areas of mathematics is discreteness. And apparently, this has a deep meaning. The fact is that physics, mechanics and other sciences leading to the formulation of problems of classical analysis are characterized by the expedient use of continuous models of the phenomena under study. In fact, the direct object of mathematical study is continuous media, continuous trajectories, continuous physical fields, etc. At the same time, biology, epidemiology or economics are characterized by the structuring of the object under study. Discrete organelles with a very diverse well-defined functioning are distinguished in the cell. Moreover, the processes of vital activity of the cell consist of the functioning of these organelles. The organism is built from various organ systems, those from individual organs, etc. A somewhat complex economic system is also built from relatively autonomous parts that interact with each other in a very specific way. All this leads to the need to take into account the various elements of the system, specific and sometimes strictly individual interactions between them, as well as taking into account the restructuring of the system as a whole in the process of its functioning. All this is difficult to describe by the methods of classical analysis and leads to the formation of completely new formulations of questions, and therefore new chapters of mathematics. Apparently, somewhat hyperbolizing, we can say that if physics in a broad sense is organically connected with continuous mathematics, then biology, in particular epidemiology in a broad sense, is just as organically connected with discrete mathematics.

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Based on this, in our work we propose a discrete model for describing the disease of sexually transmitted viral diseases. The spread of sexually transmitted diseases, such as chlamydia, syphilis, gonorrhea (Neisseria gonorrhoeae) and of course the most urgent disease is AIDS, is a very serious health problem in both developed and developing countries. There are more than a million cases of sexually transmitted infections every day. Most of them are asymptomatic. According to statistics from the World Health Organization (WHO), 374 million new cases of infection with one of the four sexually transmitted infections, such as chlamydia, syphilis, gonorrhea or trichoniasis occur every year. For example, in 2016, 376 million cases of infection with one of the four sexually transmitted infections – chlamydia (127 million), gonorrhea (87 million), syphilis (6.3 million) or trichomoniasis (156 million). In 2018, the United States became the record-breaking country for the number of sexually transmitted diseases. There are about 2.5 million people in our country who have fallen ill with these diseases. This number of cases is constantly growing. The most **harmless** diagnosis was chlamydia. 1.7 million Americans suffer from chlamydia, most of them are women from 15 to 24 years old. At the moment, about 45 percent of women in this age category are infected with chlamydia. Generally speaking, the problem is much more serious than it seems at first glance. Doctors say that the treatment of sexually transmitted diseases is necessary. Today there is a virus **super-gonorrhea**, resistant to the strongest antibiotics. Because of such viruses, the world can return to the Middle Ages, when millions of people died from such diseases. Of course, we can say that there is no such level of diagnosis in third world countries, so cases of sexually transmitted infections are simply not taken into account. As we can see, research in this area does not miss its relevance opportunity. Models considering sexually transmitted diseases are given in the monograph [1]. A simple epidemiological SI model is considered here, since the cure of, for example, gonorrhea does not lead to the development of immunity, therefore, a person eliminated for treatment becomes susceptible after recovery. Basically, many papers consider continuous dynamical systems described by differential equations.

In this paper, we will consider the discrete case of these models, which are fundamentally different from the models considered earlier. We will consider discrete dynamic Lotka-Volterra systems operating in a two-dimensional simplex, and their compositions, since they can be used in modeling the course of sexually transmitted diseases.

Let $V : S^{m-1} \to S^{m-1}$ be an arbitrary continuous mapping of the simplex into itself, acting according to the formulas [2]

$$V : x_k^{'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \ k = \overline{1,m},$$

where $|a_{ki}| \leq 1, a_{ki} = -a_{ik}$ and also

$$S^{m-1} = \left\{ x = (x_1, ..., x_m) : \ x_i \geq 0; \ \sum_{i=1}^{m} x_i = 1 \right\} \subset \mathbb{R}^m.$$

Choose an arbitrary point $x_0 \in S^{m-1}$ and build iterations $x^{(n+1)} = Vx^{(n)}, x(n)$ – the sequence will be called the trajectory of the point x_0 when displaying V.

Since the simplex is compact, the trajectory has at least one limit point. The set of all limit points is denoted by $\omega(x_0) = \{x_0, x_1, ...\}' \neq \varnothing$.

Theorem 1: ([3]-[4]). Let $A=(a_{ki})$ be a skew-symmetric matrix, in this case

$$P=\{x\in S^{m-1}:Ax\geq 0\}\neq\varnothing,\ \ Q=\{x\in S^{m-1}:Ax\leq 0\}\neq\varnothing$$

consist of from fixed points. Let $I=\{1,...,m\}$ and for an arbitrary $\alpha\subset I$ we put

Notes

$$P_\alpha=\{x\in\Gamma_\alpha:A_\alpha x\geq 0\},\quad Q_\alpha=\{x\in A_\alpha:A_\alpha x\leq 0\},$$

where A_α is the matrix resulting from the skew-symmetric matrix $A=(a_{ki})$ replacing all a_{ki} with zeros for which $(k,i)\notin\alpha\times\alpha$.

Since the narrowing of the Lotka-Volterra mapping to any face of the Γ_α simplex is also a Lotka-Volterra mapping [4], then it follows from Theorem 1 that the sets $P_\alpha\neq\varnothing$ and $Q_\alpha\neq\varnothing$ for any $\alpha\subset I$.

Let $X=\{x(\alpha):\alpha\subset I\}$ – the set of fixed points. We will say that the fixed points are $x(\alpha)$ and $x(\beta)$ form a pair (P,Q) if there exists a face Γ_γ such that $\gamma=\alpha\cup\beta$ and the following inequalities hold

$$A_\gamma x(\alpha)\geq 0,\ \ A_\gamma x(\beta)\leq 0.$$

Now the elements of $x(\alpha)$ and $x(\beta)$ we draw on the plane, then if they form a (P,Q) pair, then we connect them with an arc from $x(\alpha)$ to $x(\beta)$. The graph constructed in this way is called a card of fixed points.

II. Methods

Consider the Lotka-Volterra mappings acting in S^{m-1}, having the form:

$$V_1: x_k^{'}=x_k\left(1+\sum_{i=1}^{m}a_{ki}x_i\right),\ k=\overline{1,m},$$

$$V_2: x_k^{'}=x_k\left(1+\sum_{i=1}^{m}b_{ki}x_i\right),\ k=\overline{1,m},$$

where $S^{m-1}=\left\{x=(x_1,...,x_m):\ x_i\geq 0;\ \sum_{i=1}^{m}x_i=1\right\}\subset\mathbb{R}^m.$

Since the Lotka-Volterra mapping is an automorphism of the simplex S^{m-1} into itself, obviously the composition $V_1\circ V_2$ is also an automorphism, and it is representable as:

$$V_1\circ V_2: x_k^{'}=x_k(1+f_k(x_1,x_2,...,x_{k-1},x_{k+1},...,x_m)),\ k=\overline{1,m}$$

Let us consider a special case of compositions of Lotka-Volterra mappings, for $m=3$, with coefficients $a_{ki},b_{ki}=\pm 1$.

$$V_1:\begin{cases}x_1'=x_1(1+x_2+x_3),\\ x_2'=x_2(1-x_1+x_3),\\ x_3'=x_3(1-x_1-x_2),\end{cases}\ V_2:\begin{cases}x_1'=x_1(1+x_2-x_3),\\ x_2'=x_2(1-x_1-x_3),\\ x_3'=x_3(1+x_1+x_2),\end{cases}\qquad(1)$$

here $|a_{ki}|\leq 1, a_{ki}=-a_{ik}, |b_{ki}|\leq 1, b_{ki}=-b_{ik}, \sum_{i=1}^{3}x_i=1.$

Tournaments corresponding to these mappings are shown in Figure 1. The concept of a tournament is given in [4]-[7].

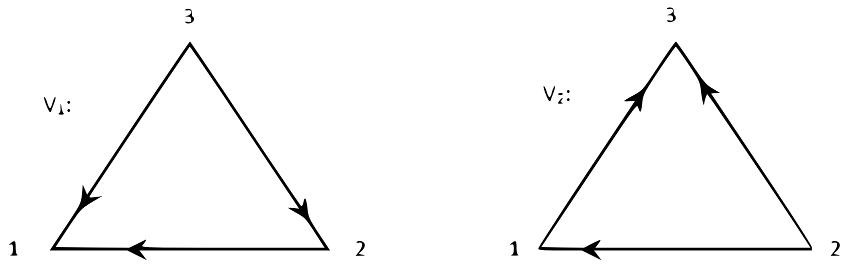

Figure 1. Tournaments corresponding to V_1 and V_2.

The compositions of these mappings are represented as:

$$V_1 \circ V_2 : \begin{cases} x_1' = x_1(1+x_2-x_3)[1+x_2(1-x_1-x_3)+x_3(1+x_1+x_2)], \\ x_2' = x_2(1-x_1-x_3)[1-x_1(1+x_2-x_3)+x_3(1+x_1+x_2)], \\ x_3' = x_3(1+x_1+x_2)[1-x_1(1+x_2-x_3)-x_2(1-x_1-x_3)], \end{cases} \tag{2}$$

or

$$V_1 \circ V_2 : \begin{cases} x_1' = -x_1^4 - 4x_1^3x_2 - 4x_1^2x_2^2 + 2x_1^2 + 4x_1x_2, \\ x_2' = -x_2^2(2x_1^2 + 4x_1x_2 + x_2^2 - 2), \\ x_3' = 1 - x_1' - x_2', \end{cases} \tag{3}$$

Since the mapping tournaments V_1 and V_2 which are shown in Figure 1 differ in the directions on two edges, i.e. the directions on two edges connecting the vertices e_1 and e_3, e_2 and e_3 are opposite, and the direction connecting the vertices e_1 and e_2 coincide, then in the end we get one more fixed point on two edges Γ_{13} and Γ_{23}. In order to determine the coordinates of these points, we take a narrowing of the composition of these mappings to these edges. For example, to find the location (coordinate) of a point belonging to the edge Γ_{13}, we take the contraction of the compositional operator on this edge:

$$\begin{cases} x_1' = x_1(1-x_2)(1+x_3(1+x_1)), \\ x_2' = 0, \\ x_3' = x_3(1+x_1)(1-x_1(1-x_3)), \\ x_1 + x_3 = 1. \end{cases} \tag{2}$$

As a result, we got a point $M_1\left(\frac{\sqrt{5}-1}{2};0;\frac{3-\sqrt{5}}{2}\right)$.

By doing the same for the edge Γ_{23}, we get $M_2\left(0;\frac{\sqrt{5}-1}{2};\frac{3-\sqrt{5}}{2}\right) \in \Gamma_{23}$.

We obtained the following theorem:

Theorem 2: If the mapping is V_1 and V_2 are represented as (1), then their composition (2) has the following fixed points:

– the five fixed points belonging to the simplex are the points –

$$e_1(1;0;0), e_2(0;1;0), e_3(0;0;1),$$

and

$$M_1 \quad \frac{\sqrt{5}-1}{2};0;\frac{3-\sqrt{5}}{2}\Bigg) \in \Gamma_{13},\ M_2 \quad 0;\frac{\sqrt{5}-1}{2};\frac{3-\sqrt{5}}{2}\Bigg) \in \Gamma_{23},$$

– two fixed points outside the simplex belonging to H are the points $N_1\left(\frac{-1-\sqrt{5}}{2};0;\frac{3+\sqrt{5}}{2}\right)$ $N_2\left(0;\frac{-1-\sqrt{5}}{2};\frac{3+\sqrt{5}}{2}\right)$, where $H=\{x\in\mathbb{R}^m:\sum_{i=1}^{m}x_i=1\}$.

Notes

Proof 2: The proof of the theorem is obtained by solving system (2) on the narrowing of the corresponding edge. This file may be formatted in both the `preprint` (the default) and `reprint` styles; the latter format may be used to mimic final journal output. Either format may be used for submission purposes. Hence, it is essential that authors check that their manuscripts format acceptably under `preprint`. Manuscripts submitted to AIP that do not format correctly under the `preprint` option may be delayed in both the editorial and production processes.

As a result, the card of fixed points of composition (2) looks like

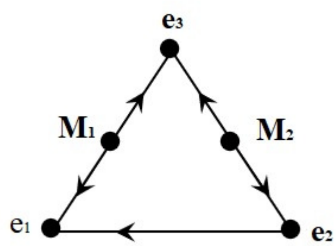

Figure 2: **Card of fxed points of composition** $V_1\circ V_2$.

Now the main task is to study the nature of the fixed points found. To do this, consider the Jacobi matrix of composition (2).

$$J(V_1\circ V_2)=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix},$$

where the coefficients of the matrix have the following form:

$$a_{11}=x_1(x_2-x_3+1)(x_3-x_2)+(x_2-x_3+1)(x_2(-x_1-x_3+1)+x_3(x_1+x_2+1)+1,$$

$$a_{12}=x_1(1-x_1)(x_2-x_3+1)+x_1(x_2(-x_1-x_3+1)+x_3(x_1+x_2+1)+1,$$

$$a_{13}=x_1(x_1+1)(x_2-x_3+1)-x_1(x_2(-x_1-x_3+1)+x_3(x_1+x_2+1)+1),$$

$$a_{21}=x_2(-x_1-x_3+1)(-x_2+2x_3-1)-x_2(-x_1(x_2-x_3+1)+x_3(x_1+x_2+1)+1),$$

$$a_{22}=x_2(-x_1-x_3+1)(x_3-x_1)+(-x_1-x_3+1)(-x_1(x_2-x_3+1)+x_3(x_1+x_2+1)+1),$$

$$a_{23}=x_2(2x_1+x_2+1)(-x_1-x_3+1)-x_2(-x_1(x_2-x_3+1)+x_3(x_1+x_2+1)+1),$$

$$a_{31}=x_3(-x_2(-x_1-x_3+1)-x_1(x_2-x_3+1)+1)+(x_3-1)x_3(x_1+x_2+1),$$

$$a_{32}=x_3(-x_2(-x_1-x_3+1)-x_1(x_2-x_3+1)+1)+(x_3-1)x_3(x_1+x_2+1),$$

$$a_{33}=(x_1+x_2+1)(-x_2(-x_1-x_3+1)-x_1(x_2-x_3+1)+1)+x_3(x_1+x_2)(x_1+x_2+1)).$$

We will find the eigenvalues of the Jacobi matrix by solving the equation

$$|J(x)-\lambda I|=0. \tag{3}$$

By the values of the eigenvalues of the Jacobi matrix, we can describe the character of fixed points. To do this, we first introduce definitions concerning the nature of fixed points from [13]:

Definition 1: A fixed point is called attracting (an attractor), if the spectrum of the Jacobian, that is, solutions of equation (3) have absolute values less than one.

Definition 2: A fixed point is called repulsive (a repeller), if the spectrum of the Jacobian, i.e. solutions of equation (3) have absolute values greater than one.

Definition 3: A fixed point is called a saddle point (i.e. it is neither a repeller nor an attractor) if among the solutions of equation (3) there are those having absolute values greater and less than 1.

Recall that it follows from the invariance of the simplex that one λ is equal to one, which we exclude from consideration, since in the simplex $\sum_{i=1}^{m} x_i = 1$.

As a result, we obtained a general form for the eigenvalues of the Jacobi matrix for system (2):

$$\lambda_1 = 1,$$

$$\lambda_2 = 2x_1^2x_2 - 2x_1^2x_3 + x_1^2 - 2x_1x_2^2 + 4x_1x_2x_3 - 4x_1x_2 - 2x_1x_3^2 + 2x_1x_3 - 2x_1 + x_2^2 + 2x_2 - x_3^2 + 1,$$

$$\lambda_3 = 2x_1^2x_3 - x_1^2 + 4x_1x_2x_3 - 2x_1x_2 - 2x_1x_3^2 + 4x_1x_3 + 2x_2^2x_3 - x_2^2 - 2x_2x_3^2 + 4x_2x_3 - x_3^2 + 1.$$

Now we calculate the eigenvalues for (each of the fixed) points:

$$e_1(1;0;0),\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0,\quad e_2(0;1;0),\ \lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 0,$$

$$e_3(0;0;1),\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0,$$

$$M_1 \left(\frac{\sqrt{5}-1}{2};0;\frac{3-\sqrt{5}}{2}\right),\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 6-2\sqrt{5},$$

$$M_2 \left(0;\frac{\sqrt{5}-1}{2};\frac{3-\sqrt{5}}{2}\right),\ \lambda_1 = 1,\ \lambda_2 = 2\sqrt{5}-1,\ \lambda_3 = 6-2\sqrt{5}.$$

$$N_1 \left(\frac{-1-\sqrt{5}}{2};0;\frac{3+\sqrt{5}}{2}\right),\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2(3+\sqrt{5}),$$

$$N_2 \left(0;\frac{-1-\sqrt{5}}{2};\frac{3+\sqrt{5}}{2}\right),\ \lambda_1 = 1,\ \lambda_2 = -2(1+\sqrt{5}),\ \lambda_3 = 2(3+\sqrt{5}).$$

As a result , we get the following corollary:

Ref

13. D. B. Eshmamatova, R. N. Ganikhodzhaev, and M. A. Tadzhieva, "Dynamics of lotka-volterra quadratic mappings with degenerate skewsymmetric matrix," Uzbek Mathematical Journal 66, 85–97 (2022), dOI: 10.29229/uzmj.2022-1-8.

Corollary 1: The fixed points defined by Theorem 2 are –

– e_1, e_3 – attracting,

– M_2, N_2 – repelling,

– points e_2, M_1, N_1 – saddle fixed points.

Notes

Now we are interested in the dynamics of the trajectories of the composition $V_1 \circ V_2$. To do this, we use Theorem 1, i.e. for operator (2), to find the sets P and Q, we solve the inequalities from this theorem.

To begin with, according to Theorem 1, we find the set P:

$$\begin{cases} x_2^2(1+x_2)+x_3(2-x_3)(1-x_3)+x_2x_3(2-x_2-x_3)+x_2-x_3 \geq 0, \\ x_1(2-x_1-2x_3)(x_1-1)+x_3(2-x_3)(1-x_3)-x_1x_3(x_1+x_3)-x_1-x_3 \geq 0, \\ -x_1(x_1+2x_2)(1+x_1)-x_2^2(1+x_2)-x_1x_2(x_1+3x_2)+x_1+x_2 \geq 0, \end{cases} \tag{4}$$

Then, outside the simplex, the set P

$$P = \begin{cases} \{x_1 \approx -3.55512,\ 3.55512 \leq x_2 \leq 4.17316,\ x_3 = 4.55512 - x_2,\}, \\ \{x_1 \approx -3.55512,\ x_2 \leq -0.0418846,\ x_3 \approx 4.55512 - x_2\}, \\ \{x_1 \approx -1.29898,\ 1.29898 \leq x_2 \leq 1.91701,\ \ x_3 \approx 2.29898 - x_2\}, \\ \{x_1 \approx -1.29898,\ x_2 \leq -0.319053,\ x_3 \approx 2.29898 - x_2\}, \\ \{x_1 \approx -3.55512,\ x_2 \approx 1.93709,\ x_3 \approx 2.61803\}, \\ \{-3.55512 < x_1 < -1.29898,\ -x_1 \leq x_2 \leq 0.5(1.23607 - 2x_1),\ x_3 = -x_1 - x_2 + 1\}. \end{cases}$$

inside the simplex, the set P is a point $M_2\left(0; \frac{\sqrt{5}-1}{2}; \frac{3-\sqrt{5}}{2}\right)$.

Now also, according to Theorem 1, we define the set Q by solving the system

$$\begin{cases} x_2^2(1+x_2)+x_3(2-x_3)(1-x_3)+x_2x_3(2-x_2-x_3)+x_2-x_3 \leq 0, \\ x_1(2-x_1-2x_3)(x_1-1)+x_3(2-x_3)(1-x_3)-x_1x_3(x_1+x_3)-x_1-x_3 \leq 0, \\ -x_1(x_1+2x_2)(1+x_1)-x_2^2(1+x_2)-x_1x_2(x_1+3x_2)+x_1+x_2 \leq 0, \end{cases} \tag{5}$$

$$Q = \begin{cases} \{x_1 \approx 0,618034,\ \ x_2 \geq 1,\ \ x_3 \approx 0,5(0,763932 - 2x_2)\}, \\ \{x_1 \approx 0,618034,\ \ -2.23607 \leq x_2 \leq -0.618034,\ \ x_3 \approx 0.5(0.763932 - 2x_2)\}, \\ \{x_1 \approx -1.29898,\ \ -0.319053 \leq x_2 \leq 0.643377,\ \ x_3 \approx 2.29898 - 2x_2\}, \\ \{x_1 \approx -3.55512,\ \ x_2 \approx 1.93709,\ \ x_3 \approx 2.61803\}, \end{cases}$$

inside the simplex , the set Q is this point $M_1\left(\frac{\sqrt{5}-1}{2}; 0; \frac{3-\sqrt{5}}{2}\right), e_1(1;0;0), e_3(0;0;1)$.

Now we present a phase portrait of the dynamics of the trajectories of each component of the composition (2).

Consider the first of them $x_1' = -x_1^4 - 4x_1^3x_2 - 4x_1^2x_2^2 + 2x_1^2 + 4x_1x_2$.

Since in this case the first component has the form $x_1 = f(x_1, x_2)$ and for the convenience of visually seeing the phase portrait, we will rewrite this equation as $x_1^4 + 4x_1^3x_2 + 4x_1^2x_2^2 - 2x_1^2 - 4x_1x_2 + x_1 = 0$. Then the phase portrait of the dynamics of the first component of the system (2) looks like this:

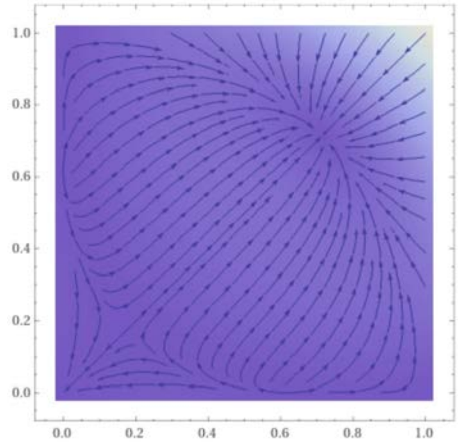

Figure 3: **Phase portrait of the dynamics of the frst component**

Now let's move on to the second component. This component is expressed as: $x_2' = -x_2^2(2x_1^2 + 4x_1x_2 + x_2^2 - 2)$. This component in this case is expressed as $x_2 = f(x_1, x_2)$, then we get the following function, as well as its phase portrait: $-x_2^2(-2 + 2x_1^2 + 4x_1x_2 + x_2^2) - x_2 = 0$.

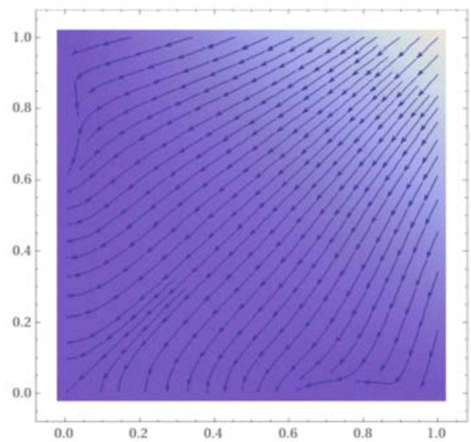

Figure 4: **Phase portrait of the dynamics of the second component**

To see the phase portrait of the third component, use the equality $x_1 + x_2 + x_3 = 1 \implies x_3' = 1 - x_1' - x_2'$. So the phase portrait of the third component looks like this:

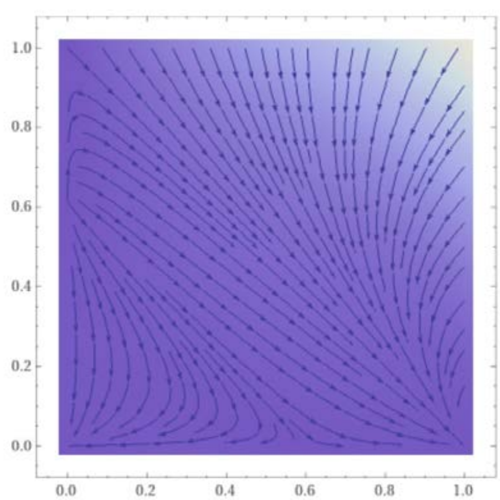

Figure 5: **Phase portrait of the dynamics of the third component, represented as the difference of the frst two.**

Notes

To explain the phase portrait of each component, we give a complete picture for the entire simplex S^2.

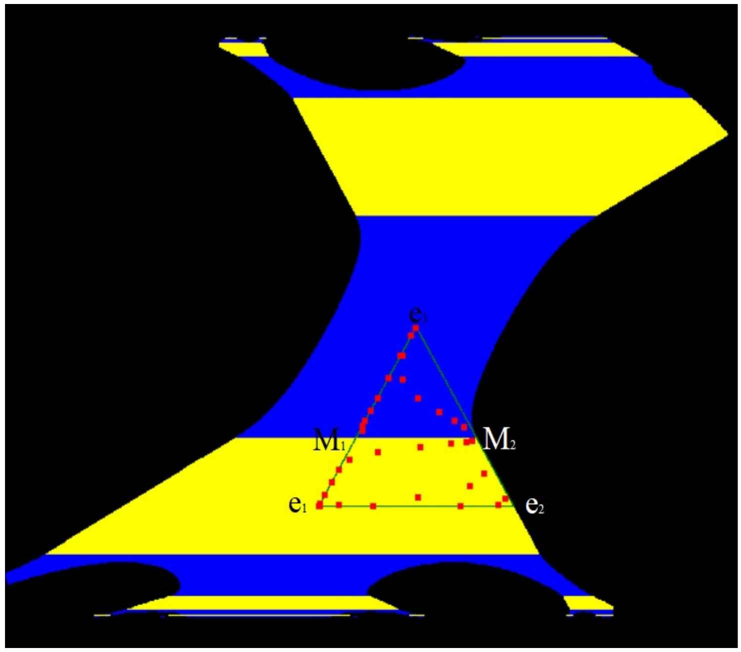

Figure 6: **A complete picture of the phase portrait for the entire simplex.**

To each mapping V_1 and V_2 correspond to a transitive tournament (see Fig.1.) and the composition of these maps (2) divides the simplex into two parts.

In Figure 6, the yellow color divides the area of the simplex in which all the trajectories of internal points are attracted to the vertex $e_1(1;0;0)$. We can see this flow of trajectories in Figure 3 and Figure 4. These phase portraits show the dynamics of the trajectories of the internal points of the simplex merging into the vertex $e_1(1;0;0)$. And the blue color divides that part of the simplex, the trajectory of the inner points, which are attracted to $e_3(0;0;1)$. The phase portrait of the trajectories of these points is shown in Figure 5.

III. Conclusion

Sexually transmitted diseases (e.g. AIDS) have spread in almost all countries of the world. For example, it has been estimated that in some regions of Central Africa up to twenty percent of the population is infected with the human immunodeficiency virus (HIV), and that in the Bronx in New York 13 percent of men and 7 percent of women aged 25-40 years are HIV-infected [18]. To prevent the further spread of these epidemics, it is important to understand how these infectious diseases are transmitted.

The transmission dynamics are complex. Many biological and sociological factors are involved. One of the main factors determining the spread of STDs is how people choose their sexual partners. Changes in sexual behavior are recorded in almost every survey of homosexuals or bisexuals and injecting drug users over the past decade [18]. These behavioral changes occur as sexually active people become more careful in their sexual activities to avoid contracting STDs such as AIDS. Understanding the consequences of these behavioral changes can help guide educational programs to prevent STD transmission.

The proposed model in this paper represents the contact of a population susceptible to infection, where the first mapping V_1 – defines a certain population (male or female), and the second V_2 – defines a certain population (male or female). The dynamics of each component of the composition determines the course of the disease (Figure 3,4 and 5), as well as a complete picture of the phase portrait (Figure 6) of the entire composition divided by two colors gives a territorial limitation of the resulting population in contact with susceptible and infected. The full interpretation in the epidemiological vocabulary of this work will be given in the next article.

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- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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ISSN 9755896