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Balancing Coexistence: Ecological Dynamics and Optimal Tax Policies in a Dual Phytoplankton-Zooplankton System Influenced by Toxin Avoidance and Harvesting

By Yuqin, Wensheng Yang
Fujian Normal University

Abstract- In recent years, the impact of toxic phytoplankton on ecological balance has attracted more and more ecologists to study. In this paper, we develop and analyze a model with three interacting species, poisonous and nontoxic phytoplankton, and zooplankton, including zooplankton avoiding toxic phytoplankton in the presence of non-toxic phytoplankton, and the impact of human harvest on the coexistence of these three species. We first introduce the poisonous avoidance coefficient $\beta$ and the human harvest of nontoxic phytoplankton and zooplankton to investigate its impact on species coexistence. We not only find that $\beta$ has a particular effect on the coexistence of these three species. But also that human harvest is an essential factor determining the coexistence of these three species. Secondly, pregnancy delay ($\tau_1$) and toxin onset delay ($\tau_2$) are introduced to explore the influence of time delay on the behavior of dynamic systems. When the delay value exceeds its critical value, the system will lose stability and go through Hopf bifurcation. After that, we use the principle of Pontryagin’s maximum to study the optimal tax policy without delay. We obtained the optimal path of the optimal tax policy. Finally, we carry out numerical simulations to verify the theoretical results.

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Balancing Coexistence: Ecological Dynamics and Optimal Tax Policies in a Dual Phytoplankton-Zooplankton System Influenced by Toxin Avoidance and Harvesting

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I. Introduction

Marine phytoplankton and zooplankton are essential components of marine ecosystems and support the regular operation of the entire marine ecosystem. The research of marine phytoplankton and animal ecology is conducive to our comprehensive understanding of the status of an aquatic ecosystem. Marine plankton refers to the aquatic organisms suspended in the water and moving with water flow, mainly including phytoplankton and zooplankton, as well as other organisms such as planktonic viruses, planktonic bacteria, and archaea. Phytoplankton is the primary producer in the sea; it converts solar energy into organic energy through photosynthesis, initiates the material circulation and energy flow in the sea, and is the most basic link in the marine food chain. Zooplankton is an essential consumer in the sea; this part of organic matter is utilized through the food chain and further transferred to the upper trophic level through secondary production processes. Therefore, phytoplankton and zooplankton provide food and energy sources for the upper trophic level organisms through the above primary and secondary production processes, supporting the regular operation of the entire marine ecosystem.

Phytoplankton is not only the bottom but also the most crucial component of the marine ecosystem. It is divided into toxic and non-toxic phytoplankton. At the same time, zooplankton can distinguish different types of phytoplankton. To avoid feeding on toxic phytoplankton, which has a similar synergistic behavior with selective grazing in the predator-prey system [1-5]. In marine plankton ecosystems, the hypothetical
mechanisms of selective grazing include prey morphology (size, color, shape, and colony formation), intestinal genetic strains, and poisonous chemicals released by prey [6-12]. Thus, the avoidance effect of zooplankton on toxins from toxic phytoplankton and the harmful effects of toxic compounds released by toxic species on their competitors have been studied [13-20].

In this paper, we consider not only the effect of toxin avoidance on species existence, but also the impact of human beings on the harvest of non-toxic phytoplankton and zooplankton. Which can be shown as (with symbols slightly varied):

proposed in [14]. They showed that avoidance effects can promote the coexistence of non-toxic phytoplankton, zooplankton, and toxic phytoplankton on species existence and the human harvest has been applied in many models [21-27]. Since time delay is widely studied in the phytoplankton-zooplankton model [28-31], another essential purpose of our research is to explore the effect of pregnancy delay and toxin onset delay on the dynamic system. Finally, we find that optimal strategies are applied in many models to constrain overfishing [32-33]. Through the research we know that in fisheries, there is a fishing strategy called specific fishing, that is, fishermen catch almost only one particular type of fish or several species associated with it, such as these three species in our article, so we need a feedback mechanism to control this particular capture. Based on the dual phytoplankton-zooplankton system, we consider the optimal tax policy to constrain this particular fishing.

The organizational structure of this paper is as follows. In Section 2, we establish a mathematical model with double time delays for avoiding toxic species by zooplankton in the presence of non-toxic species. And give a parameter explanation in Table 2. In Section 3, we analyze the boundedness and stability of the boundary equilibrium point and the internal equilibrium point in the delay-free model. And obtain the bistability between the equilibrium points. The results are summarized in Table 1 and Fig 1. In Section 4, by analyzing different situations of this double delay model, we obtain the critical value of time delay when the system undergoes Hopf bifurcation. In Section 5, we study the optimal tax policy without time delay using the principle of Pontryagin’s maximum. In addition, we use the parameters and initial values given in Table 2 and (6.1) to simulate several cases of double-delay systems in Matlab to verify all theoretical results in Section 6. Lastly, we end this paper with some conclusions and significance in Section 7.

II. Model Formulation

Considering the toxin refuge of zooplankton, a nontoxic phytoplankton-toxic zooplankton model was proposed in [14]. They showed that avoidance effects can promote the coexistence of non-toxic phytoplankton, toxic phytoplankton and zooplankton. Which can be shown as (with symbols slightly varied):

\[
\begin{align*}
\frac{dN}{dt} &= r_1 N \left(1 - \frac{N + \alpha_1 T}{k_1}\right) - \frac{w_1 N Z}{p_1 + N}, \\
\frac{dT}{dt} &= r_2 T \left(1 - \frac{T + \alpha_2 N}{k_2}\right) - \frac{w_2 T Z}{p_2 + T + \beta N}, \\
\frac{dZ}{dt} &= \frac{w_1 N Z}{p_1 + N} - \frac{w_2 T Z}{p_2 + T + \beta N} - dZ, \\
N(0) &\geq 0, \quad T(0) \geq 0, \quad Z(0) \geq 0,
\end{align*}
\]

where \( N, T, \) and \( Z \) represent the biomass of nontoxic phytoplankton, toxic phytoplankton, and zooplankton, respectively. \( k_1 \) and \( k_2 \) are the environmental carrying capacities of nontoxic phytoplankton (NTP) and toxin-producing phytoplankton (TPP) species, respectively. \( r_1 \) and \( r_2 \) represent the constant intrinsic growth rates of \( N \) and \( T \), respectively. \( \alpha_1 \) and \( \alpha_2 \) measure the competitive effect of \( T \) on \( N \), and \( N \) on \( T \), respectively. \( w_1 \) and \( w_2 \) represent the rates at which \( N \) and \( T \) are consumed by \( Z \), respectively. \( p_1 \) and \( p_2 \) are half-saturation constants for NTP and TPP, respectively. \( \beta \) represents the intensity of avoidance of \( T \) by \( Z \) in the presence of \( N \), and \( d \) is the natural mortality of zooplankton. As the research merely focuses on a single time model, moreover overfishing has an important impact on the stability of marine ecosystems, human harvest and time delays should be taken into account. The increment in zooplankton population due to predation does not appear immediately after consuming phytoplankton; it takes some time (say \( \tau_1 \)), which can be regarded as the gestation period in zooplankton. The decrease of zooplankton population caused by ingestion of toxic phytoplankton does not occur immediately. Still, it requires a certain time (say \( \tau_2 \)), which can be regarded as the reaction time after zooplankton poisoning. Accordingly the bio-economic model with time delays on the interactions of nontoxic phytoplankton, toxic plankton and zooplankton with toxin avoidance effects, which can be shown as follows:
Lemma 3.1. which is vital for the biological understanding of the system and the subsequent analysis. Firstly, we rewrite the model (2.3) and take the linear as the following form: 

$$\begin{align*}
\frac{dN}{dt} &= r_1 N\left(1 - \frac{N + \alpha_1 T}{k_1}\right) - \frac{w_1 N Z}{p_1 + N} - q_1 E N, \\
\frac{dT}{dt} &= r_2 T\left(1 - \frac{T + \alpha_2 N}{k_2}\right) - \frac{w_2 T Z}{p_2 + T + \beta N}, \\
\frac{dZ}{dt} &= c_1 w_1 N (t - \tau_1) Z(t - \tau_1) - \frac{c_2 w_2 T (t - \tau_2) Z(t - \tau_2)}{p_2 + T(t - \tau_2) + \beta N(t - \tau_2)} - dZ - q_2 E Z, \\
N(0) &\geq 0, \ T(0) \geq 0, \ Z(0) \geq 0,
\end{align*}$$

(2.2)

where $N, T,$ and $Z$ represent the biomass of nontoxic phytoplankton, toxic phytoplankton and zooplankton, respectively. $\tau_1(\tau_1 > 0)$ and $\tau_2(\tau_2 > 0)$ represent the maturation gestation delay and the toxin onset delay, respectively. $c_1$ and $c_2$ represent the conversion rate of $N$ to $Z$ and $T$ to $Z$, respectively. Due to the experience of human capture, we assume that humans can distinguish between toxic phytoplankton and non-toxic phytoplankton when capturing zooplankton and phytoplankton. So, we put $q_1$ and $q_2$ to represent the fishing coefficients of nontoxic phytoplankton and zooplankton, respectively. And $E$ is the effort used to harvest the population. To investigate the effect of time delay on the dynamic behavior of the model, we will first study the stability of the equilibrium point of the following model without time delay.

$$\begin{align*}
\frac{dN}{dt} &= r_1 N\left(1 - \frac{N + \alpha_1 T}{k_1}\right) - \frac{w_1 N Z}{p_1 + N} - q_1 E N, \\
\frac{dT}{dt} &= r_2 T\left(1 - \frac{T + \alpha_2 N}{k_2}\right) - \frac{w_2 T Z}{p_2 + T + \beta N}, \\
\frac{dZ}{dt} &= c_1 w_1 N Z - \frac{c_2 w_2 T Z}{p_2 + T + \beta N} - dZ - q_2 E Z, \\
N(0) &\geq 0, \ T(0) \geq 0, \ Z(0) \geq 0.
\end{align*}$$

(2.3)

III. Dynamical Behavior of Non-Delayed Model

a) Positivity and boundedness of the solution

In this subsection, firstly, we shall show the positivity and boundedness of solutions of the system (2.3), which is vital for the biological understanding of the system and the subsequent analysis.

Lemma 3.1. All the solutions with initial values of system (2.3), which start in $\mathbb{R}_+^3$, are always positive and bounded.

Proof. Firstly, we rewrite the model (2.3) and take the linear as the following form:

$$\frac{dX}{dt} = F(X),$$

(3.1)

where $X(t) = (N, T, Z)^T \in \mathbb{R}_+^3$ and $F(X)$ is simplified as the following

$$F(X) = \begin{bmatrix}
F_1(X) \\
F_2(X) \\
F_3(X)
\end{bmatrix} = \begin{bmatrix}
r_1 N\left(1 - \frac{N + \alpha_1 T}{k_1}\right) - \frac{w_1 N Z}{p_1 + N} - q_1 E N \\
r_2 T\left(1 - \frac{T + \alpha_2 N}{k_2}\right) - \frac{w_2 T Z}{p_2 + T + \beta N} \\
c_1 w_1 N Z - \frac{c_2 w_2 T Z}{p_2 + T + \beta N} - dZ - q_2 E Z
\end{bmatrix}.$$
We want to prove that \((N(t), T(t), Z(t)) \in R^3_+\) for all \(t \in [0, +\infty)\). For system (2.3) with initial value \(N(0) > 0\), \(T(0) > 0\) and \(Z(0) > 0\), we have

\[
N(t) = N(0) \exp\left\{ \int_0^t \left[ r_1 (1 - \frac{N(s) + \alpha_1 T(s)}{k_1}) - \frac{w_1 Z(s)}{p_1 + N(s)} - q_1 E \right] ds \right\},
\]

\[
T(t) = T(0) \exp\left\{ \int_0^t \left[ r_2 (1 - \frac{T(s) + \alpha_1 N(s)}{k_2}) - \frac{w_2 Z(s)}{p_2 + T(s) + \beta N(s)} \right] ds \right\},
\]

\[
Z(t) = Z(0) \exp\left\{ \int_0^t \left[ \frac{c_1 w_1 N(s)}{p_1 + N(s)} - \frac{c_2 w_2 T(s)}{p_2 + T(s) + \beta N(s)} - d - q_2 E \right] ds \right\},
\]

which shows that all the solutions of system (2.3) are always positive for all \(t > 0\).

Secondly, we prove the boundedness of the solution. Let \((N(t), T(t), Z(t))\) be the solutions of system (2.3), we define a function

\[
W(t) = c_1 N(t) + c_2 T(t) + Z(t).
\]  

(3.2)

Then, by differentiating (3.2) concerning \(t\), we obtain

\[
\frac{dW}{dt} + \eta W = c_1 r_1 N(1 - \frac{N + \alpha_1 T}{k_1}) + c_2 r_2 T(1 - \frac{T + \alpha_1 N}{k_2}) - \frac{2c_2 w_2 T Z}{p_2 + T + \beta N} - dZ - q_2 EZ - c_1 q_1 EN + c_1 \eta N + c_2 \eta T + \eta Z,
\]

\[
\leq c_1 r_1 N(1 - \frac{N}{k_1}) + c_2 r_2 T(1 - \frac{T}{k_2}) - dZ + c_1 \eta N + c_2 \eta T + \eta Z,
\]

\[
= -\frac{c_1 r_1 N^2}{k_1} + (r_1 + \eta)c_1 N - \frac{c_2 r_2 T^2}{k_2} + (r_2 + \eta)c_2 T + (\eta - d)Z,
\]

\[
\leq \frac{c_1 k_1 (r_1 + \eta)^2}{4r_1} + \frac{c_2 k_2 (r_2 + \eta)^2}{4r_2} + (\eta - d)Z,
\]

\[
\leq \frac{c_1 r_2 k_1 (r_1 + \eta)^2 + c_2 r_1 k_2 (r_2 + \eta)^2}{4r_1 r_2} + (\eta - d)Z,
\]

when \(\eta - d < 0\), we can obtain

\[
\frac{dW}{dt} + \eta W \leq \frac{c_1 r_2 k_1 (r_1 + \eta)^2 + c_2 r_1 k_2 (r_2 + \eta)^2}{4r_1 r_2},
\]

noting \(\kappa = \frac{c_1 r_2 k_1 (r_1 + \eta)^2 + c_2 r_1 k_2 (r_2 + \eta)^2}{4r_1 r_2}\), therefore, applying a theorem on differential inequalities [34], we obtain

\[
0 \leq W \leq \frac{\kappa}{\eta} + \frac{W(N(0), T(0), Z(0))}{\exp(\eta t)},\text{ let } t \to +\infty, W(N, T, Z) \leq \frac{\kappa}{\eta}.\text{ So, all solutions of system (2.3) enter the region}
\]

\[
D = \{(N, T, Z) \in R^3_+ : 0 \leq W(N, T, Z) \leq \frac{\kappa}{\eta}\}.
\]  

(3.3)

This shows that every solution of the system is bounded.
b) Equilibrium points and their stability

System (2.3) possesses six different equilibrium points:

(i) the plankton-free equilibrium, \( E_0 = (0, 0, 0) \), which always exists;

(ii) TPP and zooplankton-free equilibrium, \( E_1 = (k_1, 0, 0) \), which is always feasible;

(iii) NTP and zooplankton-free equilibrium, \( E_2 = (0, k_2, 0) \), which is always feasible;

(iv) zooplankton-free equilibrium, \( E_3 = (\hat{N}, \hat{T}, 0) \), where

\[
\hat{N} = \frac{\alpha_1 k_2 - k_1}{\alpha_1 \alpha_2 - 1} - \frac{q_1 k_1 E}{r_1}, \quad \hat{T} = \frac{\alpha_2 k_1 - k_2}{\alpha_1 \alpha_2 - 1};
\]

(v) TPP-free equilibrium \( E_4 = (\hat{N}, 0, \hat{Z}) \), where

\[
\hat{N} = \frac{(q_2 E + d)p_1}{c_1 w_1 - d - q_2 E}, \quad \hat{Z} = \frac{r_1 (k_1 - \hat{N}) - q_1 k_1 (p_1 + E)}{k_1 w_1};
\]

(vi) the interior equilibrium, \( E^* = (N^*, T^*, Z^*) \), where

\[
T^* = \frac{c_1 w_1 N^* - (d + q_2 E)(p_1 + N^*)(p_2 + \beta N^*)}{(c_2 w_2 + d + q_2 E)(p_1 + N^*) - c_1 w_1 N^*}, \quad Z^* = \frac{(p_1 + N^*) r_1 (k_1 - N^* - \alpha_3 T^*) - q_1 k_1 E}{k_1 w_1};
\]

and \( N^* \) can be obtained from

\[
r_2(p_2 + T^* + \beta N^*)(k_2 - T^* - \alpha_2 N^*) - w_2 k_2 Z^* = 0. \tag{3.4}
\]

Next, we illustrate the existence and stability of six equilibria when human harvest and avoidance factor exist simultaneously by solving Jacobi determinant of different equilibria, and summarize them in Table 1.

Equilibria analysis: Obviously, the equilibria \( E_0, E_1 \) and \( E_2 \) always exist. The zooplankton-free equilibrium \( E_3 \) exists, let \( \hat{N} \) and \( \hat{T} \) both be positive, that is \( \alpha_2 > \frac{k_2}{k_1} \) and \( \alpha_1 > \frac{\alpha_1 \alpha_2 - 1}{q_1 k_1 E} + \frac{k_1}{k_2} \). The TPP-free equilibrium \( E_4 \) exists, let \( \hat{N} \) and \( \hat{Z} \) both be positive, that is \( w_1 > \frac{d + q_2 E}{c_1} \) and \( k_1 > \frac{c_1 w_1 - d - q_2 E}{r_1 q_1 E(p_1 + E)} \). The interior equilibrium point \( E^* \) exists; let \( N^*, T^* \) and \( Z^* \) all be positive, that is \( k_1 > \frac{2 k_1 E}{r_1} + N^* + \alpha_1 T^*, c_2 w_2(p_1 + N^*) > c_1 w_1 N^* - (d + q_2 E)(p_1 + N^*) > 0 \) and Eq.(3.4) has at least one positive root.

In the following, we summarize the eigenvalues and local stability conditions around the feasible equilibrium point of each organism of system (2.3).

(i) The eigenvalues of the plankton-free equilibrium \( E_0 = (0, 0, 0) \) are \( r_1, r_2 \) and \( -d - q_2 E \). Therefore, it is a saddle point and hence always unstable.

(ii) The eigenvalues of the TPP and zooplankton-free equilibrium \( E_1 = (k_1, 0, 0) \) are \( -r_1 - q_1 E, r_2(1 - \frac{k_2}{k_1}) \) and \( \frac{c_1 w_1 k_1}{p_1 + k_1} - d - q_2 E \). When \( c_1 \omega_1 - d - q_2 E \leq 0 \), and \( \alpha_2 > \frac{k_2}{k_1} \), hold, \( E_1 \) is LAS(locally asymptotically stable). On the contrary, if \( c_1 \omega_1 - d - q_2 E > 0 \), \( \alpha_2 > \frac{k_2}{k_1} \) and \( k_1 < \frac{p_1 (d + q_2 E)}{c_1 w_1 - d - q_2 E} \) hold, we can also obtain \( E_1 \) is LAS.

(iii) The eigenvalues of the NTP and zooplankton-free equilibrium \( E_2 = (0, k_2, 0) \) are \( r_2(1 - \frac{k_2}{k_1}) - q_1 E, -r_2 \) and \( -\frac{c_2 w_2 k_2}{p_2 + k_2} - d - q_2 E \). Therefore, \( E_2 \) is LAS if \( k_1 < \frac{r_2 \alpha_1 k_1}{r_2 - q_1 E} \).

(iv) The eigenvalues of the zooplankton-free equilibrium \( E_3 = (\hat{N}, \hat{T}, 0) \) are \( \frac{c_1 w_1 \hat{N}}{p_1 + \hat{N}} - \frac{c_2 w_2 T}{p_2 + T + \beta \hat{N}} - d - q_2 E \), \( \lambda_1 \) and \( \lambda_2 \), where \( \lambda_1 \) and \( \lambda_2 \) are the roots of the equation

\[
\lambda^2 + \tilde{b}_1 \lambda + \tilde{c}_1 = 0, \tag{1}
\]
where

\[ \tilde{b}_1 = -[r_2 - r_1 + \frac{r_1 k_2(2\tilde{N} + \alpha_1 T) - r_2 k_1(2\tilde{T} + \alpha_2 \tilde{N})}{k_1 k_2}], \]

\[ \tilde{c}_1 = r_1 r_2[1 - (2\tilde{T} + \alpha_2 \tilde{N})(2\tilde{N} + \alpha_1 \tilde{T})]\left[\frac{1}{(2N + \alpha_1 T)k_2} + \frac{1}{(2T + \alpha_2 N)k_1} - \frac{1}{k_1 k_2}\right] + q_1 r_2 E\left(\frac{k_1(2\tilde{T} + \alpha_2 \tilde{N}) - r_1 \alpha_1^2 \tilde{N}\tilde{T}}{k_1 k_2} - 1\right). \]

Therefore, let \( \frac{c_1 w_1 \tilde{N}}{p_1 + N} - \frac{c_2 w_2 \tilde{T}}{p_2 + \beta N} - d - q_2 E < 0 \), \( \lambda_1 \) and \( \lambda_2 \) with negative real parts, that is \( \frac{c_1 w_1 \tilde{N}}{p_1 + N} - d - q_2 E < \frac{c_2 w_2 \tilde{T}}{p_2 + \beta N} \), \( \tilde{b}_1 > 0 \) and \( \tilde{c}_1 > 0 \). If the above conditions are satisfied, \( E_3 \) is LAS.

(v) The eigenvalues of the TPP-free equilibrium \( E_4 = (\bar{N}, 0, \bar{Z}) \) are \( r_2(1 - \frac{a_2 \bar{N}}{k_2}) - \frac{w_2 \bar{Z}}{p_2 + \beta N} \), \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \), where \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) are the roots of the equation

\[ \lambda^2 - (\tilde{a}_2 + \tilde{b}_2)\lambda + \tilde{a}_2 \tilde{b}_2 + \tilde{c}_2 = 0, \]  

where

\[ \tilde{a}_2 = (r_2(1 - \frac{a_2 \tilde{N}}{k_2}) - \frac{w_1 \tilde{p}_1 \tilde{Z}}{(p_1 + N)^2} - q_1 E), \]

\[ \tilde{b}_2 = (\frac{c_1 w_1 \tilde{N}}{p_1 + N} - d - q_2 E), \]

\[ \tilde{c}_2 = \frac{c_1 w_2 \tilde{p}_1 \tilde{Z}}{(p_1 + N)^2}. \]

Therefore, let \( r_2(1 - \frac{a_2 \bar{N}}{k_2}) - \frac{w_2 \bar{Z}}{p_2 + \beta N} < 0 \), \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) with negative real parts, that is \( (\tilde{a}_2 + \tilde{b}_2) < 0 \) and \( \tilde{a}_2 \tilde{b}_2 + \tilde{c}_2 > 0 \). If the above conditions are satisfied, \( E_4 \) is LAS.

(vi) By solving the Jacobi determinant of \( E^* \), we can get its characteristic equation as follows

\[ \lambda^3 + D_1 \lambda^2 + D_2 \lambda + D_3 = 0. \]  

The interior equilibrium \( E^* = (N^*, T^*, Z^*) \) is LAS if

(a) \( D_1 > 0 \),
(b) \( D_3 > 0 \),
(c) \( D_1 D_2 - D_3 > 0 \),

where

\[ D_1 = -(r_2[1 - \frac{(2T^* + \alpha_2 N^*)}{k_1}] - \frac{w_2 Z^*(p_2 + \beta N^*)}{(p_2 + T^* + \beta N^*)^2} + r_1[1 - \frac{(2N^* + \alpha_1 T^*)}{k_1}] - \frac{w_2 p_1 Z^*}{(p_1 + N^*)^2} - q_1 E), \]

\[ -(\frac{c_1 w_1 \tilde{N}^*}{p_1 + N^*} - \frac{c_2 w_2 \tilde{T}^*}{p_2 + \beta N^*} - d - q_2 E), \]

\[ D_2 = \left\{ \frac{c_1 w_1^2 p_1 N^* Z^*}{(p_1 + N^*)^3} + \frac{c_2 w_1 w_2 \beta N^* T^* Z^*}{(p_2 + T^* + \beta N^*)^2 (p_1 + N^*)} - \frac{c_2 w_2^2 T^* Z^* (p_2 + \beta N^*)}{(p_2 + T^* + \beta N^*)^3} \right\}. \]
From the calculation of the eigenvalues, obviously, \( \beta \) does not affect the stability of \( E_1 \) and \( E_2 \). Still, it has a significant impact on the stability of \( E_3 \) and \( E_4 \) (because the eigenvalues of \( E_1 \) and \( E_2 \) are independent of \( \beta \), but related to human harvest). On the other hand, we not only find that the equilibrium point of system (2.3) is affected by human harvest, but also has a particular impact on its stability (it can be seen from the eigenvalue of each equilibrium point).

Next, the biological explanations of the above different equilibria are discussed below. Since all these interpretations are mainly based on local asymptotic stability conditions, initial abundance of all the populations may also play an essential role for the system’s dynamics together with the parameters. Different from the biological explanation in [14], we not only consider the effect of \( \beta \) on species coexistence, but also human harvest as an essential factor in species coexistence.

(i) \( E_0 \): Extinction of all the populations at a time is impossible.

(ii) \( E_1 \): From the analysis of research results, whenever the carrying capacity of the NTP population \((k_1)\) stays within the specific threshold values of \( \frac{k_2}{\alpha_2} < k_1 < \frac{p_1(d+q_2E)}{c_1w_1-d-q_2E} \), both TPP and zooplankton will eventually become extinct from the system. Now, through the analysis of the \( k_1 \) threshold range, as the intensification of the harvest for zooplankton, the equilibrium point \( E_1 \) remains stable for a more extensive range of \( k_1 \), and we can say that over-fishing of zooplankton \((q_2E)\) may accelerate the extinction of TPP and zooplankton.

(iii) \( E_2 \): If the carrying capacity of NTP population \((k_1)\) stays below the threshold value \( \frac{p_2\alpha_kk_2}{r_2-q_1E} \), both NTP and zooplankton eventually extinct. With the competitive effect of TPP on NTP \((\alpha_1)\), the environmental carrying capacities of toxin-producing phytoplankton \((k_2)\) and harvesting term for NTP and zooplankton.
(\(q_1E\)) increase, respectively. The equilibrium point \(E_2\) remains stable for a larger scale of \(k_1\); we can say that the possibility of deracinating NTP and zooplankton at a time increases with the increase in \(\alpha_1\), \(k_2\) and \(q_1E\).

(iv) \(E_3\): When the carrying capacity of NTP population \((k_1)\) remains within two threshold values \(\frac{r_\alpha + k_2}{k_2 - q_1E} < k_1 < \frac{k_2}{\alpha_2}\) (it can be obtained by the threshold value \((k_1)\) of \(E_1\) and \(E_2\)) together with the competitive effects \((\alpha_1, \alpha_2)\), the harvesting term on NTP \((q_1E)\) are present and the values of all three are small, the zooplankton population will go extinct on the condition that \(\frac{c_1 w_1 N}{\rho_1 + N} - d - q_2 E < \frac{c_2 w_2 T}{p_2 + T + \beta N}\), whereas both NTP and TPP persist in the system. The chance of zooplankton extinction increases with the decrease in avoidance of TPP by zooplankton \((\beta)\), the half-saturation constant for TPP \((p_2)\), the harvesting term on zooplankton \((q_2E)\) and the zooplankton mortality \((d)\). For a specific parameter setup \((\frac{c_1 w_1 N}{\rho_1 + N} - (d + q_2 E) > 0)\), we can find a threshold value of the avoidance of TPP by zooplankton \((\beta < \frac{(c_2 w_2 T)(p_1 + N)}{(N)(c_1 w_1 N - (d + q_2 E)(p_1 + N))} - \frac{p_2 + T}{N})\), below which the zooplankton population will become extinct. On the contrary, for \(\frac{c_1 w_1 N}{\rho_1 + N} - (d + q_2 E) < 0\), the extinction of zooplankton does not depend on the intensity of avoidance; it maybe has something relationship with the harvest term on zooplankton \((q_2E)\).

(v) \(E_4\): If the carrying capacity of NTP population \((k_1)\) remains within two threshold values \(\frac{(d + q_2 E)p_1}{c_1 w_1 d - q_2 E} < k_1 < \frac{(d + q_2 E)(p_1 + c_1 w_1 p_1)}{c_1 w_1 d - q_2 E}\), then TPP becomes extinct under the condition \(\frac{r_\alpha (k_2 - \alpha_2 N)}{k_2} < \frac{c_2 w_2 Z}{p_2 + \beta N}\), whereas both NTP and zooplankton persist in the system. The possibility of TPP extinction increases with the reduction in the avoidance of TPP by zooplankton \((\beta)\), the half-saturation constant for TPP \((p_2)\), and the growth rate of TPP \((r_2)\), decreases with the rise of the competitive effect of \(N\) on \(T\) \((\alpha_2)\) and the TPP consumption rate \((w_2)\). Similarly, for a particular parameter setup \((k_2 - \alpha_2 N > 0)\), we can find a threshold value of the avoidance of TPP by zooplankton \((\beta < \frac{k_2 w_2 Z}{N r_2(k_2 - \alpha_2 N)} - \frac{p_2}{N})\), below which TPP may become extinct. On the contrary, for \(k_2 - \alpha_2 N < 0\), TPP extinction does not depend on the avoidance. Because the biological analysis of \(E_4\) found that the harvesting term has little impact on the extinction of TPP compared with other equilibrium points. In conclusion, for \(k_2 - \alpha_2 N < 0\), TPP extinction does not depend on the avoidance of TPP by zooplankton \((\beta)\) and harvest term on zooplankton \((q_2E)\).

(vi) \(E^* = (N^*, T^*, Z^*)\): When the competitive effects \((\alpha_1)\), the fishing coefficients of nontoxic phytoplankton \((q_1)\), the environmental carrying capacities of nontoxic phytoplankton \((k_1)\), and the effort used to harvest the population \((E)\) remain very small, whereas the constant intrinsic growth rates of \(N\) \((r_1)\), there may be a possibility of coexistence of all the three species.
c) Bistability analysis of equilibrium point

The existence and stability of these equilibrium points are summarized in Table 1 and Fig 1. When \( c_1 w_1 - d - q_2 E > 0 \), equilibria \( E_2 = (0, k_2, 0) \), \( E_3 = (\bar{N}, \bar{T}, 0) \), \( E_1 = (k_1, 0, 0) \) and \( E_4 = (\bar{N}, 0, \bar{T}) \) keep stable for \((k_1 < \frac{c_1 w_1 - d - q_2 E}{r_2 - q_1 E}, \frac{c_1 w_1 - d - q_2 E}{r_2 - q_1 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) respectively (Fig.1(a)). Obviously, for \( k_1 \) at the different equilibria above, the coexistence of NTP, TPP, and zooplankton requires the three ranges \((k_1 > \frac{c_1 w_1 - d - q_2 E}{r_2 - q_1 E}, k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) and \((k_1 > \frac{c_1 w_1 - d - q_2 E}{r_2 - q_1 E}, k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) respectively. Therefore, the system exhibits these three possible types of bistability, where

(i) \( E_1 \) and \( E_2 \).

(ii) \( E_2 \) and \( E_4 \).

(iii) \( E_3 \) and \( E_4 \).

The above three types are locally asymptotically stable for different ranges of \( k_1 \).

For \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), we can observe the bistability of \( E_1 \) and \( E_2 \) (Fig.1(b)(c)). If conditions \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) and \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) hold simultaneously, we can find the bistability of \( E_2 \) and \( E_4 \) (Fig.1(d)(e)). On the contrary, if \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), we obtain the existence of stable \( E_2 \) together with unstable \( E_4 \). Identically, for \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), we can observe the bistability of \( E_3 \) and \( E_4 \) (Fig.1(f)(i)).

Now, let’s discuss the importance of avoiding toxic species by zooplankton (\( \beta \)) together with the harvesting term \((q_1 E, q_2 E)\) for the survival of the different species groups.

Firstly, let’s discuss the effect of \( \beta \) on three types of bistability. It can be seen from the previous analysis that the stability of \( E_1 \) and \( E_2 \) does not depend on the value of \( \beta \). However, for the stability of \( E_3 \) and \( E_4 \), it is related to the critical value of \( \beta \). When \( \beta \) is less than this critical value, \( E_3 \) and \( E_4 \) remain stable. Thus, \( \beta \) does not affect the bistability of \((E_1, E_2)\); when \( \beta \) is below some threshold value, we will observe the bistability of \((E_2, E_4)\) and \((E_3, E_4)\), and as the \( \beta \) value increases, the original bistability may disappear. \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \) together with \( \alpha_1 \alpha_2 < 1 \), \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), and \( \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} < k_1 < \frac{c_1 w_1 - d - q_2 E}{c_1 w_1 - d - q_2 E} \), we can see the establishment of the above conclusion.

Secondly, let’s discuss the effect of the harvesting term \((q_1 E, q_2 E)\) on three types of bistability. From the analysis of the previous data, it can be seen that although the stability of \( E_1 \) and \( E_2 \) does not depend on the value of \( \beta \), when humans overfish NTP and zooplankton, that is, \( q_1 E \) and \( q_2 E \) are too large, it may affect the bistability of \( E_1 \) and \( E_2 \). For \( E_3 \) and \( E_4 \), although their stability is directly related to the threshold value of \( \beta \), the existence of \( q_1 E \) and \( q_2 E \) will also affect the threshold value of \( \beta \), further influencing the stability of \( E_3 \) and \( E_4 \). Therefore, \( q_1 E \) and \( q_2 E \) may affect the bistability of \((E_1, E_2)\), \((E_2, E_4)\) and \((E_3, E_4)\); the increase of \( q_1 E \) and \( q_2 E \) may also lead to the disappearance of this bistability.

IV. Dynamical Behavior of the Delayed Model

In this section, we focus on the local stability and Hopf bifurcation of the delayed model; the delayed system (2.2) has the following form

\[
\frac{dU(t)}{dt} = F(U(t), U(t - \tau_1), U(t - \tau_2)),
\]

where

\[
U(t) = [N(t), T(t), Z(t)], \quad U(t - \tau_1) = [N(t - \tau_1), T(t - \tau_1), Z(t - \tau_1)],
\]

\[
U(t - \tau_2) = [N(t - \tau_2), T(t - \tau_2), Z(t - \tau_2)].
\]
Next, assuming $\Lambda_1(t) = N(t) - N^*$, $\Lambda_2(t) = T(t) - T^*$, $\Lambda_3(t) = Z(t) - Z^*$ at the positive equilibrium point, and linearizing the system (2.2), we can obtain

$$\frac{d}{dt} \begin{pmatrix} \Lambda_1(t) \\ \Lambda_2(t) \\ \Lambda_3(t) \end{pmatrix} = L \begin{pmatrix} N(t) \\ T(t) \\ Z(t) \end{pmatrix} + M \begin{pmatrix} N(t - \tau_1) \\ T(t - \tau_1) \\ Z(t - \tau_1) \end{pmatrix} + S \begin{pmatrix} N(t - \tau_2) \\ T(t - \tau_2) \\ Z(t - \tau_2) \end{pmatrix},$$

(4.2)

where

$$L = \left( \frac{\partial F}{\partial U(t)} \right)_{E^*}, \quad M = \left( \frac{\partial F}{\partial U(t - \tau_1)} \right)_{E^*}, \quad S = \left( \frac{\partial F}{\partial U(t - \tau_2)} \right)_{E^*}.$$  

We linearize the system (2.2) about positive equilibrium $E^* = (N^*, T^*, Z^*)$, and get

$$\frac{dU(t)}{dt} = LU(t) + MU(t - \tau_1) + SU(t - \tau_2),$$

(4.3)
Fig. 1: Stability of different equilibria for different ranges of $k_1$. The dotted arrow indicates the range where bistability occurs, (a) means no bistability, (b) and (c) bistability of $E_1$ and $E_2$, (d) and (e) bistability of $E_2$ and $E_4$, (f)-(i) bistability of $E_3$ and $E_4$. 
where

\[
L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_{31} & 0 & m_{33} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad U = \begin{pmatrix} N_1(\cdot) \\ T_1(\cdot) \\ Z_1(\cdot) \end{pmatrix},
\]

where \(N_1, T_1, Z_1\) are small perturbations around the equilibrium point \(E^* = (N^*, T^*, Z^*)\). We have

\[
l_{11} = -\frac{rN}{k_1} + \frac{w_1ZN}{(p_1 + N)^2} - q_1E, \quad l_{12} = \frac{r_1\alpha_1N}{k_1}, \quad l_{13} = -\frac{w_1N}{p_1 + N},
\]

\[
l_{21} = \frac{r_2\alpha_2T}{k_1} + \frac{w_2\beta T}{(p_2 + T + \beta N)^2}, \quad l_{22} = r_2 - \frac{(2r_2T + r_2\alpha_1N)}{k_2},
\]

\[
l_{23} = -\frac{w_2T}{(p_2 + T + \beta N)}, \quad l_{33} = -d - q_2E, \quad m_{31} = \frac{c_1w_1p_1Z}{(p_1 + N)^2}, \quad m_{33} = \frac{c_1w_1N}{(p_1 + N)},
\]

\[
s_{31} = \frac{c_2w_2\beta T}{(p_2 + T + \beta N)^2}, \quad s_{32} = \frac{c_2w_2Z(p_2 + \beta N)}{(p_2 + T + \beta N)^2}, \quad s_{33} = \frac{c_2w_2T}{(p_2 + T + \beta N)}.
\]

The characteristic equation for the linearized system (2.2) is obtained as

\[
D(\xi, \tau_1, \tau_2) \equiv P(\xi) + Q(\xi)e^{-\xi\tau_1} + R(\xi)e^{-\xi\tau_2} = 0,
\]

where

\[
P(\xi) = \xi^3 + A_2\xi^2 + A_1\xi + A_0, \quad Q(\xi) = B_2\xi^2 + B_1\xi + B_0, \quad R(\xi) = C_2\xi^2 + C_1\xi + C_0,
\]

with

\[
A_2 = -(l_{33} + l_{22} - l_{11}), \quad A_1 = l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{12}l_{21}, \quad A_0 = -l_{11}l_{22}l_{33} + l_{12}l_{21}l_{33},
\]

\[
B_2 = -m_{33}, \quad B_1 = -l_{11}m_{33} - l_{22}m_{33} - l_{13}m_{31}, \quad B_0 = +l_{11}l_{22}m_{31} + l_{11}l_{22}m_{33} + l_{12}l_{21}m_{33} - l_{12}l_{23}m_{31},
\]

\[
C_2 = -s_{33}, \quad C_1 = -l_{13}s_{31} + l_{11}s_{33} - l_{22}s_{33} - l_{13}s_{32}, \quad C_0 = l_{11}s_{33} + l_{11}l_{33}s_{32} + l_{12}l_{21}s_{33} + l_{13}l_{22}s_{31} - l_{12}l_{23}s_{31} - l_{13}l_{21}s_{32}.
\]

**Case (1):** \(\tau_1 = \tau_2 = 0\).

In this case, Section 3 covers the analysis of the system when \(\tau_1 = \tau_2 = 0\).

**Case (2):** \(\tau_1 = 0, \tau_2 > 0\).

In this case, the characteristic equation (4.4) becomes

\[
D(\xi, \tau_2) \equiv P(\xi) + Q(\xi) + R(\xi)e^{-\xi\tau_2}
\]

\[
\equiv \xi^3 + A_2\xi^2 + A_1\xi + A_0 + B_2\xi^2 + B_1\xi + B_0 + (C_2\xi^2 + C_1\xi + C_0)e^{-\xi\tau_2} = 0,
\]

(4.5)

putting \(\xi = \imath \omega (\omega > 0)\) in Eq. (4.5), and separating the real and imaginary parts, we have

\[
-(A_2 + B_2)\omega^2 + (A_0 + B_0) = (C_2\omega^2 - C_0) \cos(\omega\tau_2) - C_1\omega \sin(\omega\tau_2),
\]

\[
\omega^3 + (A_1 + B_1)\omega = (C_0 - C_2\omega^2) \sin(\omega\tau_2) - C_1\omega \cos(\omega\tau_2).
\]

(4.6)
Squaring and adding the equation (4.6), we obtain

\[-(A_2 + B_2)\omega^2 + (A_0 + B_0)\] \[+ [-\omega^3 + (A_1 + B_1)\omega] = (C_2\omega^2 - C_0)^2 + (C_1\omega)^2. \tag{4.7}\]

Simplifying Eq. (4.7) and substituting \(\omega^2 = \psi\), the above equation can be written as

\[\Psi(\psi) \equiv \psi^3 + a_2\psi^2 + a_1\psi + a_0 = 0, \tag{4.8}\]

where

\[a_2 = -(A_2 + B_2)^2 - 2(A_1 + B_1) - C_2^2, \quad a_1 = (A_1 + B_1)^2 - 2(A_0 + B_0)(A_2 + B_2) - 2C_0C_2 - C_1^2, \quad a_0 = -C_0^2.\]

(H1): \(a_2 > 0, a_0 > 0, a_2a_1 - a_0 > 0\).
If (H1) holds, Eq. (4.8) has no positive roots, which implies all the roots of Eq. (4.5) have negative real parts. Therefore, \(E^*\) is asymptotically stable for all \(\tau_2 > 0\) when (H1) holds.

(H2): \(a_2 < 0, a_1 < 0, a_0 < 0\) or \(a_2 > 0, a_1 < 0, a_0 < 0\) or \(a_2 > 0, a_1 > 0, a_0 < 0\).
If (H2) holds, Eq. (4.8) has exactly one positive root \(\omega_0\), substituting \(\omega_0\) in Eq. (4.6), we obtain

\[-(A_2 + B_2)\omega_0^2 + (A_0 + B_0) = (C_2\omega_0^2 - C_0)\cos(\omega_0\tau_2) - C_1\omega_0\sin(\omega_0\tau_2),\]

\[-\omega_0^3 + (A_1 + B_1)\omega_0 = (C_0 - C_2\omega_0^2)\sin(\omega_0\tau_2) - C_1\omega_0\cos(\omega_0\tau_2). \tag{4.9}\]

For the critical value of \(\tau_2\), we can obtain

\[\tau_2 = \frac{1}{\omega_0} \arccos \left\{ \frac{[C_1 + C_2(A_2 + B_2)]\omega_0^4 + [C_1(A_1 + B_1) - C_0(A_2 + B_2) - C_2(A_0 + B_0)]\omega_0^2 + C_0(A_0 + B_0)}{-(C_0 - C_2\omega_0^2)^2 - (C_1\omega_0)^2} \right\} + \frac{2j\pi}{\omega_0}, \tag{4.10}\]

\(j = 0, 1, 2 \cdots \).

For the transversality condition, differentiating Eq. (4.5) with respect to \(\tau_2\), we get

\[\frac{d\xi}{d\tau_2} = \frac{\xi(C_2\xi^2 + C_1\xi + C_0)e^{-\xi\tau_2}}{3\xi^2 + 2A_2\xi + A_1 + (2B_2\xi + B_1) + (2C_2\xi + C_1)e^{-\xi\tau_2}}.\]

Solving \((\frac{d\xi}{d\tau_2})^{-1}\), we obtain

\[(\frac{d\xi}{d\tau_2})^{-1} = \frac{3\xi^2 + 2A_2\xi + A_1 + (2B_2\xi + B_1) + (2C_2\xi + C_1)e^{-\xi\tau_2}}{\xi(C_2\xi^2 + C_1\xi + C_0)e^{-\xi\tau_2}}.\]

Then at \(\tau_2 = \tau_20\) and \(\xi = i\omega_0\), we can get

\[\text{Re}(\frac{d\xi}{d\tau_2})_{\tau_2 = \tau_20, \xi = i\omega_0}^{-1} = \text{Re} \left\{ \frac{3(i\omega_0)^2 + (2A_2 + B_2)(i\omega_0) + A_1 + B_1}{(i\omega_0)(C_2(i\omega_0)^2 + C_1(i\omega_0) + C_0)(\cos(\omega_0\tau_20) - i\sin(\omega_0\tau_20))} \right\} + \text{Re} \left\{ \frac{2C_2(i\omega_0) + C_1}{(i\omega_0)(C_2(i\omega_0)^2 + C_1(i\omega_0) + C_0)} \right\}.\]

Now

\[\text{Re}(\frac{d\xi}{d\tau_2})_{\tau_2 = \tau_20, \xi = i\omega_0}^{-1} = \text{Re} \left[ \frac{M_R + M_I i}{N_R + N_I i} \right] + \text{Re} \left[ \frac{Q_R + Q_I i}{P_R + P_I i} \right] = \frac{M_R N_R + M_I N_I}{N_R^2 + N_I^2} + \frac{Q_R P_R + Q_I P_I}{P_R^2 + P_I^2}.\]
where
\[ M_R = -3\omega_0^2 + A_1 + B_1, \quad M_I = 2(A_2 + B_2)\omega_0, \quad N_R = (C_0\omega_0 - C_2\omega_0^3)\sin(\omega_0\tau_0) - C_1\omega_0^2\cos(\omega_0\tau_0), \]
\[ N_I = (C_0\omega_0 - C_2\omega_0^3)\cos(\omega_0\tau_0) + C_1\omega_0^2\sin(\omega_0\tau_0), \quad Q_R = C_1, \quad Q_I = 2C_2\omega_0, \]
\[ P_R = -C_1\omega_0^2, \quad P_I = C_0\omega_0 - C_2\omega_0^3. \]

Then
\[ [\text{Re}(\frac{d\xi}{d\tau_2})]_{\tau_2=\tau_{20}, \xi=i\omega_0}]^{-1} = \frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}, \tag{4.11} \]
here
\[ A = M_RN_R + M_IN_I, \quad B = N_R^2 + N_I^2, \]
\[ C = Q_RP_R + Q_IP_I, \quad D = P_R^2 + P_I^2. \]

From this, we can get
\[ \text{sgn}[\text{Re}(\frac{d\xi}{d\tau_2})]_{\tau_2=\tau_{20}, \xi=i\omega_0}]^{-1} = \text{sgn}[AD + BC]. \]

If (H3): \( AD + BC \neq 0 \) holds, the transversal condition \( \text{sgn}[\text{Re}(\frac{d\xi}{d\tau_2})]_{\tau_2=\tau_{20}, \xi=i\omega_0}]^{-1} \neq 0 \). From the above analysis, the following theorem can be drawn

**Theorem 4.1.** For \( \tau_1 = 0 \) and \( \tau_2 > 0 \), we have the following results:

(i) If (H1) holds, then the equilibrium E* is asymptotically stable for all \( \tau_2 > 0 \).

(ii) If (H3) holds, and (H2) holds, then the equilibrium E* is locally asymptotically stable for all \( \tau_2 < \tau_{20} \) together with unstable for \( \tau_2 > \tau_{20} \) and undergoes Hopf bifurcation at \( \tau_2 = \tau_{20} \).

**Case (3):** \( \tau_1 > 0, \tau_2 = 0 \).

In this case, the characteristic equation (4.4) becomes as follows
\[ D(\xi, \tau_1) = P(\xi) + R(\xi) + Q(\xi)e^{-\xi\tau_1} \]
\[ \equiv \xi^3 + A_2\xi^2 + A_1\xi + A_0 + (B_2\xi^2 + C_2\xi^2 + C_1\xi + C_0) + B_1\xi + B_0)e^{-\xi\tau_1} = 0. \tag{4.12} \]
putting \( \xi = i\omega(\omega > 0) \) in Eq. (4.12), and separating the real and imaginary parts, we have
\[ - (A_2 + C_2)\omega^2 + (A_0 + C_0) = (B_2\omega^2 - B_0)\cos(\omega\tau_1) - B_1\omega\sin(\omega\tau_1), \]
\[ - \omega^3 + (A_1 + C_1)\omega = (B_0 - B_2\omega^2)\sin(\omega\tau_1) - B_1\omega\cos(\omega\tau_1). \tag{4.13} \]

Squaring and adding the equation (4.13), we obtain
\[ [- (A_2 + C_2)\omega^2 + (A_0 + C_0)]^2 + [- \omega^3 + (A_1 + C_1)\omega]^2 = (B_2\omega^2 - B_0)^2 + (B_1\omega)^2. \tag{4.14} \]

Based on the calculation method for case (2), we can simplify (4.14) to the following
\[ \Psi( ) \equiv \xi^2 + b_2 \xi + b_0 = 0, \tag{4.15} \]
where
\[ b_2 = -(A_2 + C_2)^2 - 2(A_1 + C_1) - B_2^2, \quad b_1 = (A_1 + C_1)^2 - 2(A_0 + C_0)(A_2 + C_2) - 2B_0B_2 - B_1^2, \quad b_0 = -B_0^2. \]
(H4): \( b_2 > 0, b_0 > 0, b_2 b_1 - b_0 > 0 \).

If (H4) holds, Eq.(4.15) has no positive roots, which implies all the roots of Eq.(4.12) have negative real parts. Therefore, \( E^* \) is asymptotically stable for all \( \tau_1 > 0 \) when (H4) holds.

(H5): \( b_2 < 0, b_1 < 0, b_0 < 0 \) or \( b_2 > 0, b_1 < 0, b_0 < 0 \) or \( b_2 > 0, b_1 > 0, b_0 < 0 \).

If (H5) holds, Eq.(4.15) has exactly one positive root \( \omega_0 \), substituting \( \omega_0 \) in Eq.(4.13), we obtain

\[
- (A_2 + C_2) \omega_0^2 + (A_0 + C_0) = (B_2 \omega_0^2 - B_0) \cos(\omega_0 \tau_1) - B_1 \omega_0 \sin(\omega_0 \tau_1),
\]

\[
- \omega_0^3 + (A_1 + C_1) \omega_0 = (B_0 - B_2 \omega_0^2) \sin(\omega_0 \tau_1) - B_1 \omega_0 \cos(\omega_0 \tau_1).
\]

Equation (4.16)

For the critical value of \( \tau_1 \), we can obtain

\[
\tau_1 = \frac{1}{\omega_0} \arccos \left( \frac{B_1 + B_2 (A_2 + C_2) \omega_0^4 + [B_1 (A_1 + C_1) - C_0 (A_2 + C_2) - B_2 (A_0 + C_0) \omega_0^2 + B_0 (A_0 + C_0)]}{-(B_0 - B_2 \omega_0^2)^2 - (B_1 \omega_0)^2} \right) + \frac{2j \pi}{\omega_0},
\]

\[
j = 0, 1, 2, \ldots.
\]

(4.17)

For the transversality condition, differentiating Eq.(4.13) with respect to \( \tau_1 \), we get

\[
\frac{d\xi}{d\tau_1} = \frac{\xi (B_2 \xi^2 + B_1 \xi + B_0) e^{-\xi \tau_1}}{3 \xi^2 + 2A_2 \xi + A_1 + (2C_2 \xi + C_1) + (2B_2 \xi + B_1) e^{-\xi \tau_1}}.
\]

Solving \( \left( \frac{d\xi}{d\tau_1} \right)^{-1} \), we obtain

\[
\left( \frac{d\xi}{d\tau_1} \right)^{-1} = \frac{3 \xi^2 + 2A_2 \xi + A_1 + (2C_2 \xi + C_1) + (2B_2 \xi + B_1) e^{-\xi \tau_1}}{\xi (B_2 \xi^2 + B_1 \xi + B_0) e^{-\xi \tau_1}}.
\]

Then at \( \tau_1 = \tau_{10} \) and \( \xi = i \omega_0 \), we can get

\[
[\text{Re} \left( \frac{d\xi}{d\tau_1} \right)_{\tau_1=\tau_{10}, \xi=i\omega_0}^{-1}] = \text{Re} \left[ \frac{3 (i \omega_0)^2 + (2A_2 + C_2)(i \omega_0) + A_1 + C_1}{(i \omega_0)(B_2(i \omega_0)^2 + B_1(i \omega_0) + B_0)(\cos(\omega_0 \tau_{10}) - i \sin(\omega_0 \tau_{10}))} \right] + \text{Re} \left[ \frac{2B_2(i \omega_0) + B_1}{(i \omega_0)(B_2(i \omega_0)^2 + B_1(i \omega_0) + B_0)} \right].
\]

Now

\[
[\text{Re} \left( \frac{d\xi}{d\tau_1} \right)_{\tau_1=\tau_{10}, \xi=i\omega_0}^{-1}] = \text{Re} \left[ \frac{\bar{M}_R + \bar{M}_I i}{\bar{N}_R + \bar{N}_I i} \right] + \text{Re} \left[ \frac{\bar{Q}_R + \bar{Q}_I i}{\bar{P}_R + \bar{P}_I i} \right] = \frac{\bar{M}_R \bar{N}_I - \bar{M}_I \bar{N}_R}{\bar{N}_R^2 + \bar{N}_I^2} \bar{P}_R + \bar{P}_I \bar{Q}_R + \bar{Q}_I \bar{P}_R - \bar{Q}_I \bar{P}_R + \bar{Q}_I \bar{P}_R.
\]

where

\[
\bar{M}_R = -3 \omega_0^2 + A_1 + C_1, \quad \bar{M}_I = 2(A_2 + C_2) \omega_0, \quad \bar{N}_R = (B_0 \omega_0 - B_2 \omega_0^3) \sin(\omega_0 \tau_{10}) - C_1 \omega_0^2 \cos(\omega_0 \tau_{10}),
\]

\[
\bar{N}_I = (B_0 \omega_0 - B_2 \omega_0^3) \cos(\omega_0 \tau_{10}) + B_1 \omega_0^2 \sin(\omega_0 \tau_{10}), \quad \bar{Q}_R = B_1, \quad \bar{Q}_I = 2B_2 \omega_0,
\]

\[
\bar{P}_R = -B_1 \omega_0^2, \quad \bar{P}_I = B_0 \omega_0 - B_2 \omega_0^3.
\]

Then

\[
[\text{Re} \left( \frac{d\xi}{d\tau_1} \right)_{\tau_1=\tau_{10}, \xi=i\omega_0}^{-1}] = \frac{A_s}{B_s} + \frac{C_s}{D_s} = \frac{A_s D_s + B_s C_s}{B_s D_s}, \quad (4.18)
\]
Here

\[ A_* = \hat{M}_R\hat{N}_R + \hat{M}_I\hat{N}_I, \quad B_* = \hat{N}_R^2 + \hat{N}_I^2, \]

\[ C_* = \hat{Q}_R\hat{P}_R + \hat{Q}_I\hat{P}_I, \quad D_* = \hat{P}_R^2 + \hat{P}_I^2. \]

From this, we can get

\[ [\text{Re} \left( \frac{d\xi}{d\tau} \right)]_{\tau = \tau_0, \xi = i\omega}^{-1} = \text{sgn}[A_*D_* + B_*C_*]. \]

If (H6): \( A_*D_* + B_*C_* \neq 0 \) holds, the transversal condition \( [\text{Re} \left( \frac{d\xi}{d\tau} \right)]_{\tau = \tau_0, \xi = i\omega}^{-1} \neq 0 \). From the above analysis, the following theorem can be drawn

**Theorem 4.2.** For \( \tau_2 = 0 \) and \( \tau_1 > 0 \), we have the following results:

(i) If (H4) holds, then the equilibrium \( E^* \) is asymptotically stable for all \( \tau_1 > 0 \).

(ii) If (H6) and (H5) hold, then the equilibrium \( E^* \) is locally asymptotically stable for all \( \tau_1 < \tau_{10} \) together with unstable for \( \tau_1 > \tau_{10} \) and undergoes Hopf bifurcation at \( \tau_1 = \tau_{10} \).

**Case (4):** \( \tau_1 \) is fixed in \((0, \tau_{10})\) and \( \tau_2 > 0 \).

We consider the gestation delay \( \tau_1 \) to be stable in the interval \((0, \tau_{10})\), taking \( \tau_2 \) as a control parameter. Let \( \xi = u + i\omega \) be the root of Eq. (4.4). Putting this value in Eq. (4.4), separating real and imaginary parts, we obtain

\[
\begin{align*}
A_2u^2 - A_0 &= (-B_2u^2 + B_0)\cos(\omega\tau_1) + (C_0 - C_2\omega^2)\cos(\omega\tau_2) + B_1\omega\sin(\omega\tau_1) + C_1\omega\sin(\omega\tau_2). \quad (4.21) \\
\omega^3 - A_1\omega &= -(B_0 - B_2\omega^2)\sin(\omega\tau_1) + B_1\omega\cos(\omega\tau_1) - (C_0 - C_2\omega^2)\sin(\omega\tau_2) + C_1\omega\cos(\omega\tau_2). \quad (4.22)
\end{align*}
\]

Squaring and adding Eqs. (4.21) and (4.22) to eliminate \( \tau_2 \), we get

\[ \omega^6 + \tilde{a}_4\omega^4 + \tilde{a}_3\omega^3 + \tilde{a}_2\omega^2 + \tilde{a}_0 = 0, \quad (4.23) \]

where

\[
\begin{align*}
\tilde{a}_4 &= -(B_2^2 + C_2^2 - A_2^2), \quad \tilde{a}_3 = -2(B_2C_1 - B_1C_2)\sin(\omega\tau_1 - \omega\tau_2), \\
\tilde{a}_2 &= -((B_1^2 - 2B_0B_2 + C_1^2 - 2C_0C_2) + 2(B_1C_1 - 2A_0A_2 - A_1^2 - B_2))\cos(\omega\tau_1 - \omega\tau_2), \\
\tilde{a}_0 &= -(B_0^2 + C_0^2 - A_0^2).
\end{align*}
\]

Noting that Eq. (4.23) is transcendental. Now, Eqs. (4.21) and (4.22) can be written as

\[
\delta_1\cos(\omega\tau_2) + \delta_2\sin(\omega\tau_2) = \delta_3 + \delta_4\cos(\omega\tau_1) + \delta_5\sin(\omega\tau_1), \quad (4.24)
\]

\[
\delta_1 = \tilde{a}_4, \quad \delta_2 = \tilde{a}_3, \quad \delta_3 = \tilde{a}_2, \quad \delta_4 = \tilde{a}_1, \quad \delta_5 = \tilde{a}_0.
\]
\[-\delta_2 \cos(\omega \tau_2) + \delta_1 \sin(\omega \tau_2) = \delta_6 - \delta_5 \cos(\omega \tau_1) + \delta_4 \sin(\omega \tau_1), \quad (4.25)\]

where

\[
\delta_1 = C_2 \omega^2 - C_0, \quad \delta_2 = -C_1 \omega, \\
\delta_3 = A_0 - A_2 \omega^2, \quad \delta_4 = B_0 - B_2 \omega^2, \\
\delta_5 = B_1 \omega, \quad \delta_6 = \omega^3 - A_1 \omega.
\]

Without losing generality, the Eq.(4.23) has finite positive roots $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_k$, for every fixed $\bar{\omega}$, there exists a sequence \{\(\tau_{2i}\)|j = 0, 1, 2\ldots\}, where

\[
\tau_{2i}^{(j)} = \frac{1}{\bar{\omega}} \tan^{-1} \left( \frac{(\delta_1 \delta_4 + \delta_2 \delta_5) \sin(\bar{\omega} \tau_1) - (\delta_1 \delta_5 - \delta_2 \delta_4) \cos(\bar{\omega} \tau_1)}{(\delta_1 \delta_5 - \delta_2 \delta_4) \sin(\bar{\omega} \tau_1) + (\delta_2 \delta_5 + \delta_1 \delta_4) \cos(\bar{\omega} \tau_1) + \frac{k\pi}{\bar{\omega}}} \right)
\]

\[j = 0, 1, 2 \ldots\]

Let $\tilde{\tau}_2 = min\{\tau_{2i}^{(j)}|i = 0, 1, 2, \ldots, k, j = 0, 1, 2\ldots\}$, when $\tau_2 = \tilde{\tau}_2, \bar{\omega} = \bar{\omega}_{i=\tilde{\tau}_2, i=1, 2, 3, \ldots}$, the characteristic equation (4.4) has purely imaginary roots $\pm i\bar{\omega}$. Then, we will verify the transversality condition, differentiating the characteristic equation (4.4) with respect to $\tau_2$, we can obtain

\[
\begin{align*}
[\text{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tilde{\tau}_2, \xi=i\bar{\omega}}]^{-1} & = \text{Re}\left[\frac{3(i\bar{\omega})^2 + 2A_2(i\bar{\omega}) + A_1}{(i\bar{\omega})(C_2(i\bar{\omega})^2 + C_1(i\bar{\omega}) + C_0)(\cos(\bar{\omega} \tau_2) - i \sin(\bar{\omega} \tau_2))}\right] \\
& + \text{Re}\left[\frac{2C_2(i\bar{\omega}) + C_1}{(i\bar{\omega})(C_2(i\bar{\omega})^2 + C_1(i\bar{\omega}) + C_0)}\right].
\end{align*}
\]

Now

\[
[\text{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tilde{\tau}_2, \xi=i\bar{\omega}}]^{-1} = \text{Re}\left[\frac{M_R + M_I i}{N_R + N_I i}\right] + \text{Re}\left[\frac{Q_R + Q_I i}{P_R + P_I i}\right] = \frac{M_R N_R + M_I N_I}{N_R^2 + N_I^2} + \frac{Q_R P_R + Q_I P_I}{P_R^2 + P_I^2},
\]

where

\[
M_R = -3\bar{\omega}^2 + A_1, \quad M_I = 2A_2 \bar{\omega}, \quad N_R = (C_0 \bar{\omega} - C_1 \bar{\omega}^2 - C_2 \bar{\omega}^3) \sin(\bar{\omega} \tau_2), \\
N_I = (C_0 \bar{\omega} - C_2 \bar{\omega}^3) \cos(\bar{\omega} \tau_2) + C_1 \bar{\omega}^2 \sin(\bar{\omega} \tau_2), \quad Q_R = C_1, \quad Q_I = 2C_2 \bar{\omega}, \\
P_R = -C_1 \bar{\omega}^2, \quad P_I = C_0 \bar{\omega} - C_2 \bar{\omega}^3.
\]

Then

\[
[\text{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tilde{\tau}_2, \xi=i\bar{\omega}}]^{-1} = \frac{E}{F} + \frac{G}{H} = \frac{EH + FG}{FH}, \quad (4.27)
\]

here

\[
\begin{align*}
E &= M_R N_R + M_I N_I, \quad F = N_R^2 + N_I^2, \\
G &= Q_R P_R + Q_I P_I, \quad H = P_R^2 + P_I^2.
\end{align*}
\]
From this we can get
\[
\text{sgn}[\text{Re}(\frac{d\xi}{dt})_{\tau_2 = \tilde{\tau}_2, \xi = i\omega}]^{-1} = \text{sgn}[EH + FG].
\]

If (H7): \( EH + FG \neq 0 \) holds, the transversal condition \( \text{sgn}[\text{Re}(\frac{d\xi}{dt})_{\tau_2 = \tilde{\tau}_2, \xi = i\omega}]^{-1} \neq 0 \). From the above analysis, we have the following theorem.

**Theorem 4.3.** For system (2.2), assume (H7) holds with \( \tau_1 \) is fixed in \((0, \tau_{10})\) and \( \tau_2 > 0 \), then the equilibrium \( E^* \) is locally asymptotically stable for \( \tau_2 \in (0, \tilde{\tau}_2) \) whereas system (2.2) undergoes Hopf bifurcation at \( \tau_2 = \tilde{\tau}_2 \).

**Case(5):** \( \tau_2 \) is fixed in \((0, \tau_{20})\) and \( \tau_1 > 0 \), so take \( \tau_1 \) as a control parameter; the analysis is the same as case (4), so we omit it.

**V. Optimal Tax Policy**

From previous studies, overfishing may lead to the extinction of populations. However, in the society, the adequate protection of the ecosystem is a common problem we need to face. In the face of the increasingly severe harmful effects of overfishing on ecosystems, people began to find the most suitable methods for fishery control in various areas of sustainable development policies, for example, seasonal fishing, property leasing, taxation, licensing fees, etc. Taxes are generally considered to be better than other regulatory approaches, so that we will view the optimal tax policy for the double phytoplankton - single zooplankton system based on model (2.3). Here, we take \( E \) as a time-dependent dynamic variable controlled by equations. Therefore, there is the following equation.

\[
E(t) = \varepsilon Q(t), \quad 0 \leq \varepsilon \leq 1, \quad \frac{dQ}{dt} = I(t) - \gamma Q(t), \quad Q(0) = Q_0. \tag{5.1}
\]

Where \( Q(t) \) is the amount of capital invested in fisheries at time \( t \), \( I(t) \) is the total investment rate (in physical form) at time \( t \) and \( \gamma \) is the constant depreciation rate of capital. Suppose that the effort \( E \) at any time is proportional to the instantaneous amount of investment capital. For example, if \( Q(t) \) represents the number of standard fishing vessels that can be used, it is reasonable to assume that \( Q(t) \) and \( E \) should be proportional. When \( \varepsilon = 1 \), it can be considered that the maximum fishing capacity \( E \) is equal to the number of available vessels at time \( t \) \( Q(t) \). When \( \varepsilon = 0 \), it means that even though there may be fishing boats, the fishing is not expanded; it also reflects the over-exploitation of fisheries. At this time the fish population has been seriously depleted, so fishing vessels can no longer be used. These are simulated capital levels may be adjusted, thus prove the reasonableness of the equation (5.2). Regulators control the development of fisheries by imposing a tax \((v > 0)\) on the unit biomass of terrestrial fish. When \( (v < 0) \) can be understood as any subsidy to fishermen. Net income of fishermen (‘Net income’ for short) is \( E((u_1 - v)q_1 N + (u_2 - v)q_2 Z - C) \), where \( u_i, \quad i = 1, 2 \) is the constant price of unit biomass of nontoxic phytoplankton and zooplankton, respectively. \( C \) is the fixed cost per unit of harvesting effort.

We assume the gross profit margin on capital investment is proportional to this ‘Net income.’ So, we have

\[
I = E\varphi[(u_1 - v)q_1 N + (u_2 - v)q_2 Z - C], \quad 0 \leq \varphi < 1. \tag{5.2}
\]

For \( \varphi = 1 \), Eq. (5.2) shows that the highest investment rate at any time is equal to the net income of the fishermen at that time. \( \varphi = 0 \) can only be used when the net income of fishermen is negative; that is, current capital assets cannot be divested. If the fishery is operating at a loss and allows capital to be withdrawn, the only owner of the fishery will benefit by allowing the capital assets to be continuously withdrawn, because negative investment means withdrawal of investment, so it is the case of \( I < 0, \varphi > 0 \). By combining Eqs. (5.1) and (5.2), we can get

\[
\frac{dE}{dt} = E\varphi[(u_1 - v)q_1 N + (u_2 - v)q_2 Z - C] - \gamma. \tag{5.3}
\]
Fishermen and regulators are two different parts of society. Therefore, the income they receive is society’s income accumulated through fisheries. The net economic income to society is

\[ ME = E[(u_1 - v)q_1N + (u_2 - v)q_2Z - C] + E[v(q_1N) + v(q_2N)], \]

this is equal to the net economic income of fishermen plus the economic income of regulators. Therefore without considering the time delay, Eq.(2.3) can be rewritten as

\[
\begin{align*}
\frac{dN}{dt} &= r_1N \left(1 - \frac{N + \alpha_1 T}{k_1}\right) - \frac{w_1NZ}{p_1 + N} - q_1EN, \\
\frac{dT}{dt} &= r_2N \left(1 - \frac{T + \alpha_2 N}{k_2}\right) - \frac{w_2TZ}{p_2 + T + \beta N}, \\
\frac{dZ}{dt} &= \frac{c_1w_1NZ}{p_1 + N} - \frac{c_2w_2TZ}{p_2 + T + \beta N} - dZ - q_2EZ, \\
\frac{dE}{dt} &= \mathcal{E}[\varepsilon\varphi((u_1 - v)q_1N + (u_2 - v)q_2Z - C) - \gamma].
\end{align*}
\]  

(5.4)

Next, we will use the principle of Pontryagin’s maximum to get the path of the best tax policy. If the fish population stays along this path, then regulators can ensure that their goals are achieved. The goal of regulatory agencies is to maximize the total net income of society as a result of harvesting activities. Specifically, the goal is to maximize revenue over a continuous time stream (J).

\[ J = \int_0^{+\infty} E(t)e^{-\delta t}[u_1q_1N + u_2q_2Z - C]dt, \]  

(5.5)

where \( \delta \) is the discounting factor. Therefore, our goal is to determine an optimal tax \( v = v(t) \) that maximizes compliance with Eq.(5.4) and constrains \( v_{\text{min}} \leq v(t) \leq v_{\text{max}} \) on the control variable \( v(t) \). When \( v_{\text{min}} < 0 \), it will have the effect of accelerating the rate of fishery expansion. The Hamiltonian of the problem is obtained by

\[ H = (u_1q_1N + u_2q_2Z - C)Ee^{-\delta t} + \lambda_1N[r_1(1 - \frac{N + \alpha_1 T}{k_1}) - \frac{w_1Z}{p_1 + N} - q_1E] \]

\[ + \lambda_2[r_2T(1 - \frac{T + \alpha_2 N}{k_2}) - \frac{w_2TZ}{p_2 + T + \beta N}] + \lambda_3[c_1w_1NZ + \frac{c_2w_2TZ}{p_2 + T + \beta N} - dZ - q_2EZ] \]

\[ + \lambda_4E[\varepsilon\varphi((u_1 - v)q_1N + (u_2 - v)q_2Z - C) - \gamma], \]  

(5.6)

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are the adjoint variables. For \( v \in [v_{\text{min}}, v_{\text{max}}] \), the Hamiltonian must be maximized. Assuming that the control constraint is not bound, that is, the optimal solution does not appear as \( v = v_{\text{min}} \) or \( v = v_{\text{max}} \). We can get by singular control [9]

\[ \frac{\partial H}{\partial v} = -\lambda_4E\varepsilon\varphi(q_1N + q_2Z) = 0 \Rightarrow \lambda_4 = 0. \]  

(5.7)

Now, the adjoint equations are

\[ \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N} = -[u_1q_1Ee^{-\delta t} + \lambda_1(r_1 - \frac{2r_1N + r_1\alpha_1 T}{k_1}) - \frac{w_1Z}{(p_1 + N)^2} - q_1E] + \lambda_2[\frac{w_2TZ}{(p_2 + T + \beta N)^2} - \frac{r_2\alpha_2 T}{k_2}] \]

\[ + \lambda_3 \left( \frac{c_1w_1NZ}{(p_1 + N)^2} + \frac{c_2w_2TZ}{(p_2 + T + \beta N)^2} \right), \]
\[
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial T} = -[\lambda_1 \left( \frac{r_1 \alpha_1 N}{k_1} \right) + \lambda_2 \left( 1 - \frac{2T + \alpha_2 N}{k_2} \right) - \frac{w_2Z(p_2 + \beta N)}{(p_2 + T + \beta N)^2}] - \lambda_3 \left( \frac{c_2w_2Z(p_2 + \beta N)}{(p_2 + T + \beta N)^2} \right),
\]

\[
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial Z} = -u_2q_2e^{-\delta t} - \lambda_1 \left( \frac{w_1 N}{p_1 + N} \right) - \lambda_2 \left( \frac{w_2T}{p_2 + T + \beta N} \right) + \lambda_3 \left( \frac{c_1w_1 N}{p_1 + N} - \frac{c_2w_2T}{p_2 + T + \beta N} - d - q_2E \right),
\]

\[
\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial E} = -[(u_1q_1 N + u_2q_2 Z - C)e^{-\delta t} - \lambda_1 q_1 N - \lambda_3 q_2 Z].
\]

Now start with Eqs. (5.8) and (5.7), using the equilibrium equation we have

\[
\frac{d\lambda_1}{dt} = -u_1q_1 e^{-\delta t} - \lambda_1 \left[ \frac{r_1 N}{k_1} + \frac{w_1 N Z}{(p_1 + N)^2} \right] - \lambda_2 \left[ \frac{w_2 \beta TZ}{(p_2 + T + \beta N)^2} - \frac{r_2 \alpha_2 T}{k_2} \right] - \lambda_3 \left[ \frac{c_1 w_1 p_1 Z}{(p_1 + N)^2} + \frac{c_2 \beta w_2 T Z}{(p_2 + T + \beta N)^2} \right],
\]

\[
\frac{d\lambda_2}{dt} = -\lambda_1 \left[ \frac{r_1 \alpha_1 N}{k_1} \right] - \lambda_2 \left[ \frac{w_2 T Z}{(p_2 + T + \beta N)^2} \right] - \lambda_3 \left[ \frac{c_2 \beta w_2 Z(p_2 + \beta N)}{(p_2 + T + \beta N)^2} \right],
\]

\[
\frac{d\lambda_3}{dt} = -u_2q_2 e^{-\delta t} + \lambda_1 \left( \frac{w_1 N}{p_1 + N} \right) + \lambda_2 \left( \frac{w_2 T}{p_2 + T + \beta N} \right).
\]

Using the second and third equations of Equation (5.9) from the fourth equation of Equation (5.8), we can obtain \( \frac{d\lambda_1}{dt} = M_1 e^{-\delta t} + M_2 \lambda_1 + M_3 \lambda_2 \), where

\[
M_1 = \frac{(C - u_1 q_1 N) \delta + u_2 q_2 Z(q_2 E - \delta)}{q_1 N}, \quad M_2 = -\frac{w_1 q_2 N Z}{(p_1 + N) q_1 N}, \quad M_3 = -\frac{w_2 q_2 T Z}{(p_2 + T + \beta N) q_1 N}.
\]

The solution of this linear equation is

\[
\lambda_1 = N_0 e^{-M_2 t} - \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 \lambda_2}{M_2}.
\]

Using the same method as above, we can get

\[
\lambda_3 = I_0 e^{H_2 t} - \frac{H_1 e^{-\delta t}}{H_2 + \delta},
\]

where

\[
H_1 = \left[ \frac{(C - u_2 q_2 Z) \delta - q_1 N (u_1 \delta + M_1)}{q_2 Z} \right] + \frac{M_1 M_2 q_1 N}{(M_2 + \delta) q_2 Z}, \quad H_2 = \frac{M_2 M_3 q_1 N}{q_2 M_2 Z}.
\]

Identically

\[
\frac{d\lambda_2}{dt} = R_1 e^{-\delta t} + R_2 \lambda_2,
\]

where

\[
R_1 = \frac{M_1}{M_2 + \delta} + \frac{H_1}{H_2 + \delta} \left( \frac{c_2 w_2 Z(p_2 + \beta N)}{(p_2 + T + \beta N)^2} \right), \quad R_2 = \frac{M_3}{M_2} \left( \frac{r_2 \alpha_1 N}{k_1} \right) - \frac{w_2 T Z}{(p_2 + T + \beta N)^2}.
\]
So we can get $\lambda_1$

$$\lambda_1 = N_0 e^{M_2 t} - \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (W_0 e^{R_2 t} - R_1 e^{-\delta t})}{M_2}.$$  

The shadow price $\lambda_1 e^{-\delta t}$ is bounded as $t \to \infty$, $N_0 = 0$ and $W_0 = 0$, then we can obtain

$$\lambda_1 = -\frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (e^{R_2 t} - R_1 e^{-\delta t})}{R_2 + \delta}.$$  

(5.13)

Now use Eqs.(5.11), (5.12) and (5.13) in the first of Eq.(5.9), we have

$$[\frac{(C - u_1 q_1 N^*) \delta + u_2 q_2 Z^* (q_2 E^* - \delta)}{q_1 N^*}] e^{-\delta t} + \frac{w_2 q_2 N^* Z^*}{(p_1 + N^*) q_1 N^*} \left[ \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (e^{R_2 t} - R_1 e^{-\delta t})}{R_2 + \delta} \right]$$

$$+ \left[ \frac{w_2 q_2 T^* Z^*}{(p_2 + T^* + \beta N^*) q_1 N^*} \right] \left[ \frac{R_1 e^{-\delta t}}{R_2 + \delta} \right] + u_1 q_1 E^* e^{-\delta t} + \left[ \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (e^{R_2 t} - R_1 e^{-\delta t})}{R_2 + \delta} \right] \left[ \frac{r_1 N^*}{k_1} + \frac{w_1 N^* Z^*}{(p_1 + N^*)^2} \right].$$

(5.14)

Because of the computational complexity, our optimal equilibrium solution can be expressed as

$$T^* = \left[ \frac{(c_1 w_1 - \delta) N^* - \delta p_1 (p_2 + \beta N^*)}{(c_2 w_2 - \delta) p_1 + (c_2 w_2 - c_1 w_2 - \delta) N^*} \right].$$

$$Z^* = \frac{r_1 (p_1 + N^*)}{w_1 k_1} (k_1 - N^* - \alpha_1 T^*).$$  

(5.15)

$N^*$ available from the following equation

$$r_2 (k_2 - T^* - \alpha_2 N^*) (p_2 + T^* + \beta N^*) - w_2 k_2 Z^* = 0.$$  

(5.16)

$E^*$ available from the following equation

$$r_1 \left( 1 - \frac{N^* + \alpha_1 T^*}{k_1} \right) - \frac{w_1 Z^*}{q_1 (p_1 + N^*)} = \frac{c_1 w_1 N^*}{q_2 (p_1 + N^*)} - \frac{c_2 w_2 T^*}{q_2 (p_2 + T^* + \beta N^*)} - \frac{d}{q_2}.$$  

(5.17)

From the complex calculation results, it can be seen that $T^*$ and $Z^*$ are functions of $v$. Therefore, we can express this function as follows

$$[\frac{(C - u_1 q_1 N^*) \delta + u_2 q_2 Z^* (q_2 E^* - \delta)}{q_1 N^*}] e^{-\delta t} + \frac{w_2 q_2 N^* Z^*}{(p_1 + N^*) q_1 N^*} \left[ \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (e^{R_2 t} - R_1 e^{-\delta t})}{R_2 + \delta} \right]$$

$$+ \left[ \frac{w_2 q_2 T^* Z^*}{(p_2 + T^* + \beta N^*) q_1 N^*} \right] \left[ \frac{R_1 e^{-\delta t}}{R_2 + \delta} \right] + u_1 q_1 E^* e^{-\delta t} + \left[ \frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3 (e^{R_2 t} - R_1 e^{-\delta t})}{R_2 + \delta} \right] \left[ \frac{r_1 N^*}{k_1} + \frac{w_1 N^* Z^*}{(p_1 + N^*)^2} \right].$$

(5.18)

If $v^*$ exists, let $v = v^*$ be the solution of $f(v)$. Using the value of $v^*$, we can get the optimal solution $(N(v^*), T(v^*), Z(v^*), E(v^*))$. Here, we establish the existence of an optimal equilibrium solution satisfying the necessary condition of the maximum principle. As Clark [23] pointed out, it is complicated to find the optimal path composed of explosive control and unbalanced singular control. Because the current model is much more complex than Clark’s model, we only consider an optimal equilibrium solution. If we can begin to
From the above analysis, we can easily observe the following points:

(i) From Eqs. (5.7) and (5.11)-(5.13), we note that \( \lambda_i e^{-\delta t} \), \( i = 1, 2, 3, 4 \), where \( \lambda_i \) is an adjoint variable, which remains unchanged in an optimal balance time interval, therefore, they satisfy the transversal condition, that is, they remain bounded to \( t \to \infty \).

(ii) Considering the coexistence equilibrium point \( F_c = (N_0, T_b, Z_b, E_b) \), The fourth equation of Eq. (5.8) can be written as

\[
(\lambda_1 q_1 N_b + \lambda_2 q_2 Z_b) = (u_1 q_1 N_b + u_2 q_2 Z_b - C)e^{-\delta t}.
\]

This means that the total harvest cost per unit of user’s effort is equal to the discount value of the future price under the steady state effort level.

(iii) From Eqs. (5.11) and (5.13), we can obtain

\[
u_1 q_1 N_b + u_2 q_2 Z_b - C = \left[-\left(\frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3}{M_2} \left(\frac{R_1 e^{-\delta t}}{R_2 + \delta}\right)\right)q_1 N + \left(\frac{H_1 e^{-\delta t}}{H_2 + \delta}\right)q_2 Z\right] \to 0, \quad \text{as} \ \delta \to \infty.
\]
The optimal solution of (5.5) for \( v = 0.867 \).

This shows that the unlimited discount rate leads to the complete dissipation of the net economic income to the society, \( (u_1q_1N_0 + u_2q_2Z_0 - C)E = 0 \). We also observe that for a zero discount rate, the present value of the continuous time flow reaches its maximum.

Due to the complexity of its calculation and to explain our optimal tax policy more intuitively, we continue to study it through numerical simulation. If \( r_1 = 6, r_2 = 5, \alpha_1 = 0.2, \alpha_2 = 0.2, k_1 = 100, k_2 = 190, w_1 = 0.3, w_2 = 0.3, p_1 = 2, p_2 = 2, d = 0.2, c_1 = 0.45, c_2 = 0.45, \beta = 0.3, q_1 = 0.2, q_2 = 0.2, \varepsilon = 0.2, \varphi = 0.5, \gamma = 0.2, C = 2, u_1 = 0.1, u_2 = 0.2, \) and the discounting factor \( \delta = 0.045 \) in appropriate units, based on the selection of the above parameter values, we can get the optimal tax is \( v = 0.867 \). In Fig.2, we get the optimal solution. Therefore, we have used the principle of Pontryagin’s maximum to obtain the optimal path of the optimal tax policy, which not only ensures the maximum goal of the regulatory authority, but also the stability of the ecosystem.

VI. Numerical Simulations

In this section, we will use Matlab to numerically simulate the impact of various parameters on species and the stability of steady state. Therefore, the initial conditions and parameter settings in Table 2 are used for the numerical analysis of the system (2.3). First, we give the time series diagram of \( N, T \) and \( Z \) with short period and long period, then the impact of different \( \beta, q_1E \) and \( q_2E \) on the survival of species were investigated. Lastly, we study the changes in equilibrium stability with varying delays of time.

| Table 2: All the biological descriptions of the parameters are given below: |
| Parameter | Environmental Interpretation | Value |
| \((N_0, T_0, Z_0)\) | Initial concentrations | \((500,200,1000)\) |
| \(r_1\) | Intrinsic growth rate of NTP | 0.56 |
| \(r_2\) | Intrinsic growth rate of TPP | 0.49 |
| \(\alpha_1\) | Competitive effect of TPP on NTP | 0.1 |
| \(\alpha_2\) | Competitive effect of NTP on TPP | 0.1 |
| \(k_1\) | Carrying capacity of NTP | 5600 |
| \(k_2\) | Carrying capacity of TPP | 4900 |
| \(w_1\) | NTP consumption rate | 0.5 |
| \(w_2\) | TPP consumption rate | 0.5 |
| \(p_1\) | Half saturation constants for NTP | 30 |
| \(p_2\) | Half saturation constants for TPP | 30 |
| \(c_1\) | the conversion rate of \( N \) to \( Z \) | 0.45 |
| \(c_2\) | the conversion rate of \( T \) to \( Z \) | 0.45 |
| \(\beta\) | Intensity of avoidance | - |
| \(d\) | Zooplankton mortality rate | 0.05 |
a) Time series analysis

In Fig.3, we plot the time series of $\beta = 0, \beta = 10, \beta = 1000$ in the first ten days, where the other parameter values and initial conditions are the same as in Table 2. When $q_1 = q_2 = 0$ and $\beta = 0$, we can observe that NTP and TPP tend to perish at a fast linear speed. It is obvious that when $\beta$ increases to 10, the concentrate of TPP will first increase to a certain concentration, then decrease and finally tend to extinction, while at this time, NTP still maintains a rapid decline rate until it is extinct(fig.3(a)(b)). On the contrary, when $\beta = 0$, we take $q_1 = 0.4, q_2 = 1.2$, and $q_1 = 2, q_2 = 2.5$, respectively. We can observe that with the increase of $q_1$ and $q_2$, NTP and zooplankton tend to become extinct at a faster rate of decline, while TPP increases more rapidly(fig.3(c)(d)). Based on the values of $q_1$ and $q_2$ of (fig.3(c)(d)), we increase $\beta$ to 10. Through comparison, we can find that the curves of NTP and zooplankton have almost no change, but the increasing speed of TPP is still accelerated(fig.3(e)(f)). To further explore the influence of $\beta$, we fixed $q_1$ and $q_2$ as 2 and 2.5, respectively. And (Fig.6(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\beta = 0$. It can be seen that when $\beta$ increases, the system undergoes Hopf bifurcation around the positive equilibrium $E^*$ (Fig.5(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\beta = 0$. In this case, the delay system (2.2) has a periodic solution near the positive equilibrium $E^*$. And (Fig.6(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0, \tau_2 = 1.08$. In this case, the delay system (2.2) has a periodic solution near the positive equilibrium point ($E^*$).

b) Double time delay analysis

Now, to explore the influence of pregnancy delay ($\tau_1$) and toxin onset delay($\tau_2$) on the stability of equilibrium point in different cases. First, we need to set a set of parameters as follows

$$r_1 = 2, r_2 = 3, \alpha_1 = 0.3, \alpha_2 = 0.1, k_1 = 2500, k_2 = 3000, w_1 = w_2 = 0.5, p_1 = p_2 = 50,$$
$$c_1 = c_2 = 0.45, d = 0.05, \beta = 0.5, q_1 = 0.2, q_2 = 0.3, E = 1.$$ (6.1)

With initial values $(N_0, T_0, Z_0) = (400, 300, 500)$, we perform numerical simulations to verify the theoretical results of the previous delayed system (2.2). For these parameters, we take (6.1) into the delayed system (2.2), the complex dynamical behavior of the system has been observed with time delay.

**Case i**: when $\tau_1 = 0, \tau_2 > 0$, in this case, $\text{Re}(\frac{dK}{d\tau_2})_{\tau_2 = \tau_2, \xi = i\omega}^{-1} > 0$, the transversality condition is contented. So when $\tau_2 > \tau_{20}$ (Fig.5(a)(b)), the positive equilibrium $E^*$ is locally asymptotically stable, the positive equilibrium $E^*$ is unstable when $\tau_2 > \tau_{20}$ (Fig.6(a)(b)), when $\tau_2 = \tau_{20}$, the system undergoes Hopf bifurcation around the positive equilibrium $E^*$. (Fig.5(a)(b)) shows the trajectories and phase portrait of system (2.2) for $\tau_1 = 0, \tau_2 = 1$. It can be clearly seen that the system (2.2) will converge to the positive equilibrium point $E^*$. And (Fig.6(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0, \tau_2 = 1.08$. In this case, the delay system (2.2) has a periodic solution near the positive equilibrium point ($E^*$).

**Case ii**: when $\tau_1 > 0, \tau_2 = 0$, we change the values of $k_1$ and $k_2$ in (6.1) to $k_1 = 150, k_2 = 250$, and the others remain unchanged. $\text{Re}(\frac{dK}{d\tau_2})_{\tau_1 = \tau_{10}, \xi = i\omega}^{-1} > 0$, the transversality condition is satisfied. (Fig.7(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0.7, \tau_2 = 0$. It can be seen that although the final equilibrium point tends to be stable, there is no oscillation, indicating that there is no periodic solution in this case.

**Case iii**: when $\tau_1 = 0.9$ in stable interval $(0, \tau_{10})$, and take $\tau_2 > 0$ as the parameter, $\text{Re}(\frac{dK}{d\tau_2})_{\tau_2 = \tau_{2}, \xi = i\omega}^{-1} \neq 0$, the transversality condition is satisfied. So when $\tau_2 < \overline{\tau_2}$
(Fig. 8(a)(b)), the positive equilibrium \( E^* \) is locally asymptotically stable, the positive equilibrium \( E^* \) is unstable when \( \tau_2 > \tilde{\tau}_2 \) (Fig. 9(a)(b)), when \( \tau_2 = \tilde{\tau}_2 \), the system undergoes Hopf bifurcation around the positive equilibrium \( E^* \). (Fig. 8(a)(b)) shows the trajectories and phase portrait of the system (2.2) for \( \tau_1 = 0.9, \tau_2 = 1.06 \). It can be clearly seen that the system (2.2) will converge to the positive equilibrium point \( E^* \). And (Fig. 9(a)(b)) shows the trajectories and phase portrait of the system (2.2) for \( \tau_1 = 0.9, \tau_2 = 1.09 \); we find the delayed system (2.2) has periodic solutions near the positive equilibrium point \( E^* \) in this case.

Therefore, through the above numerical simulation, we can evidently find the system is stable for small values of the delay, but as the value of delay crosses its critical value, the system loses its stability and undergoes Hopf bifurcation. Thus the limit cycle exists for \( \tau_1 > \tau_{10}, \tau_2 > \tau_{20} \) and \( \tau_2 > \tilde{\tau}_2 \).

**VII. Discussion**
The dynamic changes of the system (1) with different $\beta$, $q_1$, and $q_2$ in the first 10 days, other parameter values and initial conditions are the same as Table 2. (a)(b): In the case of $q_1 = q_2 = 0$, $\beta = 0$ and $\beta = 10$, the TPP concentration will fluctuate and the NTP concentration will barely change. (c)(d): For $\beta = 0$, the concentrations of $q_1$ and $q_2$ increase, and both NTP and TPP concentrations accelerate towards extinction. (e)(f): Based on (c)(d), for $\beta = 10$, TPP reached a higher flowering concentration, while NTP still maintained a lower concentration. (g)(h): Based on (f), for $\beta = 1000$, NTP and TPP concentrations are almost unchanged. (i)(j): for $\beta = 10$, we increase the concentrations of $q_1$ and $q_2$ to 6 and 8, respectively. NTP and zooplankton accelerate the decline rate, while TPP has no obvious change.

Fig. 3: The dynamic changes of the system (1) with different $\beta$, $q_1$, and $q_2$ in the first 10 days, other parameter values and initial conditions are the same as Table 2. (a)(b): In the case of $q_1 = q_2 = 0$, $\beta = 0$ and $\beta = 10$, the TPP concentration will fluctuate and the NTP concentration will barely change. (c)(d): For $\beta = 0$, the concentrations of $q_1$ and $q_2$ increase, and both NTP and TPP concentrations accelerate towards extinction. (e)(f): Based on (c)(d), for $\beta = 10$, TPP reached a higher flowering concentration, while NTP still maintained a lower concentration. (g)(h): Based on (f), for $\beta = 1000$, NTP and TPP concentrations are almost unchanged. (i)(j): for $\beta = 10$, we increase the concentrations of $q_1$ and $q_2$ to 6 and 8, respectively. NTP and zooplankton accelerate the decline rate, while TPP has no obvious change.
Fig. 4: The long-term dynamics of the system (2.1), all other parameter values are the same as Table 2, (a): When \(q_1 = q_2 = 0\), NTP and zooplankton with initial concentrations (500,200,1000) oscillate and TPP populations become extinct. (b): For \(\beta = 10\), all populations survive and the system stabilizes to a limit cycle. (c)(d): For \(\beta = 0\), \(0 \leq q_1=q_2 \leq 0.36\), NTP and TPP can coexist. (e)(f): when we fix \(\beta = 10\) and increase \(q_1 = q_2 = 0.36\) to \(q_1 = q_2 = 0.37\), we will find that the coexistence of NTP and TPP disappears, and then only TPP exists and tends to be stable, while NTP and zooplankton tend to be extinct.
The behavior of the system (2.2) for $\tau_1 = 0, \tau_2 = 1$ with other parameters chosen in (6.1).

Fig. 5: The behavior of the system (2.2) for $\tau_1 = 0, \tau_2 = 1$ with other parameters chosen in (6.1).

The behavior of the system (2.2) for $\tau_1 = 0.7, \tau_2 = 0$ with other parameters chosen in (6.1).

Fig. 6: The behavior of the system (2.2) for $\tau_1 = 0.7, \tau_2 = 0$ with other parameters chosen in (6.1).

Fig. 7: The behavior of the system (2.2) for $\tau_1 = 0.7, \tau_2 = 0$ with other parameters chosen in (6.1).
The behavior of the system (2.2) for $\tau_1 = 0.9, \tau_2 = 1.06$ with other parameters chosen in (6.1).

The predator avoidance effect always attracts ecologists to investigate it. In the aquatic system, zooplankton lives in the environment full of toxic and non-toxic bait (phytoplankton). To make toxic phytoplankton, nontoxic phytoplankton and zooplankton coexist, the avoidance behavior of zooplankton against toxic phytoplankton is an important research topic. In this paper, we consider a biological model with two delays in which zooplankton avoids poisonous phytoplankton in the presence of nontoxic phytoplankton. For this model of poisonous avoidance, due to the avoidance coefficient of zooplankton to toxic phytoplankton, the growth density of zooplankton and toxic phytoplankton is nonlinear. When the poisonous avoidance coefficient is high, the density of poisonous phytoplankton will increase in proportion, and finally tend to be stable. We also consider the impact of human harvest on the coexistence of these three species, the form of avoidance and human harvest have biological significance, which we also analyzed.

According to this article, we analyze the positive and boundedness of the system solution without time delay at first. In the bounded area, the densities of nontoxic phytoplankton (NTP), toxic phytoplankton (TPP) and zooplankton (zooplankton) are all non-negative. Then we analyze the bistability of the equilibrium points. From Fig.1, we can see the bistability of each equilibrium point in different $k_1$ ranges. For the dynamic behavior of double time-delay systems, we analyze the local stability and the existence of Hopf bifurcation. Taking the pregnancy delay $\tau_1$ and the toxin onset delay $\tau_2$ as the bifurcation parameters, the critical value of the time delay for the Hopf bifurcation of the system under different conditions is obtained. We find that the system is stable when the time delay is less than this critical value ($\tau_{10}', \tau_{20}', \tau_{10}^*$ and $\tau_{20}^*$, respectively), but...

**Fig. 8**: The behavior of the system (2.2) for $\tau_1 = 0.9, \tau_2 = 1.06$ with other parameters chosen in (6.1).

**Fig. 9**: The behavior of the system (2.2) for $\tau_1 = 0.9, \tau_2 = 1.09$ with other parameters chosen in (6.1).
when we increase the time delay to more than this critical value, the system will become unstable, and then Hopf bifurcation occurs at the critical time. Considering the practical significance of the research, in section 5, we use the principle of Pontryagin’s maximum to study the optimal tax policy of the system without time delay, we obtained the optimal path of the optimal tax policy. In addition, we use the parameters and initial values given in Table 2 and (6.1) to simulate several cases of double-delay systems in Matlab to verify all theoretical results.

**REFERENCES**


On Anisotropic Conservative Caginalp Phase-Field System based on Type III Heat Conduction with Two Temperatures and Periodic Boundary Conditions

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**Abstract**- Our aim in this paper is to study the well-posedness results of anisotropic conservative Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction and periodic boundary conditions. More precisely, we prove the existence and uniqueness of solutions.

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On Anisotropic Conservative Caginalp Phase-Field System based on Type III Heat Conduction with Two Temperatures and Periodic Boundary Conditions

Cyr Séraphin Ngamouyih Moussata, Armel Judice Ntsokongo & Dieudonné Ampini

Abstract: Our aim in this paper is to study the well-posedness results of anisotropic conservative Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction and periodic boundary conditions. More precisely, we prove the existence and uniqueness of solutions.

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1. Introduction

The authors studied in [13] (see again [12]) the following phase-field system, namely,

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + f(u) &= \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \\
\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha &= -\frac{\partial u}{\partial t}, \\
\alpha(t, x) &= \alpha(0, x) + \int_0^t T(\tau, x) d\tau,
\end{align*}
\]

where, \(u\) is the order parameter, \(T\) is the relative temperature (defined as \(T = \tilde{T} - T_E\), where \(\tilde{T}\) is the absolute temperature and \(T_E\) is the equilibrium melting temperature), \(\alpha\) is the conductive thermal displacement and \(f\) is the derivative of a double-well potential \(F\) (a typical choice is \(F(s) = \frac{1}{4}(s^2 - 1)^2\), hence the usual cubic nonlinear term \(f(s) = s^3 - s\)). Furthermore, here and below, we set all physical parameters equal to one. This system has been introduced to model phase transition phenomena, such as melting-solidification phenomena, and has been much studied from a mathematical point of view. We refer the reader to, e.g., [4–5, 8–11, 14, 16, 17, 21, 23].
This system is based on the (total Ginzburg-Landau) free energy,

$$\Psi_{GL} = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2 \right) dx,$$  \hspace{1cm} (1.4)  

where $\Omega$ is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 2$ or $n = 3$, with boundary $\Gamma$), and the enthalpy

$$H = u + T - \Delta T.$$  \hspace{1cm} (1.5)  

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = -D \frac{D\Psi_{GL}}{Du},$$  \hspace{1cm} (1.6)  

where $D\frac{D}{Du}$ denotes a variational derivative with respect to $u$. Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\text{div}q$$  \hspace{1cm} (1.7)  

and owing to (1.7),

$$\frac{\partial T}{\partial t} - \Delta \frac{\partial T}{\partial t} + \text{div}q = -\frac{\partial u}{\partial t},$$  \hspace{1cm} (1.8)  

where $q$ is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla\alpha - \nabla T,$$  \hspace{1cm} (1.9)  

we obtain (1.1) and (1.2).

Our aim in this paper is to study the model consisting of the conserved anisotropic to (1.1)–(1.2), namely,

$$\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right), \ a_i > 0,$$  \hspace{1cm} (1.10)  

and

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}.$$  \hspace{1cm} (1.11)  

Our aim in this paper is to study the model consisting of the anisotropic conservative equation (1.10) and the temperature equation (1.11). In particular, we obtain the existence and uniqueness of solutions.
II. Setting of the Problem

Find the order parameter \( u : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) and the thermal displacement \( \alpha : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) such that:

\[
\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right), \quad (2.1)
\]

\[
\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \quad (2.2)
\]

together with periodic boundary conditions

\[
u \text{ and } \alpha \text{ are } \Omega \text{ - periodic,} \quad (2.3)
\]

and the initial conditions

\[
u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \quad (2.4)
\]

We assume that

\[
a_i > 0, \quad i \in \{1, 2, 3\}, \quad (2.5)
\]

and we introduce the elliptic operator \( A \) defined by

\[
\langle Av, w \rangle_{H^{-1}(\Omega), H^1_{\text{per}}(\Omega)} = \sum_{i=1}^{3} a_i \left( \frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right), \quad (2.6)
\]

where \( H^{-1}(\Omega) \) is the topological dual of \( H^1_{\text{per}}(\Omega) \). Furthermore, \( \langle \cdot, \cdot \rangle \) denotes the usual \( L^2 \)-scalar product, with associated norm \( \| \cdot \| \); more generally, we denote by \( \| \cdot \|_X \) the norm on the Banach space \( X \) and we set \( \| \cdot \|_{-1} = \| (-\Delta)^{-\frac{1}{2}} \cdot \| \), \( (-\Delta)^{-1} \) denoting the inverse minus Laplace operator with periodic boundary conditions and acting on functions with null average, is a norm in \( H^{-1}(\Omega) = H^1_{\text{per}}(\Omega)' \), which is equivalent to the usual \( H^{-1} \)-norm. We can note that

\[
(v, w) \in H^1_{\text{per}}(\Omega)^2 \mapsto \sum_{i=1}^{3} a_i \left( \frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right)
\]

is bilinear, symmetric, continuous and coercive, so that

\[
A : H^1_{\text{per}}(\Omega) \to H^{-1}(\Omega)
\]
is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2$ (see [1–2]) that $A$ is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A) = H^2_{\text{per}}(\Omega),$$

where, for $v \in D(A)$,

$$Av = -\sum_{i=1}^{3} a_i \frac{\partial^2 v}{\partial x_i^2}.$$

We further note that $D(A^{1/2}) = H^1_{\text{per}}(\Omega)$ and, for $v \in D(A^{1/2})$,

$$((A^{1/2}v, A^{1/2}v)) = \sum_{i=1}^{3} a_i \left\| \frac{\partial v}{\partial x_i} \right\|^2.$$

We finally note that (see, e.g., [18]) $v \mapsto (\|Av\|^2 + \langle v \rangle^2)^{1/2}$ defines a norm on $H^2_{\text{per}}(\Omega)$ which is equivalent to the usual $H^2$-norm on $D(A)$ (resp., $v \mapsto (\|A^{1/2}v\|^2 + \langle v \rangle^2)^{1/2}$ defines a norm on $H^1_{\text{per}}(\Omega)$ which is equivalent to the usual $H^1$-norm on $D(A^{1/2})$), where

$$\langle . \rangle = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} .dx,$$

being understood that, for $v \in H^{-1}(\Omega)$,

$$\langle v \rangle = \frac{1}{\text{Vol}(\Omega)} \langle v, 1 \rangle_{H^{-1}(\Omega), H^1_{\text{per}}(\Omega)},$$

and we note that

$$v \mapsto (\|v - \langle v \rangle\|^2_{-1} + \langle v \rangle^2)^{1/2}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual one. Here, $\Omega = \prod_{i=1}^{n} (0, L_i), L_i > 0, n = 2$ or $n = 3$. Furthermore, for a space $W$ we shall denote by $\tilde{W}$ the space

$$\tilde{W} = \{ v \in W, \langle v \rangle = 0 \}.$$

**Remark 2.1.** Actually, the conserved phase-field system usually is endowed with Neumann boundary conditions. In our case, these conditions read

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial A u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \quad (2.7)$$

where $\nu$ denotes the unit outer normal.
**Remark 2.2.** Note that similar properties hold for the operator $-\Delta$, with obvious changes.

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} - \Delta Au - \Delta f(u) = -\Delta \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right).$$  \tag{2.8}$$

Furthermore, we assume that the function $f$ satisfies the following conditions:

$$f \in C^2(\mathbb{R}), \quad f(0) = 0,$$  \tag{2.9}$$

$$f' \geq -c_0, \quad c_0 \geq 0,$$  \tag{2.10}$$

$$f(s)s \geq c_1 F(s) - c_2, \quad F(s) \geq -c_3, \quad c_1 > 0, \quad c_2, c_3 \geq 0, \quad s \in \mathbb{R},$$  \tag{2.11}$$

where, we denote by $F$ the primitive of $f$ vanishing at $s = 0$,

$$c_4 s^{2p-1} - c_5 \leq f''(s) \leq c_6 s^{2p-1} + c_7, \quad c_4, c_6 > 0, \quad c_5, c_7 \geq 0, \quad p \geq 1, \quad s \in \mathbb{R}.$$  \tag{2.12}$$

**Remark 2.3.** In particular, these assumptions are satisfied by function

$$f(s) = \sum_{i=1}^{2p+1} a_i s^i, \quad a_{2p+1} > 0, \quad \forall s \in \mathbb{R}$$

(and, the usual cubic nonlinear term $f(s) = s^3 - s$).

Throughout the paper, the same letters $c$, $c'$ and $c''$ denote (generally positive) constants which may vary from line to line. Similarly, the same letter $Q$ denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

### III. A Priori Estimates

The estimates below are formal, but they can also be justified within a Galerkin scheme for the approximated problem.

We first note that, integrating (formally) (2.8) over $\Omega$, we have

$$\frac{d\langle u \rangle}{dt} = 0,$$

hence

$$\langle u(t) \rangle = \langle u_0 \rangle, \quad \forall t \geq 0.$$  \tag{3.1}$$
Furthermore, integrating \((2.2)\) over \(\Omega\), we obtain, in view of \((2.7)\),

\[
\frac{d^2\langle \alpha \rangle}{dt^2} = -\frac{d\langle u \rangle}{dt}.
\]  

\(3.2\)

It thus follows from \((2.4)\) and \((3.2)\) that

\[
\frac{d\langle \alpha \rangle}{dt} = \langle u_0 + \alpha_1 \rangle - \langle u \rangle,
\]  

\(3.3\)

meaning, in particular, that \(\langle u + \frac{\partial \alpha}{\partial t} \rangle\) is a conserved quantity and from \((3.1)\) that

\[
\frac{d\langle \alpha \rangle}{dt} = \langle \alpha_1 \rangle,
\]  

\(3.4\)

so that

\[
\langle \alpha(t) \rangle = \langle \alpha_0 \rangle + \langle \alpha_1 \rangle t, \quad t \geq 0.
\]  

\(3.5\)

We now assume that

\[
|\langle u_0 \rangle| \leq M_1, \quad |\langle \alpha_1 \rangle| \leq M_2, \quad |\langle u_0 + \alpha_1 \rangle| \leq M_1 + M_2,
\]  

\(3.6\)

for fixed positive constants \(M_1\) et \(M_2\). Thus,

\[
|\langle u(t) \rangle| \leq M_1, \quad \left| \langle \frac{\partial \alpha}{\partial t}(t) \rangle \right| \leq M_2, \quad \left| \langle u + \frac{\partial \alpha}{\partial t}(t) \rangle \right| \leq M_1 + M_2, \quad t \geq 0.
\]  

\(3.7\)

Furthermore, it follows from \((3.5)\) that

\[
|\langle \alpha(t) \rangle| \leq |\langle \alpha_0 \rangle| + |\langle \alpha_1 \rangle| t, \quad t \geq 0.
\]  

\(3.8\)

We rewrite, in view of \((3.4)\), \((2.8)\) as

\[
(-\Delta)^{-1}\frac{\partial u}{\partial t} + Au + f(u) - \langle f(u) \rangle = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha_1 \rangle.
\]  

\(3.9\)
and, in view of (3.3) an (3.9) that

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + Au + f(u) - \langle f(u) \rangle = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha \rangle_1. \quad (3.10)$$

Furthermore, we deduce from (2.2) and (3.2) that

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}. \quad (3.11)$$

We first multiply (3.9) by \(\frac{\partial u}{\partial t}\) and obtain, noting that \(\langle \frac{\partial u}{\partial t} \rangle = 0\),

$$\frac{1}{2} \frac{d}{dt} \left( \|A^{1/2} u\|^2 + 2 \int_{\Omega} F(u) \, dx + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) = \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t} \right). \quad (3.12)$$

We then multiply (2.2) by \(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\) to obtain

$$\frac{1}{2} \frac{d}{dt} \left( \left\| \nabla \alpha \right\|^2 + \|\Delta \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) = - \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t} \right) \right) \quad (3.13)$$

(note indeed that \(\left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right\|^2 = \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + 2 \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2\)).

Summing finally (3.12) and (3.13), we find a differential equality

$$\frac{dE_1}{dt} + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 + 2 \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + 2 \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 = 0 \quad (3.14)$$

where

$$E_1 = \|A^{1/2} u\|^2 + 2 \int_{\Omega} F(u) \, dx + \left\| \nabla \alpha \right\|^2 + \|\Delta \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right\|^2.$$
satisfies, owing to (2.12),

\[ c \left( \| A \frac{1}{2} u \|^2 + \| u \|_{L^{2p+2} (\Omega)}^{2p+2} + \| \Delta \alpha \|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c' \leq E_1 \]

\[ \leq c'' \left( \| A \frac{1}{2} u \|^2 + \| u \|_{L^{2p+2} (\Omega)}^{2p+2} + \| \Delta \alpha \|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c'', \quad c, c'' > 0. \quad (3.15) \]

(here and below, when not specified, the sign of the constants (\( c' \) and \( c'' \) here) can be arbitrary).

Multiplying (3.9) by \( u \) and have, integrating over \( \Omega \) and by parts,

\[ \frac{1}{2} \frac{d}{dt} \| \pi \|^2_{-1} + \| A \frac{1}{2} u \|^2 + (f(u), u) = \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, u \right) \right) + ((f(u), \langle u \rangle)) - \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \langle u \rangle \right) \right). \]

It follows from (2.11) that

\[ ((f(u), u)) \geq c_2 \int_{\Omega} F(u) \, dx + c, \]

from (2.12) and (3.7) that

\[ \left| (f(u), \langle u \rangle) \right| \leq c_1 \int_{\Omega} |f(u)| \, dx \leq \frac{c_2}{2} \int_{\Omega} F(u) \, dx + c_{M_1}, \]

and from (3.7) that

\[ \left| \left( \frac{\partial \alpha}{\partial t}, \langle u \rangle \right) \right| \leq c_{M_1, M_2}. \quad (3.16) \]

Therefore, owing again to (3.7) and remembering that \( v \mapsto (\| A \frac{1}{2} v \|^2 + \langle v \rangle^2)^{\frac{1}{2}} \) is a norm in \( H_{\text{per}}^1 (\Omega) \) which is equivalent to the usual \( H^1 \)-norm,

\[ \frac{d}{dt} \| \pi \|_{-1}^2 + c \| u \|_{H_{\text{per}}^1 (\Omega)}^2 + 2 \int_{\Omega} F(u) \, dx \leq c' \left( \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c''_{M_1, M_2}, \quad c > 0. \quad (3.17) \]

Summing (3.14) and \( \delta_1 (3.17) \), where \( \delta_1 > 0 \) is small enough, we have a differential inequality of the form

\[ \frac{dE_2}{dt} + c \left( \| u \|_{H_{\text{per}}^1 (\Omega)}^2 + 2 \int_{\Omega} F(u) \, dx + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2_{H_{\text{per}}^1 (\Omega)} \right) \leq c'_{M_1, M_2}, \quad c > 0, \quad (3.18) \]
where
\[ E_2 = E_1 + \delta_1 \|u\|_{L^1}^2 \]
satisfies
\[
c \left( \|A^\frac{1}{2} u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta \alpha\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c' \leq E_2 \]
\[
\leq c'' \left( \|A^\frac{1}{2} u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta \alpha\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c''', \quad c, c'' > 0. \tag{3.19} \]

We multiply (2.8) by \(u\) to obtain, owing to (2.9) and (2.10),
\[
\frac{d}{dt} \|u\|^2 + \|\nabla A^\frac{1}{2} u\|^2 \leq c \left( \|u\|^2_{H^1_{\text{per}}(\Omega)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right). \tag{3.20} \]

Summing (3.18) and \(\delta_2(3.20)\), where \(\delta_2 > 0\) is small enough, we obtain a differential inequality of the form
\[
\frac{dE_3}{dt} + c \left( \|u\|^2_{H^1_{\text{per}}(\Omega)} + 2 \int_{\Omega} F(u) dx + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_{H^2_{\text{per}}(\Omega)}^2 \right) \leq c_{M_1,M_2}, \quad c > 0, \tag{3.21} \]
where
\[ E_3 = E_2 + \delta_2 \|u\|^2 \]
satisfies
\[
c \left( \|u\|^2_{H^1_{\text{per}}(\Omega)} + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta \alpha\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c' \leq E_3 \]
\[
\leq c'' \left( \|u\|^2_{H^1_{\text{per}}(\Omega)} + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta \alpha\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + c''', \quad c, c'' > 0. \tag{3.22} \]

Now, multiplying (3.10) by \(\frac{\partial u}{\partial t}\), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|A^\frac{1}{2} u\|^2 + 2 \int_{\Omega} F(u) dx \right) + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 = \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right) \left( \frac{\partial \alpha}{\partial t} \right) \right). \tag{3.23} \]
We then multiply (3.11) by \(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \pi}{\partial t}\)

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \alpha\|^2 + \|\Delta \pi\|^2 + \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \pi}{\partial t} \right\|^2 \right) + \left\| \nabla \frac{\partial \pi}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \pi}{\partial t} \right\|^2
\]

\[
= - \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t} \right) \right) \quad (3.24)
\]

(note indeed that \(\left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \pi}{\partial t} \right\|^2 = \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + 2 \left\| \nabla \frac{\partial \pi}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \pi}{\partial t} \right\|^2\).

Summing finally (3.21), (3.23) and (3.24), we obtain a differential inequality of the form

\[
\frac{dE_4}{dt} + c \left( \|u\|_{H^2_{per}(\Omega)}^2 + 2 \int_{\Omega} F(u) dx + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_{H^2_{per}(\Omega)}^2 + \left\| \frac{\partial \pi}{\partial t} \right\|_{H^2_{per}(\Omega)}^2 \right) \leq c'_M, c > 0, \quad (3.25)
\]

where

\[
E_4 = E_3 + \|A^{1/2} u\|^2 + 2 \int_{\Omega} F(u) dx + \|\nabla \pi\|^2 + \|\Delta \pi\|^2 + \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \pi}{\partial t} \right\|^2
\]

satisfies

\[
c \left( \|u\|_{H^2_{per}(\Omega)}^2 \right) + c' \leq E_4
\]

\[
\leq c'' \left( \|u\|_{H^2_{per}(\Omega)}^2 + \|\Delta \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \Delta \pi\right\|^2 + \left\| \Delta \frac{\partial \pi}{\partial t} \right\|^2 \right) + c''', \quad c, c'' > 0. \quad (3.26)
\]

We finally assume that \(p = 1\) when \(n = 3\) and multiply (2.8) by \(Au\) to find

\[
\frac{1}{2} \frac{d}{dt} \|A^{1/2} u\|^2 + \|\nabla Au\|^2 + ((f'(u) \nabla u, \nabla Au)) = \left( \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t}, \nabla Au \right).
\]

We assume that \(n = 3\) and \(p = 1\) (the case \(n = 2\) can be treated in a similar way) and have, owing to (2.12) and Hölder’s inequality,
\( |((f'(u)\nabla u, \nabla Au))| \leq c \int_{\Omega} (|u|^2 + 1)|\nabla u| |\nabla Au| \, dx \)

\[
\leq c(||u||^2_{L^6(\Omega)} + 1)||\nabla u||_{L^6(\Omega)} ||\nabla Au|| \leq c(||u||^2_{H^1_{\text{per}}(\Omega)} + 1)||u||_{H^2_{\text{per}}(\Omega)} ||\nabla Au||.
\]

Therefore,

\[
\frac{d}{dt} ||A^{\frac{1}{2}} u||^2 + ||\nabla Au||^2 \leq c(||u||^4_{H^1_{\text{per}}(\Omega)} + 1)||u||^2_{H^2_{\text{per}}(\Omega)} + c' \left( \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right).
\] (3.27)

IV. Well-Posedness

We have the following result.

**Theorem 4.1.** We assume that (2.9)–(2.12) hold. Then, for every \((u_0, \alpha_0, \alpha_1) \in (H^1_{\text{per}}(\Omega) \cap L^{2p+2}(\Omega)) \times H^2_{\text{per}}(\Omega) \times H^2_{\text{per}}(\Omega)\), (2.1)–(2.4) possesses at least one solution \((u, \alpha, \frac{\partial \alpha}{\partial t})\) such that

\[
u \in L^\infty(0, T; H^1_{\text{per}}(\Omega) \cap L^{2p+2}(\Omega)) \cap L^2(0, T; H^2_{\text{per}}(\Omega)),
\]

\[
\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)),
\]

\[
\alpha, \overline{\alpha} \in L^\infty(0, T; H^2_{\text{per}}(\Omega))
\]

and

\[
\frac{\partial \alpha}{\partial t}, \frac{\partial \overline{\alpha}}{\partial t} \in L^\infty(0, T; H^2_{\text{per}}(\Omega)) \cap L^2(0, T; H^2_{\text{per}}(\Omega))
\]

\(\forall T > 0\).

Furthermore, if \(p = 1\) when \(n = 3\), then

\[
u \in L^2(0, T; H^3_{\text{per}}(\Omega)).
\]

**Proof:** The proof is based on (3.8), (3.25), (3.27) and, e.g., a standard Galerkin scheme.

We have, concerning the uniqueness, the following.

**Theorem 4.2.** We assume that the assumptions of Theorem 4.1 hold and that \(p = 1\) when \(n = 3\) and \(p \in [1, 2]\) when \(n = 2\). Then, the solution obtained in Theorem 4.1 is unique.
Proof. Let \( (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) \) and \( (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}) \) be two solutions to (2.1)–(2.3) with initial data \( (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) \) and \( (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}) \), respectively, such that

\[
|\langle u_0^{(i)} \rangle| \leq M_1, \quad |\langle \alpha_1^{(i)} \rangle| \leq M_2, \quad |\langle u_0^{(i)} + \alpha_1^{(i)} \rangle| \leq M_1 + M_2, \quad i = 1, 2, \quad (4.1)
\]

for fixed positive constants \( M_1 \) and \( M_2 \). We set

\[
(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})
\]

and

\[
(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).
\]

We have

\[
(-\Delta)^{-1} \frac{\partial u}{\partial t} + Au + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha_1 \rangle, \quad (4.2)
\]

\[
\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \quad (4.3)
\]

\[
u \text{ and } \alpha \text{ are } \Omega \text{ -- periodic}, \quad (4.4)
\]

\[
u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \quad (4.5)
\]

We multiply (4.2) by \( \frac{\partial u}{\partial t} \) (note that \( \langle \frac{\partial u}{\partial t} \rangle = 0 \)) and obtain

\[
\frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|^2 + \| \frac{\partial u}{\partial t} \|_{-1}^2 = \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t} \right) \right) - \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right). \quad (4.6)
\]

We first assume that \( n = 3 \) and \( p = 1 \). We have

\[
| \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right) | = | \left( \left( f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle, \frac{\partial u}{\partial t} \right) \right) |
\]
\[ \leq \| \nabla (f(u^{(1)}) - f(u^{(2)})) \| \left\| \frac{\partial u}{\partial t} \right\|_{-1} \]

\[ = \| \nabla \int_0^1 f'(u^{(1)}) + s(u^{(2)} - u^{(1)}) ds \| \left\| \frac{\partial u}{\partial t} \right\|_{-1} \]

\[ \leq \left\| \int_0^1 f'(u^{(1)}) + s(u^{(2)} - u^{(1)}) ds \nabla u \left\| \frac{\partial u}{\partial t} \right\|_{-1} \]

\[ + \left\| u \int_0^1 f''(u^{(1)}) + s(u^{(2)} - u^{(1)}) (\nabla u^{(1)} + s \nabla (u^{(2)} - u^{(1)})) ds \left\| \frac{\partial u}{\partial t} \right\|_{-1}. \]

Furthermore, owing to Agmon's inequality,

\[ \left\| \int_0^1 f'(u^{(1)}) + s(u^{(2)} - u^{(1)}) ds \nabla u \right\|^2 \leq c \int \Omega \left( |u^{(1)}|^4 + |u^{(2)}|^4 + 1 \right) |\nabla u|^2 dx \]

\[ \leq c \left( \| u^{(1)} \|^4_{L^\infty(\Omega)} + \| u^{(2)} \|^4_{L^\infty(\Omega)} + 1 \right) \| \nabla u \|^2 \]

\[ \leq c \left( \| u^{(1)} \|^2_{H^1_{per}(\Omega)} + \| u^{(2)} \|^2_{H^1_{per}(\Omega)} + 1 \right) \| \nabla u \|^2 \times \left( \| u^{(1)} \|^2_{H^2_{per}(\Omega)} + \| u^{(2)} \|^2_{H^2_{per}(\Omega)} + 1 \right) \| \nabla u \|^2 \]

and, owing to Hölder’s inequality,

\[ \left\| u \int_0^1 f''(u^{(1)}) + s(u^{(2)} - u^{(1)}) (\nabla u^{(1)} + s \nabla (u^{(2)} - u^{(1)})) ds \right\|^2 \]

\[ \leq c \int \Omega \left( |u^{(1)}|^2 + |u^{(2)}|^2 + 1 \right) \left( |\nabla u^{(1)}|^2 + |\nabla u^{(2)}|^2 \right) |u|^2 dx \]

\[ \leq c \left( \| u^{(1)} \|^2_{H^1_{per}(\Omega)} + \| u^{(2)} \|^2_{H^1_{per}(\Omega)} + 1 \right) \left( \| u^{(1)} \|^2_{H^2_{per}(\Omega)} + \| u^{(2)} \|^2_{H^2_{per}(\Omega)} \right) \| u \|^2_{H^1_{per}(\Omega)}. \]

We now assume that \( n = 2 \) and we take the most complicated case \( p = 2 \). Then, owing to Agmon’s inequality and a proper interpolation inequality,
\[ \| f'(u^{(1)} + s(u^{(2)} - u^{(1)})) \|_{L^\infty}^2 \leq c \int_\Omega (|u^{(1)}|^8 + |u^{(2)}|^8 + 1) \| \nabla u \|^2 dx \]

\[ \leq c(\| u^{(1)} \|_{L^\infty}^8 + \| u^{(2)} \|_{L^\infty}^8 + 1) \| \nabla u \|^2 \]

\[ \leq c(\| u^{(1)} \|^4 \| u^{(1)} \|_{H^1_{\text{per}}}^4 + \| u^{(1)} \|^4 \| u^{(1)} \|_{H^1_{\text{per}}}^4 + 1) \| \nabla u \|^2 \]

\[ \leq c(\| u^{(1)} \|_{H^1_{\text{per}}}^6 + \| u^{(2)} \|_{H^1_{\text{per}}}^6 + 1)(\| u^{(1)} \|^2_{H^1_{\text{per}}} + \| u^{(2)} \|^2_{H^1_{\text{per}}}) \| \nabla u \|^2. \]

Furthermore, owing to Hölder’s inequality,

\[ \| f''(u^{(1)} + s(u^{(2)} - u^{(1)})) \|_{H^1}^2 \leq c \int_\Omega (|u^{(1)}|^6 + |u^{(2)}|^6 + 1)(|\nabla u^{(1)}|^2 + |\nabla u^{(2)}|^2) |u|^2 dx \]

\[ \leq c(\| u^{(1)} \|^6_{H^1_{\text{per}}} + \| u^{(2)} \|^6_{H^1_{\text{per}}} + 1)(\| u^{(1)} \|^2_{H^1_{\text{per}}} + \| u^{(2)} \|^2_{H^1_{\text{per}}}) \| u \|^2_{H^1_{\text{per}}}. \]

Finally, we obtain, in both cases, an inequality of the form

\[ \frac{d}{dt} \| A^{1/2} u \|^2 + \| \partial u / \partial t \|^2_1 \leq 2 \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t} \right) \right) \]

\[ + c(\| u^{(1)} \|^q_{H^1_{\text{per}}} + \| u^{(2)} \|^q_{H^1_{\text{per}}} + 1)(\| u^{(1)} \|^2_{H^1_{\text{per}}} + \| u^{(2)} \|^2_{H^1_{\text{per}}} + 1) \| \nabla u \|^2, \quad q \geq 1. \] (4.7)

Multiplying then (4.3) by \( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \), we find

\[ \frac{d}{dt} \left( \| \nabla \alpha \|^2 + \| \Delta \alpha \|^2 + \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right) \right) + 2 \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + 2 \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \]

\[ = -2 \left( \left( \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial t} \right) \right). \] (4.8)
Summing finally (4.7) and (4.8), we have, an inequality of the form
\[ \frac{dE_5}{dt} \leq c \left( \|u(1)\|_{H^q_{\text{per}}(\Omega)}^q + \|u(2)\|_{H^q_{\text{per}}(\Omega)}^q + 1 \right) \left( \|u(1)\|_{H^3_{\text{per}}(\Omega)}^2 + \|u(2)\|_{H^3_{\text{per}}(\Omega)}^2 + 1 \right) E_4, \tag{4.9} \]

where \( q \geq 1 \), and
\[ E_5 = \|A^{\frac{1}{2}}u\|^2 + \|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right\|^2. \]

Then, We deduce from (4.9), (3.8), the estimates obtained in the previous subsection and Gronwall’s lemma the uniqueness, as well as the continuous dependence with respect to the initial data. □

V. REGULARITY OF SOLUTIONS

We have the following result which gives the existence and uniqueness of more regular solutions.

**Theorem 5.1.** We assume that the assumptions of Theorem 4.1 hold and that (2.12) is replaced by
\[ |f(s)| \leq \epsilon F(s) + c_\epsilon, \quad \forall \epsilon > 0, \quad s \in \mathbb{R}. \tag{5.1} \]

Then, if \((u_0, \alpha_0, \alpha_1) \in H^2_{\text{per}}(\Omega) \times H^3_{\text{per}}(\Omega) \times H^3_{\text{per}}(\Omega)\), the problem possesses a unique solution such that
\[ u \in L^\infty(0; T; H^2_{\text{per}}(\Omega)) \]
and
\[ \alpha, \frac{\partial \alpha}{\partial t} \in L^\infty(0; T; H^3_{\text{per}}(\Omega)), \quad \forall T > 0. \]

**Proof.** The proof of uniqueness is obtained by proceeding as in that of Theorem 4.2, noting that, with the higher regularity considered here, no growth assumption on \( f \) is needed, owing to the continuous embedding \( H^2_{\text{per}}(\Omega) \subset L^\infty(\Omega) \).

We now turn to the proof of existence and, more precisely, of the further regularity of the solutions.

We multiply (2.8) by \( \frac{\partial u}{\partial t} \) we have
\[ \frac{1}{2} \frac{d}{dt} \|\nabla A^{\frac{1}{2}}u\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 = \left( \left( \Delta f(u), \frac{\partial u}{\partial t} \right) \right) + \left( \left( \nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right) \right). \tag{5.2} \]
Multiplying then \((2.2)\) by \(-\Delta \left( \partial \alpha / \partial t - \Delta \partial \alpha / \partial t \right)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta \alpha \|^2 + \| \nabla \Delta \alpha \|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right) + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 + \left\| \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2
\]

\[
= - \left( \left( \nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right) \right). \tag{5.3}
\]

Summing finally \((5.2)\) and \((5.3)\), we find

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla A^{\frac{1}{2}} u \|^2 + \| \Delta \alpha \|^2 + \| \nabla \Delta \alpha \|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right)
\]

\[
+ \| \frac{\partial u}{\partial t} \|^2 + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 + \left\| \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 = \left( \left( \Delta f(u), \frac{\partial u}{\partial t} \right) \right). \tag{5.4}
\]

It follows from \((5.4)\) and the continuous embedding \(H^2_{\text{per}}(\Omega) \subset C(\overline{\Omega})\) that

\[
\frac{d}{dt} \left( \| \nabla A^{\frac{1}{2}} u \|^2 + \| \Delta \alpha \|^2 + \| \nabla \Delta \alpha \|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \right)
\]

\[
+ \| \frac{\partial u}{\partial t} \|^2 + 2 \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 + 2 \left\| \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \leq Q(\| \Delta u \|^2). \tag{5.5}
\]

We set

\[
y = \| \nabla A^{\frac{1}{2}} u \|^2 + \| \Delta \alpha \|^2 + \| \nabla \Delta \alpha \|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \tag{5.6}
\]

and we deduce from \((5.5)\) that we have an inequality of the form

\[
y' \leq Q(y). \tag{5.7}
\]

Let \(z\) be the solution to the ordinary differential equation

\[
z' = Q(z), \quad z(0) = y(0). \tag{5.8}
\]

It follows from the comparison principle that there exists \(T_0 = T_0(\| u_0 \|_{H^2_{\text{per}}(\Omega)}, \| \alpha_0 \|_{H^3_{\text{per}}(\Omega)}, \| \alpha_1 \|_{H^3_{\text{per}}(\Omega)})\) (say, belonging to \((0, \frac{1}{2})\)) such that

\[
y(t) \leq z(t), \quad t \in [0, T_0]. \tag{5.9}
\]
from which it follows, owing also to (3.8) and (3.25), that

\[ \|u(t)\|_{H^2_{per}(\Omega)}^2 + \|\alpha(t)\|_{H^3_{per}(\Omega)}^2 + \|\frac{\partial \alpha}{\partial t}(t)\|_{H^2_{per}(\Omega)}^2 \]

\[ \leq Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega),} \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in [0,T_0]. \] (5.10)

We now differentiate (3.9) with respect to times and have, owing to (2.2),

\[ (-\Delta)^{-1} \frac{\partial}{\partial t} \frac{\partial u}{\partial_t} + A\frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial t} = \Delta \alpha + \Delta \frac{\partial \alpha}{\partial t} - \frac{\partial u}{\partial t}. \] (5.11)

We multiply (5.11) by \( t \frac{\partial u}{\partial t} \) and have

\[ \frac{1}{2} \frac{d}{dt} \left( t \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) + \left\| A^2 \frac{\partial u}{\partial t} \right\|_{-1}^2 + t \left( \left\| f'(u) \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) \]

\[ = -t \left( \left\| \nabla \alpha, \nabla \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) - t \left( \left\| \nabla \frac{\partial \alpha}{\partial t}, \nabla \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) - t \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2, \]

which yields, owing to (2.10) and a proper interpolation inequality (see the proof of Theorem 4.2),

\[ \frac{d}{dt} \left( t \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) + ct \left\| \frac{\partial u}{\partial t} \right\|_{H^2_{per}(\Omega)}^2 \leq c't \left( \left\| \nabla \alpha \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2, \quad c > 0. \] (5.12)

It follows from (3.25), (5.12) and Gronwall’s lemma that

\[ \left\| \frac{\partial u}{\partial t}(t) \right\|_{-1}^2 \leq \frac{1}{t} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega),} \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in (0,T_0]. \] (5.13)

Next, we multiply (5.11) by \( \frac{\partial u}{\partial t} \) and obtain, proceeding similarly,

\[ \frac{d}{dt} \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + c \left\| \frac{\partial u}{\partial t} \right\|_{H^2_{per}(\Omega)}^2 \leq c' \left( \left\| \nabla \alpha \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right), \quad c > 0. \] (5.14)

We deduce from (3.25), (5.14) and Gronwall’s lemma that

\[ \left\| \frac{\partial u}{\partial t}(t) \right\|_{-1}^2 \leq e^{ct} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega),} \|\alpha_1\|_{H^3_{per}(\Omega)}), \left\| \frac{\partial u}{\partial t}(T_0) \right\|_{-1}^2, \quad t \geq T_0, \]
We rewrite, for \( t \geq T_0 \) fixed, (3.9) as an elliptic equation,

\[
Au + f(u) - \langle f(u) \rangle = h_u(t), \quad u \text{ is } \Omega - \text{periodic},
\]

where

\[
h_u(t) = -(-\Delta)^{-1} \frac{\partial u}{\partial t} + \partial \alpha \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha \rangle
\]

(5.17)
satisfies, owing to (3.25) and (5.15),

\[
\|h_u(t)\| \leq e^{ct} Q M_1, M_2 (\|u_0\|_{H^2_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)}), \quad t \geq T_0.
\]

(5.18)

Multiplying (5.16) by \( Au \), we have, owing to (2.10),

\[
\|Au(t)\|^2 \leq c(\|A^{\frac{1}{2}} u(t)\|^2 + \|h_u(t)\|^2), \quad t \geq T_0.
\]

(5.20)

Combining (5.19) and (5.20), we finally obtain, owing to (3.25) and (5.18),

\[
\|u(t)\|^2_{H^2_{\text{per}}(\Omega)} \leq e^{ct} Q M_1, M_2 (\|u_0\|_{H^2_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)}), \quad t \geq T_0,
\]

(5.21)
hence, owing to (5.10),

\[
\|u(t)\|^2_{H^3_{\text{per}}(\Omega)} \leq e^{ct} Q M_1, M_2 (\|u_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)}), \quad t \geq 0.
\]

(5.22)

We now come back to (5.3), from which it follows that

\[
\frac{d}{dt} \left( \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2 + \|\nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t}\|^2 \right) \leq \|\nabla \frac{\partial u}{\partial t}\|^2.
\]

(5.23)
Noting that it follows from (3.25), (5.14) and (5.15) that

\[
\int_{T_0}^t \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \, dt \leq e^{ct} Q_{M_1,M_2} \left( \|u_0\|_{H^2_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)} \right), \quad t \geq T_0,
\]

we deduce from (3.14), (5.23) and (5.24) that

\[
\| \Delta \alpha(t) \|^2 + \| \nabla \Delta \alpha(t) \|^2 + \left\| \left( \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right) (t) \right\|^2 \leq e^{ct} Q_{M_1,M_2} \left( \|u_0\|_{H^2_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)} \right) + \| \Delta \alpha(T_0) \|^2 + \| \nabla \Delta \alpha(T_0) \|^2 + \left\| \left( \nabla \frac{\partial \alpha}{\partial t} - \nabla \Delta \frac{\partial \alpha}{\partial t} \right) (T_0) \right\|^2, \quad t \geq T_0,
\]

hence, owing to (3.8), (3.25), (5.10) and (5.22),

\[
\| u(t) \|^2_{H^2_{\text{per}}(\Omega)} + \| \alpha(t) \|^2_{H^3_{\text{per}}(\Omega)} + \left\| \frac{\partial \alpha}{\partial t} (t) \right\|^2_{H^3_{\text{per}}(\Omega)} \leq e^{ct} Q_{M_1,M_2} \left( \|u_0\|_{H^2_{\text{per}}(\Omega)}, \|\alpha_0\|_{H^3_{\text{per}}(\Omega)}, \|\alpha_1\|_{H^3_{\text{per}}(\Omega)} \right), \quad t \geq 0,
\]

which finishes the proof of the theorem. \(\square\)

**Remark 5.1.** We can note that, here, we are not able to study the asymptotic behavior of the associated dynamical system. Indeed, the estimates derived in this section are not dissipative.

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**References Références Referencias**


Effects of Heat Transfer of Natural Convection Laminar Flow of Fluid’s Velocity Components and Viscous Dissipation through a Circular Heated Plate on a Square Body

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**Abstract**

This research is on natural convection flow of a fluid through a circular plate on a two-dimensional square body. In this paper heat transfer of laminar flow will be investigated based on mesh modeling and graphs under heat flux and viscosity. Temperatures, thermal conductivity, velocity components, viscous dissipation of fluid with time are used as variables. A flat square surface with circle is considered in the investigation by using COMSOLMUltiPHYSICS. Here square body and circular plate are insulated. As material we have used water. The parameters of the fluid have inlet velocity, outlet velocity, heat flux and thermal conductivity. Using boundary condition and velocity components with temperature, we can solve governing equations numerically. The solution has been expressed as meshes and graphs in the results. Different sectors of science like magneto hydrodynamic, fluid flow, natural convection, Radiation, solar, engineering and medical science will be benefitted by this research. We construct many mesh models through our studied technique which is shown in Figure 'a' to 'o'. We will also discuss 2D graphical representation through the proposed technique which is shown in Figures 'p' to 't'. In future, we may try to apply our results in space heating, domestic hot or cold water processes, heat from electricity and many other related fields.

**Keywords:** viscous dissipation, viscosity, laminar flow, heat transfer, heat flux, thermal conductivity, insulation and mesh.

**GJSFR-F Classification:** FOR Code: 0103

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Effects of Heat Transfer of Natural Convection Laminar Flow of Fluid’s Velocity Components and Viscous Dissipation through a Circular Heated Plate on a Square Body

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1. Introduction

Natural convection laminar wave takes place in multiple scientific and industrial treatments in nature. Heat transfers take place in a low-velocity, solar receivers exposed breeze currents, nuclear reactors to be cooled during emergency shutdown; electronic devices cooled by fans and many more [1-20]. Fluid mechanics is the study of effects of force on a fluid. If the fluid is at rest, it remains static. When fluid is in motion, the studies are known as fluid dynamics. Fluid is any substance which can flow and relative change of position of particles with respect to time. Fluid has no certain shape which occupies the shape of vessels can flow under its own weight. There are no voids between the molecules of fluid. Different forces act upon fluid as surface force, body force (gravitational force). This force acts on the mass of the fluid that is the quantity of the component of the particular fluid. Due to the conservation of mass, it can’t create or

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destroy. Viscous dissipation occurs by the continuous force on the fluid where the molecule deformation happens. Fluid flows are of two types: laminar flows and turbulent flows. When the fluid flows in infinitesimal parallel layers with no disruption between them. Laminar flow occurs fluid layers flow in parallel and current is normal to the flow itself. This type of flow is also referred to as streamline flow because nature this flow does not cross the streamlines flow. Laminar flow in a straight pipe may be considered as the relative motion of a set of concentric cylinders of fluid, the outside one fixed at the pipe wall. Others movement is approached increasing speed at the center of the pipe. Regarding the smolder rising in a straight path from a burned material is creating laminar flow. With the rising of a small distance, the smoke usually changes to turbulent flow. For the laminar flow channel is relatively small, that is why the fluid moves slowly. So, its viscosity is relatively high. Oil flow through a thin pipe like blood flow is laminar. Turbulent flows are flowing near solid boundaries, where the flow is often laminar, in a thin layer just adjacent to the surface is noticeable. Laminar and turbulent flow takes place depend the velocity of the fluid flow. But viscosity of the fluid when a fluid flow spreads around the heated circular plate. At the lower velocity laminar flow takes place under an edge when the flow turns turbulent. Heat transfer is the phenomena; it conveys energy and entropy from one location to another. If any heated circular plate is placed on a square solid body than heat transfer will occur. Heat transfer has gained considerable attention due to its numerous applications in the area of energy conservations. The applications also include refrigeration of electrical devices, electronic components, design of solar collectors and heat exchangers. The determination of the velocity profile greatly influences the heat transfer process creates the main difficulty in solving natural convection problems. Heat flux is the amount of heat transferred per unit area and per unit time to or from a surface. The heat transfer results from the resulting quantity of the heat flux per unit time and per unit area to or from the surface. Viscosity is the inter frictional force between adjacent layers of fluid relative in motion. The resistance of a fluid from deformation at a definite rate is viscosity- for example syrup and water.

The continuous process like electromagnetic fields. Fluid dynamics and deformable bodies cause Navier-Stokes equations for the fluids flow. The nature of the fluid can be described by using Navier-Stoke’s partial differential equation based on continuity, momentum and conservation of energy. Effects of viscous dissipation have an important role to play in free convection in various gadgets subjected to large deceleration that operates at high rotational speeds. It is also a strong gravitational field processes on large scales in geological processes. MHD is the fluid dynamics of conducting fluid. A typical feature of MHD leads to forming a singular current density. Current sheets are associated with the breaking and reconnecting magnetic field lines due to presence of viscous dissipation. Study of heat transfer, temperature, MHD fluid flow, natural convection and viscous dissipation are very important because these are used in many branches of science. Some researchers have also showed that both the flows are affected by geometrical parameters.

Circular cavity with various heated isothermal wall has received the devotion because of it has wide application, cooling of electric devices, fire control in a building, extraction of oil from container, the construction of stars, planet and blood flow. The density and viscous dissipation effect play a vital role in free convection flow over a circular heated plate or various gadgets in geological system. Dissipation is the output for formulating a viscous material that is converted into energy. The discussion of process of free convection flows, viscosity, density and viscous dissipation on the heated
circular plate on a rectangular body are usually ignored. In this work, it is considered the effect of heat transfer through a circular heated plate on natural convection flow with time. To solve extraction one or more components of a liquid mixture are extracted by solution in a selective solvent. In humidification water is transferred from the liquid to air. The biological applications include oxygenation of blood, food and drug and respiration mechanism. In the process of analogous or digital heat diffusion resulting from temperature gradient when the fluid is in motion heat transfer take in place by molecular diffusion. Hence velocity component field is needed to the heat transfer process. So, the heat transfers equation and the coefficient of thermal conductivity obtained by COMSOL MULTIPHYSICS.

Meanwhile we considered heat transferred in forced convection in which the fluid flow imposed external by surface pressure, fan, blower and pump. Convection flow is set up within the fluid without a force where velocity \( u' \) is the free convection.

The wave velocity in free convection is much smaller than in forced convection. So, heat transfer by free convection is much smaller than forced convection.

Meshing means mesh generation which is the procedure of two-dimensional and three-dimensional grid. These divided compound geometries into elements then it can be used to discretize a domain. The most important steps of meshing procedure in performance model using finite element analysis. A mesh is the last presentation of elements. The coordinate sites in space which can vary on element nature belong to nodes that represent the contour of the geometry. The performance of mesh is to divide the model into cells in sequence behavior a simulation analysis or concentrate a digital model. Meshing is the collective of coordinates, boundaries, and faces which describe the form of aim in two or three dimensions.

Steady laminar free convection in air-filled 2D rectangular enclosures heated from below and cooled from above is studied numerically for a wide variety of thermal boundary conditions at the walls [1]. Another study shows that, complex interactions between Nano fluids and the walls of cavity. This complexity may increase with a change of geometry or orientation of the cavity. So, study of natural convection fluid flow and heat transfer in a trapezoidal geometry is more difficult than that of square or rectangular enclosures due to the presence of sloping walls [3]. A good number of investigations on natural convection fluid flows and heat transfers in trapezoidal cavities have already been published. A complete study of the laminar solution of the problem (Ra up to 105) was given by MacGregor and Emery together with experimental results covering a wide range of Prandtl numbers. Many correlations of Nusselt number (Nu) and Ra concerning experimental results can also be found in this paper. Jaluria and Gebhart have worked on vertical natural convection flows. Their studies expressed the process occurring during the transition from laminar to turbulent flow near a vertical flat plate when the surface heat flux is uniform. The interaction of the velocity and temperature fields during the transition and the effect on Nu were also investigated. Mallinson and De Vahl Davis presented detailed three-dimensional calculations for laminar flow. The calculations were performed for different values of Prandtl number and aspect ratio of cavity dimensions. Natural convection flow in a square cavity revisited: laminar and turbulent models with wall functions [2, 9]. They found that the flow and temperature fields are obstructed by the presence of the body. Oztop et al. analyzed the effects of the surface of the insulated body for partially heated enclosure [10]. Natural convection in porous triangular enclosure with a circular obstacle in presence of heat generation [11]. Compared to other study, this study includes the velocity profile of fluid particles and temperature with respect to time. The purpose of
this research is the exploration of the effect of heat transfer of natural convection laminar flow of a fluid’s velocity components and viscous dissipation through a circular heated plate on a square body with time.

II. **Aim, Findings, Future Scope**

**a) Aim**

The aim of the study is to investigate the effects of heat transfer on natural convection laminar flow of fluid’s velocity components and viscous dissipation through a circular heated plate on a square body. Specifically, the study aims to understand how the velocity and temperature fields of the fluid are affected by the presence of the circular heated plate and how the viscous dissipation affects the heat transfer characteristics. This information is important for understanding, optimizing heat transfer processes in several industrial and engineering uses, such as in heat exchangers, cooling process, and energy production policy. The study also aims to contribute to the scientific understanding of natural convection laminar flow and its relationship with heat transfer in flat media.

1. Investigation of the heat transfer characteristics of natural convection laminar flow over a circular heated plate on a square body.
2. To analysis the effects of fluid’s velocity components and viscous dissipation on the heat transfer rate.
3. To determination the optimal conditions for improving the heat transfer rate in the method, such as the best area for the circular heated plate, the optimal size of the square body and the optimal properties of the fluid flow.
4. Comparing the results in the present study compare to earlier studies on related systems and authenticate the requirement.

Overall, the research improve our understanding of the natural convection laminar flow over a circular heated plate on a square body, which has practical uses in many fields like energy, environmental science and engineering.

**b) Findings**

1. Mesh Type Impact: The choice of mesh type significantly affects the simulation results, particularly the temperature distribution and fluid flow patterns. Different types of meshes, such as triangular and free triangular meshes, yield distinct temperature profiles and flow characteristics. This highlights the importance of mesh selection in accurately capturing the physics of the problem and obtaining reliable results.
2. Shaded Area: The simulation identifies certain shaded area in the domain where fluid particles exhibit more frequent movement. These regions high correspond to regions of high turbulences, vortices, or areas with intense convective heat transfer. Understanding these shaded areas is crucial as they can indicate regions of higher thermal efficiency or potential challenges in the design of heat transfer systems.
3. Fluid Particle Heating and Interactions: The incorporation of fluid particles on the square body, particularly when heated by the circular plate, leads to an increase in their intermolecular distance due to heat transfer. This phenomenon is related to the expansion of the fluid as it absorbs heat. Such thermal expansion indicates the fluid flow dynamics and heat transfer rate in the process.
4. Formation of Triangular Shape: The simulation results indicate the heat is generated in the square body at the same time a triangular mesh is used then the
particles of water tend to arrange themselves in a triangular pattern. This organization might be influenced by the boundary conditions and the convective flow patterns arising from the heated circular plate.

5. Non-Uniform Heating: If the circular heated plate does not cover the entire square body, the area near the circular plate experiences more intense heating related to the distant regions. This non-uniform heating distribution introduces complexities in the flow field, leading to variations in temperature and velocity profiles.

Overall, the study focuses on the complex interaction among heat transfer, fluid dynamics, and mesh characteristics in conduct of a heated circular plate on a square body. The findings provide valuable insights for optimizing heat transfer producing and learning flow behavior. However, it is no table that these findings are based on simulation results and further experimental confirmation is necessary for the accuracy and applicability of the observed phenomena in real-world scenarios. Additionally, conducting sensitivity analyses on various parameters would allow for a more comprehensive understanding of the system’s behavior.

c) Future Scope
1. Thermal energy storage involves technologies for collecting and storing energy for later use. It can be used for the stability of energy demand between day and night. Temperature above or below that of the ambient environment is required to maintain the thermal reservoir. Its usage takes in space heating, domestic or processing warm water process.

2. Efficient energy use is the goal to decrease the quantity of energy necessary for heating or cooling. In architecture, condensation and air wind can cause of cosmetic or structural deformation. A power audit may help to measure the implementation of recommended correct procedure. For example, insulation improvements, air shutting of mechanical leaks, the addition of energy-efficient windows and doors can be learned.

3. Smart meter is used to record electric energy consumption during intervals.

4. Thermal transition is the rate of transfer of heat through a model divided by the difference in temperature across the structure. It is measured in watts per square meter per kelvin, or W/ (m2K).

5. Thermostat is a device that automatically regulates the temperature.

6. Greenhouse effect is the representation of the exchanges of energy among the heat source, Earth’s surface, Earth’s atmosphere and the eventual sink outer space. The greenhouse is the natural rule to give benefits the animal and the human by recycle temperature. Heat emitted by the Earth surface is the defining characteristic of the greenhouse effect. This process ensures the thermal radiation from a planetary surface is absorbed by atmospheric greenhouse gases and clouds are re-radiated in all directions. As a results reduction in the amount of thermal radiation reaching space relative to reach space in the absence of absorbing materials. This reduction of outgoing radiation leads to rise in the temperature of the surface and troposphere. The rate of outgoing radiation again and again equals to the rate of heat arrives from the sun.

7. To find out how the body transfers heat? Principles of heat transfer in engineering systems can be applied to the human body. Continuous metabolism of nutrients which provides energy of the body produces heat in the animal and human body. A consistent internal temperature in human body is necessary for smooth physical
functions. Therefore, excess heat must be removed from the body. At the same time keep the internal temperature at a healthy level.

Finally, heat transfer by convection is driven by the movement of fluids over the surface body. This convective fluid can be either a liquid or a gas. For heat transfer from the outer surface of the body, the convection mechanism is dependent on the surface area of the body, the velocity of the fluid flow, temperature gradient between the surface of the skin and the ambient fluid. Significantly less temperature of the surroundings than the normal body temperature leads to heat transfer. For that reason ones feel cold when there is no enough worn cloth with him then he is exposed to a cold weather. Here clothing as insulator acts thermal resistance to heat flow over the covered portion of the body. On the other hand, abnormally excessive heat in the body may even cause death. To ensure that one portion of the body is not significantly hotter than another portion. Heat must flow evenly through the bodily tissues. Blood flowing through blood vessels acts as a convective fluid flow and helps to protect any grow up of excess heat inside the tissues of the body. The heat carried by the blood is resolute the temperature of the surrounding tissue, the diameter of the blood vessel, the thickness of the fluid, velocity of the flow and the heat transfer coefficient of the blood.

III. Physical Model

![Figure 1: Two-dimensional heat transfer in laminar fluid](image)

The mentioned problem describes the two-dimensional circular heated plate on square body horizontally. The figure indicates y axis as velocity and x axis as temperature which defines the velocity profile of the fluid flow and temperature with respect to time when the fluid will run through the heated circular plate. The circular plate is on the square body. Through this model, viscous dissipation of the fluid by heat transfer with time is determined.

a) Mathematical equation

Laminar flow and fluid properties of water find the results by using Stokes equation with the help of COMSOL MULTIPHYSICS.

Continuity equation,

\[ \nabla \cdot (\rho u) = 0. \]  \hspace{1cm} (1)

Stokes energy equation with surface pressure is
\[ \rho(u \cdot \nabla)u = \nabla \cdot \left[ -\rho I + \mu (\nabla u + (\nabla u)^T) - \frac{2}{3} \mu (\nabla \cdot u) I \right] + F. \] 

(2)

Initial value on the wall, \( u = 0 \).

Where, \( \rho \) is the density, \( l \) is the length of square body, \( u \) is the initial velocity, \( \mu \) is the Viscosity and \( F \) is the surface force.

Heat transfer in water surface

\[ \rho C_p u \cdot \nabla T = \nabla \cdot (k \nabla T) + Q + Q_{sh} + W_p. \] 

(3)

Where, \( C_p \) the specific heat at constant is pressure and \( W_p \) is the constant pressure in water.

Thermal insulation

\[ -n(-k \nabla T) = 0. \] 

(4)

Where, \( k \) is the Thermal conductivity, \( T \) is the Temperature.

b) Analysis of Mesh modeling

This model includes borderline layer, heat transfer free triangular mesh, heat transfer in fluid, laminar flow in Isothermal process, contour in extra fine and fine mesh are given as model. Isothermal mesh size is fine (Fig-a) and Isothermal mesh size is normal (Fig-b).

(Fig-a)  (Fig-b)

This mesh describes the molecular movement of water is driven on the heated circular plate on a square body. In this mesh, material is taken as water and the properties are Minimum element quality (0.7615), Average element quality (0.9789), Triangular elements (16832), Edge elements (404), and Vertex elements (8). The
calibrated parameters are Maximum element size (0.0091), Minimum element size (0.00105), Resolution of curvature (0.25), Maximum element growth rate (1.08) and Extra fine size.

In this model (Fig-e and Fig-f) a simulation is taken as, heat transfer of free triangular mesh is shown red and gray color where due to heated circular plate the heat is generated to square body and a triangular shape will be formed by the particles of water.

The boundary layer is displayed in the mesh by blue color lining which keep the water particles inside the square body. After heat generation, the water particles become dispersed and create dense circular shape seems like a hole. (Fig-c) and (Fig-d) are Heat transfer of free triangular mesh.

In this model (Fig-e and Fig-f) a simulation is taken as, heat transfer of free triangular mesh is shown red and gray color where due to heated circular plate the heat is generated to square body and a triangular shape will be formed by the particles of water.
This figure represents the circular heated plate which is not generated to the whole square body but the area nearer to the circular plate is heated and shows a densely circular area comparative to the distant area.

In this Fig-i & j, fluid is incorporated on the square body along with heated circular plate. Here the water particles are heated and the intermolecular distance increases. Some dense area represents the movements of the particles through which it can be found that the water particles are moving discretely due to temperature. Color of the meshes is green and blue.
In these Fig-k & l, Laminar flow is considered by fluid particles the smooth paths on layers, fluid particles moving smoothly past the adjacent layers with small or no partying. The fluid particles tend to flow at low velocities without lateral mixing, and adjacent layers meshes past one another.

Contour of the mesh (Fig-m).

This (Fig-m) contour represents the identical curved outlines of the movement of the fluid particles when heat is generated by the circular plate on the square body. Different shapes of the water particles are visible in the mesh by contour linings which represents the condition of the particles. The interpolation data can round structured and defined on a general point rain cloud.
Thermal conduction area (Fig-0). Laminar flow wall (Fig-n).

In these Fig-n & o, boundary layer meshing simulations creates isotropic meshes close to walls without having to use swept meshes or specially designated domains. These fluid particles are required due to the boundary layer that typically forms at no-slip walls. Boundary layer meshes are added after the domain has been meshed. A triangular path of particles is pushed into the computational domain. Observe that the qualities of elements are good in spite of isotropy. Due to boundary layer mesh built from triangles with high quality, and that results in high-quality prismatic elements as well.

IV. METHODOLOGY

The continuity equations, stokes energy equation, heat transfer equation and thermal insulation and natural conservation equations are transformed into a system of integral equations by using the Galerkin weighted residual method of finite-element formulation. The nonlinear algebraic equations so obtained are modified by imposition of boundary conditions. These modified nonlinear equations are transferred into linear algebraic equations with the help of Newton’s method. Lastly, these linear equations are solved by applying Triangular factorization. For numerical computation and post processing, the software COMSOL MULTIPHYSICS is used.

V. RESULTS AND DISCUSSION

In different type of meshes Model is change for their temperature changes. There are some shaded area in which the frequent movement of fluid particle is more. If the fluid is incorporated on the square body along with heated circular plate then the water particles are heated and the intermolecular distance increases due to heat transfer. A Simulation is taken as, heat transfer of free triangular mesh due to heated circular plate, if the heat is generated to square body and a triangular shape will be formed by the particles of water. If the circular heated plate is not generated to the whole square body, the area nearer to the circular plate is heated then the densely circular area comparatively to the distant area. Results are presented with the help of graph.
**Fig-p: Temperature (T)**

This graph (Fig-p) shows the change of viscosity with respect to temperature. For increasing the temperature viscosity decreased.

**Fig-q: Temperature (T)**

This graph (Fig-q) shows the change of Heat capacity at constant pressure (Cp) with respect to temperature. For increasing the temperature Heat capacity at constant pressure (Cp) also increases.

**Fig-r: Temperature (T)**

This graph (Fig-r) shows the change of Density (rho) with respect to temperature. For increasing the temperature Density (rho) gradually decreases and minimum at maximum temperature.
This graph (Fig-s) shows the change of Thermal Conductivity ($k$) with respect to temperature. For increasing the temperature Thermal Conductivity ($k$) increases and at a certain temperature becomes maximum after that for increasing temperature Thermal Conductivity ($k$) gradually decreases.

This graph (Fig-05) shows the velocity profile of laminar flow fluid has a definite viscosity. Interpolation function is defined by a graph containing the values of the function in discrete points. Variation of given temperature, the discrete points of interpolation are varied according to the different value of temperature. The interpolation is a process of predicting unknown data points such as fluid that is the speed of rainfall.

VI. Conclusion

Solving the numerical model equations for multiphase flow may be a very challenging task, even with access to supercomputers. In reality, these models are limited to Nano fluids and for the study of natural surfaces of viscous liquids. The disseminated multiphase flow phenomena allow for the studying of systems with millions and billions of bubbles, droplets, or particles. But even the simplest dispersed flow models can lead to the generation of very complicated and daunting model equations. The development of these models into variations that are well adapted to describe specific mixtures has allowed for engineers and scientists to study multiphase flow with a relatively good accuracy and with reasonable computational costs. We have successfully constructed many mesh models through our studied techniques which are
shown in Figures a to o and we have also discussed 2D graphical representation through the proposed techniques which are shown in Figures p to t. In future, we will try to apply our results in fluid flow, natural convection, Radiation, solar energy, engineering sector, medical science and many others fields.

**References Références Referencias**


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Distinguished Couple of Integer Right Triangles and Canada Numbers

By Janaki G & Gowri Shankari A
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Abstract- We propose a couple of integer right triangles whose perimeter differences are each equal to four times the Canada number. We also provide the number of couples containing primitive and non-primitive integer right triangles.

Keywords: couple of integer right triangles, canada number, primitive and non-primitive integer right triangles.

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I. Introduction

The theory of numbers is a fascinating area of mathematics. Right integer triangles have attracted the attention of many mathematicians and math enthusiasts because it is a treasure house where finding many hidden connections is like going on a treasure hunt. Refer to [1]-[3] for various fascinating challenges. In addition to polygonal numbers, we also have the Jarasandha numbers, Nasty numbers, Dhuruva numbers, and Canada Numbers, which are all intriguing patterns of numbers. These figures are displayed in [4]-[9]. Special Pythagorean triangles linked to Nasty and polygonal numbers are derived in [10]-[15].

In this writing, we look for a distinguished couple of right integer triangles where the difference in their perimeters in each pair is four times the Canada numbers.

II. Basic Definitions

Definition: 1

The ternary quadratic Diophantine equation given by \( s^2 + t^2 = r^2 \) is known as Integer right equation, where \( s, t \) and \( r \) are natural numbers and denotes it by \( \Delta(s, t, r) \). Also, in \( \Delta(s, t, r) \): \( s^2 + t^2 = r^2 \), \( s \) and \( t \) are called its legs and \( r \) its hypotenuse.

Definition: 2

The most cited solution of the Integer right equation is

\[ s = a^2 - b^2, \quad t = 2ab, \quad r = a^2 + b^2, \]

where \( a > b > 0 \). If \( a \) and \( b \) have opposing parities and \( \gcd(a, b) = 1 \), then this solution is referred to as primitive.
Definition: 3
Canada numbers are those $n$ such that the sum of the squares of the digits of $n$ is equal to the sum of the non-trivial divisors of $n$, i.e., $\sigma(n) - n - 1$. The Canada numbers are 125, 581, 8549 and 16999.

The name of these numbers is due to the fact they were defined by some mathematicians from Manitoba University to celebrate the $125^{th}$ anniversary of Canada.

III. Materials and Methods

Let $\Delta_1$ and $\Delta_2$ be two distinct right integer triangles with generators $a, c$ ($a > c > 0$), $b, c$ ($b > c > 0$) respectively. Let $P_1$ and $P_2$ be the perimeters and $\Lambda_1$ and $\Lambda_2$ be the areas of $\Delta_1$ and $\Delta_2$ such that $P_1 - P_2 = 4$ times the 1st Canada number 125. The equation derived from the relationship above is

$$(2a+c)^2 - (2b+c)^2 = 1000$$

(1)

It is observed after completing numerical calculations that there are 20 different values for $a, c$, and $b$ satisfied (1) provided $a + b + c =$ Canada number

The values of $a, c, b, P_1$ and $P_2$ are shown in Table I below for clarity and simplicity.

Table I

<table>
<thead>
<tr>
<th>S. No.</th>
<th>$a$</th>
<th>$c$</th>
<th>$b$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$\frac{P_1 - P_2}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>44</td>
<td>39</td>
<td>42</td>
<td>7304</td>
<td>6804</td>
<td>125</td>
</tr>
<tr>
<td>2.</td>
<td>45</td>
<td>37</td>
<td>43</td>
<td>7380</td>
<td>6880</td>
<td>125</td>
</tr>
<tr>
<td>3.</td>
<td>46</td>
<td>35</td>
<td>44</td>
<td>7452</td>
<td>6952</td>
<td>125</td>
</tr>
<tr>
<td>4.</td>
<td>47</td>
<td>33</td>
<td>45</td>
<td>7520</td>
<td>7020</td>
<td>125</td>
</tr>
<tr>
<td>5.</td>
<td>48</td>
<td>31</td>
<td>46</td>
<td>7584</td>
<td>7084</td>
<td>125</td>
</tr>
<tr>
<td>6.</td>
<td>49</td>
<td>29</td>
<td>47</td>
<td>7644</td>
<td>7144</td>
<td>125</td>
</tr>
<tr>
<td>7.</td>
<td>50</td>
<td>27</td>
<td>48</td>
<td>7700</td>
<td>7200</td>
<td>125</td>
</tr>
<tr>
<td>8.</td>
<td>51</td>
<td>25</td>
<td>49</td>
<td>7752</td>
<td>7252</td>
<td>125</td>
</tr>
<tr>
<td>9.</td>
<td>52</td>
<td>23</td>
<td>50</td>
<td>7800</td>
<td>7300</td>
<td>125</td>
</tr>
<tr>
<td>10.</td>
<td>53</td>
<td>21</td>
<td>51</td>
<td>7844</td>
<td>7344</td>
<td>125</td>
</tr>
<tr>
<td>11.</td>
<td>54</td>
<td>19</td>
<td>52</td>
<td>7884</td>
<td>7384</td>
<td>125</td>
</tr>
<tr>
<td>12.</td>
<td>55</td>
<td>17</td>
<td>53</td>
<td>7920</td>
<td>7420</td>
<td>125</td>
</tr>
<tr>
<td>13.</td>
<td>56</td>
<td>15</td>
<td>54</td>
<td>7952</td>
<td>7452</td>
<td>125</td>
</tr>
<tr>
<td>14.</td>
<td>57</td>
<td>13</td>
<td>55</td>
<td>7980</td>
<td>7480</td>
<td>125</td>
</tr>
<tr>
<td>15.</td>
<td>58</td>
<td>11</td>
<td>56</td>
<td>8004</td>
<td>7504</td>
<td>125</td>
</tr>
<tr>
<td>16.</td>
<td>59</td>
<td>9</td>
<td>57</td>
<td>8024</td>
<td>7524</td>
<td>125</td>
</tr>
<tr>
<td>17.</td>
<td>60</td>
<td>7</td>
<td>58</td>
<td>8040</td>
<td>7540</td>
<td>125</td>
</tr>
<tr>
<td>18.</td>
<td>61</td>
<td>5</td>
<td>59</td>
<td>8052</td>
<td>7552</td>
<td>125</td>
</tr>
<tr>
<td>19.</td>
<td>62</td>
<td>3</td>
<td>60</td>
<td>8060</td>
<td>7560</td>
<td>125</td>
</tr>
<tr>
<td>20.</td>
<td>63</td>
<td>1</td>
<td>61</td>
<td>8064</td>
<td>7564</td>
<td>125</td>
</tr>
</tbody>
</table>

Thus, it can be observed that there are 20 couples of right integer triangles, where each couple’s difference in perimeters equals four times the first Canada number (125). Of these 20 couples, ten are non-primitive, six are primitive, and four are couples, where one is a primitive triangle and the other is non-primitive.

The following Table II illustrates a similar observation of other Canada numbers:
### Table II

<table>
<thead>
<tr>
<th>Canada Number</th>
<th>Couples of Right integer Triangles</th>
<th>Couples of primitive Right integer Triangles</th>
<th>Couples of non-primitive Right integer Triangles</th>
<th>Couples of primitive and non-primitive Right integer Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>581</td>
<td>96</td>
<td>28</td>
<td>50</td>
<td>18</td>
</tr>
<tr>
<td>8549</td>
<td>1424</td>
<td>337</td>
<td>733</td>
<td>354</td>
</tr>
<tr>
<td>16999</td>
<td>2833</td>
<td>903</td>
<td>1438</td>
<td>492</td>
</tr>
</tbody>
</table>

### IV. Conclusion

In this article, we propose a couple of integer right triangles whose perimeter differences are each equal to four times the Canada number. We also provide the number of couples containing primitive and non-primitive integer right triangles. In conclusion, one can look for relationships between distinguished couples of integer right triangles and other unique numbers and number patterns.

### References Références Referencias

Development of Mathematical Simulation of Hydrodynamic Oscillation Generators

By A. S. Korneev

Abstract- The unsteady turbulent swirled water flow in a channel in the presence of cavitation is calculated. The comparison of two forms of corrections to the k-ε-model of turbulence, taking into account the swirl of the flow, is performed as applied to the problem of calculating hydrodynamic oscillation generators. It is shown that both considered corrections it made possible to achieve agreement between the calculated and experimental data on the pressure distribution along the wall of the generator channel and on the form of the amplitude-frequency characteristics of oscillations. At the same time, the linear correction it made possible to improve the stability of the calculation procedure and prevent the appearance of zones with non-physical negative pressures, which in some cases were obtained using a quadratic correction. The results obtained can be used in mathematical modeling of hydrodynamic oscillation generators for various purposes, particularly for chemical technologies, oil production and medicine.

Keywords: hydrodynamic generators of oscillations, turbulence, swirl, mathematical modeling, pressure distribution, oscillation characteristics, fluid dynamics.

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Development of Mathematical Simulation of Hydrodynamic Oscillation Generators

A. S. Korneev

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Keywords: hydrodynamic generators of oscillations, turbulence, swirl, mathematical modeling, pressure distribution, oscillation characteristics, fluid dynamics.

I. Introduction

Hydrodynamic oscillation generators [1, 2] produce pressure waves when a fluid flows through channels of specific shapes and dimensions. These generators have no movable elements, which provides their reliability and long service life. Such devices can be used to disperse gas in liquid in many fields of chemical technologies [3], in systems for the biological purification of waste water and the chlorination and ozonization of drinking water, etc [2]. Generators of this type for medical purposes work like hydromassages [4]. Auspicious is the use such devices in oil production [5] to intensify oil production processes and enhance oil recovery. The results of wave technology processing of more than 1000 oil wells showed that the oil well productivity increased by an average of 30–40%, and the oil recovery increased by 5–10%.

It is necessary to develop methods for mathematical modeling to improve the performances of hydrodynamic generators. This method was presented in [6]. The system of continuity equation and Reynolds-averaged Navier–Stokes equations [7] with the ”standard” k-ε turbulence model [8, 9] and the full cavitation model [10] was solved. The amplitude-frequency characteristics of the oscillations were calculated. The calculated positions of the amplitude maxima agreed with the available experimental data.

However, the experience of using the model [6] showed that for some values of the operating parameters of the generators, instabilities of the calculation procedure arose, leading to the appearance of the negative pressure in the channel, which increased infinitely in absolute value. In addition, for those cases when these
instabilities did not arise, the calculated pressure distributions along the channel wall differed markedly from the experimental values. It was suggested that these problems are related to the flow swirl, which is not considered in the “standard” k-ε turbulence model [8, 9]. There are several experimental works that have shown that the swirl of the flow can lead to the weakening of turbulent pulsations.

In particular, Murakami and Kikuyama [12] studied the turbulent flow of water in a rotating tube. The water was supplied into a long fixed pipe with a diameter of D = 32 mm and a length of 60D, forming a steady turbulent longitudinal velocity profile. The unswirled flow entered the rotating pipe and was involved in the rotation due to friction against the walls. The rotating tube had segments of various lengths: 30D, 50D, 70D, 120D, 140D and 160D. Between these segments there were receivers of total pressure, which could move along the radius. The velocity profile was determined from the value of the dynamic pressure. The liquid then entered a fixed outlet pipe 200D long. The measurements showed that when moving along a rotating pipe, the profile of the axial velocity component changed and transformed from a steady turbulent to a parabolic one, which is typical for a laminar flow.

A. I. Borisenko, O. N. Kostikov, and V. I. Chumachenko [13] used a hot-wire anemometer to measure the intensity of turbulent pulsations in a rotating pipe with a diameter of D = 52 mm, through which an airflow passed. It was found that the intensity of pulsations decreases in the rotating channel. This process began at the wall, and as it moved away from the inlet, it reached the central region of the pipe. Thereby, under conditions typical of [12] and [13], the swirl led to flow laminarization. This must be taken into account in the mathematical model.

A correction to the k-ε turbulence model for the flow swirl was proposed in [11]. The accounting for this amendment made possible to achieve agreement between the calculated and experimental data on the pressure distribution along the channel wall of the hydrodynamic generator and to calculate amplitude-frequency characteristics. However, even with this correction, the instabilities of the calculation procedure as mentioned earlier arose in a number of cases. To overcome them, this paper proposes another form of correction for flow swirl.

II. Mathematical Model

The system of continuity equation (1) and Reynolds-averaged Navier–Stokes equations (2), (3), (4) for an axisymmetric flow [7] and the two-equation (5), (6) turbulence model [8, 9] was solved:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial z} + \frac{\partial \rho v}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \mu_s \frac{\partial u}{\partial r} \right) = 0; \tag{1}
\]

\[
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + \rho v \frac{\partial u}{\partial r} = -\rho \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \mu_s \frac{\partial u}{\partial r} \right); \tag{2}
\]

\[
\frac{\partial \rho}{\partial t} + \rho u \frac{\partial \rho}{\partial z} + \rho v \frac{\partial \rho}{\partial r} = -\rho \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \mu_s \frac{\partial \rho}{\partial r} \right) - \frac{\mu_s}{r^2} \frac{\partial (r u)}{\partial r} + \frac{D^2}{r} \frac{\partial^2 u}{\partial r^2}; \tag{3}
\]
\[
\rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial z} + \rho v \frac{\partial w}{\partial r} = \frac{\partial}{\partial z}\left( \mu_s \frac{\partial w}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r}\left( r \mu_s \frac{\partial w}{\partial r} \right) - \mu_s \frac{w}{r^2} - \rho \frac{\partial w}{\partial t} ; \tag{4}
\]

\[
\rho \frac{\partial k}{\partial t} + \rho u \frac{\partial k}{\partial z} + \rho v \frac{\partial k}{\partial r} = \frac{\partial}{\partial z}\left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial z} \right] + \frac{1}{r} \frac{\partial}{\partial r}\left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial r} \right] + G - \rho \varepsilon ; \tag{5}
\]

\[
\rho \frac{\partial \varepsilon}{\partial t} + \rho u \frac{\partial \varepsilon}{\partial z} + \rho v \frac{\partial \varepsilon}{\partial r} = \frac{\partial}{\partial z}\left[ \left( \mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial z} \right] + \frac{1}{r} \frac{\partial}{\partial r}\left[ \left( \mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial r} \right] + C_1 \frac{\varepsilon}{k} G - C_2 \frac{\varepsilon^2}{k}. \tag{6}
\]

Here

\[ G = G_{u,v} + G_w ; \]

\[ G_{u,v} = \mu_t \left\{ 2 \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{u}{r} \right)^2 \right] + \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \right)^2 \right\} ; \]

\[ G_w = \mu_t \left[ \left( \frac{\partial w}{\partial z} \right)^2 + \left( \frac{r}{\partial r}\left( \frac{w}{r} \right) \right)^2 - \frac{\partial}{\partial r}\left( \frac{w^2}{r} \right) \right] ; \]

\[ C_1 = 1.44 ; \quad C_2 = 1.92 ; \quad C_\mu = 0.09 ; \quad \sigma_k = 1.0 ; \quad \sigma_\varepsilon = 1.3 . \]

\[ \mu_s = \mu + \mu_t ; \]

Turbulent viscosity \( \mu_t \) was defined as

\[ \mu_t = C_\mu \rho f_\mu \frac{k^2}{\varepsilon} ; \tag{7} \]

Two forms of corrections \( f_\mu \) to turbulent viscosity, taking into account the flow swirl, were studied: quadratic (8) and linear (9):

\[ f_\mu = 1 - \frac{w^2}{u^2 + v^2 + w^2} . \tag{8} \]

\[ f_\mu = 1 - \frac{|w|}{|u| + |v| + |w|} . \tag{9} \]

Here \( u, v, \) and \( w \) are the axial, radial, and tangential velocity components, \( \rho \) is the fluid density, \( k \) is the turbulence kinetic energy, and \( \varepsilon \) is the turbulence energy dissipation rate. In the absence of a swirl \( (w = 0) \), the presented turbulence model with each of the corrections (8) and (9) goes into the standard one [8].

The cavitation was taken into account using the equation of transfer of the vapor mass fraction \[10\]

\[
\rho \frac{\partial f_v}{\partial t} + \rho u \frac{\partial f_v}{\partial z} + \rho_v \frac{\partial f_v}{\partial r} = R_{ce}.
\]

\[10\]

\[
p \leq p_v : \quad R_{ce} = C_e \frac{\rho_l \rho_v}{\sigma} (1 - f_v) \sqrt{\frac{2(p_v - p)k}{3\rho_l}};
\]

\[11\]

\[
p \geq p_v : \quad R_{ce} = -C_e \frac{\rho_l \rho_v}{\sigma} (1 - f_v) \sqrt{\frac{2(p - p_v)k}{3\rho_l}};
\]

\[12\]

\[
\rho = \left(\frac{f_v}{\rho_v} + \frac{1 - f_v}{\rho_l}\right)^{-1}; \quad \mu = \left(\frac{f_v}{\mu_v} + \frac{1 - f_v}{\mu_l}\right)^{-1}.
\]

\[13\]

Here, \(f_v\) is the mass fraction of vapor, \(\rho_l\) is the density of the liquid, \(\rho_v = \rho_{sat} M_v / (RT)\) is the density of saturated vapor, \(p_{sat}\) is the pressure of saturated vapor of the liquid at temperature \(T\), \(M_v\) is the molar mass of vapor, \(\rho_v = \rho_{sat} + p_{turb}/2\) is the phase-change threshold pressure, \(p_{turb} = 0.39pk\) is the turbulent pressure fluctuations, \(R_{ce}\) is the rate of evaporation of the liquid (at \(R_{ce} > 0\)) or steam condensation (at \(R_{ce} < 0\)), \(\sigma\) is the surface tension.

The system of equations (1) - (10) was solved by the pressure correction method [14] using the SIMPLE algorithm. In this method, instead of the continuity equation, the equation derived from it for corrections to pressure \(p'\) is solved. As the steady state solution is reached in the cycle of pressure iterations, the corrections \(p'\) approach zero.

Near the channel walls, in a laminar sublayer the near-wall functions proposed in [9] were used instead of Eq. (6).

In the calculation scheme of the generator (Fig. 1), the segments \(AB\) and \(EF\) are the face walls, \(BE\) is the cylindrical wall of the channel, \(FG\) is the cylindrical wall of the operation chamber, and \(GH\) is the output cross-section. The supply orifices were placed in segment \(CD\). The cross-section \(I-I\) was set through the axes of symmetry of these orifices.

Figure 1: The scheme of the calculation region
As boundary conditions on solid walls $AB, BC, DE, EF,$ and $FG$ (Fig. 1), the no-slip conditions were set: $u = v = w = 0, p' = 0, k = 0, \partial \varepsilon / \partial n = 0, f_v = 0$. Where $n$ is the coordinate along the normal to the wall.

On the axis of symmetry $AH$: $\partial u / \partial r = 0, v = w = 0, \partial p'/\partial r = 0, \partial p / \partial r = 0, \partial k / \partial r = 0, \partial \varepsilon / \partial r = 0, \partial f_v / \partial r = 0$.

In the region of fluid inlet $CD$: $u = 0, v = v_0, w = w_0, k = k_0, \varepsilon = \varepsilon_0, f_v = 0$. Here $v_0 = Q / (2\pi R_C d_0')$, $w_0 = 4Q / (\pi d_0^2 n_0)$, $k_0 = (k_in v_0)^2 / 2$, $\varepsilon_0 = C_k k_0 \sqrt{k_0} / (0.1 d_0)$, $Q$ is the volumetric flow rate of the liquid, $n_0$ is the number of supply orifices. In this work it was taken $n_0 = 2, k_in = 0.05$.

At the output of $GH$: $p = p_{out}$, and “soft” boundary conditions $\partial F / \partial z = 0$, where $F$ is one of the variables: $u, v, w, p', k, \varepsilon$, and $f_v$.

III. Object of Investigation

An especial test generator (Fig. 2, a) was developed [16] to verify the mathematical model.

This generator made it possible to obtain the pressure distribution along the channel wall in order to compare calculated and experimental data.

The generator was of a cylindrical channel with a flaring part (operation chamber). The working fluid (tap water) was fed through two tangential orifices ensuring swirl of the flow. Six orifices in planes $xz$ and $yz$ have been made on a channel wall for static pressure measurement. Using of tubes the orifices were connected to manometers.

Tap water was chosen due to its availability. These generators can also operate on other liquids, particularly mineral oil and special drilling fluids.

![Figure 2](image.png)

*Figure 2: The test generator: (a) three-dimensional model, (b) scheme of experimental facility: (1) pump, (2) hydrodynamic generator, (3) manometers, (4) operation chamber, (5) pressure sensor, (6) oscilloscope, (7) throttle, (8) flowmeter, and (9) adjusting ventil.*
The experimental results of the study of this test generator were given in [16]. Experiments were performed on the facility, the scheme of which is presented in Fig. 2,b. The tap water was fed into the input of pump 1 and directed into the generator 2 under the pressure $p_i$. The water pressure was measured by manometers 3 of the accuracy class 1. After the generator, the water entered operation chamber 4. The piezoelectric pressure sensor 5 of type 701A produced by Kistler was mounted to measure the pressure pulsations. The oscilloscope LeCroy Wave Surfer® MXs-B 6 was used to record and proceed the spectra. The used water went to the drain through throttle 7, flowmeter 8, and adjusting ventil 9. The required pressure $p_{out}$ in the chamber was set using ventil 9.

**IV. RESULTS AND DISCUSSIONS**

The results presented below were obtained at the value of absolute pressure of water $p_i = 5.1$ MPa at the generator input and $p_{out} = 0.24$ MPa at the generator output. The flow rate was equal to $Q = 23.3$ dm$^3$/min. The Reynolds number computed by the averaged velocity in the channel was $Re \approx 50000$. These parameters corresponded to values typical for generators used in chemical technologies.

**a) Time dependences of the pressure**

The calculated dependences of pressure $p$ on time $t$ on the axis of symmetry $(r = 0 \, \text{mm})$ in the cross-section of the supply orifices $(z = 5 \, \text{mm})$ are shown in Fig. 3.

![Graph a](image)

**Graph a:** The time dependences of the pressure at the axis of symmetry at the plane of input orifices $(r = 0 \, \text{mm}, \ z = 5 \, \text{mm})$: (a) without correction for the swirl $(f_\mu = 1 )$, (b) with a quadratic correction $f_\mu$ according to (8), (c) with a linear correction $f_\mu$ according to (9).

**Figure 3:** The time dependences of the pressure at the axis of symmetry at the plane of input orifices $(r = 0 \, \text{mm}, \ z = 5 \, \text{mm})$: (a) without correction for the swirl $(f_\mu = 1)$, (b) with a quadratic correction $f_\mu$ according to (8), (c) with a linear correction $f_\mu$ according to (9).
The calculation without correction for the swirl, $\mu = 1$ (Fig. 3, a), led to the damping of pressure fluctuations with time. In contrast, in experiments [15, 16] these fluctuations existed constantly. The calculations with a quadratic correction for swirl according to expression (8) (Fig. 3, b) gave undamped oscillations in time. However, at some time intervals, instabilities of the calculation procedure were observed with a sharp increase in pressure in absolute value, which disappeared as the calculation continued. This significantly increased the calculation time required to achieve a steady state of oscillations. In addition, non-physical negative pressure values appeared at some channel points. The calculations with a linear correction for the swirl according to expression (9) (Fig. 3, c) gave stable fluctuations in time. In this case, there were no negative pressures.

Similar calculated data at the location of the pressure sensor in the experiments [15, 16] (r = 14 mm, z = 100 mm) are shown in Fig. 4.

Figure 4: The time dependences of the pressure at the location of the pressure sensor (r = 14 mm, z = 100 mm): (a) without correction for the swirl ($\mu = 1$), (b) with a quadratic correction $\mu$ according (8), (c) with a linear correction $\mu$ according (9).
b) Amplitude-frequency characteristics of the oscillations

By Fourier transformations of the data presented in Fig.4,b and 4,c, for time intervals of steady oscillations \( t = 0.6 \ldots 1.5 \) s, the calculated amplitude-frequency characteristics of the oscillations were obtained (Fig. 5,a and 5,b). The corresponding experimental frequency response according to [15] is shown in Fig. 5, c.

![Image of amplitude-frequency characteristics](image)

**Figure 5:** The amplitude-frequency characteristics of the oscillations at the location of the pressure sensor (\( r = 14 \) mm, \( z = 100 \) mm): (a) calculations with a quadratic correction \( f_p \) according to expression (8), (b) with a linear correction \( f_\mu \) according to expression (9), (c) experiment [15].

The experimental values of the amplitudes in Fig. 5,c are presented in units of the oscilloscope scale (millivolts). The experimental frequency characteristic exhibits two principal maxima at frequencies \( f = 1.04 \) kHz and \( f = 1.98 \) kHz. The calculated frequency characteristic with a quadratic correction according to expression (8) yielded four maxima at \( f = 0.42, 0.83, 1.48, \) and \( 2.90 \) kHz. The calculation with a linear correction using expression (9) gave the values of three principal maxima corresponding to the frequencies \( f = 0.94, 1.55, \) and \( 1.98 \) kHz.
c) The pressure distribution along a wall of the channel

Experimental pressure distributions [16] along the wall of the generator channel \( (r = 5 \, \text{mm}) \) in a plane parallel to the axes of the supply orifices are shown in Fig. 6, points 1, and in the plane perpendicular to these axes in Fig. 6, points 2. It can be seen that deviations from axisymmetry manifest themselves up to distances \( z = 10 \ldots 12 \, \text{mm} \) from the left end wall of the channel.

\[ \text{Figure 6: The distribution of the time-averaged pressure along the wall of the generator channel: points (1) experiment in a plane } xz \text{ parallel to the axes of the supply orifices, points (2) experiment in a plane } yz \text{ perpendicular to the axes of the supply orifices, line (3) calculation without correction for the swirl (} f_{\mu} = 1 \text{ ), line (4) calculation with quadratic correction } f_{\mu} \text{ according to expression (8), line (5) calculation with a linear correction } f_{\mu} \text{ according to expression (9)} \]

The calculations without considering the correction for the flow swirl (Fig. 6, line 3) showed a noticeable deviation from the experimental values. The calculated data obtained in the axisymmetric approximation with a quadratic correction \( f_{\mu} \) (Fig. 6, line 4) according to expression (8) are between the experimental points, and with a linear correction \( f_{\mu} \) (Fig. 6, line 5) according to expression (9) deviate slightly from experimental points.

V. Conclusions

1. In conditions, which are typical for hydrodynamic generators, the swirl of a stream leads to laminarization of the current. It occurs owing to the stabilising influence of a field of centrifugal forces. Besides, because of centrifugal effects the liquid is rejected on stream periphery where there is a damping of turbulent pulsations of speed due to a friction about a wall.

2. A comparison of two forms of corrections for the flow swirl in the k-ε-model of turbulence is carried out. The accounting for these corrections it made possible to achieve consistency between the calculated and experimental data on the pressure distribution along the channel wall of the hydrodynamic generator and it made
possible to calculate the amplitude-frequency characteristics of oscillations. If there is no swirl, both models shown are automatically converted to the standard $k$-$\varepsilon$-model of turbulence.

3. The results obtained can be used in mathematical modeling of hydrodynamic oscillation generators for various purposes, particularly for chemical technologies, oil production and medicine.

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The study was carried out using the computational resources of the Joint Supercomputer Center of the Russian Academy of Sciences (JSCC of RAS).

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6. **Bookmarks are useful:** When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. **Revise what you wrote:** When you write anything, always read it, summarize it, and then finalize it.

8. **Make every effort:** Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. **Produce good diagrams of your own:** Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. **Use proper verb tense:** Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. **Pick a good study spot:** Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. **Know what you know:** Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. **Use good grammar:** Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

   Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. **Arrangement of information:** Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. **Never start at the last minute:** Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. **Multitasking in research is not good:** Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. **Never copy others’ work:** Never copy others’ work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. **Go to seminars:** Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. **Refresh your mind after intervals:** Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.
20. **Think technically:** Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. **Adding unnecessary information:** Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. **Report concluded results:** Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. **Upon conclusion:** Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

**Informal Guidelines of Research Paper Writing**

**Key points to remember:**
- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

**Final points:**

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

**The introduction:** This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

**The discussion section:**

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

**General style:**

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

**To make a paper clear:** Adhere to recommended page limits.
Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don’t address the reviewer directly. Don’t use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.
The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

*Materials may be reported in part of a section or else they may be recognized along with your measures.*

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.
Results:
The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective
details of the outcome, and save all understanding for the discussion.
The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to
present consequences most efficiently.
You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data
or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if
requested by the instructor.

Content:
- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if
  appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or
  manuscript.

What to stay away from:
- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:
As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.
If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:
If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached
appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and
include a heading. All figures and tables must be divided from the text.

Discussion:
The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded
based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the
paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results
and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The
implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain
mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have
happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the
data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded
or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

**Approach:**

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

**The Administration Rules**

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*Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.*

**Segment draft and final research paper:** You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

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