Online ISSN : 2249-4626
Print ISSN : 0975-5896
DOI : 10.17406/GJSFR

Global Journal

of Science Frontier Research: F

Mathematics and Decision Science

Highlights

- Fluid's Velocity Components
- Toxin Avoidance and Harvesting
- Transfer of Natural Convection
- Couple of Integer Right Triangles

Discovering Thoughts, Inventing Future

VOLUME 23 ISSUE 8 VERSION 1.0

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Volume 23 Issue 8 (Ver. 1.0)

Open Association of Research Society

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

Global Journals Inc.

*(A Delaware USA Incorporation with "Good Standing"; **Reg. Number: 0423089**)*

Sponsors: *Open Association of Research Society*
Open Scientific Standards

Publisher's Headquarters office

Global Journals® Headquarters
945th Concord Streets,
Framingham Massachusetts Pin: 01701,
United States of America
USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

Global Journals Incorporated
2nd, Lansdowne, Lansdowne Rd., Croydon-Surrey,
Pin: CR9 2ER, United Kingdom

Packaging & Continental Dispatching

Global Journals Pvt Ltd
E-3130 Sudama Nagar, Near Gopur Square,
Indore, M.P., Pin:452009, India

Find a correspondence nodal officer near you

To find nodal officer of your country, please email us at *local@globaljournals.org*

eContacts

Press Inquiries: *press@globaljournals.org*
Investor Inquiries: *investors@globaljournals.org*
Technical Support: *technology@globaljournals.org*
Media & Releases: *media@globaljournals.org*

Pricing (Excluding Air Parcel Charges):

Yearly Subscription (Personal & Institutional)
250 USD (B/W) & 350 USD (Color)

Contents of the Issue

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Balancing Coexistence: Ecological Dynamics and Optimal Tax Policies in a Dual Phytoplankton-Zooplankton System Influenced by Toxin Avoidance and Harvesting

By Yuqin, Wensheng Yang

Fujian Normal University

Abstract- In recent years, the impact of toxic phytoplankton on ecological balance has attracted more and more ecologists to study. In this paper, we develop and analyze a model with three interacting species, poisonous and nontoxic phytoplankton, and zooplankton, including zooplankton avoiding toxic phytoplankton in the presence of non-toxic phytoplankton, and the impact of human harvest on the coexistence of these three species. We first introduce the poisonous avoidance coefficient β and the human harvest of nontoxic phytoplankton and zooplankton to investigate its impact on species coexistence. We not only find that β has a particular effect on the coexistence of these three species. But also that human harvest is an essential factor determining the coexistence of these three species. Secondly, pregnancy delay (τ_1) and toxin onset delay (τ_2) are introduced to explore the influence of time delay on the behavior of dynamic systems. When the delay value exceeds its critical value, the system will lose stability and go through Hopf bifurcation. After that, we use the principle of Pontryagin's maximum to study the optimal tax policy without delay. We obtained the optimal path of the optimal tax policy. Finally, we carry out numerical simulations to verify the theoretical results.

Keywords: toxic phytoplankton; human harvest; time delay; optimal tax policy; hopf bifurcation.

GJSFR-F Classification: (LCC): QH541.5.P4

BALANCINGCOEXISTENCEECOLOGICALDYNAMICSANDOPTIMALTAXPOLICIESINADUALPHYTOPLANKTONZOOPLANKTONSYSTEMINFLUENCEDBYTOXINAVOIDANCEANDHARVESTING

Strictly as per the compliance and regulations of:

Balancing Coexistence: Ecological Dynamics and Optimal Tax Policies in a Dual Phytoplankton-Zooplankton System Influenced by Toxin Avoidance and Harvesting

Yuqin, Wensheng Yang

Abstract- In recent years, the impact of toxic phytoplankton on ecological balance has attracted more and more ecologists to study. In this paper, we develop and analyze a model with three interacting species, poisonous and nontoxic phytoplankton, and zooplankton, including zooplankton avoiding toxic phytoplankton in the presence of non-toxic phytoplankton, and the impact of human harvest on the coexistence of these three species. We first introduce the poisonous avoidance coefficient $\boldsymbol{\beta}$ and the human harvest of nontoxic phytoplankton and zooplankton to investigate its impact on species coexistence. We not only find that $\boldsymbol{\beta}$ has a particular effect on the coexistence of these three species. But also that human harvest is an essential factor determining the coexistence of these three species. Secondly, pregnancy delay (τ_1) and toxin onset delay (τ_2) are introduced to explore the influence of time delay on the behavior of dynamic systems. When the delay value exceeds its critical value, the system will lose stability and go through Hopf bifurcation. After that, we use the principle of Pontryagin's maximum to study the optimal tax policy without delay. We obtained the optimal path of the optimal tax policy. Finally, we carry out numerical simulations to verify the theoretical results.

Keywords: toxic phytoplankton; human harvest; time delay; optimal tax policy; hopf bifurcation.

I. Introduction

Marine phytoplankton and zooplankton are essential components of marine ecosystems and support the regular operation of the entire marine ecosystem. The research of marine phytoplankton and animal ecology is conducive to our comprehensive understanding of the status of an aquatic ecosystem. Marine plankton refers to the aquatic organisms suspended in the water and moving with water flow, mainly including phytoplankton and zooplankton, as well as other organisms such as planktonic viruses, planktonic bacteria ,and archaea. Phytoplankton is the primary producer in the sea; it converts solar energy into organic energy through photosynthesis, initiates the material circulation and energy flow in the sea, and is the most basic link in the marine food chain. Zooplankton is an essential consumer in the sea; this part of organic matter is utilized through the food chain and further transferred to the upper trophic level through secondary production processes. Therefore, phytoplankton and zooplankton provide food and energy sources for the upper trophic level organisms through the above primary and secondary production processes, supporting the regular operation of the entire marine ecosystem.

Phytoplankton is not only the bottom but also the most crucial component of the marine ecosystem. It is divided into toxic and non-toxic phytoplankton. At the same time, zooplankton can distinguish different types of phytoplankton. To avoid feeding on toxic phytoplankton, which has a similar synergistic behavior with selective grazing in the predator-prey system [1-5]. In marine plankton ecosystems, the hypothetical

Author: e-mail: yyuqin0207@126.com

Ref

[5] Y.L. Zheng, X. Gong, H.W. Gao, Selective grazing of zooplankton on phytoplankton defines rapid algal succession and blooms in oceans, Ecol.l Modelling. 468 (2022) 109947.

mechanisms of selective grazing include prey morphology (size, color, shape, and colony formation), intestinal genetic strains, and poisonous chemicals released by prey [6-12]. Thus, the avoidance effect of zooplankton on toxins from toxic phytoplankton and the harmful effects of toxic compounds released by toxic species on their competitors have been studied [13-20].

In this paper, we consider not only the effect of toxin avoidance on species existence, but also the impact of human beings on the harvest of non-toxic phytoplankton and zooplankton is considered, whereas non-toxic phytoplankton on species existence and the human harvest has been applied in many models [21-27]. Since time delay is widely studied in the phytoplankton-zooplankton model [28-31], another essential purpose of our research is to explore the effect of pregnancy delay and toxin onset delay on the dynamic system. Finally, we find that optimal strategies are applied in many models to constrain overfishing [32-33]. Through the research we know that in fisheries, there is a fishing strategy called specific fishing, that is, fishermen catch almost only one particular type of fish or several species associated with it, such as these three species in our article, so we need a feedback mechanism to control this particular capture. Based on the dual phytoplankton-zooplankton system, we consider the optimal tax policy to constrain this particular fishing.

The organizational structure of this paper is as follows. In Section 2, we establish a mathematical model with double time delays for avoiding toxic species by zooplankton in the presence of non-toxic species. And give a parameter explanation in Table 2. In Section 3, we analyze the boundedness and stability of the boundary equilibrium point and the internal equilibrium point in the delay-free model. And obtain the bistability between the equilibrium points. The results are summarized in Table 1 and Fig 1. In Section 4, by analyzing different situations of this double delay model, we obtain the critical value of time delay when the system undergoes Hopf bifurcation. In Section 5, we study the optimal tax policy without time delay using the principle of Pontryagin's maximum. In addition, we use the parameters and initial values given in Table 2 and (6.1) to simulate several cases of double-delay systems in Matlab to verify all theoretical results in Section 6. Lastly, we end this paper with some conclusions and significance in Section 7.

II. Model Formulation

Considering the toxin refuge of zooplankton, a nontoxic phytoplankton-toxic zooplankton model was proposed in [14]. They showed that avoidance effects can promote the coexistence of non-toxic phytoplankton, toxic phytoplankton and zooplankton. Which can be shown as(with symbols slightly varied):

$$\begin{cases} \dfrac{dN}{dt} = r_1 N(1 - \dfrac{N + \alpha_1 T}{k_1}) - \dfrac{w_1 NZ}{p_1 + N}, \\ \dfrac{dT}{dt} = r_2 T(1 - \dfrac{T + \alpha_2 N}{k_2}) - \dfrac{w_2 TZ}{p_2 + T + \beta N}, \\ \dfrac{dZ}{dt} = \dfrac{w_1 NZ}{p_1 + N} - \dfrac{w_2 TZ}{p_2 + T + \beta N} - dZ, \\ N(0) \geq 0, \quad T(0) \geq 0, \quad Z(0) \geq 0, \end{cases} \tag{2.1}$$

where N, T ,and Z represent the biomass of nontoxic phytoplankton, toxic phytoplankton ,and zooplankton, respectively. k_1 and k_2 are the environmental carrying capacities of nontoxic phytoplankton (NTP) and toxin-producing phytoplankton (TPP) species, respectively. r_1 and r_2 represent the constant intrinsic growth rates of N and T, respectively. α_1 and α_2 measure the competitive effect of T on N, and N on T, respectively. w_1 and w_2 represent the rates at which N and T are consumed by Z, respectively. p_1 and p_2 are half- saturation constants for NTP and TPP, respectively. β represents the intensity of avoidance of T by Z in the presence of N, and d is the natural mortality of zooplankton. As the research merely focuses on a single time model, moreover overfishing has an important impact on the stability of marine ecosystems, human harvest and time delays should be taken into account. The increment in zooplankton population due to predation does not appear immediately after consuming phytoplankton; it takes some time(say τ_1), which can be regarded as the gestation period in zooplankton. The decrease of zooplankton population caused by ingestion of toxic phytoplankton does not occur immediately. Still, it requires a certain time(say τ_2), which can be regarded as the reaction time after zooplankton poisoning. Accordingly the bio-economic model with time delays on the interactions of nontoxic phytoplankton, toxic plankton and zooplankton with toxin avoidance effects, which can be shown as follows:

Ref

[6] K.G. Porter, Selective grazing and differential digestion of algae by zooplankton, Nature. 244 (1973) 179-180.

$$\begin{cases} \frac{dN}{dt} = r_1 N(1 - \frac{N + \alpha_1 T}{k_1}) - \frac{w_1 NZ}{p_1 + N} - q_1 EN, \\ \frac{dT}{dt} = r_2 T(1 - \frac{T + \alpha_2 N}{k_2}) - \frac{w_2 TZ}{p_2 + T + \beta N}, \\ \frac{dZ}{dt} = \frac{c_1 w_1 N(t-\tau_1) Z(t-\tau_1)}{p_1 + N(t-\tau_1)} - \frac{c_2 w_2 T(t-\tau_2) Z(t-\tau_2)}{p_2 + T(t-\tau_2) + \beta N(t-\tau_2)} - dZ - q_2 EZ, \\ N(0) \geq 0, \ \ T(0) \geq 0, \quad Z(0) \geq 0, \end{cases} \tag{2.2}$$

Notes

where N, T, and Z represent the biomass of nontoxic phytoplankton, toxic phytoplankton and zooplankton, respectively. $\tau_1(\tau_1 > 0)$ and $\tau_2(\tau_2 > 0)$ represent the maturation gestation delay and the toxin onset delay, respectively. c_1 and c_2 represent the conversion rate of N to Z and T to Z, respectively. Due to the experience of human capture, we assume that humans can distinguish between toxic phytoplankton and non-toxic phytoplankton when capturing zooplankton and phytoplankton. So, we put q_1 and q_2 to represent the fishing coefficients of nontoxic phytoplankton and zooplankton, respectively. And E is the effort used to harvest the population. To investigate the effect of time delay on the dynamic behavior of the model, we will first study the stability of the equilibrium point of the following model without time delay.

$$\begin{cases} \frac{dN}{dt} = r_1 N(1 - \frac{N + \alpha_1 T}{k_1}) - \frac{w_1 NZ}{p_1 + N} - q_1 EN, \\ \frac{dT}{dt} = r_2 T(1 - \frac{T + \alpha_2 N}{k_2}) - \frac{w_2 TZ}{p_2 + T + \beta N}, \\ \frac{dZ}{dt} = \frac{c_1 w_1 NZ}{p_1 + N} - \frac{c_2 w_2 TZ}{p_2 + T + \beta N} - dZ - q_2 EZ, \\ N(0) \geq 0, \ \ T(0) \geq 0, \quad Z(0) \geq 0. \end{cases} \tag{2.3}$$

III. Dynamical Behavior of Non-Delayed Model

a) Positivity and boundedness of the solution

In this subsection, firstly, we shall show the positivity and boundedness of solutions of the system (2.3), which is vital for the biological understanding of the system and the subsequent analysis.

Lemma 3.1. All the solutions with initial values of system (2.3), which start in R^3_+, are always positive and bounded.

Proof. Firstly, we rewrite the model (2.3) and take the linear as the following form:

$$\frac{dX}{dt} = F(X), \tag{3.1}$$

where $X(t) = (N, T, Z)^T \in R^3_+$ and $F(X)$ is simplified as the following

$$F(X) = \begin{bmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{bmatrix} = \begin{bmatrix} r_1 N(1 - \frac{N+\alpha_1 T}{k_1}) - \frac{w_1 NZ}{p_1+N} - q_1 EN \\ r_2 T(1 - \frac{T+\alpha_2 N}{k_2}) - \frac{w_2 TZ}{p_2+T+\beta N} \\ \frac{c_1 w_1 NZ}{p_1+N} - \frac{c_2 w_2 TZ}{p_2+T+\beta N} - dZ - q_2 EZ \end{bmatrix}.$$

We want to prove that $(N(t), T(t), Z(t)) \in R_+^3$ for all $t \in [0, +\infty)$. For system (2.3) with initial value $N(0) > 0$, $T(0) > 0$ and $Z(0) > 0$, we have

$$N(t) = N(0)\exp\{\int_0^t [r_1(1 - \frac{N(s)+\alpha_1 T(s)}{k_1}) - \frac{w_1 Z(s)}{p_1+N(s)} - q_1 E]ds\},$$

$$T(t) = T(0)\exp\{\int_0^t [r_2(1 - \frac{T(s)+\alpha_1 N(s)}{k_2}) - \frac{w_2 Z(s)}{p_2+T(s)+\beta N(s)}]ds\},$$

$$Z(t) = Z(0)\exp\{\int_0^t [\frac{c_1 w_1 N(s)}{p_1+N(s)} - \frac{c_2 w_2 T(s)}{p_2+T(s)+\beta N(s)} - d - q_2 E]ds\},$$

which shows that all the solutions of system (2.3) are always positive for all $t > 0$.

Secondly, we prove the boundedness of the solution. Let $(N(t),\ T(t),\ Z(t))$ be the solutions of system (2.3), we define a function

$$W(t) = c_1 N(t) + c_2 T(t) + Z(t). \tag{3.2}$$

Then, by differentiating (3.2) concerning t, we obtain

$$\begin{aligned}\frac{dW}{dt} + \eta W &= c_1 r_1 N(1 - \frac{N+\alpha_1 T}{k_1}) + c_2 r_2 T(1 - \frac{T+\alpha_1 N}{k_2}) - \frac{2c_2 w_2 TZ}{p_2 + T + \beta N} - dZ - q_2 EZ - c_1 q_1 EN \\ &+ c_1\eta N + c_2\eta T + \eta Z, \\ &\le c_1 r_1 N(1 - \frac{N}{k_1}) + c_2 r_2 T(1 - \frac{T}{k_2}) - dZ + c_1\eta N + c_2\eta T + \eta Z, \\ &= -\frac{c_1 r_1 N^2}{k_1} + (r_1+\eta)c_1 N - \frac{c_2 r_2 T^2}{k_2} + (r_2+\eta)c_2 T + (\eta - d)Z, \\ &\le \frac{c_1 k_1 (r_1+\eta)^2}{4r_1} + \frac{c_2 k_2 (r_2+\eta)^2}{4r_2} + (\eta - d)Z, \\ &\le \frac{c_1 r_2 k_1 (r_1+\eta)^2 + c_2 r_1 k_2 (r_2+\eta)^2}{4 r_1 r_2} + (\eta - d)Z,\end{aligned}$$

when $\eta - d < 0$, we can obtain

$$\frac{dW}{dt} + \eta W \le \frac{c_1 r_2 k_1 (r_1+\eta)^2 + c_2 r_1 k_2 (r_2+\eta)^2}{4 r_1 r_2},$$

noting $\kappa = \frac{c_1 r_2 k_1 (r_1+\eta)^2 + c_2 r_1 k_2 (r_2+\eta)^2}{4 r_1 r_2}$, therefore, applying a theorem on differential inequalities [34], we obtain $0 \le W \le \frac{\kappa}{\eta} + \frac{W(N(0),T(0),Z(0))}{e^{\eta t}}$, let $t \to +\infty$, $W(N,T,Z) \le \frac{\kappa}{\eta}$. So, all solutions of system (2.3) enter the region

$$D = \{(N,T,Z) \in \mathbb{R}_+^3 : 0 \le W(N,T,Z) \le \frac{\kappa}{\eta}\}. \tag{3.3}$$

This shows that every solution of the system is bounded.

Global Journal of Science Frontier Research (F) Volume XXIII Issue VIII Version I Year 2023

Ref

[34] J. Sotomayor, Generic bifurcations of dynamical systems, In Dynamical systems, Academic Press. 1973.

Notes

b) Equilibrium points and their stability

System (2.3) possesses six different equilibrium points:

(i) the plankton-free equilibrium, $E_0 = (0, 0, 0)$, which always exists;

(ii) TPP and zooplankton-free equilibrium, $E_1 = (k_1, 0, 0)$, which is always feasible;

(iii) NTP and zooplankton-free equilibrium, $E_2 = (0, k_2, 0)$, which is always feasible;

(iv) zooplankton-free equilibrium, $E_3 = (\widehat{N}, \widehat{T}, 0)$, where

$$\widehat{N} = \frac{\alpha_1 k_2 - k_1}{\alpha_1\alpha_2 - 1} - \frac{q_1 k_1 E}{r_1}, \quad \widehat{T} = \frac{\alpha_2 k_1 - k_2}{\alpha_1\alpha_2 - 1};$$

(v)TPP-free equilibrium $E_4 = (\bar{N}, 0, \bar{Z})$, where

$$\bar{N} = \frac{(q_2E + d)p_1}{c_1w_1 - d - q_2E}, \quad \bar{Z} = \frac{r_1(k_1 - \bar{N}) - q_1k_1E(p_1 + E)}{k_1w_1};$$

(vi)the interior equilibrium, $E^* = (N^*, T^*, Z^*)$, where

$$T^* = \frac{c_1w_1N^* - (d + q_2E)(p_1 + N^*)(p_2 + \beta N^*)}{(c_2w_2 + d + q_2E)(p_1 + N^*) - c_1w_1N^*}, \quad Z^* = \frac{(p_1 + N^*)r_1(k_1 - N^* - \alpha_1T^*) - q_1k_1E}{k_1w_1};$$

and N^* can be obtained from

$$r_2(p_2 + T^* + \beta N^*)(k_2 - T^* - \alpha_2N^*) - w_2k_2Z^* = 0. \tag{3.4}$$

Next, we illustrate the existence and stability of six equilibria when human harvest and avoidance factor exist simultaneously by solving Jacobi determinant of different equilibria, and summarize them in Table 1.

Equilibria analysis: Obviously, the equilibria E_0, E_1 and E_2 always exist. The zooplankton-free equilibrium E_3 exists, let $\hat{N}$ and $\hat{T}$ both be positive, that is $\alpha_2 > \frac{k_2}{k_1}$ and $\alpha_1 > \frac{(\alpha_1\alpha_2-1)q_1k_1E}{r_1k_1} + \frac{k_1}{k_2}$. The TPP-free equilibrium E_4 exists, let $\bar{N}$ and $\bar{Z}$ both be positive, that is $w_1 > \frac{d+q_2E}{c_1}$ and $k_1 > \frac{r_1N}{r_1-q_1E(p_1+E)}$. The interior equilibrium point E^* exists; let N^*, T^* and Z^* all be positive, that is $k_1 > \frac{q_1k_1E}{r_1} + N^* + \alpha_1T^*$, $c_2w_2(p_1 + N^*) > c_1w_1N^* - (d + q_2E)(p_1 + N^*) > 0$ and Eq.(3.4) has at least one positive root.

In the following, we summarize the eigenvalues and local stability conditions around the feasible equilibrium point of each organism of system (2.3).

(i) The eigenvalues of the plankton-free equilibrium $E_0 = (0, 0, 0)$ are r_1, r_2 and $-d - q_2E$. Therefore, it is a saddle point and hence always unstable.

(ii) The eigenvalues of the TPP and zooplankton-free equilibrium $E_1 = (k_1, 0, 0)$ are $-r_1 - q_1E$, $r_2(1 - \frac{k_1\alpha_2}{k_2})$ and $\frac{c_1w_1k_1}{p_1+k_1} - d - q_2E$. When $c_1\omega_1 - d - q_2E \leq 0$, and $\alpha_2 > \frac{k_2}{k_1}$ hold, E_1 is LAS(locally asymptotically stable). On the contrary, if $c_1\omega_1 - d - q_2E > 0$, $\alpha_2 > \frac{k_2}{k_1}$ and $k_1 < \frac{p_1(d+q_2E)}{c_1w_1-d-q_2E}$ hold, we can also obtain E_1 is LAS.

(iii) The eigenvalues of the NTP and zooplankton-free equilibrium $E_2 = (0, k_2, 0)$ are $r_2(1 - \frac{k_2\alpha_1}{k_1}) - q_1E$, $-r_2$ and $-\frac{c_2w_2k_2}{p_2+k_2} - d - q_2E$, Therefore, E_2 is LAS if $\text{k}_1 < \frac{r_2\alpha_1k_2}{r_2-q_1E}$.

(iv) The eigenvalues of the zooplankton-free equilibrium $E_3 = (\hat{N}, \hat{T}, 0)$ are $\frac{c_1w_1\hat{N}}{p_1+\hat{N}} - \frac{c_2w_2\hat{T}}{p_2+\hat{T}+\beta\hat{N}} - d - q_2E$, λ_1 and λ_2, where λ_1 and λ_2 are the roots of the equation

$$\lambda^2 + \bar{b}_1\lambda + \bar{c}_1 = 0, \tag{1}$$

where

$$\bar{b}_1 = -[r_2 - r_1 + \frac{r_1k_2(2\hat{N} + \alpha_1\hat{T}) - r_2k_1(2\hat{T} + \alpha_2\hat{N})}{k_1k_2}],$$

$$\bar{c}_1 = r_1r_2[1 - (2\hat{T} + \alpha_2\hat{N})(2\hat{N} + \alpha_1\hat{T})][\frac{1}{(2\hat{N} + \alpha_1\hat{T})k_2} + \frac{1}{(2\hat{T} + \alpha_2\hat{N})k_1} - \frac{1}{k_1k_2}]$$

$$+ q_1r_2E(\frac{k_1(2\hat{T} + \alpha_2\hat{N}) - r_1{\alpha_1}^2\hat{N}\hat{T}}{k_1k_2} - 1).$$

Therefore, let $\frac{c_1w_1\hat{N}}{p_1+\hat{N}} - \frac{c_2w_2\hat{T}}{p_2+\hat{T}+\beta\hat{N}} - d - q_2E < 0$, λ_1 and λ_2 with negative real parts, that is $\frac{c_1w_1\hat{N}}{p_1+\hat{N}} - d - q_2E < \frac{c_2w_2\hat{T}}{p_2+\hat{T}+\beta\hat{N}}$, $\bar{b}_1 > 0$ and $\bar{c}_1 > 0$. If the above conditions are satisfied, E_3 is LAS.

(v) The eigenvalues of the TPP-free equilibrium $E_4 = (\bar{N}, 0, \bar{Z})$ are $r_2(1 - \frac{\alpha_2\bar{N}}{k_2}) - \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$, where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are the roots of the equation

$$\lambda^2 - (\tilde{a}_2 + \tilde{b}_2)\lambda + \tilde{a}_2\tilde{b}_2 + \tilde{c}_2 = 0, \tag{2}$$

where

$$\tilde{a}_2 = (r_1(1 - \frac{2\bar{N}}{k_1}) - \frac{w_1p_1\bar{Z}}{(p_1+\bar{N})^2} - q_1E),$$

$$\tilde{b}_2 = (\frac{c_1w_1\bar{N}}{p_1+\bar{N}} - d - q_2E), \quad \tilde{c}_2 = \frac{c_1{w_1}^2p_1\bar{N}\bar{Z}}{(p_1+\bar{N})^3}.$$

Therefore, let $r_2(1 - \frac{\alpha_2\bar{N}}{k_2}) - \frac{w_2\bar{Z}}{p_2+\beta\bar{N}} < 0$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ with negative real parts, that is $(\tilde{a}_2 + \tilde{b}_2) < 0$ and $\tilde{a}_2\tilde{b}_2 + \tilde{c}_2 > 0$. If the above conditions are satisfied, E_4 is LAS.
(vi)By solving the Jacobi determinant of E^*, we can get its characteristic equation as follows

$$\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 = 0. \tag{3}$$

The interior equilibrium $E^* = (N^*, T^*, Z^*)$ is LAS if
(a) $D_1 > 0$,
(b) $D_3 > 0$,
(c) $D_1D_2 - D_3 > 0$,

where

$$D_1 = -\{r_2[1 - \frac{(2T^* + \alpha_2N^*)}{k_1}] - \frac{w_2Z^*(p_2 + \beta N^*)}{(p_2 + T^* + \beta N^*)^2} + r_1[1 - \frac{(2N^* + \alpha_1T^*)}{k_1}] - \frac{w_2p_1Z^*}{(p_1 + N^*)^2} - q_1E\}$$

$$-(\frac{c_1w_1N^*}{p_1 + N^*} - \frac{c_2w_2T^*}{p_2 + T^* + \beta N^*} - d - q_2E),$$

$$D_2 = \{\frac{c_1{w_1}^2p_1N^*Z^*}{(p_1 + N^*)^3} + \frac{c_2w_1w_2\beta N^*T^*Z^*}{(p_2 + T^* + \beta N^*)^2(p_1 + N^*)} - \frac{c_2{w_2}^2T^*Z^*(p_2 + \beta N^*)}{(p_2 + T^* + \beta N^*)^3}\}$$

Notes

$$+\{r_1[1-\frac{(2N^*+\alpha_1T^*)}{k_1}]-\frac{w_2p_1Z^*}{(p_1+N^*)^2}-q_1E\}\times\{r_2[1-\frac{(2T^*+\alpha_2N^*)}{k_1}]-\frac{w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2}\}$$

$$+\frac{r_1\alpha_1N^*}{k_1}(\frac{r_1\alpha_1T^*}{k_2}+\frac{w_2\beta T^*Z^*)}{(p_2+T^*+\beta N^*)^2})+\{r_2[1-\frac{(2T^*+\alpha_2N^*)}{k_1}]-\frac{w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2}$$

$$+r_1[1-\frac{(2N^*+\alpha_1T^*)}{k_1}]-\frac{w_2p_1Z^*}{(p_1+N^*)^2}-q_1E\}\times\{\frac{c_1w_1N^*}{p_1+N^*}-\frac{c_2w_2T^*}{p_2+T^*+\beta N^*}-d-q_2E\},$$

$$D_3=-\{\frac{c_1w_1p_1Z^*}{(p_1+N^*)^2}-\frac{c_2w_2\beta T^*Z^*}{(p_2+T^*+\beta N^*)^2}\}\times\{-\frac{r_1\alpha_1w_2T^*}{k_1(p_2+T^*+\beta N^*)}+\frac{w_1N^*}{p_1+N^*}\times(r_2(1-\frac{(2T^*+\alpha_2N^*)}{k_2})$$

$$-\frac{w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2})-(\frac{c_2w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2})\times(\frac{w_2T^*}{p_2+T^*+\beta N^*})\times[-r_1(1-\frac{(2N^*+\alpha_1T^*)}{k_1})+\frac{w_1p_1Z^*}{(p_1+N^*)^2}+q_1E]\}$$

$$+\frac{w_1N^*}{p_1+N^*}\times(\frac{r_1\alpha_1T^*}{k_2}+\frac{w_2\beta T^*Z^*}{(p_2+T^*+\beta N^*)^2})-(\frac{c_2w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2})\times\{-\frac{r_1w_2T^*}{p_2+T^*+\beta N^*}+\frac{r_1w_2(2N^*+\alpha_1T^*)T^*}{k_1(p_2+T^*+\beta N^*)}$$

$$+\frac{w_1w_2p_1T^*Z^*}{(p_2+T^*+\beta N^*)(p_1+N^*)^2}+\frac{w_2q_1ET^*}{p_2+T^*+\beta N^*}+\frac{r_1\alpha_1w_1N^*T^*}{k_2(p_1+N^*)}+\frac{w_1w_2\beta N^*T^*Z^*}{(p_2+T^*+\beta N^*)^2(p_1+N^*)}\}$$

$$+\{r_1(1-\frac{(2N^*+\alpha_1T^*)}{k_1})-\frac{w_2p_1Z^*}{(p_1+N^*)^2}-q_1E\}\times\{r_2(1-\frac{(2T^*+\alpha_2N^*)}{k_1})-\frac{w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2}\}$$

$$+\frac{r_1\alpha_1N^*}{k_1}\times(\frac{r_1\alpha_1T^*}{k_2}+\frac{w_2\beta T^*Z^*}{(p_2+T^*+\beta N^*)^2}).$$

From the calculation of the eigenvalues, obviously, β does not affect the stability of E_1 and E_2. Still, it has a significant impact on the stability of E_3 and E_4 (because the eigenvalues of E_1 and E_2 are independent of β, but related to human harvest). On the other hand, we not only find that the equilibrium point of system (2.3) is affected by human harvest, but also has a particular impact on its stability(it can be seen from the eigenvalue of each equilibrium point).

Next, the biological explanations of the above different equilibria are discussed below. Since all these interpretations are mainly based on local asymptotic stability conditions, initial abundance of all the populations may also play an essential role for the system's dynamics together with the parameters. Different from the biological explanation in [14], we not only consider the effect of β on species coexistence, but also human harvest as an essential factor in species coexistence.

(i)E_0: Extinction of all the populations at a time is impossible.

(ii)E_1: From the analysis of research results, whenever the carrying capacity of the NTP population (k_1) stays within the specific threshold values of $\frac{k_2}{\alpha_2}<k_1<\frac{p_1(d+q_2E)}{c_1w_1-d-q_2E}$, both TPP and zooplankton will eventually become extinct from the system. Now, through the analysis of the k_1 threshold range, as the intensification of the harvest for zooplankton, the equilibrium point E_1 remains stable for a more extensive range of k_1, and we can say that over-fishing of zooplankton (q_2E) may accelerate the extinction of TPP and zooplankton.

(iii)E_2: If the carrying capacity of NTP population (k_1) stays below the threshold value $\frac{r_2\alpha_1k_2}{r_2-q_1E}$, both NTP and zooplankton eventually extinct. With the competitive effect of TPP on NTP (α_1), the environmental carrying capacities of toxin-producing phytoplankton (k_2) and harvesting term for NTP and zooplankton

Ref

[14] S. Chakraborty, S. Bhattacharya, U. Feudel, J. Chattopadhyay, The role of avoidance by zooplankton for survival and dominance of toxic phytoplankton, Ecol. Complexity 11 (2012) 144-153.

(q_1E) increase, respectively. The equilibrium point E_2 remains stable for a larger scale of k_1; we can say that the possibility of deracinating NTP and zooplankton at a time increases with the increase in α_1, k_2 and q_1E.

(iv)E_3: When the carrying capacity of NTP population (k_1) remains within two threshold values $\frac{r_2\alpha_1k_2}{r_2-q_1E} < k_1 < \frac{k_2}{\alpha_2}$ (it can be obtained by the threshold value (k_1) of E_1 and E_2) together with the competitive effects $(\alpha_1,\ \alpha_2)$, the harvesting term on NTP (q_1E) are present and the values of all three are small, the zooplankton population will go extinct on the condition that $\frac{c_1w_1\widehat{N}}{p_1+\widehat{N}} - d - q_2E < \frac{c_2w_2\widehat{T}}{p_2+\widehat{T}+\beta\widehat{N}}$, whereas both NTP and TPP persist in the system. The chance of zooplankton extinction increases with the decrease in avoidance of TPP by zooplankton (β), TPP consumption rate (w_1), the half-saturation constant for TPP (p_2), the harvesting term on zooplankton (q_2E) and the zooplankton mortality(d). For a specific parameter setup $(\frac{c_1w_1\widehat{N}}{p_1+\widehat{N}} - (d+q_2E) > 0)$, we can find a threshold value of the avoidance of TPP by zooplankton $(\beta < \frac{(c_2w_2\widehat{T})(p_1+\widehat{N})}{(\widehat{N})(c_1w_1\widehat{N}-(d+q_2E)(p_1+\widehat{N}))} - \frac{p_2+\widehat{T}}{\widehat{N}})$, below which the zooplankton population will become extinct. On the contrary, for $\frac{c_1w_1\widehat{N}}{p_1+\widehat{N}} - (d+q_2E) < 0$, the extinction of zooplankton dose not depend on the intensity of avoidance; it maybe has something relationship with the harvest term on zooplankton (q_2E).

(v)E_4: If the carrying capacity of NTP population (k_1) remains within two threshold values $(\frac{(d+q_2E)p_1}{c_1w_1-d-q_2E} < k_1 < \frac{(d+q_2E)(p_1)+c_1w_1p_1}{c_1w_1-d-q_2E})$, then TPP becomes extinct under the condition $(\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} < \frac{w_2\bar{Z}}{p_2+\beta\bar{N}})$, whereas both NTP and zooplankton persist in the system. The possibility of TPP extinction increases with the reduction in the avoidance of TPP by zooplankton (β), the half-saturation constant for TPP (p_2), and the growth rate of TPP (r_2), decreases with the rise of the competitive effect of N on T (α_2) and the TPP consumption rate (w_2). Similarly, for a particular parameter setup $(k_2 - \alpha_2\bar{N} > 0)$, we can find a threshold value of the avoidance of TPP by zooplankton $(\beta < \frac{k_2w_2\bar{Z}}{\bar{N}r_2(k_2-\alpha_2\bar{N})} - \frac{p_2}{\bar{N}})$, below which TPP may become extinct. On the contrary, for $k_2 - \alpha_2\bar{N} < 0$, TPP extinction dose not depend on the avoidance. Because the biological analysis of E_4 found that the harvesting term has little impact on the extinction of TPP compared with other equilibrium points. In conclusion, for $k_2 - \alpha_2\bar{N} < 0$, TPP extinction dose not depend on the avoidance of TPP by zooplankton (β) and harvest term on zooplankton (q_2E).

(vi)$E^* = (N^*, T^*, Z^*)$: When the competitive effects (α_1), the fishing coefficients of nontoxic phytoplankton (q_1), the environmental carrying capacities of nontoxic phytoplankton (k_1), and the effort used to harvest the population (E) remain very small, whereas the constant intrinsic growth rates of N (r_1), there may be a possibility of coexistence of all the three species.

Table 1: Existence and stability conditions of the equilibrium points.

Equilibrium	Existence conditions	Stability conditions
$E_0 = (0,0,0)$	Always exist	Always unstable
$E_1 = (k_1,0,0)$	Always exist	(i) $c_1w_1 - d - q_2E > 0,\ \alpha_2 > \frac{k_2}{k_1}$, $k_1 < \frac{p_1(d+q_2E)}{c_1w_1-d-q_2E}$, or (ii) $c_1w_1 - d - q_2E \le 0,\ \alpha_2 > \frac{k_2}{k_1}$
$E_2 = (0,k_2,0)$	Always exist	(i) $k_1 < \frac{r_2\alpha_1k_2}{r_2-q_1E}$
$E_3 = (\hat{N},\hat{T},0)$	(i) $\alpha_2 > \frac{k_2}{k_1}$, (ii) $\alpha_1 > \frac{(\alpha_1\alpha_2-1)q_1k_1E}{r_1k_1} + \frac{k_1}{k_2}$	(i) $\frac{c_1w_1\hat{N}}{p_1+\hat{N}} - d - q_2E < \frac{c_2w_2\hat{T}}{p_2+\hat{T}+\beta\hat{N}}$, (ii) $\bar{b}_1 > 0,\ \bar{c}_1 > 0$
$E_4 = (\bar{N},0,\bar{Z})$	(i) $w_1 > \frac{d+q_2E}{c_1}$, (ii) $k_1 > \frac{r_1N}{r_1-q_1E(p_1+E)}$	(i) $r_2(1-\frac{\alpha_2\bar{N}}{k_2}) < \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$, (ii) $\tilde{a}_2 + \tilde{b}_2 < 0,\ \tilde{a}_2\tilde{b}_2 + \tilde{c}_2 > 0$
$E^* = (N^*,T^*,Z^*)$	(i) $k_1 > \frac{q_1k_1E}{r_1} + N^* + \alpha_1T^*$, (ii) $c_2w_2(p_1+N^*) > c_1w_1N^* - (d+q_2E)(p_1+N^*) > 0$, (iii) positive root of Eq.(3.4) exists	(i) $D_1 > 0$, (ii) $D_3 > 0$, (iii) $D_1D_2 - D_3 > 0$

Notes

Global Journal of Science Frontier Research (F) Volume XXIII Issue VIII Version I Year 2023

c) Bistability analysis of equilibrium point

The existence and stability of these equilibrium points are summarized in Table 1 and Fig 1. When $c_1w_1 - d - q_2E > 0$, equilibria $E_2 = (0, k_2, 0)$, $E_3 = (\widehat{N}, \widehat{T}, 0)$, $E_1 = (k_1, 0, 0)$ and $E_4 = (\bar{N}, 0, \bar{Z})$ keep stable for $(0 < k_1 < \frac{r_2\alpha_1k_2}{r_2-q_1E})$, $(\frac{r_2\alpha_1k_2}{r_2-q_1E} < k_1 < \frac{k_2}{\alpha_2})$, $(\frac{k_2}{\alpha_2} < k_1 < \frac{p_1(d+q_2E)}{c_1w_1-d-q_2E})$ and $(\frac{(d+q_2E)p_1}{c_1w_1-d-q_2E} < k_1 < \frac{(d+q_2E)(p_1)+c_1w_1p_1}{c_1w_1-d-q_2E})$, respectively(Fig.1(a)). Obviously, for k_1 at the different equilibria above, the coexistence of NTP, TPP, and zooplankton requires the three ranges $(k_1 > \frac{r_2\alpha_1k_2}{r_2-q_1E})$, $(k_1 < \frac{k_2}{\alpha_2})$, and $(k_1 > \frac{(d+q_2E)p_1}{c_1w_1-d-q_2E})$, respectively. Therefore, the system exhibits these three possible types of bistability, where

Notes

(i)E_1 and E_2.

(ii)E_2 and E_4.

(iii)E_3 and E_4.

The above three types are locally asymptotically stable for different ranges of k_1.

For $\frac{k_2}{\alpha_2} < k_1 < \min\{\frac{r_2\alpha_1k_2}{r_2-q_1E}, \frac{(d+q_2E)p_1}{c_1w_1-d-q_2E}\}$, we can observe the bistability of E_1 and E_2 (Fig.1(b)(c)). If conditions $\frac{(d+q_2E)p_1}{c_1w_1-d-q_2E} < k_1 < \min\{\frac{r_2\alpha_1k_2}{r_2-q_1E}, \frac{(d+q_2E)p_1+c_1w_1p_1}{c_1w_1-d-q_2E}\}$ and $(\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} < \frac{w_2\bar{Z}}{p_2+\beta\bar{N}})$ hold simultaneous, we can find the bistability of E_2 and E_4 (Fig.1(d)(e)). On the contrary, if $\frac{(d+q_2E)p_1}{c_1w_1-d-q_2E} < k_1 < \frac{r_2\alpha_1k_2}{r_2-q_1E}$ holds, for either $k_1 > \frac{(d+q_2E)(p_1)+c_1w_1p_1}{c_1w_1-d-q_2E}$ or $\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} > \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$, we'll get the existence of stable E_2 together with unstable E_4. Identically, for $\max\{\frac{r_2\alpha_1k_2}{r_2-q_1E}, \frac{(d+q_2E)p_1}{c_1w_1-d-q_2E}\} < k_1 < \min\{\frac{k_2}{\alpha_2}, \frac{(d+q_2E)p_1+c_1w_1p_1}{c_1w_1-d-q_2E}\}$ together with $\alpha_1\alpha_2 < 1$, $\frac{c_1w_1\widehat{N}}{p_1+\widehat{N}} - d - q_2E < \frac{c_2w_2\widehat{T}}{p_2+\widehat{T}+\beta\widehat{N}}$ and $\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} < \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$, we can observe the bistability of E_3 and E_4 (Fig.1(f)-(i)).

Now, let's discuss the importance of avoiding toxic species by zooplankton (β) together with the harvesting term (q_1E, q_2E) for the survival of the different species groups.

Firstly, let's discuss the effect of β on three types of bistability. It can be seen from the previous analysis that the stability of E_1 and E_2 does not depend on the value of β. However, for the stability of E_3 and E_4, it is related to the critical value of β. When β is less than this critical value, E_3 and E_4 remain stable. Thus, β does not affect the bistability of (E_1, E_2); when β is below some threshold value, we will observe the bistability of (E_2, E_4) and (E_3, E_4), and as the β value increases, the original bistability may disappear. ($\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} > \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$, $\frac{c_1w_1\widehat{N}}{p_1+\widehat{N}} - d - q_2E < \frac{c_2w_2\widehat{T}}{p_2+\widehat{T}+\beta\widehat{N}}$ and $\frac{r_2(k_2-\alpha_2\bar{N})}{k_2} < \frac{w_2\bar{Z}}{p_2+\beta\bar{N}}$. From these conditions, we can see the establishment of the above conclusion.)

Secondly, let's discuss the effect of the harvesting term (q_1E, q_2E) on three types of bistability. From the analysis of the previous data, it can be seen that although the stability of E_1 and E_2 does not depend on the value of β, when humans overfish NTP and zooplankton, that is, q_1E and q_2E are too large, it may affect the bistability of E_1 and E_2. For E_3 and E_4, although their stability is directly related to the threshold value of β, the existence of q_1E and q_2E will also affect the threshold value of β, further influencing the stability of E_3 and E_4. Therefore, q_1E and q_2E may affect the bistability of (E_1, E_2), (E_2, E_4) and (E_3, E_4); the increase of q_1E and q_2E may also lead to the disappearance of this bistability.

IV. Dynamical Behavior of the Delayed Model

In this section, we focus on the local stability and Hopf bifurcation of the delayed model; the delayed system (2.2) has the following form

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = F(U(t), U(t-\tau_1), U(t-\tau_2)), \tag{4.1}$$

where

$$U(t) = [N(t), T(t), Z(t)], \quad U(t-\tau_1) = [N(t-\tau_1), T(t-\tau_1), Z(t-\tau_1)],$$

$$U(t-\tau_2) = [N(t-\tau_2), T(t-\tau_2), Z(t-\tau_2)].$$

Next, assuming $\Lambda_1(t) = N(t) - N^*$, $\Lambda_2(t) = T(t) - T^*$, $\Lambda_3(t) = Z(t) - Z^*$ at the positive equilibrium point, and linearizing the system (2.2), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Lambda_1(t)\\ \Lambda_2(t)\\ \Lambda_3(t)\end{pmatrix} = L\begin{pmatrix}N(t)\\ T(t)\\ Z(t)\end{pmatrix} + M\begin{pmatrix}N(t-\tau_1)\\ T(t-\tau_1)\\ Z(t-\tau_1)\end{pmatrix} + S\begin{pmatrix}N(t-\tau_2)\\ T(t-\tau_2)\\ Z(t-\tau_2)\end{pmatrix}, \tag{4.2}$$

where

$$L = \left(\frac{\partial F}{\partial U(t)}\right)_{E^*}, \quad M = \left(\frac{\partial F}{\partial U(t-\tau_1)}\right)_{E^*}, \quad S = \left(\frac{\partial F}{\partial U(t-\tau_2)}\right)_{E^*}.$$

We linearize the system(2.2) about positive equilibrium $E^* = (N^*, T^*, Z^*)$, and get

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = LU(t) + MU(t-\tau_1) + SU(t-\tau_2), \tag{4.3}$$

Notes

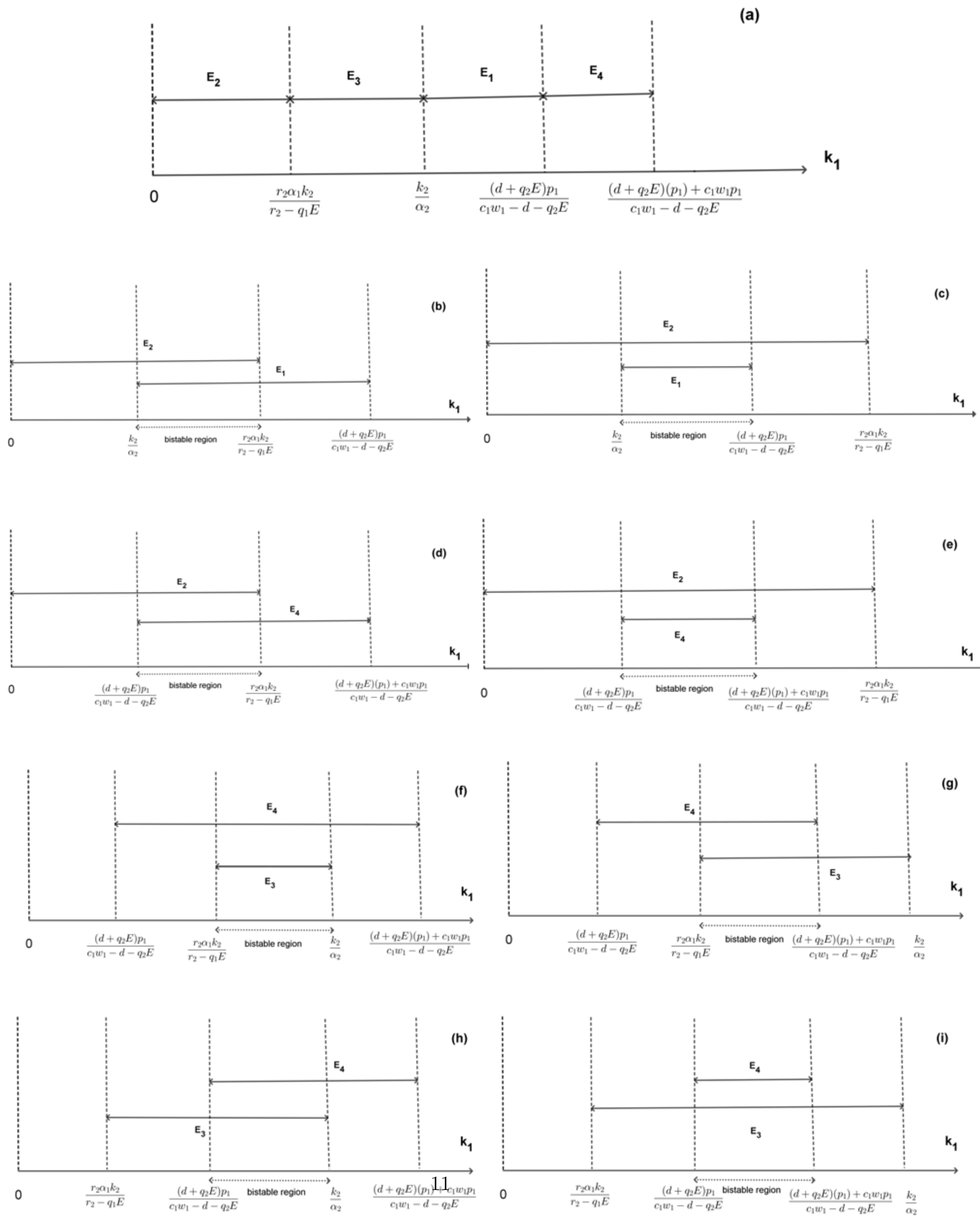


Fig. 1: Stability of different equilibria for different ranges of k_1. The dotted arrow indicates the range where bistability occurs, (a) means no bistability, (b) and (c) bistability of E_1 and E_2, (d) and (e) bistability of E_2 and E_4, (f)-(i) bistability of E_3 and E_4.

where

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_{31} & 0 & m_{33} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad U = \begin{pmatrix} N_1(\cdot) \\ T_1(\cdot) \\ Z_1(\cdot) \end{pmatrix},$$

where N_1 , T_1 , Z_1 are small perturbations around the equilibrium point $E^* = (N^*, T^*, Z^*)$. We have

$$l_{11} = \frac{-rN}{k_1} + \frac{w_1 ZN}{(p_1+N)^2} - q_1 E, \;\; l_{12} = \frac{r_1\alpha_1 N}{k_1}, \;\; l_{13} = -\frac{w_1 N}{p_1+N},$$

$$l_{21} = \frac{r_2\alpha_2 T}{k_1} + \frac{w_2\beta TZ}{(p_2+T+\beta N)^2}, \;\; l_{22} = r_2 - \frac{(2r_2T + r_2\alpha_1 N)}{k_2},$$

$$l_{23} = -\frac{w_2 T}{(p_2+T+\beta N)}, \;\; l_{33} = -d - q_2E, \quad m_{31} = \frac{c_1w_1p_1Z}{(p_1+N)^2}, \;\; m_{33} = \frac{c_1w_1N}{(p_1+N)},$$

$$s_{31} = \frac{c_2w_2\beta TZ}{(p_2+T+\beta N)^2}, \;\; s_{32} = \frac{c_2w_2Z(p_2+\beta N)}{(p_2+T+\beta N)^2}, \;\; s_{33} = \frac{c_2w_2T}{(p_2+T+\beta N)}.$$

The characteristic equation for the linearized system (2.2) is obtained as

$$D(\xi,\tau_1,\tau_2) \equiv P(\xi) + Q(\xi)\mathrm{e}^{-\xi\tau_1} + R(\xi)\mathrm{e}^{-\xi\tau_2} = 0, \tag{4.4}$$

where

$$P(\xi) = \xi^3 + A_2\xi^2 + A_1\xi + A_0, \quad Q(\xi) = B_2\xi^2 + B_1\xi + B_0, \quad R(\xi) = C_2\xi^2 + C_1\xi + C_0,$$

with

$$A_2 = -(l_{33} + l_{22} - l_{11}), \quad A_1 = l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{12}l_{21}, \quad A_0 = -l_{11}l_{22}l_{33} + l_{12}l_{21}l_{33}$$

$$B_2 = -m_{33}, \quad B_1 = -l_{11}m_{33} - l_{22}m_{33} - l_{13}m_{31}, \quad B_0 = +l_{13}l_{22}m_{31} + l_{11}l_{22}m_{33} + l_{12}l_{21}m_{33} - l_{12}l_{23}m_{31},$$

$$C_2 = -s_{33}, \quad C_1 = -l_{13}s_{31} + l_{11}s_{33} - l_{22}l_{23}s_{32} - l_{22}s_{33},$$

$$C_0 = l_{11}s_{33} + l_{11}l_{23}s_{32} + l_{12}l_{21}s_{33} + l_{13}l_{22}s_{31} - l_{12}l_{23}s_{31} - l_{13}l_{21}s_{32}.$$

Case (1): $\tau_1 = \tau_2 = 0$.

In this case, Section 3 covers the analysis of the system when $\tau_1 = \tau_2 = 0$.

Case (2): $\tau_1 = 0, \tau_2 > 0$.

In this case, the characteristic equation(4.4) becomes

$$\begin{aligned} D(\xi,\tau_2) &\equiv P(\xi) + Q(\xi) + R(\xi)\mathrm{e}^{-\xi\tau_2} \\ &\equiv \xi^3 + A_2\xi^2 + A_1\xi + A_0 + B_2\xi^2 + B_1\xi + B_0 + (C_2\xi^2 + C_1\xi + C_0)\mathrm{e}^{-\xi\tau_2} = 0, \end{aligned} \tag{4.5}$$

putting $\xi = i\omega(\omega > 0)$ in Eq.(4.5), and separating the real and imaginary parts, we have

$$\begin{aligned} &-(A_2+B_2)\omega^2 + (A_0+B_0) = (C_2\omega^2 - C_0)\cos(\omega\tau_2) - C_1\omega\sin(\omega\tau_2), \\ &-\omega^3 + (A_1+B_1)\omega = (C_0 - C_2\omega^2)\sin(\omega\tau_2) - C_1\omega\cos(\omega\tau_2). \end{aligned} \tag{4.6}$$

Notes

Squaring and adding the equation(4.6), we obtain

$$[-(A_2+B_2)\omega^2+(A_0+B_0)]^2+[-\omega^3+(A_1+B_1)\omega]^2=(C_2\omega^2-C_0)^2+(C_1\omega)^2. \tag{4.7}$$

Simplifying Eq.(4.7) and substituting $\omega^2=\psi$, the above equation can be written as

Notes

$$\Psi(\psi)\equiv\psi^3+a_2\psi^2+a_1\psi+a_0=0, \tag{4.8}$$

where

$$a_2=-(A_2+B_2)^2-2(A_1+B_1)-C_2^2,\quad a_1=(A_1+B_1)^2-2(A_0+B_0)(A_2+B_2)-2C_0C_2-C_1^2,\quad a_0=-C_0^2.$$

(H1): $a_2>0, a_0>0, a_2a_1-a_0>0$.
If (H1) holds, Eq.(4.8) has no positive roots, which implies all the roots of Eq.(4.5) have negative real parts. Therefore, E^* is asymptotically stable for all $\tau_2>0$ when (H1) holds.

(H2): $a_2<0, a_1<0, a_0<0$ or $a_2>0, a_1<0, a_0<0$ or $a_2>0, a_1>0, a_0<0$.
If (H2) holds, Eq.(4.8) has exactly one positive root ω_0, substituting ω_0 in Eq.(4.6), we obtain

$$-(A_2+B_2){\omega_0}^2+(A_0+B_0)=(C_2{\omega_0}^2-C_0)\cos(\omega_0\tau_2)-C_1\omega_0\sin(\omega_0\tau_2),$$

$$-{\omega_0}^3+(A_1+B_1)\omega_0=(C_0-C_2{\omega_0}^2)\sin(\omega_0\tau_2)-C_1\omega_0\cos(\omega_0\tau_2). \tag{4.9}$$

For the critical value of τ_2, we can obtain

$$\tau_{2\mathrm{j}}=\frac{1}{\omega_0}\arccos\{\frac{[C_1+C_2(A_2+B_2)]{\omega_0}^4+[C_1(A_1+B_1)-C_0(A_2+B_2)-C_2(A_0+B_0)]{\omega_0}^2+C_0(A_0+B_0)}{-(C_0-C_2{\omega_0}^2)^2-(C_1\omega_0)^2}\}+\frac{2j\pi}{\omega_0},$$

$$j=0,1,2\cdots. \tag{4.10}$$

For the transversality condition, differentiating Eq.(4.5) with respect to τ_2, we get

$$\frac{d\xi}{d\tau_2}=\frac{\xi(C_2\xi^2+C_1\xi+C_0)e^{-\xi\tau_2}}{3\xi^2+2A_2\xi+A_1+(2B_2\xi+B_1)+(2C_2\xi+C_1)e^{-\xi\tau_2}}.$$

Solving $(\frac{d\xi}{d\tau_2})^{-1}$, we obtain

$$(\frac{d\xi}{d\tau_2})^{-1}=\frac{3\xi^2+2A_2\xi+A_1+(2B_2\xi+B_1)+(2C_2\xi+C_1)e^{-\xi\tau_2}}{\xi(C_2\xi^2+C_1\xi+C_0)e^{-\xi\tau_2}}.$$

Then at $\tau_2=\tau_{20}$ and $\xi=i\omega_0$, we can get

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1}=\mathrm{Re}[\frac{3(i\omega_0)^2+(2A_2+B_2)(i\omega_0)+A_1+B_1}{(i\omega_0)(C_2(i\omega_0)^2+C_1(i\omega_0)+C_0)(\cos(\omega_0\tau_{20})-i\sin(\omega_0\tau_{20}))}]$$

$$+\mathrm{Re}[\frac{2C_2(i\omega_0)+C_1}{(i\omega_0)(C_2(i\omega_0)^2+C_1(i\omega_0)+C_0)}].$$

Now

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1}=\mathrm{Re}[\frac{M_R+M_Ii}{N_R+N_Ii}]+\mathrm{Re}[\frac{Q_R+Q_Ii}{P_R+P_Ii}]=\frac{M_RN_R+M_IN_I}{{N_R}^2+{N_I}^2}+\frac{Q_RP_R+Q_IP_I}{{P_R}^2+{P_I}^2},$$

where

$$M_R = -3{\omega_0}^2 + A_1 + B_1, \quad M_I = 2(A_2 + B_2)\omega_0, \quad N_R = (C_0\omega_0 - C_2{\omega_0}^3)\sin(\omega_0\tau_{20}) - C_1{\omega_0}^2\cos(\omega_0\tau_{20}),$$

$$N_I = (C_0\omega_0 - C_2{\omega_0}^3)\cos(\omega_0\tau_{20}) + C_1{\omega_0}^2\sin(\omega_0\tau_{20}), \quad Q_R = C_1, \quad Q_I = 2C_2\omega_0,$$

$$P_R = -C_1{\omega_0}^2, \quad P_I = C_0\omega_0 - C_2{\omega_0}^3.$$

Then

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1} = \frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}, \tag{4.11}$$

here

$$A = M_R N_R + M_I N_I, \quad B = {N_R}^2 + {N_I}^2,$$

$$C = Q_R P_R + Q_I P_I, \quad D = {P_R}^2 + {P_I}^2.$$

From this, we can get

$$\mathrm{sgn}[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1} = \mathrm{sgn}[AD+BC].$$

If (H3): $AD+BC \neq 0$ holds, the transversal condition $\mathrm{sgn}[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1} \neq 0$. From the above analysis, the following theorem can be drawn

Theorem 4.1. For $\tau_1 = 0$ and $\tau_2 > 0$, we have the following results:
(i)If (H1) holds, then the equilibrium E^* is asymptotically stable for all $\tau_2 > 0$.
(ii)If (H3) holds, and (H2) holds, then the equilibrium E^* is locally asymptotically stable for all $\tau_2 < \tau_{20}$ together with unstable for $\tau_2 > \tau_{20}$ and undergoes Hopf bifurcation at $\tau_2 = \tau_{20}$.

Case (3): $\tau_1 > 0, \tau_2 = 0$.

In this case, the characteristic equation(4.4) becomes as follows

$$\begin{aligned} D(\xi,\tau_1) &\equiv P(\xi) + R(\xi) + Q(\xi)e^{-\xi\tau_1} \\ &\equiv \xi^3 + A_2\xi^2 + A_1\xi + A_0 + (B_2\xi^2 + (C_2\xi^2 + C_1\xi + C_0) + B_1\xi + B_0)e^{-\xi\tau_1} = 0. \end{aligned} \tag{4.12}$$

putting $\xi = i\omega(\omega > 0)$ in Eq.(4.12), and separating the real and imaginary parts, we have

$$\begin{aligned} &-(A_2 + C_2)\omega^2 + (A_0 + C_0) = (B_2\omega^2 - B_0)\cos(\omega\tau_1) - B_1\omega\sin(\omega\tau_1), \\ &-\omega^3 + (A_1 + C_1)\omega = (B_0 - B_2\omega^2)\sin(\omega\tau_1) - B_1\omega\cos(\omega\tau_1). \end{aligned} \tag{4.13}$$

Squaring and adding the equation(4.13), we obtain

$$[-(A_2 + C_2)\omega^2 + (A_0 + C_0)]^2 + [-\omega^3 + (A_1 + C_1)\omega]^2 = (B_2\omega^2 - B_0)^2 + (B_1\omega)^2. \tag{4.14}$$

Based on the calculation method for case (2), we can simplify (4.14) to the following

$$\Psi(\ ) \equiv \ ^3 + b_2 \ ^2 + b_1 \ + b_0 = 0, \tag{4.15}$$

where

$$b_2 = -(A_2 + C_2)^2 - 2(A_1 + C_1) - B_2^2, \quad b_1 = (A_1 + C_1)^2 - 2(A_0 + C_0)(A_2 + C_2) - 2B_0B_2 - B_1^2, \quad b_0 = -B_0^2.$$

Notes

(H4): $b_2 > 0, b_0 > 0, b_2 b_1 - b_0 > 0$.

If (H4) holds, Eq.(4.15) has no positive roots, which implies all the roots of Eq.(4.12) have negative real parts. Therefore, E^* is asymptotically stable for all $\tau_1 > 0$ when (H4) holds.

(H5): $b_2 < 0, b_1 < 0, b_0 < 0$ or $b_2 > 0, b_1 < 0, b_0 < 0$ or $b_2 > 0, b_1 > 0, b_0 < 0$.

If (H5) holds, Eq.(4.15) has exactly one positive root $\hat{\omega}_0$, substituting $\hat{\omega}_0$ in Eq.(4.13), we obtain

$$\begin{aligned} &-(A_2 + C_2)\hat{\omega}_0{}^2 + (A_0 + C_0) = (B_2\hat{\omega}_0{}^2 - B_0)\cos(\hat{\omega}_0\tau_1) - B_1\hat{\omega}_0\sin(\hat{\omega}_0\tau_1), \\ &-\hat{\omega}_0{}^3 + (A_1 + C_1)\hat{\omega}_0 = (B_0 - B_2\hat{\omega}_0{}^2)\sin(\hat{\omega}_0\tau_1) - B_1\hat{\omega}_0\cos(\hat{\omega}_0\tau_1). \end{aligned} \tag{4.16}$$

For the critical value of τ_1, we can obtain

$$\tau_{1\mathrm{j}} = \frac{1}{\hat{\omega}_0}\arccos\{\frac{[B_1 + B_2(A_2 + C_2)]\hat{\omega}_0{}^4 + [B_1(A_1 + C_1) - C_0(A_2 + C_2) - B_2(A_0 + C_0)]\hat{\omega}_0{}^2 + B_0(A_0 + C_0)}{-(B_0 - B_2\hat{\omega}_0{}^2)^2 - (B_1\hat{\omega}_0)^2}\} + \frac{2j\pi}{\hat{\omega}_0},$$

$$j = 0, 1, 2\cdots. \tag{4.17}$$

For the transversality condition, differentiating Eq.(4.13) with respect to τ_1, we get

$$\frac{d\xi}{d\tau_1} = \frac{\xi(B_2\xi^2 + B_1\xi + B_0)e^{-\xi\tau_1}}{3\xi^2 + 2A_2\xi + A_1 + (2C_2\xi + C_1) + (2B_2\xi + B_1)e^{-\xi\tau_1}}.$$

Solving $(\frac{d\xi}{d\tau_1})^{-1}$, we obtain

$$(\frac{d\xi}{d\tau_1})^{-1} = \frac{3\xi^2 + 2A_2\xi + A_1 + (2C_2\xi + C_1) + (2B_2\xi + B_1)e^{-\xi\tau_1}}{\xi(B_2\xi^2 + B_1\xi + B_0)e^{-\xi\tau_1}}.$$

Then at $\tau_1 = \tau_{10}$ and $\xi = i\hat{\omega}_0$, we can get

$$\begin{aligned} [\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}_0}]^{-1} = &\mathrm{Re}[\frac{3(i\hat{\omega}_0)^2 + (2A_2 + C_2)(i\hat{\omega}_0) + A_1 + C_1}{(i\hat{\omega}_0)(B_2(i\hat{\omega}_0)^2 + B_1(i\hat{\omega}_0) + B_0)(\cos(\hat{\omega}_0\tau_{10}) - i\sin(\hat{\omega}_0\tau_{10}))}] \\ &+ \mathrm{Re}[\frac{2B_2(i\hat{\omega}_0) + B_1}{(i\hat{\omega}_0)(B_2(i\hat{\omega}_0)^2 + B_1(i\hat{\omega}_0) + B_0)}]. \end{aligned}$$

Now

$$[\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}_0}]^{-1} = \mathrm{Re}[\frac{\widehat{M_R} + \widehat{M_I}i}{\widehat{N_R} + \widehat{N_I}i}] + \mathrm{Re}[\frac{\widehat{Q_R} + \widehat{Q_I}i}{\widehat{P_R} + \widehat{P_I}i}] = \frac{\widehat{M_R}\widehat{N_R} + \widehat{M_I}\widehat{N_I}}{\widehat{N_R}^2 + \widehat{N_I}^2} + \frac{\widehat{Q_R}\widehat{P_R} + \widehat{Q_I}\widehat{P_I}}{\widehat{P_R}^2 + \widehat{P_I}^2},$$

where

$$\begin{aligned} &\widehat{M_R} = -3\hat{\omega}_0{}^2 + A_1 + C_1, \quad \widehat{M_I} = 2(A_2 + C_2)\hat{\omega}_0, \quad \widehat{N_R} = (B_0\hat{\omega}_0 - B_2\hat{\omega}_0{}^3)\sin(\hat{\omega}_0\tau_{10}) - C_1\hat{\omega}_0{}^2\cos(\hat{\omega}_0\tau_{10}), \\ &\widehat{N_I} = (B_0\hat{\omega}_0 - B_2\hat{\omega}_0{}^3)\cos(\hat{\omega}_0\tau_{10}) + B_1\hat{\omega}_0{}^2\sin(\hat{\omega}_0\tau_{10}), \quad \widehat{Q_R} = B_1, \quad \widehat{Q_I} = 2B_2\hat{\omega}_0, \\ &\widehat{P_R} = -B_1\hat{\omega}_0{}^2, \quad \widehat{P_I} = B_0\hat{\omega}_0 - B_2\hat{\omega}_0{}^3. \end{aligned}$$

Then

$$[\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}}]^{-1} = \frac{A_*}{B_*} + \frac{C_*}{D_*} = \frac{A_*D_* + B_*C_*}{B_*D_*}, \tag{4.18}$$

here

$$A_* = \widehat{M_R}\widehat{N_R} + \widehat{M_I}\widehat{N_I}, \quad B_* = \widehat{N_R}^2 + \widehat{N_I}^2,$$

$$C_* = \widehat{Q_R}\widehat{P_R} + \widehat{Q_I}\widehat{P_I}, \quad D_* = \widehat{P_R}^2 + \widehat{P_I}^2.$$

From this, we can get

$$[\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}}]^{-1} = \mathrm{sgn}[A_*D_* + B_*C_*].$$

If (H6): $A_*D_* + B_*C_* \neq 0$ holds, the transversal condition $[\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}}]^{-1} \neq 0$. From the above analysis, the following theorem can be drawn

Theorem 4.2. For $\tau_2 = 0$ and $\tau_1 > 0$, we have the following results:
(i)If (H4) holds, then the equilibrium E^* is asymptotically stable for all $\tau_1 > 0$.
(ii)If (H6) and (H5) hold, then the equilibrium E^* is locally asymptotically stable for all $\tau_1 < \tau_{10}$ together with unstable for $\tau_1 > \tau_{10}$ and undergoes Hopf bifurcation at $\tau_1 = \tau_{10}$.

Case (4): τ_1 is fixed in $(0, \tau_{10}]$ and $\tau_2 > 0$.

We consider the gestation delay τ_1 to be stable in the interval $(0, \tau_{10}]$, taking τ_2 as a control parameter. Let $\xi = u + i\omega$ be the root of Eq.(4.4). Putting this value in Eq.(4.4), separating real and imaginary parts, we obtain

$$\begin{aligned}&u^3 - 3u\omega^2 + A_2(u^2-\omega^2) + A_1u + A_0 + (B_2u^2 - B_2\omega^2 + B_1u + B_0)e^{-u\tau_1}\cos(\omega\tau_1)\\&+(2B_2u\omega + B_1\omega)e^{-u\tau_1}\sin(\omega\tau_1) + (C_2u^2 - C_2\omega^2 + C_1u + C_0)e^{-u\tau_1}\\&\cos(\omega\tau_2) + (2C_2u\omega + C_1\omega)\sin(\omega\tau_2) = 0.\end{aligned} \tag{4.19}$$

$$\begin{aligned}&3u^2\omega - \omega^3 + 2A_2u\omega + A_1\omega - (B_2u^2 - B_2\omega^2 + B_1u + B_0)\sin(\omega\tau_1) + (2B_2u\omega\\&+B_1\omega)e^{-u\tau_1}\cos(\omega\tau_1) - (C_2u^2 - C_2\omega^2 + C_1u + C_0)\sin(\omega\tau_2) + (2C_2u\omega\\&+C_1\omega)e^{-u\tau_2}\cos(\omega\tau_2) = 0.\end{aligned} \tag{4.20}$$

Putting $u = 0$ in Eqs.(4.19) and (4.20), we obtain

$$A_2\omega^2 - A_0 = (-B_2\omega^2 + B_0)\cos(\omega\tau_1) + (C_0 - C_2\omega^2)\cos(\omega\tau_2) + B_1\omega\sin(\omega\tau_1) + C_1\omega\sin(\omega\tau_2). \tag{4.21}$$

$$\omega^3 - A_1\omega = -(B_0 - B_2\omega^2)\sin(\omega\tau_1) + B_1\omega\cos(\omega\tau_1) - (C_0 - C_2\omega^2)\sin(\omega\tau_2) + C_1\omega\cos(\omega\tau_2). \tag{4.22}$$

Squaring and adding Eqs.(4.21) and (4.22) to eliminate τ_2, we get

$$\omega^6 + \tilde{a}_4\omega^4 + \tilde{a}_3\omega^3 + \tilde{a}_2\omega^2 + \tilde{a}_0 = 0, \tag{4.23}$$

where

$$\tilde{a}_4 = -({B_2}^2 + {C_2}^2 - {A_2}^2), \quad \tilde{a}_3 = -2(B_2C_1 - B_1C_2)\sin(\omega\tau_1 - \omega\tau_2),$$

$$\tilde{a}_2 = -(({B_1}^2 - 2B_0B_2 + {C_1}^2 - 2C_0C_2) + 2(B_1C_1 - 2A_0A_2 - A_1^2 - B_2))\cos(\omega\tau_1 - \omega\tau_2),$$

$$\tilde{a}_0 = -({B_0}^2 + {C_0}^2 - {A_0}^2).$$

Noting that Eq.(4.23) is transcedental. Now, Eqs.(4.21) and (4.22) can be written as

$$\delta_1\cos(\omega\tau_2) + \delta_2\sin(\omega\tau_2) = \delta_3 + \delta_4\cos(\omega\tau_1) + \delta_5\sin(\omega\tau_1), \tag{4.24}$$

Notes

$$-\delta_2\cos(\omega\tau_2)+\delta_1\sin(\omega\tau_2)=\delta_6-\delta_5\cos(\omega\tau_1)+\delta_4\sin(\omega\tau_1), \tag{4.25}$$

where

$$\delta_1=C_2\omega^2-C_0,\quad \delta_2=-C_1\omega,$$

$$\delta_3=A_0-A_2\omega^2,\quad \delta_4=B_0-B_2\omega^2,$$

$$\delta_5=B_1\omega,\quad \delta_6=\omega^3-A_1\omega.$$

Notes

Without losing generality, the Eq.(4.23) has finite positive roots $\widetilde{\omega_1},\widetilde{\omega_2},\cdots,\widetilde{\omega_k}$, for every fixed $\widetilde{\omega}$, there exists a sequence $\{\tau_{2i}^{j}|j=0,1,2...\}$, where

$$\tau_{2i}^{(j)}=\frac{1}{\widetilde{\omega}_i}\tan^{-1}[\frac{(\delta_1\delta_4+\delta_2\delta_4)\sin(\widetilde{\omega}_i\tau_1)-(\delta_1\delta_5-\delta_2\delta_4)\cos(\widetilde{\omega}_i\tau_1)+\delta_1\delta_6+\delta_2\delta_3}{(\delta_1\delta_5-\delta_2\delta_4)\sin(\widetilde{\omega}_i\tau_1)+(\delta_2\delta_5+\delta_1\delta_4)\cos(\widetilde{\omega}_i\tau_1)+\delta_1\delta_3-\delta_2\delta_4}+\frac{k\pi}{\widetilde{\omega}_i} \tag{4.26}$$

$$j=0,1,2,\cdots$$

let $\widetilde{\tau}_2=min\{\tau_{2i}^{(j)}|i=0,1,2,...k,j=0,1,2...\}$, when $\tau_2=\widetilde{\tau}_2,\widetilde{\omega}=\widetilde{\omega}_i|_{\tau_2=\widetilde{\tau}_2}, i=1,2,3,...$, the characteristic equation (4.4) has purely imaginary roots $\pm i\widetilde{\omega}$. Then, we will verify the transversality condition, differentiating the characteristic equation (4.4) with respect to τ_2, we can obtain

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1}=\mathrm{Re}[\frac{3(i\widetilde{\omega})^2+2A_2(i\widetilde{\omega})+A_1}{(i\widetilde{\omega})(C_2(i\widetilde{\omega})^2+C_1(i\widetilde{\omega})+C_0)(\cos(\widetilde{\omega}\widetilde{\tau}_2)-i\sin(\widetilde{\omega}\widetilde{\tau}_2))}]$$

$$+\mathrm{Re}[\frac{2C_2(i\widetilde{\omega})+C_1}{(i\widetilde{\omega})(C_2(i\widetilde{\omega})^2+C_1(i\widetilde{\omega})+C_0)}].$$

Now

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1}=\mathrm{Re}[\frac{M_R+M_Ii}{N_R+N_Ii}]+\mathrm{Re}[\frac{Q_R+Q_Ii}{P_R+P_Ii}]=\frac{M_RN_R+M_IN_I}{{N_R}^2+{N_I}^2}+\frac{Q_RP_R+Q_IP_I}{{P_R}^2+{P_I}^2},$$

where

$$M_R=-3\widetilde{\omega}^2+A_1,\quad M_I=2A_2\widetilde{\omega},\quad N_R=(C_0\widetilde{\omega}-C_1\widetilde{\omega}^2-C_2\widetilde{\omega}^3)\sin(\widetilde{\omega}\bar{\tau}_2)$$

$$N_I=(C_0\widetilde{\omega}-C_2\widetilde{\omega}^3)\cos(\widetilde{\omega}\bar{\tau}_2)+C_1\widetilde{\omega}^2\sin(\widetilde{\omega}\bar{\tau}_2),\quad Q_R=C_1,\quad Q_I=2C_2\widetilde{\omega},$$

$$P_R=-C_1\widetilde{\omega}^2,\quad P_I=C_0\widetilde{\omega}-C_2\widetilde{\omega}^3.$$

Then

$$[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1}=\frac{E}{F}+\frac{G}{H}=\frac{EH+FG}{FH}, \tag{4.27}$$

here

$$E=M_RN_R+M_IN_I,\quad F={N_R}^2+{N_I}^2,$$

$$G=Q_RP_R+Q_IP_I,\quad H={P_R}^2+{P_I}^2.$$

From this we can get

$$\operatorname{sgn}[\operatorname{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1} = \operatorname{sgn}[EH+FG].$$

If (H7): $EH+FG \neq 0$ holds, the transversal condition $\operatorname{sgn}[\operatorname{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1} \neq 0$. From the above analysis, we have the following theorem.

Theorem 4.3. *For system(2.2), assume (H7) holds with τ_1 is fixed in $(0,\tau_{10}]$ and $\tau_2>0$, then the equilibrium E^* is locally asymptotically stable for $\tau_2 \in (0,\widetilde{\tau}_2)$ whereas system (2.2) undergoes Hopf bifurcation at $\tau_2=\widetilde{\tau}_2$.*

Case(5): τ_2 is fixed in $(0,\tau_{20}]$ and $\tau_1>0$, so take τ_1 as a control parameter; the analysis is the same as case(4), so we omit it.

V. Optimal Tax Policy

From previous studies, overfishing may lead to the extinction of populations. However, in the society, the adequate protection of the ecosystem is a common problem we need to face. In the face of the increasingly severe harmful effects of overfishing on ecosystems, people began to find the most suitable methods for fishery control in various areas of sustainable development policies, for example, seasonal fishing, property leasing, taxation, licensing fees, etc. Taxes are generally considered to be better than other regulatory approaches, so that we will view the optimal tax policy for the double phytoplankton - single zooplankton system based on model (2.3). Here, we take E as a time-dependent dynamic variable controlled by equations. Therefore, there is the following equation.

$$E(t)=\varepsilon Q(t), \quad 0\leq\varepsilon\leq 1, \quad \frac{dQ}{dt}=I(t)-\gamma Q(t), \quad Q(0)=Q_0. \tag{5.1}$$

Where $Q(t)$ is the amount of capital invested in fisheries at time t, $I(t)$ is the total investment rate(in physical form) at time t and γ is the constant depreciation rate of capital. Suppose that the effort E at any time is proportional to the instantaneous amount of investment capital. For example, if $Q(t)$ represents the number of standard fishing vessels that can be used, it is reasonable to assume that $Q(t)$ and E should be proportional. When $\varepsilon=1$, it can be considered that the maximum fishing capacity(E)is equal to the number of available vessels at time t ($Q(t)$). When $\varepsilon=0$, it means that even though there may be fishing boats, the fishing is not expanded; it also reflects the over-exploitation of fisheries. At this time the fish population has been seriously depleted, so fishing vessels can no longer be used. These are simulated capital levels may be adjusted, thus prove the reasonableness of the equation (5.2). Regulators control the development of fisheries by imposing a tax ($v>0$) on the unit biomass of terrestrial fish. When ($v<0$) can be understood as any subsidy to fishermen. Net income of fishermen('Net income' for short) is $E[(u_1-v)q_1N+(u_2-v)q_2N-C]$, where u_i, $i=1,2$ is the constant price of unit biomass of nontoxic phytoplankton and zooplankton, respectively. C is the fixed cost per unit of harvesting effort.

We assume the gross profit margin on capital investment is proportional to this 'Net income.' So, we have

$$I=E\varphi[(u_1-v)q_1N+(u_2-v)q_2Z-C], \quad 0\leq\varphi<1. \tag{5.2}$$

For $\varphi=1$, Eq.(5.2) shows that the highest investment rate at any time is equal to the net income of the fishermen at that time. $\varphi=0$ can only be used when the net income of fishermen is negative; that is, current capital assets cannot be divested. If the fishery is operating at a loss and allows capital to be withdrawn, the only owner of the fishery will benefit by allowing the capital assets to be continuously withdrawn, because negative investment means withdrawal of investment, so it is the case of $I<0,\varphi>0$. By combining Eqs.(5.1) and (5.2), we can get

$$\frac{dE}{dt}=E\{\varepsilon\varphi[(u_1-v)q_1N+(u_2-v)q_2Z-C]-\gamma\}. \tag{5.3}$$

Notes

Fishermen and regulators are two different parts of society. Therefore, the income they receive is society's income accumulated through fisheries. The net economic income to society is

$$ME = E[(u_1 - v)q_1N + (u_2 - v)q_2Z - C] + E[v(q_1N) + v(q_2N)],$$

this is equal to the net economic income of fishermen plus the economic income of regulators. Therefore without considering the time delay, Eq.(2.3) can be rewritten as

$$\begin{cases} \frac{dN}{dt} = r_1N(1 - \frac{N+\alpha_1T}{k_1}) - \frac{w_1NZ}{p_1+N} - q_1EN, \\ \frac{dT}{dt} = r_2N(1 - \frac{T+\alpha_2N}{k_2}) - \frac{w_2TZ}{p_2+T+\beta N}, \\ \frac{dZ}{dt} = \frac{c_1w_1NZ}{p_1+N} - \frac{c_2w_2TZ}{p_2+T+\beta N} - dZ - q_2EZ, \\ \frac{dE}{dt} = E\{\varepsilon\varphi[(u_1-v)q_1N + (u_2-v)q_2Z - C] - \gamma\}. \end{cases} \tag{5.4}$$

Next, we will use the principle of Pontryagin's maximum to get the path of the best tax policy. If the fish population stays along this path, then regulators can ensure that their goals are achieved. The goal of regulatory agencies is to maximize the total net income of society as a result of harvesting activities. Specifically, the goal is to maximize revenue over a continuous time stream (J).

$$J = \int_0^{+\infty} E(t)e^{-\delta t}[u_1q_1N + u_2q_2Z - C]dt, \tag{5.5}$$

where δ is the discounting factor. Therefore, our goal is to determine an optimal tax $v = v(t)$ that maximizes compliance with Eq.(5.4) and constrains $v_{\min} \le v(t) \le v_{\max}$ on the control variable $v(t)$. When $v_{\min} < 0$, it will have the effect of accelerating the rate of fishery expansion. The Hamiltonian of the problem is obtained by

$$\begin{aligned} H = {} & (u_1q_1N + u_2q_2Z - C)Ee^{-\delta t} + \lambda_1N[r_1(1 - \frac{N+\alpha_1T}{k_1}) - \frac{w_1Z}{p_1+N} - q_1E] \\ & +\lambda_2[r_2T(1 - \tfrac{T+\alpha_1N}{k_2}) - \tfrac{w_2TZ}{p_2+T+\beta N}] + \lambda_3[\tfrac{c_1w_1NZ}{p_1+N} - \tfrac{c_2w_2TZ}{p_2+T+\beta N} - dZ - q_2EZ] \\ & +\lambda_4E\{\varepsilon\varphi[(u_1-v)q_1N + (u_2-v)q_2Z - C] - \gamma\}, \end{aligned} \tag{5.6}$$

where λ_1, λ_2, λ_3 and λ_4 are the adjoint variables. For $v \in [v_{\min}, v_{\max}]$, the Hamiltonian must be maximized. Assuming that the control constraint is not bound, that is, the optimal solution does not appear as $v = v_{\min}$ or $v = v_{\max}$. We can get by singular control [9]

$$\frac{\partial H}{\partial v} = -\lambda_4E\varepsilon\varphi(q_1N + q_2Z) = 0 \Rightarrow \lambda_4 = 0. \tag{5.7}$$

Now, the adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} = {} & -\frac{\partial H}{\partial N} = -[u_1q_1Ee^{-\delta t} + \lambda_1(r_1 - \frac{2r_1N + r_1\alpha_1T}{k_1} - \frac{w_1p_1Z}{(p_1+N)^2} - q_1E) + \lambda_2[\frac{w_2\beta TZ}{(p_2+T+\beta N)^2} - \frac{r_2\alpha_2T}{k_2}] \\ & + \lambda_3(\frac{c_1w_1p_1Z}{(p_1+N)^2} + \frac{c_2w_2\beta TZ}{(p_2+T+\beta N)^2}), \end{aligned}$$

Ref

[9] C.J. Zilverberg, J. Angerer, J. Williams, L.J. Metz, K. Harmoney, Sensitivity of diet choices and environmental outcomes to a selective grazing algorithm, Ecol. Model. 390 (2018) 10-22.

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial T} = -[\lambda_1(\frac{r_1\alpha_1 N}{k_1}) + \lambda_2[r_2(1-\frac{2T+\alpha_2 N}{k_2}) - \frac{w_2 Z(p_2+\beta N)}{(p_2+T+\beta N)^2}] - \lambda_3(\frac{c_2 w_2 Z(p_2+\beta N)}{(p_2+T+\beta N)^2}),$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial Z} = -[u_2 q_2 E e^{-\delta t} - \lambda_1(\frac{w_1 N}{p_1+N}) - \lambda_2(\frac{w_2 T}{p_2+T+\beta N}) + \lambda_3(\frac{c_1 w_1 N}{p_1+N} - \frac{c_2 w_2 T}{p_2+T+\beta N} - d - q_2 E)],$$

$$\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial E} = -[(u_1 q_1 N + u_2 q_2 Z - C)e^{-\delta t} - \lambda_1 q_1 N - \lambda_3 q_2 Z]. \tag{5.8}$$

Notes

Now start with Eqs.(5.8) and (5.7), using the equilibrium equation we have

$$\frac{d\lambda_1}{dt} = -u_1 q_1 E e^{-\delta t} - \lambda_1[-\frac{r_1 N}{k_1} + \frac{w_1 N Z}{(p_1+N)^2}] - \lambda_2[\frac{w_2 \beta T Z}{(p_2+T+\beta N)^2} - \frac{r_2\alpha_2 T}{k_2}] - \lambda_3[\frac{c_1 w_1 p_1 Z}{(p_1+N)^2} + \frac{c_2 w_2 \beta T Z}{(p_2+T+\beta N)^2}],$$

$$\frac{d\lambda_2}{dt} = -\lambda_1[\frac{r_1\alpha_1 N}{k_1}] - \lambda_2[\frac{w_2 T Z}{(p_2+T+\beta N)^2}] - \lambda_3[\frac{c_2 w_2 Z(p_2+\beta N)}{(p_2+T+\beta N)^2}],$$

$$\frac{d\lambda_3}{dt} = -u_2 q_2 E e^{-\delta t} + \lambda_1(\frac{w_1 N}{p_1+N}) + \lambda_2(\frac{w_2 T}{p_2+T+\beta N}). \tag{5.9}$$

Using the second and third equations of Equation (5.9) from the fourth equation of Equation (5.8), we can obtain $\frac{d\lambda_1}{dt} = M_1 e^{-\delta t} + M_2\lambda_1 + M_3\lambda_2$, where

$$M_1 = \frac{(C-u_1 q_1 N)\delta + u_2 q_2 Z(q_2 E - \delta)}{q_1 N}, \quad M_2 = -\frac{w_1 q_2 N Z}{(p_1+N)q_1 N}, \quad M_3 = -\frac{w_2 q_2 T Z}{(p_2+T+\beta N)q_1 N}.$$

The solution of this linear equation is

$$\lambda_1 = N_0 e^{-M_2 t} - \frac{M_1 e^{-\delta t}}{M_2+\delta} - \frac{M_3\lambda_2}{M_2}. \tag{5.10}$$

Using the same method as above, we can get

$$\lambda_3 = I_0 e^{H_2 t} - \frac{H_1 e^{-\delta t}}{H_2+\delta}, \tag{5.11}$$

where

$$H_1 = [\frac{(C-u_2 q_2 Z)\delta - q_1 N(u_1\delta + M_1)}{q_2 Z} + \frac{M_1 M_2 q_1 N}{(M_2+\delta)q_2 Z}], \quad H_2 = \frac{M_2 M_3 q_1 N}{q_2 M_2 Z}.$$

Identically

$$\frac{d\lambda_2}{dt} = R_1 e^{-\delta t} + R_2\lambda_2, \tag{5.12}$$

where

$$R_1 = \frac{M_1}{M_2+\delta} + \frac{H_1}{H_2+\delta}(\frac{c_2 w_2 Z(p_2+\beta N)}{(p_2+T+\beta N)^2}), \quad R_2 = \frac{M_3}{M_2}(\frac{r_2\alpha_1 N}{k_1}) - \frac{w_2 T Z}{(p_2+T+\beta N)^2}.$$

So we can get λ_1

$$\lambda_1 = N_0 e^{M_2 t} - \frac{M_1 e^{-\delta t}}{M_2+\delta} - \frac{M_3(W_0 e^{R_2 t} - \frac{R_1 e^{-\delta t}}{R_2+\delta})}{M_2}.$$

The shadow price $\lambda_1 e^{-\delta t}$ is bounded as $t \to \infty$, $N_0 = 0$ and $W_0 = 0$, then we can obtain

$$\lambda_1 = -\frac{M_1 e^{-\delta t}}{M_2+\delta} - \frac{M_3}{M_2}(e^{R_2 t} - \frac{R_1 e^{-\delta t}}{R_2+\delta}). \tag{5.13}$$

Now use Eqs.(5.11), (5.12) and (5.13) in the first of Eq.(5.9), we have

$$\begin{aligned}
&[\frac{(C-u_1q_1N^*)\delta + u_2q_2Z^*(q_2E^*-\delta)}{q_1N^*}]e^{-\delta t} + \frac{w_2q_2N^*Z^*}{(p_1+N^*)q_1N^*}[\frac{M_1e^{-\delta t}}{M_2+\delta} - \frac{M_3}{M_2}(e^{R_2t} - \frac{R_1e^{-\delta t}}{R_2+\delta})] \\
&+[\tfrac{w_2q_2T^*Z^*}{(p_2+T^*+\beta N^*)q_1N^*}][\tfrac{R_1e^{-\delta t}}{R_2+\delta}] + u_1q_1E^*e^{-\delta t} + [\tfrac{M_1e^{-\delta t}}{M_2+\delta} - \tfrac{M_3}{M_2}(e^{R_2t} - \tfrac{R_1e^{-\delta t}}{R_2+\delta})][-\tfrac{r_1N^*}{k_1} + \tfrac{w_1N^*Z^*}{(p_1+N^*)^2}] \\
&= (\tfrac{R_1e^{-\delta t}}{R_2+\delta})[\tfrac{w_2\beta T^*Z^*}{(p_2+T^*+\beta N^*)^2} - \tfrac{r_2\alpha_2T^*}{k_2}] + (\tfrac{H_1e^{-\delta t}}{H_2+\delta})[\tfrac{c_2w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2}].
\end{aligned} \tag{5.14}$$

Because of the computational complexity, our optimal equilibrium solution can be expressed as

$$\begin{aligned}
T^* &= \frac{[(c_1w_1-\delta)N^* - \delta p_1](p_2+\beta N^*)}{[(c_2w_2-\delta)p_1 + (c_2w_2-c_1w_2-\delta)N^*]}, \\
Z^* &= r_1(\tfrac{p_1+N^*}{w_1k_1})(k_1 - N^* - \alpha_1T^*).
\end{aligned} \tag{5.15}$$

N^* available from the following equation

$$r_2(k_2 - T^* - \alpha_2N^*)(p_2 + T^* + \beta N^*) - w_2k_2Z^* = 0. \tag{5.16}$$

E^* available from the following equation

$$\frac{r_1}{q_1}(1 - \frac{N^*+\alpha_1T^*}{k_1}) - \frac{w_1Z^*}{q_1(p_1+N^*)} = \frac{c_1w_1N^*}{q_2(p_1+N^*)} - \frac{c_2w_2T^*}{q_2(p_2+T^*+\beta N^*)} - \frac{d}{q_2}. \tag{5.17}$$

From the complex calculation results, it can be seen that T^* and Z^* are functions of v. Therefore, we can express this function as follows

$$\begin{aligned}
&[\frac{(C-u_1q_1N^*)\delta + u_2q_2Z^*(q_2E^*-\delta)}{q_1N^*}]e^{-\delta t} + \frac{w_2q_2N^*Z^*}{(p_1+N^*)q_1N^*}[\frac{M_1e^{-\delta t}}{M_2+\delta} - \frac{M_3}{M_2}(e^{R_2t} - \frac{R_1e^{-\delta t}}{R_2+\delta})] \\
&+[\tfrac{w_2q_2T^*Z^*}{(p_2+T^*+\beta N^*)q_1N^*}][\tfrac{R_1e^{-\delta t}}{R_2+\delta}] + u_1q_1E^*e^{-\delta t} + [\tfrac{M_1e^{-\delta t}}{M_2+\delta} - \tfrac{M_3}{M_2}(e^{R_2t} - \tfrac{R_1e^{-\delta t}}{R_2+\delta})][-\tfrac{r_1N^*}{k_1} + \tfrac{w_1N^*Z^*}{(p_1+N^*)^2}] \\
&-(\tfrac{R_1e^{-\delta t}}{R_2+\delta})[\tfrac{w_2\beta T^*Z^*}{(p_2+T^*+\beta N^*)^2} - \tfrac{r_2\alpha_2T^*}{k_2}] - (\tfrac{H_1e^{-\delta t}}{H_2+\delta})[\tfrac{c_2w_2Z^*(p_2+\beta N^*)}{(p_2+T^*+\beta N^*)^2}] = f(v).
\end{aligned} \tag{5.18}$$

If v^* exists, let $v = v^*$ be the solution of $f(v)$. Using the value of v^*, we can get the optimal solution $(N(v^*), T(v^*), Z(v^*), E(v^*))$. Here, we establish the existence of an optimal equilibrium solution satisfying the necessary condition of the maximum principle. As Clark [23] pointed out, it is complicated to find the optimal path composed of explosive control and unbalanced singular control. Because the current model is much more complex than Clark's model, we only consider an optimal equilibrium solution. If we can begin to

Ref

[23] C.W. Clark, Bioeconomic Modelling and Fisheries Management, New York (USA) Wiley. 1985.

get ${F^*}_c = (N(v^*), T(v^*), Z(v^*), E(v^*))$ at any initial state in $[0, S]$ to reach its maximum benefit in a limited time S_0. The period $[0, S]$ may be a planning cycle, or it may be the shortest cycle closest to ${F^*}_c$, so we take S to be the shortest time to reach ${F^*}_c$. Let $(N_0, T_0, Z_0, E_0) \in R_+^4/\{0\}$, $(N(v^*), T(v^*), Z(v^*), E(v^*)) \in R_+^4/\{0\}$ be the optimal equilibrium. Now, we seek min $S_0(v)$ subject to the solution to Eq.(5.5).

$$N(0) = N_0, \quad T(0) = T_0, \quad Z(0) = Z_0, \quad E(0) = E_0, \quad N(S_0) = N(v^*), \quad T(S_0) = T(v^*), \quad Z(S_0) = Z(v^*),$$

$$E(S_0) = E(v^*), \quad (N, T, Z, E) \in R_+^4/\{0\}, \quad t \in [0, S_0]. \tag{5.19}$$

Using the maximum principle, we will get the adjoint variables λ_1, λ_2, λ_3 and λ_4 as

$${\lambda_1}' = -\frac{\partial H}{\partial N}, \quad {\lambda_2}' = -\frac{\partial H}{\partial T}, \quad {\lambda_3}' = -\frac{\partial H}{\partial Z}, \quad {\lambda_4}' = -\frac{\partial H}{\partial E}. \tag{5.20}$$

The adjoint variables λ_1, λ_2, λ_3 and λ_4 satisfies the another condition

$$M\{N(t), T(t), Z(t), E(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)\} = 0, \quad t \in [0, S_0], \tag{5.21}$$

where

$$M\{N(t), T(t), Z(t), E(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), t\} = \left[\sup_{v_{\min}, v_{\max}}\right] H(N(t), T(t), Z(t), E(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), v).$$

Eq.(5.19) specifies a set of initial conditions for λ_1, λ_2, λ_3 and λ_4, and Eq.(5.20) uses these initial conditions to determine the unique solution of λ_1, λ_2, λ_3 and λ_4. Therefore, it is easy to obtain the optimal tax policy as follows:

$$\bar{v}(t) = \begin{cases} v_{\max}, & for \ \ all \ \ t \in [0, S_0] \ \ if \ \ \frac{\partial H}{\partial v} > 0, \\ v_{\min}, & for \ \ all \ \ t \in [0, S_0] \ \ if \ \ \frac{\partial H}{\partial v} < 0. \end{cases} \tag{5.22}$$

The optimal path in $[0, S]$ is the solution of Eq.(5.5) in its elementary state. We will now combine these two stages to obtain the optimal tax policy and optimal path in an infinite range:

$$v(t) = \bar{v}(t), \ \ t \in [0, S_0], \ \ v(t) = v^*, \ \ t > S_0, \ \ \Gamma(t) = \bar{\Gamma}(t), \ \ t \in [0, S_0], \ \ \Gamma(t) = F_c = (N_b, T_b, Z_b, E_b), \ \ t > S_0.$$

From the above analysis, we can easily observe the following points:

(i) From Eqs.(5.7) and (5.11)-(5.13), we note that $\lambda_i e^{-\delta t}$, $(i = 1, 2, 3, 4)$, where λ_i is an adjoint variable, which remains unchanged in an optimal balance time interval, therefore, they satisfy the transversal condition, that is, they remain bounded to $t \to \infty$.

(ii) Considering the coexistence equilibrium point $F_c = (N_b, T_b, Z_b, E_b)$, The fourth equation of Eq.(5.8) can be written as

$$(\lambda_1 q_1 N_b + \lambda_2 q_2 Z_b) = (u_1 q_1 N_b + u_2 q_2 Z_b - C)e^{-\delta t}.$$

This means that the total harvest cost per unit of user's effort is equal to the discount value of the future price under the steady state effort level.

(iii) From Eqs.(5.11) and (5.13), we can obtain

$$u_1 q_1 N_b + u_2 q_2 Z_b - C = -[(\frac{M_1 e^{-\delta t}}{M_2 + \delta} - \frac{M_3}{M_2}(\frac{R_1 e^{-\delta t}}{R_2 + \delta}))q_1 N + (\frac{H_1 e^{-\delta t}}{H_2 + \delta})q_2 Z] \to 0, \quad as \ \delta \to \infty.$$

Notes

Notes

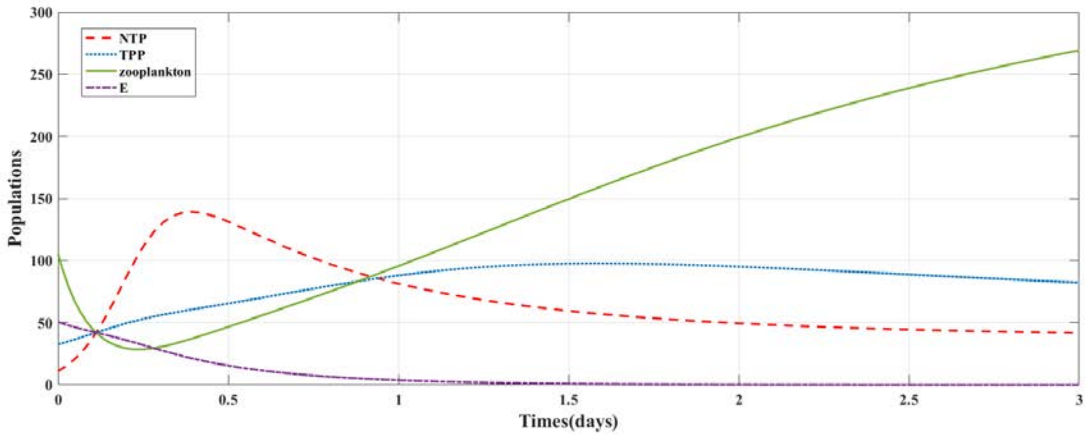


Fig. 2: The optimal solution of (5.5) for $v = 0.867$.

This shows that the unlimited discount rate leads to the complete dissipation of the net economic income to the society, $(u_1q_1N_b + u_2q_2Z_b - C)E = 0$. We also observe that for a zero discount rate, the present value of the continuous time flow reaches its maximum.

Due to the complexity of its calculation and to explain our optimal tax policy more intuitively, we continue to study it through numerical simulation. If $r_1 = 6, r_2 = 5, \alpha_1 = 0.2, \alpha_2 = 0.2, k_1 = 100, k_2 = 190, w_1 = 0.3, w_2 = 0.3, p_1 = 50, p_2 = 50, d = 0.2, c_1 = 0.45, c_2 = 0.45, \beta = 0.3, q_1 = 0.2, q_2 = 0.2, \varepsilon = 0.2, \varphi = 0.5, \gamma = 0.2, C = 2, u_1 = 0.1, u_2 = 0.2$, and the discounting factor $\delta = 0.045$ in appropriate units, based on the selection of the above parameter values, we can get the optimal tax is $v = 0.867$. In Fig.2, we get the optimal solution. Therefore, we have used the principle of Pontryagin's maximum to obtain the optimal path of the optimal tax policy, which not only ensures the maximum goal of the regulatory authority, but also the stability of the ecosystem.

VI. Numerical Simulations

In this section, we will use Matlab to numerically simulate the impact of various parameters on species and the stability of steady state. Therefore, the initial conditions and parameter settings in Table 2 are used for the numerical analysis of the system (2.3). First, we give the time series diagram of N, T and Z with short period and long period, then the impact of different β, q_1E and q_2E on the survival of species were investigated. Lastly, we study the changes in equilibrium stability with varying delays of time.

Table 2: All the biological descriptions of the parameters are given below:

Parameter	Environmental Interpretation	Value
(N^0, T^0, Z^0)	Initial concentrations	(500,200,1000)
r_1	Intrinsic growth rate of NTP	0.56
r_2	Intrinsic growth rate of TPP	0.49
α_1	Competitive effect of TPP on NTP	0.1
α_2	Competitive effect of NTP on TPP	0.1
k_1	Carrying capacity of NTP	5600
k_2	Carrying capacity of TPP	4900
w_1	NTP consumption rate	0.5
w_2	TPP consumption rate	0.5
p_1	Half saturation constants for NTP	30
p_2	Half saturation constants for TPP	30
c_1	the conversion rate of N to Z	0.45
c_2	the conversion rate of T to Z	0.45
β	Intensity of avoidance	-
d	Zooplankton mortality rate	0.05

a) *Time series analysis*

In Fig.3, we plot the time series of $\beta = 0, \beta = 10, \beta = 1000$ in the first ten days, where the other parameter values and initial conditions are the same as in Table 2. When $q_1 = q_2 = 0$ and $\beta = 0$, we can observe that NTP and TPP tend to perish at a fast linear speed. It is obvious that when β increases to 10, the concentrate of TPP will first increase to a certain concentration, then decrease and finally tend to extinction, while at this time, NTP still maintains a rapid decline rate until it is extinct(fig.3(a)(b)). On the contrary, when $\beta = 0$, we take $q_1 = 0.4$, $q_2 = 1.2$, and $q_1 = 2$, $q_2 = 2.5$, respectively. We can observe that with the increase of q_1 and q_2, NTP and zooplankton tend to become extinct at a faster rate of decline, while TPP increases more rapidly(fig.3(c)(d)). Based on the values of q_1 and q_2 of (fig.3(c)(d)), we increase β to 10. Through comparison, we can find that the curves of NTP and zooplankton have almost no change, but the increasing speed of TPP is still accelerated(fig.3(e)(f)). To further explore the influence of β, we fixed q_1 and q_2 as 2 and 2.5, respectively. And increased the value of β from 10 to 1000. At this time, We can observe that the concentration of NTP, TPP and zooplankton has almost no change(fig.3(g)(h)). Finally, when β exists and is fixed at 10, we increase the concentrations of q_1 and q_2 to 6 and 8, respectively. At this time, we can observe that NTP and zooplankton accelerate the decline rate, while TPP has no obvious change(fig.3(i)(j)).

In Fig.4, we draw a long-term time series diagram of the system (2.3). We fixed that q_1 and q_2 are both 0. In fig.4(a)(b), we can observe the dynamic change of β from 0 to 10. First, we take $\beta = 0$, in fig.4(a), we will find the extinction of TPP, while NTP and zooplankton oscillate in the form of limit cycles. Next, we increase β to 10, observe the fig.4(b), all species are in a coexistence state, and the system is stabilized to a periodic orbit. These periods show large oscillations of all populations. Secondly, when we fix $\beta = 0$ and increase $q_1 = q_2 = 0.1$ to $q_1 = q_2 = 0.36$, we can find that when q_1 and q_2 are within a certain range, NTP and TPP will coexist, and zooplankton will tend to become extinct(fig.4(c)(d)). Finally, when we fix $\beta = 10$ and increase $q_1 = q_2 = 0.36$ to $q_1 = q_2 = 0.37$, we will find that the coexistence of NTP and TPP disappears, and then only TPP exists and tends to be stable, while NTP and zooplankton tend to be extinct(fig.4(e)(f)).

b) *Double time delay analysis*

Now, to explore the influence of pregnancy delay (τ_1) and toxin onset delay(τ_2) on the stability of equilibrium point in different cases. First, we need to set a set of parameters as follows

$$r_1 = 2,\ r_2 = 3,\ \alpha_1 = 0.3,\ \alpha_2 = 0.1,\ k_1 = 2500,\ k_2 = 3000,\ w_1 = w_2 = 0.5,\ p_1 = p_2 = 50,$$

$$c_1 = c_2 = 0.45,\ d = 0.05,\ \beta = 0.5,\ q_1 = 0.2,\ q_2 = 0.3,\ E = 1. \tag{6 1}$$

With initial values $(N_0, T_0, Z_0) = (400, 300, 500)$, we perform numerical simulations to verify the theoretical results of the previous delayed system (2.2). For these parameters, we take (6.1) into the delayed system (2.2), the complex dynamical behavior of the system has been observed with time delay.

Case i: when $\tau_1 = 0,\ \tau_2 > 0$, in this case, $[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\tau_{20},\xi=i\omega_0}]^{-1} > 0$, the transversality condition is contented. So when $\tau_2 < \tau_{20}$(Fig.5(a)(b)), the positive equilibrium E^* is locally asymptotically stable, the positive equilibrium E^* is unstable when $\tau_2 > \tau_{20}$(Fig.6(a)(b)), when $\tau_2 = \tau_{20}$, the system undergoes Hopf bifurcation around the positive equilibrium E^*. (Fig.5(a)(b)) shows the trajectories and phase portrait of system (2.2) for $\tau_1 = 0,\ \tau_2 = 1$. It can be clearly seen that the system (2.2) will converge to the positive equilibrium point E^*. And (Fig.6(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0,\ \tau_2 = 1.08$. In this case, the delay system (2.2) has a periodic solution near the positive equilibrium point (E^*).

Case ii : when $\tau_1 > 0,\ \tau_2 = 0$, we change the values of k_1 and k_2 in (6.1) to $k_1 = 150,\ k_2 = 250$, and the others remain unchanged. $[\mathrm{Re}(\frac{d\xi}{d\tau_1})_{\tau_1=\tau_{10},\xi=i\hat{\omega}_0}]^{-1} > 0$, the transversality condition is satisfied. (Fig.7(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0.7,\ \tau_2 = 0$. It can be seen that although the final equilibrium point tends to be stable, there is no oscillation, indicating that there is no periodic solution in this case.

Case iii : when $\tau_1 = 0.9$ in stable interval $(0, \tau_{10})$, and take $\tau_2 > 0$ as the parameter, $[\mathrm{Re}(\frac{d\xi}{d\tau_2})_{\tau_2=\widetilde{\tau}_2,\xi=i\widetilde{\omega}}]^{-1} \neq 0$, the transversality condition is satisfied. So when

$$\tau_2 < \widetilde{\tau}_2$$

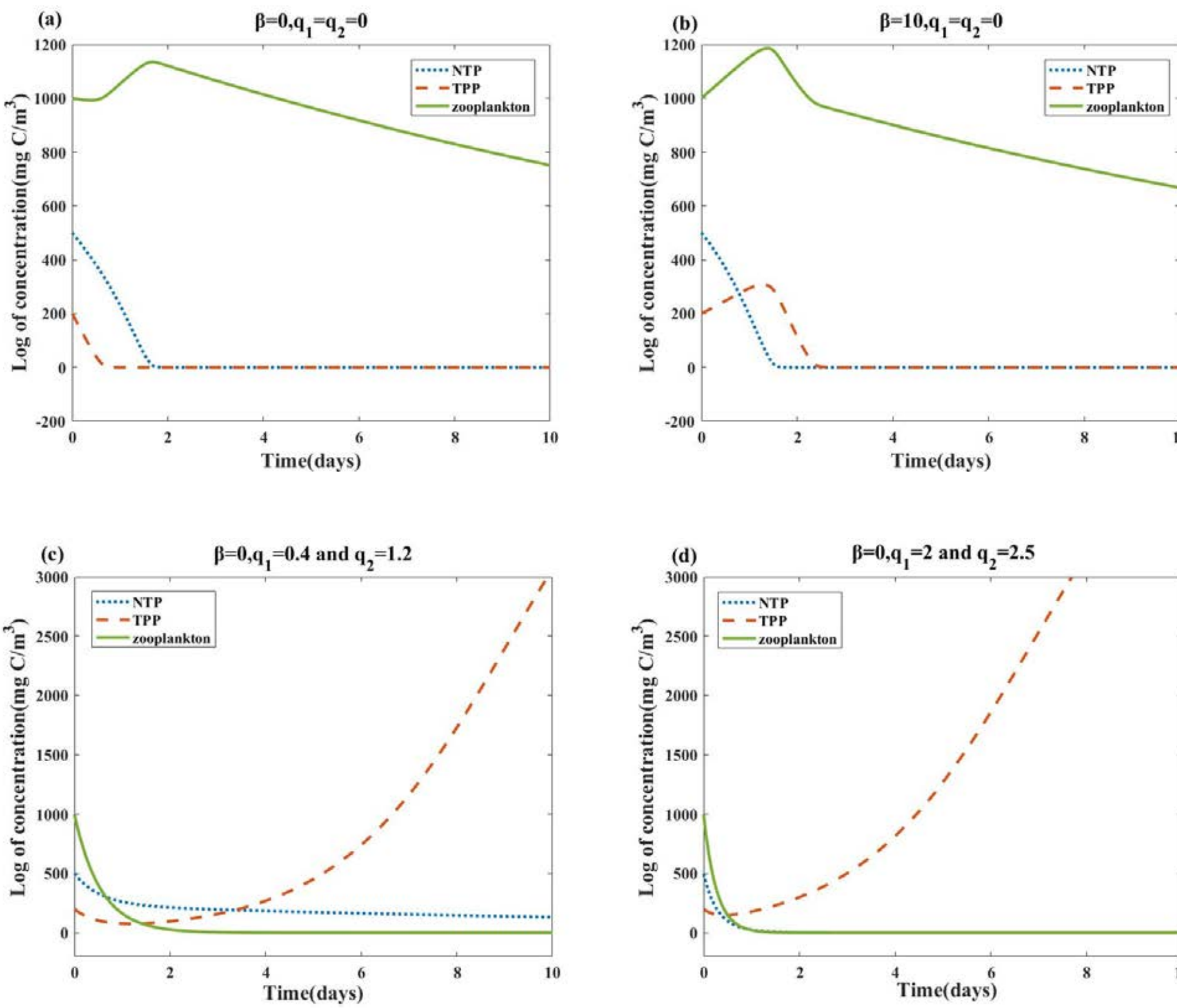
Notes

(Fig.8(a)(b)), the positive equilibrium E^* is locally asymptotically stable, the positive equilibrium E^* is unstable when $\tau_2 > \widetilde{\tau}_2$(Fig.9(a)(b)), when $\tau_2 = \widetilde{\tau}_2$, the system undergoes Hopf bifurcation around the positive equilibrium E^*. (Fig.8(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0.9$, $\tau_2 = 1.06$. It can be clearly seen that the system (2.2) will converge to the positive equilibrium point E^*. And (Fig.9(a)(b)) shows the trajectories and phase portrait of the system (2.2) for $\tau_1 = 0.9$, $\tau_2 = 1.09$; we find the delayed system (2.2) has periodic solutions near the positive equilibrium point E^* in this case.

Notes

Therefore, through the above numerical simulation, we can evidently find the system is stable for small values of the delay, but as the value of delay crosses its critical value, the system loses its stability and undergoes Hopf-bifurcation. Thus the limit cycle exists for $\tau_1 > \tau_{10}$, $\tau_2 > \tau_{20}$ and $\tau_2 > \widetilde{\tau}_2$.

VII. Discussion



Notes

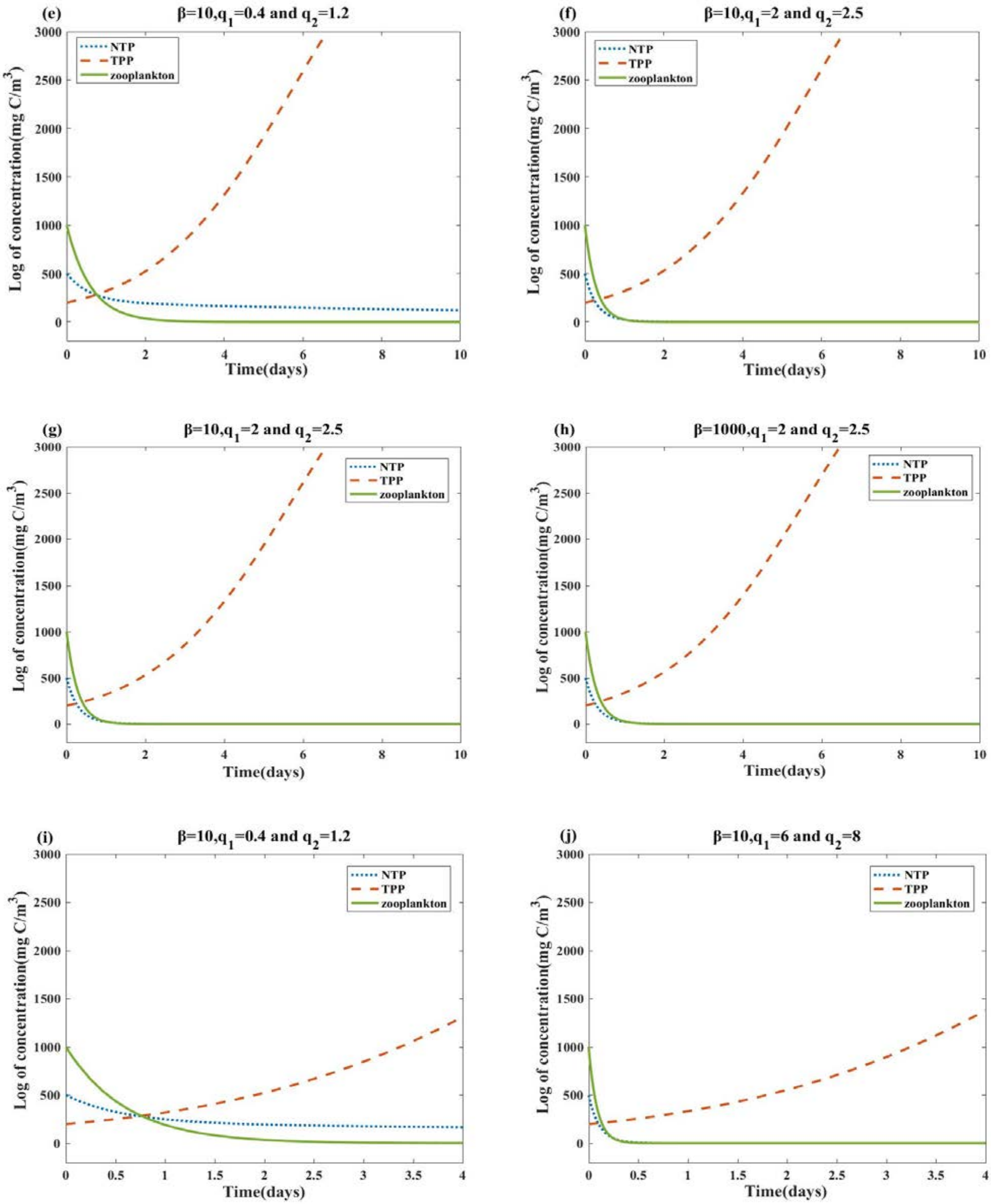


Fig. 3: The dynamic changes of the system (1) with different β, q_1 and q_2 in the first 10 days, other parameter values and initial conditions are the same as Table 2. (a)(b) : In the case of $q_1 = q_2 = 0$, $\beta = 0$ and $\beta = 10$, the TPP concentration will fluctuate and the NTP concentration will barely change. (c)(d) : For $\beta = 0$, the concentrations of q_1 and q_2 increase, and both NTP and TPP concentrations accelerate towards extinction. (e)(f) : Based on (c)(d), for $\beta = 10$, TPP reached a higher flowering concentration, while NTP still maintained a lower concentration. (g)(h) : Based on (f), for $\beta = 1000$, NTP and TPP concentrations are almost unchanged. (i)(j): for $\beta = 10$, we increase the concentrations of q_1 and q_2 to 6 and 8, respectively. NTP and zooplankton accelerate the decline rate, while TPP has no obvious change.

Notes

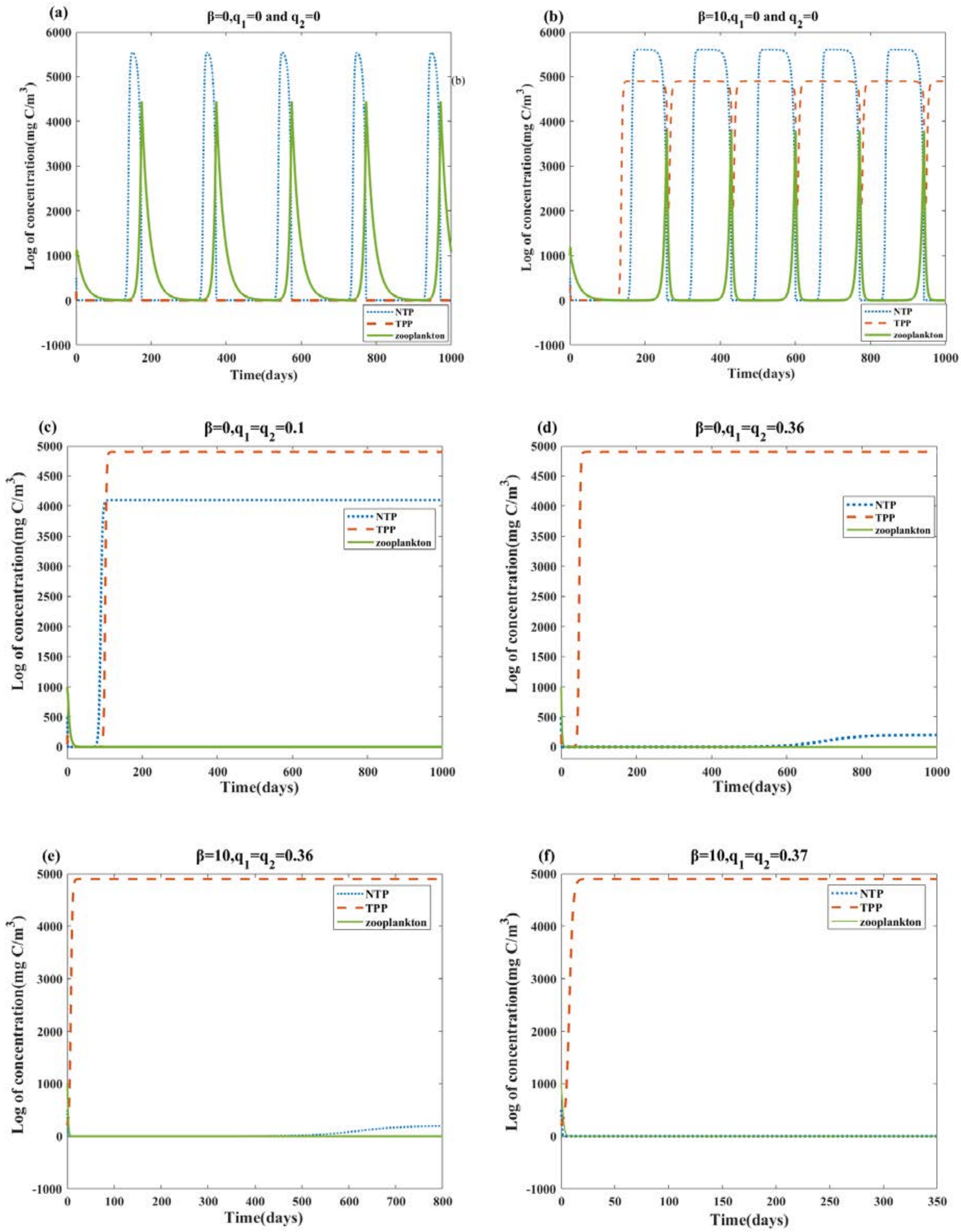


Fig. 4: The long-term dynamics of the system (2.1), all other parameter values are the same as Table 2. (a): When $q_1 = q_2 = 0$, NTP and zooplankton with initial concentrations (500,200,1000) oscillate and TPP populations become extinct. (b): For $\beta = 10$, all populations survive and the system stabilizes to a limit cycle. (c)(d) : For $\beta = 0$, $0 \le q_1 = q_2 \le 0.36$, NTP and TPP can coexist. (e)(f): when we fix $\beta = 10$ and increase $q_1 = q_2 = 0.36$ to $q_1 = q_2 = 0.37$, we will find that the coexistence of NTP and TPP disappears, and then only TPP exists and tends to be stable, while NTP and zooplankton tend to be extinct.

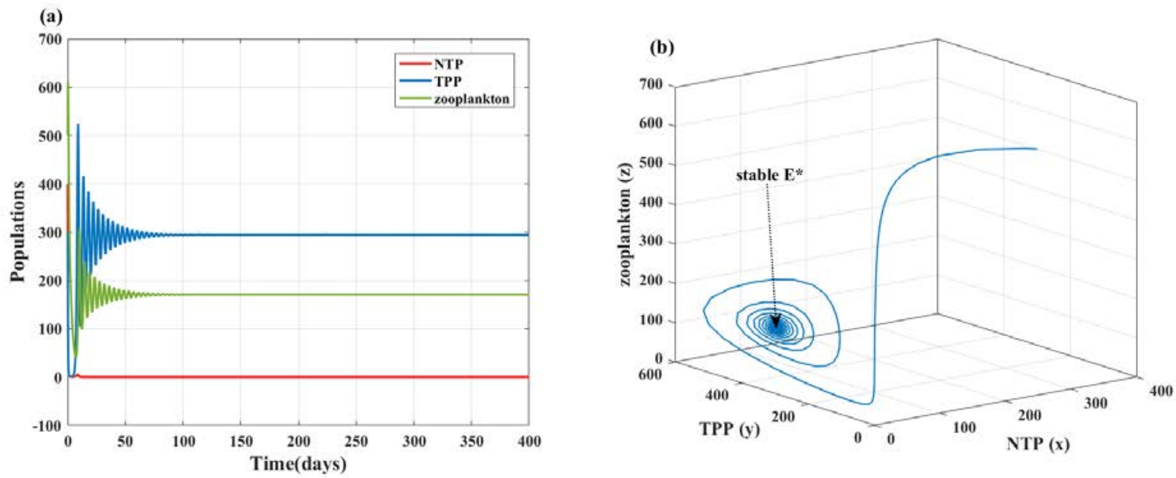


Fig. 5: The behavior of the system(2.2) for $\tau_1 = 0, \tau_2 = 1$ with other parameters chosen in (6.1).

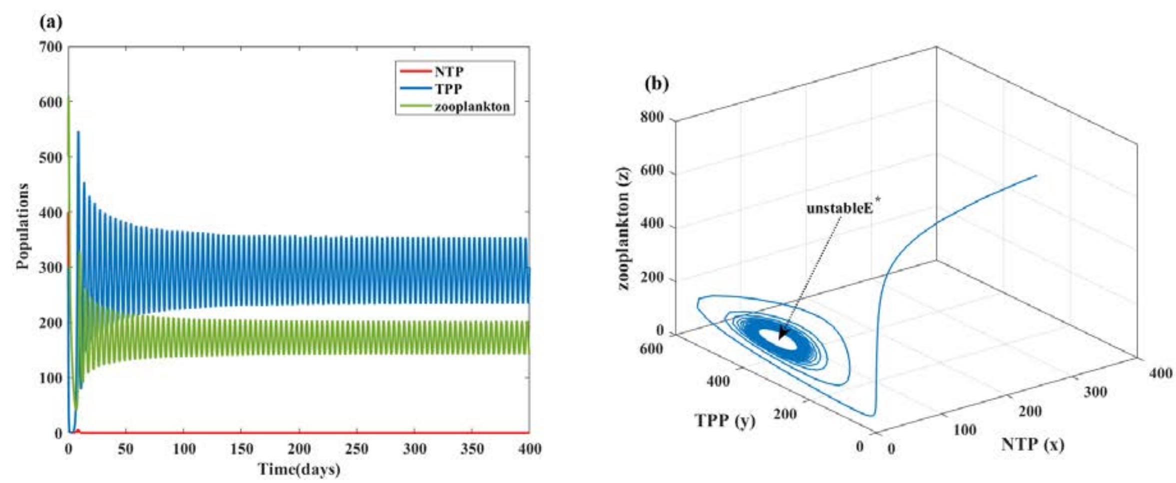


Fig. 6: The behavior of the system(2.2) for $\tau_1 = 0, \tau_2 = 1.08$ with other parameters chosen in (6.1).

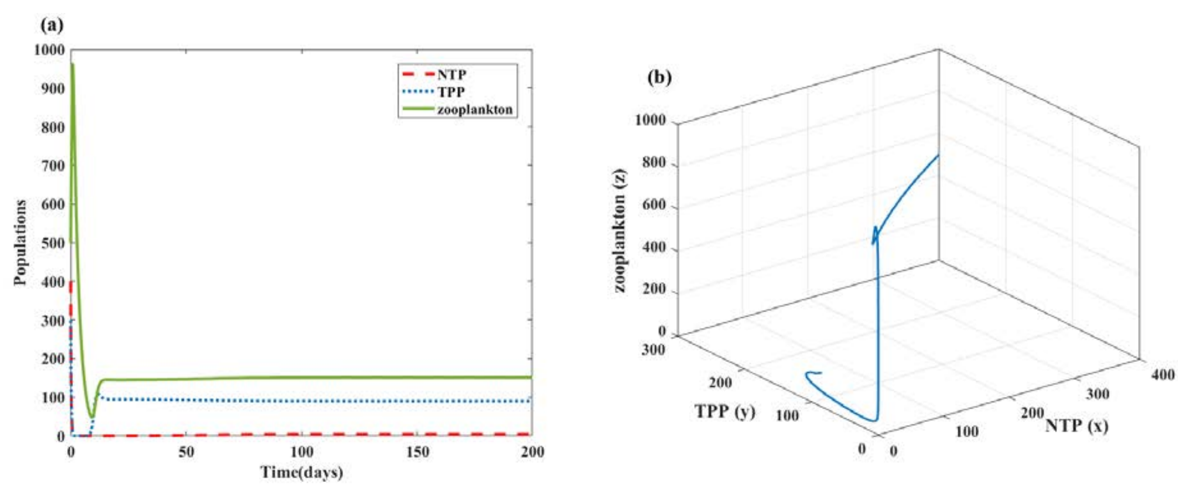


Fig. 7: The behavior of the system(2.2) for $\tau_1 = 0.7, \tau_2 = 0$ with other parameters chosen in (6.1).

Notes

Notes

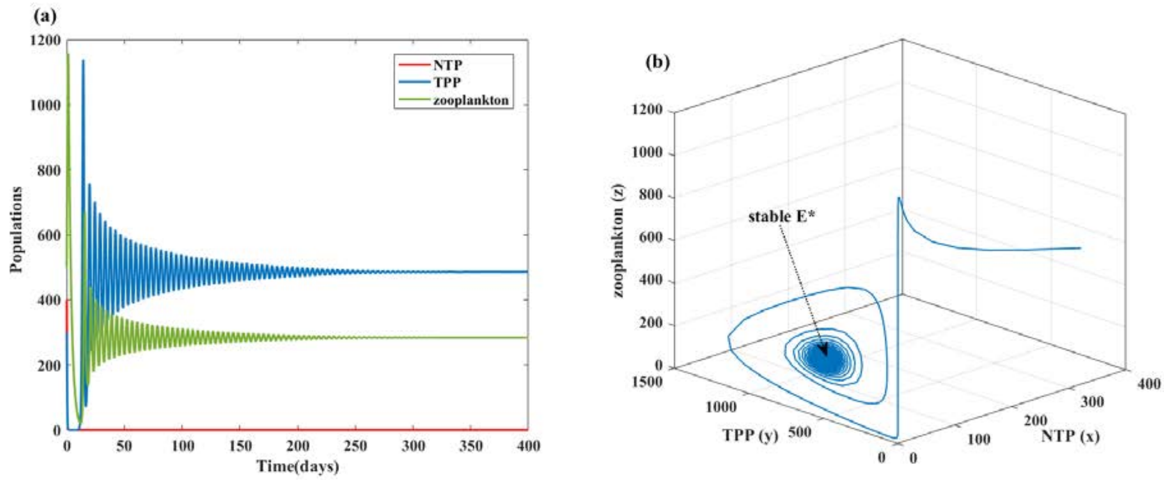


Fig. 8: The behavior of the system(2.2) for $\tau_1 = 0.9$,$\tau_2 = 1.06$ with other parameters chosen in (6.1).

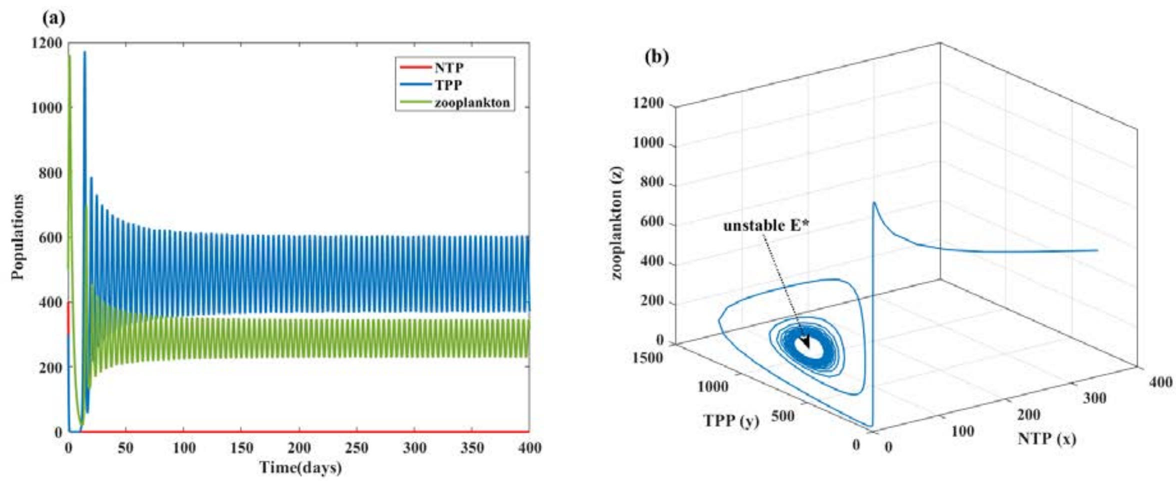


Fig. 9: The behavior of the system(2.2) for $\tau_1 = 0.9$,$\tau_2 = 1.09$ with other parameters chosen in (6.1).

The predator avoidance effect always attracts ecologists to investigate it. In the aquatic system, zooplankton lives in the environment full of toxic and non-toxic bait (phytoplankton). To make toxic phytoplankton, nontoxic phytoplankton and zooplankton coexist, the avoidance behavior of zooplankton against toxic phytoplankton is an important research topic. In this paper, we consider a biological model with two delays in which zooplankton avoids poisonous phytoplankton in the presence of nontoxic phytoplankton. For this model of poisonous avoidance, due to the avoidance coefficient of zooplankton to toxic phytoplankton, the growth density of zooplankton and toxic phytoplankton is nonlinear. When the poisonous avoidance coefficient is high, the density of poisonous phytoplankton will increase in proportion, and finally tend to be stable. we also consider the impact of human harvest on the coexistence of these three species, the form of avoidance and human harvest have biological significance, which we also analyzed.

According to this article, we analyze the positive and boundedness of the system solution without time delay at first. In the bounded area, the densities of nontoxic phytoplankton (NTP), toxic phytoplankton (TPP) and zooplankton (zooplankton) are all non negative. Then we analyze the bistability of the equilibrium points. From fig.1, we can see the bistability of each equilibrium point in different k_1 ranges. For the dynamic behavior of double time-delay systems, we analyze the local stability and the existence of Hopf bifurcation. Taking the pregnancy delay τ_1 and the toxin onset delay τ_2 as the bifurcation parameters, the critical value of the time delay for the Hopf bifurcation of the system under different conditions is obtained. We find that the system is stable when the time delay is less than this critical value(τ_1^0, τ_2^0, τ_{10}^* and τ_{20}^*, respectively), but when we increase the time delay to more than this critical value, the system will become unstable, and then Hopf bifurcation occurs at the critical time. Considering the practical significance of the research, in section 5, we use the principle of Pontryagin's maximum to study the optimal tax policy of the system without time delay, we obtained the optimal path of the optimal tax policy. In addition, we use the parameters and initial values given in Table 2 and (6.1) to simulate several cases of double-delay systems in Matlab to verify all theoretical results.

References Références Referencias

[1] A. Mitra, K.J. Flynn, Accounting for variation in prey selectivity by zooplankton, Ecol. Model. 199 (2006) 82-92.

[2] S.B. Linhart, J.D. Roberts, S.A. Shumake, R. Johnson, Avoidance of prey by captive coyotes punished with electric shock, In: Proceedings of the Vertebrate Pest Conference, escholarship, pp. 7 (1976) 302-330.

[3] S. Ghorai, B. Chakraborty, N. Bairagi, Preferential selection of zooplankton and emergence of spatiotemporal patterns in plankton population, Chaos. Soliton. Fract. 153 (2021) 111471.

[4] X.J. Liu, C.H. Tang, C.K. Wong, Microzooplankton grazing and selective feeding during bloom periods in the Tolo Harbour area as revealed by HPLC pigment analysis, J. Sea Res. 90 (2014) 83-94.

[5] Y.L. Zheng, X. Gong, H.W. Gao, Selective grazing of zooplankton on phytoplankton defines rapid algal succession and blooms in oceans, Ecol.l Modelling. 468 (2022) 109947.

[6] K.G. Porter, Selective grazing and differential digestion of algae by zooplankton, Nature. 244 (1973) 179-180.

[7] B.W. Frost, Effect of size and density of food particles on the feeding behaviour of the marine planktonic copepod Calanuspacificus, Limnol. Oceanogr. 17 (1972) 1752-1765.

[8] Q.Y. Zhao, S.T. Liu, D.D. Tian, Dynamic behavior analysis of phytoplanktonzooplankton system with cell size and time delay, Chaos. Solitons. Fractals. 113 (2018) 160-168.

[9] C.J. Zilverberg, J. Angerer, J. Williams, L.J. Metz, K. Harmoney, Sensitivity of diet choices and environmental outcomes to a selective grazing algorithm, Ecol. Model. 390 (2018) 10-22.

[10] S. Uye, K. Takamatsu, Feeding interactions between planktonic copepods and red-tide flagellates from Japanese coastal waters, Mar. Ecol. Prog. Ser. 59 (1990) 97-107.

[11] J. Sole, E. Garcia-Ladona, M. Estrada, The role of selective predation in harmful algal blooms, J. Mar. Syst. 62 (2006) 46-54.

[12] K. Agnihotri, H. Kaur, Optimal control of harvesting effort in a phytoplanktonzooplankton model with infected zooplankton under the influence of toxicity, Math. Comput. Simul. 19 (2021) 946-964.

[13] J. Chattopadhyay, Effect of toxic substances on a two-species competitive system, Ecol. Model. 84 (13) (1996) 287-289.

[14] S. Chakraborty, S. Bhattacharya, U. Feudel, J. Chattopadhyay, The role of avoidance by zooplankton for survival and dominance of toxic phytoplankton, Ecol. Complexity 11 (2012) 144-153.

[15] S. Khare, O.P. Misra, J. Dhar, Role of toxin producing phytoplankton on a plankton ecosystem, Nonlinear Anal. 4 (3) (2010) 496-502.

[16] N. Turriff, J.A. Runge, A.D. Cembella, Toxin accumulation and feeding behaviour of the planktonic-copepod Calanus jinmarchicus exposed to the red -tide dinoflagellate Alexandrium excavatum, Mar. Biol. 123 (1995) 55-64.

[17] S.J. Jang, J. Baglama, J. Rick, Nutrient-phytoplankton-zooplankton models with a toxin, Math. Comput. Modelling. 43 (12) (2006) 105-118.

[18] B. Dubey, J. Hussain, A model for the allelopathic effect on two competing species, Ecol. Model. 129 (23) (2000) 195-207.

[19] G.P. Samanta, A stochastic two species competition model: Nonequilibrium fluctuation and stability, Int. J. Stoch. Anal. 2011 (2011) 1-7.

[20] S. Roy, J. Chattopadhyay, Toxin-allelopathy among phytoplankton species prevents competitive exclusion, J. Biol. Syst. 15 (01) (2007) 73-93.

[21] Y. Pei, Y. Lv, C. Li, Evolutionary consequences of harvesting for a two-zooplankton one-phytoplankton system, Appl. Math. Model. 36 (4) (2012) 1752-1765.

Notes

Notes

[22] K. Chakraborty, S. Das, T. Kar, Optimal control of effort of a stage structured preypredator fishery model with harvesting, Nonlinear Anal. RWA. 12 (6) (2011) 3452-3467.
[23] C.W. Clark, Bioeconomic Modelling and Fisheries Management, New York (USA) Wiley. 1985.
[24] P. Panja, S.K. Mondal, D.K. Jana, Effects of toxicants on Phytoplankton-Zooplankton-Fish dynamics and harvesting, Chaos. Solitons. Fractals. 104 (2017) 389-399.
[25] F.F. Zhang, J.M. Sun, W. Tian, Spatiotemporal pattern selection in a nontoxic-phytoplankton- toxic-phytoplankton - zooplankton model with toxin avoidance effects, App. Math. Comput. 423 (2022) 127007.
[26] D. Xiao, L. Jennings, Bifurcations of a ratio-dependent predatorprey system with constant rate harvesting, Siam. Appl. Math. 65 (2005) 737-753.
[27] Y. Lv, Y. Pei, S. Gao, C. Li, Harvesting of a phytoplanktonzooplankton model, Nonlinear Anal: Real World Appl. 11 (2010) 3608-3619.
[28] A.K. Sharma, A. Sharma, K. Agnihotri, Bifurcation behaviors analysis of a plankton model with multiple delays, Int. J. Biomath. 9 (06) (2016) 1650086.
[29] K. Agnihotri, H. Kaur, The dynamics of viral infection in toxin producing phytoplankton and zooplankton system with time delay, Chaos Solitons. Fractals. 118 (2019) 122-133.
[30] A. Mondal, A.K. Pal, G.P. Samanta, Rich dynamics of non-toxic phytoplankton, toxic phytoplankton and zooplankton system with multiple gestation delays, Int. J. Dyn. Control. 8 (2020) 112-131.
[31] P. Panday, S. Samanta, N. Pal, Delay induced multiple stability switch and chaos in a predatorprey model with fear effect, Math. Comput. Simulation. 172 (2020) 134-158.
[32] T. Kar, Conservation of a fishery through optimal taxation: a dynamic reaction model, Commun. Nonlinear Sci. Numer. Simul. 10 (2005) 121-131.
[33] S. Sarkar, Optimal fishery harvesting rules under uncertainty, Resour. Energy Econ. 31 (2009) 272-286.
[34] J. Sotomayor, Generic bifurcations of dynamical systems, In Dynamical systems, Academic Press. 1973.

This page is intentionally left blank

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Anisotropic Conservative Caginalp Phase-Field System based on Type III Heat Conduction with Two Temperatures and Periodic Boundary Conditions

By Cyr Séraphin Ngamouyih Moussata, Armel Judice Ntsokongo & Dieudonné Ampini
Fujian Normal University

Abstract- Our aim in this paper is to study the well-posedness results of anisotropic conservative Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction and periodic boundary conditions. More precisely, we prove the existence and uniqueness of solutions.

Keywords: conserved phase-field system, anisotropy, type III thermomechanics, two temperatures, well-posedness.

GJSFR-F Classification: LCC Code: QA801-939

Strictly as per the compliance and regulations of:

Ref
[13] A. Miranville and R. Quintanilla, *A Caginalp phase-field system based on type III heat conduction with two temperatures*, Quart. Appl. Math., 2016, 74, 375–398.

On Anisotropic Conservative Caginalp Phase-Field System based on Type III Heat Conduction with Two Temperatures and Periodic Boundary Conditions

Cyr Séraphin Ngamouyih Moussata [α], Armel Judice Ntsokongo [σ] & Dieudonné Ampini [ρ]

Abstract- Our aim in this paper is to study the well-posedness results of anisotropic conservative Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction and periodic boundary conditions. More precisely, we prove the existence and uniqueness of solutions.

Keywords: conserved phase-field system, anisotropy, type III thermomechanics, two temperatures, well-posedness.

I. Introduction

The authors studied in [13] (see aigain [12]) the following phase-field system, namely,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{1.2}$$

$$\alpha(t,x) = \alpha(0,x) + \int_0^t T(\tau,x)d\tau, \tag{1.3}$$

where, u is the order parameter, T is the relative temperature (defined as $T = \widetilde{T} - T_E$, where $\widetilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature), α is the conductive thermal displacement and f is the derivative of a double-well potential F (a typical choice is $F(s) = \frac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, here and below, we set all physical parameters equal to one. This system has been introduced to model phase transition phenomena, such as melting-solidication phenomena, and has been much studied from a mathematical point of view. We refer the reader to, e.g., [4–5, 8–11, 14, 16, 17, 21, 23].

Author α σ ρ: Faculté des Sciences et Techniques, Université Marien Ngouabi. B.P. 69 Congo-Brazzaville.
e-mail: armeljudice@gmail.com

This system is based on the (total Ginzburg-Landau) free energy,

$$\Psi_{\mathrm{GL}} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \tag{1.4}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 2$ or $n = 3$, with boundary Γ), and the enthalpy

$$H = u + T - \Delta T. \tag{1.5}$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial u} = -\frac{D\Psi_{GL}}{Du}, \tag{1.6}$$

where $\frac{D}{Du}$ denotes a variational derivative with respect to u. Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\mathrm{divq} \tag{1.7}$$

and owing to (1.7),

$$\frac{\partial T}{\partial t} - \Delta\frac{\partial T}{\partial t} + \mathrm{divq} = -\frac{\partial u}{\partial t}, \tag{1.8}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla\alpha - \nabla T, \tag{1.9}$$

we obtain (1.1) and (1.2).

Our aim in this paper is to study the model consisting of the conserved anisotropic to (1.1)–(1.2), namely,

$$\frac{\partial u}{\partial t} + \Delta\sum_{i=1}^{3} a_i\frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta\left(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right),\ a_i > 0, \tag{1.10}$$

$$\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}. \tag{1.11}$$

Our aim in this paper is to study the model consisting of the anisotropic conservative equation (1.10) and the temperature equation (1.11). In particular, we obtain the existence and uniqueness of solutions.

Notes

Notes

II. Setting of the Problem

Find the order parameter u $:\Omega \times \mathbb{R}^+ \to \mathbb{R}$ and the termal displacement $\alpha : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ such that:

$$\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right), \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

together with periodic boundary conditions

$$u \quad \text{and} \quad \alpha \quad \text{are} \quad \Omega - \text{periodic}, \tag{2.3}$$

and the initial conditions

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We assume that

$$a_i > 0, \quad i \in \{1, 2, 3\}, \tag{2.5}$$

and we introduce the elliptic operator A defined by

$$\langle Av, w \rangle_{H^{-1}(\Omega), H^1_{per}(\Omega)} = \sum_{i=1}^{3} a_i \left(\left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right) \right), \tag{2.6}$$

where $H^{-1}(\Omega)$ is the topological dual of $H^1_{per}(\Omega)$. Furthermore, $((.,.))$ denotes the usual L^2-scalar product, with associated norm $\|.\|$; more generally, we denote by $\|.\|_X$ the norm on the Banach space X and we set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, $(-\Delta)^{-1}$ denoting the inverse minus Laplace operator with periodic boundary conditions and acting on functions with null average, is a norm in $H^{-1}(\Omega) = H^1_{per}(\Omega)'$ which is equivalent to the usual H^{-1}-norm. We can note that

$$(v, w) \in H^1_{per}(\Omega)^2 \mapsto \sum_{i=1}^{3} a_i \left(\left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right) \right)$$

is bilinear, symmetric, continuous and coercive, so that

$$A : H^1_{per}(\Omega) \to H^{-1}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order 2 (see [1–2]) that A is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A) = H^2_{per}(\Omega),$$

where, for $v \in D(A)$,

$$Av = -\sum_{i=1}^{3} a_i \frac{\partial^2 v}{\partial x_i^2}.$$

We further note that $D(A^{\frac{1}{2}}) = H^1_{per}(\Omega)$ and, for $v \in D(A^{\frac{1}{2}})$,

$$((A^{\frac{1}{2}}v, A^{\frac{1}{2}}v)) = \sum_{i=1}^{3} a_i \Big\| \frac{\partial v}{\partial x_i} \Big\|^2.$$

We finally note that (see, e.g., [18]) $v \mapsto (\|Av\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$ defines a norm on $H^2_{per}(\Omega)$ which is equivalent to the usual H^2-norm on $D(A)$ (resp., $v \mapsto (\|A^{\frac{1}{2}}v\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$ defines a norm on $H^1_{per}(\Omega)$ which is equivalent to the usual H^1-norm on $D(A^{\frac{1}{2}})$), where

$$\langle . \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega .\mathrm{d}x,$$

being understood that, for $v \in H^{-1}(\Omega)$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle v, 1 \rangle_{H^{-1}(\Omega), H^1_{per}(\Omega)},$$

and we note that

$$v \mapsto (\|v - \langle v \rangle\|^2_{-1} + \langle v \rangle^2)^{\frac{1}{2}}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual one. Here, $\Omega = \prod_{i=1}^{n}(0, L_i)$, $L_i > 0$, $n = 2$ or $n = 3$. Furthermore, for a space W we shall dénote by $\dot{W}$ the space

$$\dot{W} = \{v \in W,\ \langle v \rangle = 0\}.$$

Remark 2.1. *Actually, the conseved phase-field system usually is endowed with Neumann boundary conditions. In our case, these conditions read*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} \quad \Big(= \frac{\partial A u}{\partial \nu}\Big) = \frac{\partial \alpha}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{2.7}$$

where ν denotes the unit outer normal.

Ref

[18] J.E. Taylor, Mean curvature and weighted mean curvature, Acta Metall. Mater., 1992, 40, 1475–1495.

Notes

Remark 2.2. *Note that similar properties hold for the operator $-\Delta$, with obvious changes.*

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} - \Delta A u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right). \tag{2.8}$$

Furthermore, we assume that the function f satisfies the following conditions:

$$f \in C^2(\mathbb{R}), \quad f(0) = 0, \tag{2.9}$$

$$f' \geqslant -c_0, \quad c_0 \geqslant 0, \tag{2.10}$$

$$f(s)s \geqslant c_1 F(s) - c_2, \quad F(s) \geqslant -c_3, \quad c_1 > 0, \quad c_2, c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.11}$$

where, we denote by F the primitive of f vanishing at $s = 0$,

$$c_4 s^{2p-1} - c_5 \leqslant f''(s) \leqslant c_6 s^{2p-1} + c_7, \quad c_4, c_6 > 0, \quad c_5, c_7 \geqslant 0, \quad p \geqslant 1, \quad s \in \mathbb{R}. \tag{2.12}$$

Remark 2.3. *In particular, these assumptions are satisfied by function*

$$f(s) = \sum_{i=1}^{2p+1} a_i s^i, \quad a_{2p+1} > 0, \quad \forall s \in \mathbb{R}$$

(and, the usual cubic nonlinear term $f(s) = s^3 - s$).

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similaly, the same letter Q denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

III. A Priori Estimates

The estimates below are formal, but they can also be justified within a Galerkin scheme for the approximated problem.

We first note that, integrating (formally) (2.8) over Ω, we have

$$\frac{d\langle u\rangle}{dt} = 0,$$

hence

$$\langle u(t)\rangle = \langle u_0\rangle, \quad \forall t \geqslant 0. \tag{3.1}$$

Furthermore, integrating (2.2) over Ω, we obtain, in view of (2.7),

$$\frac{d^2\langle\alpha\rangle}{dt^2} = -\frac{d\langle u\rangle}{dt}. \tag{3.2}$$

It thus follows from (2.4) and (3.2) that

$$\frac{d\langle\alpha\rangle}{dt} = \langle u_0 + \alpha_1\rangle - \langle u\rangle, \tag{3.3}$$

meaning, in particular, that $\left\langle u + \frac{\partial\alpha}{\partial t}\right\rangle$ is a conserved quantity and from (3.1) that

$$\frac{d\langle\alpha\rangle}{dt} = \langle\alpha_1\rangle, \tag{3.4}$$

so that

$$\langle\alpha(t)\rangle = \langle\alpha_0\rangle + \langle\alpha_1\rangle t, \quad t \geqslant 0. \tag{3.5}$$

We now assume that

$$|\langle u_0\rangle| \leqslant M_1, \quad |\langle\alpha_1\rangle| \leqslant M_2, \quad |\langle u_0 + \alpha_1\rangle| \leqslant M_1 + M_2, \tag{3.6}$$

for fixed positive constants M_1 et M_2. Thus,

$$|\langle u(t)\rangle| \leqslant M_1, \quad \left|\left\langle\frac{\partial\alpha}{\partial t}(t)\right\rangle\right| \leqslant M_2, \quad \left|\left\langle u + \frac{\partial\alpha}{\partial t}\right\rangle(t)\right| \leqslant M_1 + M_2, \quad t \geqslant 0. \tag{3.7}$$

Furthemore, it follows from (3.5) that

$$|\langle\alpha(t)\rangle| \leqslant |\langle\alpha_0\rangle| + |\langle\alpha_1\rangle| t, \quad t \geqslant 0. \tag{3.8}$$

We rewrite, in view of (3.4), (2.8) as

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + Au + f(u) - \langle f(u)\rangle = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t} + \langle\alpha_1\rangle \tag{3.9}$$

Notes

and, in view of (3.3) an (3.9) that

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + Au + f(u) - \langle f(u)\rangle = \frac{\partial \overline{\alpha}}{\partial t} - \Delta\frac{\partial \overline{\alpha}}{\partial t} + \langle \alpha_1\rangle. \tag{3.10}$$

Notes

Furthermore, we deduce from (2.2) and (3.2) that

$$\frac{\partial^2\overline{\alpha}}{\partial t^2} - \Delta\frac{\partial^2\overline{\alpha}}{\partial t^2} - \Delta\frac{\partial\overline{\alpha}}{\partial t} - \Delta\overline{\alpha} = -\frac{\partial \overline{u}}{\partial t}, \tag{3.11}$$

We first multiply (3.9) by $\frac{\partial u}{\partial t}$ and obtain, noting that $\left\langle \frac{\partial u}{\partial t}\right\rangle = 0$,

$$\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 = \left(\left(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}, \frac{\partial u}{\partial t}\right)\right). \tag{3.12}$$

We then multiply (2.2) by $\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2$$

$$= -\left(\left(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}, \frac{\partial u}{\partial t}\right)\right) \tag{3.13}$$

(note indeed that $\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|^2 = \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2$).

Summing finally (3.12) and (3.13), we find a differential equality

$$\frac{dE_1}{dt} + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + 2\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 = 0 \tag{3.14}$$

where

$$E_1 = \|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|^2$$

satisfies, owing to (2.12),

$$c\left(\|A^{\frac{1}{2}}u\|^2+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+c'\leqslant E_1$$

$$\leqslant c''\left(\|A^{\frac{1}{2}}u\|^2+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+c''',\quad c,c''>0. \tag{3.15}$$

(here and below, when not specified, the sign of the constants (c' and c''' here) can be arbitrary).

Multiplying (3.9) by $\overline{u}$ and have, integrating over Ω and by parts,

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|^2_{-1}+\|A^{\frac{1}{2}}u\|^2+((f(u),u))=\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},u\right)\right)+((f(u),\langle u\rangle))-\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\langle u\rangle\right)\right).$$

It follows from (2.11) that

$$((f(u),u))\geqslant c_2\int_\Omega F(u)\mathrm{d}x+c,$$

from (2.12) and (3.7) that

$$|((f(u),\langle u\rangle))|\leqslant cM_1\int_\Omega|f(u)|\mathrm{d}x\leqslant\frac{c_2}{2}\int_\Omega F(u)\mathrm{d}x+c_{M_1}$$

and from (3.7) that

$$\left|\left(\left(\frac{\partial\alpha}{\partial t},\langle u\rangle\right)\right)\right|\leqslant c_{M_1,M_2}. \tag{3.16}$$

Therefore, owing again to (3.7) and remembering that $v\mapsto(\|A^{\frac{1}{2}}v\|^2+\langle v\rangle^2)^{\frac{1}{2}}$ is a norm in $H^1_{per}(\Omega)$ which is equivalent to the usual H^1-norm,

$$\frac{d}{dt}\|\overline{u}\|^2_{-1}+c(\|u\|^2_{H^1_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x)\leqslant c'\left(\|\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\right)+c''_{M_1,M_2},\quad c>0. \tag{3.17}$$

Summing (3.14) and δ_1(3.17), where $\delta_1>0$ is small enough, we have a differential inequality of the form

$$\frac{dE_2}{dt}+c\left(\|u\|^2_{H^1_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial\alpha}{\partial t}\right\|^2_{H^2_{per}(\Omega)}\right)+\leqslant c'_{M_1,M_2},\quad c>0, \tag{3.18}$$

Notes

Notes

where

$$E_2 = E_1 + \delta_1 \|\overline{u}\|_{-1}^2$$

satisfies

$$c\left(\|A^{\frac{1}{2}}u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c' \leqslant E_2$$

$$\leqslant c''\left(\|A^{\frac{1}{2}}u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c''', \quad c, c'' > 0. \tag{3.19}$$

We multiply (2.8) by u to obtain, owing to (2.9) and (2.10),

$$\frac{d}{dt}\|u\|^2 + \|\nabla A^{\frac{1}{2}}u\|^2 \leqslant c\left(\|u\|_{H_{per}^1(\Omega)}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right). \tag{3.20}$$

Summing (3.18) and δ_2(3.20), where $\delta_2 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{dE_3}{dt} + c\left(\|u\|_{H_{per}^2(\Omega)}^2 + 2\int_\Omega F(u)\mathrm{d}x + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|_{H_{per}^2(\Omega)}^2\right) \leqslant c'_{M_1,M_2}, \quad c > 0, \tag{3.21}$$

where

$$E_3 = E_2 + \delta_2\|u\|^2$$

satisfies

$$c\left(\|u\|_{H_{per}^1(\Omega)}^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c' \leqslant E_3$$

$$\leqslant c''\left(\|u\|_{H_{per}^1(\Omega)}^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c''', \quad c, c'' > 0. \tag{3.22}$$

Now, multiplying (3.10) by $\frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 = \left(\left(\frac{\partial\overline{\alpha}}{\partial t} - \Delta\frac{\partial\overline{\alpha}}{\partial t}, \frac{\partial\overline{u}}{\partial t}\right)\right). \tag{3.23}$$

We then multiply (3.11) by $\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}$

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\overline{\alpha}\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$$

$$=-\left(\left(\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t},\frac{\partial\overline{u}}{\partial t}\right)\right)\tag{3.24}$$

(note indeed that $\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2=\left\|\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+2\left\|\nabla\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$).

Summing finally (3.21), (3.23) and (3.24), we obtaine a differential inequality of the form

$$\frac{dE_4}{dt}+c\left(\|u\|^2_{H^2_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial\alpha}{\partial t}\right\|^2_{H^2_{per}(\Omega)}+\left\|\frac{\partial\overline{\alpha}}{\partial t}\right\|^2_{H^2_{per}(\Omega)}\right)\leqslant c'_{M_1,M_2},\quad c>0,\tag{3.25}$$

where

$$E_4=E_3+\|A^{\frac{1}{2}}u\|^2+2\int_\Omega F(u)\mathrm{d}x+\|\nabla\overline{\alpha}\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$$

satisfies

$$c\left(\|u\|^2_{H^1_{per}(\Omega)}+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+c'\leqslant E_4$$

$$\leqslant c''\left(\|u\|^2_{H^1_{per}(\Omega)}+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+c''',\quad c,c''>0.\tag{3.26}$$

We finally assume that $p=1$ when $n=3$ and multiply (2.8) by Au to find

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2+\|\nabla Au\|^2+((f'(u)\nabla u,\nabla Au))=\left(\left(\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t},\nabla Au\right)\right).$$

We assume that $n=3$ and $p=1$ (the case $n=2$ can be treated in a similar way) and have, owing to (2.12) and Hölder's inequality,

$$|((f'(u)\nabla u, \nabla Au))| \leqslant c\int_\Omega (|u|^2+1)|\nabla u||\nabla Au|\mathrm{d}x$$

$$\leqslant c(\|u\|^2_{L^6(\Omega)}+1)\|\nabla u\|_{L^6(\Omega)}\|\nabla Au\| \leqslant c(\|u\|^2_{H^1_{per}(\Omega)}+1)\|u\|_{H^2_{per}(\Omega)}\|\nabla Au\|.$$

Notes

Therefore,

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2+\|\nabla Au\|^2 \leqslant c(\|u\|^4_{H^1_{per}(\Omega)}+1)\|u\|^2_{H^2_{per}(\Omega)}+c'\left(\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right). \tag{3.27}$$

IV. Well-Posedness

We have the following result.

Theorem 4.1. *We assume that (2.9)–(2.12) hold. Then, for every $(u_0,\alpha_0,\alpha_1) \in (H^1_{per}(\Omega)\cap L^{2p+2}(\Omega))\times H^2_{per}(\Omega)\times H^2_{per}(\Omega))$, (2.1)–(2.4) possesses at least one solution $(u,\alpha,\frac{\partial\alpha}{\partial t})$ such that*

$$u \in L^\infty(0,T;H^1_{per}(\Omega)\cap L^{2p+2}(\Omega))\cap L^2(0,T;H^2_{per}(\Omega)),$$

$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(\Omega)),$$

$$\alpha,\overline{\alpha}\in L^\infty(0,T;H^2_{per}(\Omega))$$

and

$$\frac{\partial\alpha}{\partial t},\frac{\partial\overline{\alpha}}{\partial t}\in L^\infty(0,T;H^2_{per}(\Omega))\cap L^2(0,T;H^2_{per}(\Omega))$$

$\forall T>0.$

Furthermore, if $p=1$ when $n=3$, then

$$u\in L^2(0,T;H^3_{per}(\Omega)).$$

Proof. The proof is based on (3.8), (3.25), (3.27) and, e.g., a standard Galerkin scheme. □

We have, concerning the uniqueness, the following.

Theorem 4.2. *We assume that the assumptions of Theorem 4.1 hold and that $p=1$ when $n=3$ and $p\in[1,2]$ when $n=2$. Then, the solution obtained in Theorem 4.1 is unique.*

Proof. Let $\left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$ be two solutions to (2.1)–(2.3) with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively, such that

$$|\langle u_0^{(i)} \rangle| \leqslant M_1, \quad |\langle \alpha_1^{(i)} \rangle| \leqslant M_2, \quad |\langle u_0^{(i)} + \alpha_1^{(i)} \rangle| \leqslant M_1 + M_2, \quad i = 1, 2, \tag{4.1}$$

for fixed positive constants M_1 and M_2. We set

$$\left(u, \alpha, \frac{\partial \alpha}{\partial t}\right) = \left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right) - \left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

We have

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} + Au + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha_1 \rangle, \tag{4.2}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{4.3}$$

$$u \quad \text{and} \quad \alpha \quad \text{are} \quad \Omega - \text{periodic}, \tag{4.4}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{4.5}$$

We multiply (4.2) by $\frac{\partial u}{\partial t}$ (note that $\left\langle \frac{\partial u}{\partial t} \right\rangle = 0$) and obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2 = \left(\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right)\right) - \left(\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t}\right)\right). \tag{4.6}$$

We first assume that $n = 3$ and $p = 1$. We have

$$\left| \left(\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t}\right)\right) \right| = \left| \left(\left(f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle, \frac{\partial u}{\partial t}\right)\right) \right|$$

Notes

$$\leqslant \|\nabla(f(u^{(1)}) - f(u^{(2)}))\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$= \Big\|\nabla(\int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))\mathrm{d}su)\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$\leqslant \Big\|\int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))\mathrm{d}s\nabla u\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$+ \Big\|u\int_0^1 f''(u^{(1)} + s(u^{(2)} - u^{(1)}))(\nabla u^{(1)} + s\nabla(u^{(2)} - u^{(1)}))\mathrm{d}s\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}.$$

Furthermore, owing to Agmon's inequality,

$$\Big\|\int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))\mathrm{d}s\nabla u\Big\|^2 \leqslant c\int_\Omega (|u^{(1)}|^4 + |u^{(2)}|^4 + 1)|\nabla u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^4_{L^\infty(\Omega)} + \|u^{(2)}\|^4_{L^\infty(\Omega)} + 1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^2_{H^1_{per}(\Omega)} + \|u^{(2)}\|^2_{H^1_{per}(\Omega)} + 1)$$

$$\times (\|u^{(1)}\|^2_{H^2_{per}(\Omega)} + \|u^{(2)}\|^2_{H^2_{per}(\Omega)} + 1)\|\nabla u\|^2$$

and, owing to Hölder's inequality,

$$\Big\|u\int_0^1 f''(u^{(1)} + s(u^{(2)} - u^{(1)}))(\nabla u^{(1)} + s\nabla(u^{(2)} - u^{(1)}))\mathrm{d}s\Big\|^2$$

$$\leqslant c\int_\Omega (|u^{(1)}|^2 + |u^{(2)}|^2 + 1)(|\nabla u^{(1)}|^2 + |\nabla u^{(2)}|^2)|u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^2_{H^1_{per}(\Omega)} + \|u^{(2)}\|^2_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^2_{per}(\Omega)} + \|u^{(2)}\|^2_{H^2_{per}(\Omega)})\|u\|^2_{H^1_{per}(\Omega)}.$$

We now assume that $n = 2$ and we take the most complicated case $p = 2$. Then, owing to Agmon's inequality and a proper interpolation inequality,

Notes

$$\Big\| \int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))\mathrm{d}s \nabla u \Big\|^2 \leqslant c \int_\Omega (|u^{(1)}|^8 + |u^{(2)}|^8 + 1)|\nabla u|^2 \mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^8_{L^\infty(\Omega)} + \|u^{(2)}\|^8_{L^\infty(\Omega)} + 1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^4 \|u^{(1)}\|^4_{H^2_{per}(\Omega)} + \|u^{(1)}\|^4 \|u^{(1)}\|^4_{H^2_{per}(\Omega)} + 1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^6_{H^1_{per}(\Omega)} + \|u^{(2)}\|^6_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)}$$

$$+ \|u^{(2)}\|^2_{H^3_{per}(\Omega)} + 1)\|\nabla u\|^2.$$

Furthermore, owing to Hölder's inequality,

$$\Big\| u \int_0^1 f''(u^{(1)} + s(u^{(2)} - u^{(1)}))(\nabla u^{(1)} + s\nabla(u^{(2)} - u^{(1)}))\mathrm{d}s \Big\|^2$$

$$\leqslant c \int_\Omega (|u^{(1)}|^6 + |u^{(2)}|^6 + 1)(|\nabla u^{(1)}|^2 + |\nabla u^{(2)}|^2)|u|^2 \mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^6_{H^1_{per}(\Omega)} + \|u^{(2)}\|^6_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^2_{per}(\Omega)} + \|u^{(2)}\|^2_{H^1_{per}(\Omega)})\|u\|^2_{H^1_{per}(\Omega)}.$$

Finally, we obtain, in both cases, an inequality of the form

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2 + \|\frac{\partial u}{\partial t}\|^2_{-1} \leqslant 2\left(\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right)\right)$$

$$+ c(\|u^{(1)}\|^q_{H^1_{per}(\Omega)} + \|u^{(2)}\|^q_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)} + \|u^{(2)}\|^2_{H^3_{per}(\Omega)} + 1)\|\nabla u\|^2, \quad q \geqslant 1. \quad (4.7)$$

Multiplying then (4.3) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$, we find

$$\frac{d}{dt}\left(\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \Big\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big\|^2\right) + 2\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 + 2\Big\|\Delta\frac{\partial \alpha}{\partial t}\Big\|^2$$

$$= -2\left(\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right)\right). \quad (4.8)$$

Notes

Summing finally (4.7) and (4.8), we have, an inequality of the form

$$\frac{dE_5}{dt} \leqslant c(\|u^{(1)}\|^q_{H^1_{per}(\Omega)} + \|u^{(2)}\|^q_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)} + \|u^{(2)}\|^2_{H^3_{per}(\Omega)} + 1)E_4, \tag{4.9}$$

Notes

$q \geqslant 1$, where

$$E_5 = \|A^{\frac{1}{2}}u\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|^2.$$

Then, We deduce from (4.9), (3.8), the estimates obtained in the previous subsection and Gronwall's lemma the uniqueness, as well as the continuous dependence with respect to the initial data. □

V. Regularity of Solutions

We have the following result which gives the existence and uniqueness of more regular solutions.

Theorem 5.1. We assume that the assumptions of Theorem 4.1 hold and that (2.12) is replaced by

$$|f(s)| \leqslant \epsilon F(s) + c_\epsilon, \quad \forall \epsilon > 0, \quad s \in \mathbb{R}. \tag{5.1}$$

Then, if $(u_0, \alpha_0, \alpha_1) \in H^2_{per}(\Omega) \times H^3_{per}(\Omega) \times H^3_{per}(\Omega)$, the problem possesses a unique solution such that

$$u \in L^\infty(0, T; H^2_{per}(\Omega))$$

and

$$\alpha, \frac{\partial\alpha}{\partial t} \in L^\infty(0, T; H^3_{per}(\Omega)), \quad \forall T > 0.$$

Proof. The proof of uniqueness is obtained by proceeding as in that of Theorem 4.2, noting that, with the higher regularity considered here, no growth assumption on f is needed, owing to the continuous embedding $H^2_{per}(\Omega) \subset L^\infty(\Omega)$.

We now turn to the proof of existence and, more precisely, of the further regularity of the solutions.

We multiply (2.8) by $\frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla A^{\frac{1}{2}}u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 = \left(\left(\Delta f(u), \frac{\partial u}{\partial t}\right)\right) + \left(\left(\nabla\frac{\partial u}{\partial t}, \nabla\frac{\partial\alpha}{\partial t} - \nabla\Delta\frac{\partial\alpha}{\partial t}\right)\right). \tag{5.2}$$

Multiplying then (2.2) by $-\Delta\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}-\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right)+\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2$$

$$=-\left(\left(\nabla\frac{\partial u}{\partial t},\nabla\frac{\partial \alpha}{\partial t}-\nabla\Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{5.3}$$

Summing finally (5.2) and (5.3), we find

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}-\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right)$$

$$+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2=\left(\left(\Delta f(u),\frac{\partial u}{\partial t}\right)\right). \tag{5.4}$$

It follows from (5.4) and the continuous embedding $H^2_{per}(\Omega)\subset\mathcal{C}(\overline{\Omega})$ that

$$\frac{d}{dt}\left(\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}-\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right)$$

$$+\left\|\frac{\partial u}{\partial t}\right\|^2+2\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2+2\left\|\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2\leqslant Q(\|\Delta u\|^2). \tag{5.5}$$

We set

$$y=\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}-\nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2 \tag{5.6}$$

and we deduce from (5.5) that we have an inequality of the form

$$y'\leqslant Q(y). \tag{5.7}$$

Let z be the solution to the ordinary differential equation

$$z'=Q(z),\quad z(0)=y(0). \tag{5.8}$$

It follows from the comparison principle that there exists $T_0=T_0(\|u_0\|_{H^2_{per}(\Omega)},\|\alpha_0\|_{H^3_{per}(\Omega)},\|\alpha_1\|_{H^3_{per}(\Omega)})$ (say, belongin to $(0,\frac{1}{2})$) such that

$$y(t)\leqslant z(t),\quad t\in[0,T_0], \tag{5.9}$$

Notes

Notes

from which it follows, owing also to (3.8) and (3.25), that

$$\|u(t)\|^2_{H^2_{per}(\Omega)} + \|\alpha(t)\|^2_{H^3_{per}(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^3_{per}(\Omega)}$$

$$\leqslant Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in [0, T_0]. \tag{5.10}$$

We now differentiate (3.9) with respect to times and have, owing to (2.2),

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} - \left\langle f'(u)\frac{\partial u}{\partial t}\right\rangle = \Delta\alpha + \Delta\frac{\partial \alpha}{\partial t} - \frac{\partial u}{\partial t}. \tag{5.11}$$

We multiply (5.11) by $t\dfrac{\partial u}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + t\left\|A^{\frac{1}{2}}\frac{\partial u}{\partial t}\right\|^2 + t\left(\left(f'(u)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right)\right)$$

$$= -t\left(\left(\nabla\alpha, \nabla\frac{\partial u}{\partial t}\right)\right) - t\left(\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\frac{\partial u}{\partial t}\right)\right) - t\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1},$$

which yields, owing to (2.10) and a proper interpolation inequality (see the proof of Theorem 4.2),

$$\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + ct\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1_{per}(\Omega)} \leqslant c't\left(\|\nabla\alpha\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}, \quad c > 0. \tag{5.12}$$

It follows from (3.25), (5.12) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2_{-1} \leqslant \frac{1}{t}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in (0, T_0]. \tag{5.13}$$

Next, we multiply (5.11) by $\dfrac{\partial u}{\partial t}$ and obtain, proceeding similarly,

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + c\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1_{per}(\Omega)} \leqslant c'\left(\|\nabla\alpha\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right), \quad c > 0. \tag{5.14}$$

We deduce from (3.25), (5.14) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2_{-1} \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)})\left\|\frac{\partial u}{\partial t}(T_0)\right\|^2_{-1}, \quad t \geqslant T_0,$$

hence, owing to (5.13),

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0. \tag{5.15}$$

We rewrite, for $t \geqslant T_0$ fixed, (3.9) as an elliptic equation,

$$Au + f(u) - \langle f(u)\rangle = h_u(t), \quad u \quad \text{is} \quad \Omega-\text{periodic}, \tag{5.16}$$

where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} + \langle \alpha_1\rangle \tag{5.17}$$

satisfies, owing to (3.25) and (5.15),

$$\|h_u(t)\| \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0. \tag{5.18}$$

Multiplying (5.16) by $\overline{u}$, we find, owing to (2.11) and (5.1),

$$\|A^{\frac{1}{2}}u(t)\|^2 + c\int_\Omega F(u(t))\mathrm{d}x \leqslant c'_{M_1}(\|h_u(t)\|^2+1), \quad c>0, \quad t \geqslant T_0. \tag{5.19}$$

Multiplying then (5.16) by Au, we have, owing to (2.10),

$$\|Au(t)\|^2 \leqslant c(\|A^{\frac{1}{2}}u(t)\|^2 + \|h_u(t)\|^2), \quad t \geqslant T_0. \tag{5.20}$$

Combining (5.19) and (5.20), we finally obtain, owing to (3.25) and (5.18),

$$\|u(t)\|^2_{H^2_{per}(\Omega)} \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0, \tag{5.21}$$

hence, owing to (5.10),

$$\|u(t)\|^2_{H^2_{per}(\Omega)} \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant 0. \tag{5.22}$$

We now come back to (5.3), from which it follows that

$$\frac{d}{dt}\left(\|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t} - \nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) \leqslant \left\|\nabla\frac{\partial u}{\partial t}\right\|^2. \tag{5.23}$$

Notes

Notes

Noting that it follows from (3.25), (5.14) and (5.15) that

$$\int_{T_0}^{t}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 d\tau \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)},\|\alpha_0\|_{H^3_{per}(\Omega)},\|\alpha_1\|_{H^3_{per}(\Omega)}),\quad t\geqslant T_0, \tag{5.24}$$

we deduce from (3.14), (5.23) and (5.24) that

$$\|\Delta\alpha(t)\|^2+\|\nabla\Delta\alpha(t)\|^2+\left\|\left(\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right)(t)\right\|^2$$

$$\leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)},\|\alpha_0\|_{H^3_{per}(\Omega)},\|\alpha_1\|_{H^3_{per}(\Omega)})$$

$$+\|\Delta\alpha(T_0)\|^2+\|\nabla\Delta\alpha(T_0)\|^2+\left\|\left(\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right)(T_0)\right\|^2,\quad t\geqslant T_0,$$

hence, owing to (3.8), (3.25), (5.10) and (5.22),

$$\|u(t)\|^2_{H^2_{per}(\Omega)}+\|\alpha(t)\|^2_{H^3_{per}(\Omega)}+\left\|\frac{\partial\alpha}{\partial t}(t)\right\|^2_{H^3_{per}(\Omega)}$$

$$\leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)},\|\alpha_0\|_{H^3_{per}(\Omega)},\|\alpha_1\|_{H^3_{per}(\Omega)}),\quad t\geqslant 0, \tag{5.25}$$

which finishes the proof of the theorem. □

Remark 5.1. We can note that, here, we are not able to study the asymptotic behavior of the associated dynamical system. Indeed, the estimates derived in this section are not dissipative.

Acknowledgements

The author is thankful to Alain Miranville for helpful discussions. He also wishes to thank an anonymous referee for her/his careful reading of the paper and useful comments.

References Références Referencias

[1] S. Agmon, *Lectures on elliptic boundary value problems*, Mathematical Studies. Van Nostrand, New York, 1965.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations*, I, Commun. Pure Appl. Math., 1959, 12, 623–727.

[3] G. Caginalp, *Conserved-phase field system: implications for kinetic undercooling*, phys. Rev. B, 1988, 38, 789–791.

[4] G. Caginalp, *An analysis of a phase-field model of a free boundary*, Arch. Rational Mech. Anal., 1986, 92, 205–245.

[5] G. Caginalp and E. Esenturk, *Anisotropic phase-field equations of arbitrary order*, Discrete Contin. Dyn. Systems S, 2011, 4, 311–350.

[6] X. Chen, G. Caginalp and E. Esenturk *Interface conditions for a phase field model with anisotropic and non-local interactions*, Arch. Rational Mech. Anal., 2011, 202, 349–372.

[7] G. Giacomin and J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interaction I. Macroscopic limits*, J. Statist. Phys., 1997, 87, 37–61.

[8] L. Cherfils, A. Miranville, S. Peng and W. Zhang *Higher-order generalized Cahn-Hilliard equation*, Electron. J. Qualitative Theory Diff. Eqns., 2017, 9, 1–22.

[9] L. Cherfils, A. Miranville and S.Peng, *Higher-order models in phase separation*, J. Appl. Anal. Comput., 2017, 7, 39–56.

[10] L. Cherfils, A. Miranville and S. Peng, *Higher-order Allen-Cahn models with logarithmic nonlinear terms*, Advances in dynamical systems and control, 69, 247–263, 2016.

[11] A. Miranville, *Higher-order Anisotropic Caginalp Phase-Field systems*, Mediterr. J. Math., 2016, 13, 4519–4535.

[12] A. Miranville and A.J. Ntsokongo, *On the anisotropic Caginalp phase-field type models with singular nonlinear terms*, Journal of Applied Analysis and Computation, 655–274, 8, 2018.

[13] A. Miranville and R. Quintanilla, *A Caginalp phase-field system based on type III heat conduction with two temperatures*, Quart. Appl. Math., 2016, 74, 375–398.

[14] A. Miranville and A. Piétrus, *A new formulation of the Cahn-Hilliard equation*, Nonlinear Analysis: Real World Applications, 2006, 7, 285–307.

[15] A.J. Ntsokongo, *On higher-order anisotropic Caginalp phase-field systems with polynomial nonlinear terms*, J. Appl. Anal. Comput., 2017, 7, 992–1012.

[16] A. Makki and A. Miranville, *Existence of solutions for anisotropic Cahn-Hilliard and Allen-Cahn systems in higher space dimensions*, Discrete and Continuous Dynamical Systems Series S, 2016, 9, 759–775.

[17] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics, Second edition, Applied Mathematical Sciences*, vol. **68**, Springer-Verlag, New York, 1997.

[18] J.E. Taylor, Mean curvature and weighted mean curvature, Acta Metall. Mater., 1992, 40, 1475–1495.

[19] J.W. Cahn and J. E Hilliard, *Free energy of a nonuniform I. Interfacial free energy*, J. Chem. Phys., 1958, 2, 258–267.

[20] R. Kobayishi, *Modelling and numerical simulations of dentritic chrystal growth*, Phys. D, 1993, 63, 410–423.

[21] A. Kostianko and S. Zelik, *Inertial manifolds for the 3D Cahn-Hilliard equation with periodic boundary conditions*, Commun. Pure Appl. Anal, 2015, 14, 2069–2094.

[22] C.I. Christov and P.M. Jordan, *Heat conduction paradox involving second-sound propagation in moving media*, Phys. Review Letters, 2005, 94, 154–301.

[23] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis.*, Math. Nachr., 1998, 195, 77–114.

[24] P.J. Chen, M.E. Gurtin and W.O. Williams, *A note on non-simple heat conduction*, J. Appl. Math. Psys. (ZAMP) 1968, 19, 969–970.

[25] R. Quintanilla, *A well-posed problem for the three-dual-phase-lag heat conduction*, J. Thermal Stresses, 2009, 32, 1270-1278.

Notes

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Effects of Heat Transfer of Natural Convection Laminar Flow of Fluid's Velocity Components and Viscous Dissipation through a Circular Heated Plate on a Square Body

By Md Saiful Islam, Md. Mahmud Alam & Somaiya Islam Shuchy
Daffodil International University

Abstract- This research is on natural convection flow of a fluid through a circular plate on a two-dimensional square body. In this paper heat transfer of laminar flow will be investigated based on mesh modeling and graphs under heat flux and viscosity. Temperatures, thermal conductivity, velocity components, viscous dissipation of fluid with time are used as variables. A flat square surface with circle is considered in the investigation by using COMSOLMULTIPHYSICS. Here square body and circular plate are insulated. As material we have used water. The parameters of the fluid have inlet velocity, outlet velocity, heat flux and thermal conductivity. Using boundary condition and velocity components with temperature, we can solve governing equations numerically. The solution has been expressed as meshes and graphs in the results. Different sectors of science like magneto hydrodynamic, fluid flow, natural convection, Radiation, solar, engineering and medical science will be benefitted by this research. We construct many mesh models through our studied technique which is shown in Figure 'a' to'o'. We will also discuss 2D graphical representation through the proposed technique which is shown in Figures 'p' to 't'. In future, we may try to apply our results in space heating, domestic hot or cold water processes, heat from electricity and many other related fields.

Keywords: *viscous dissipation, viscosity, laminar flow, heat transfer, heat flux, thermal conductivity, insulation and mesh.*

GJSFR-F Classification: *FOR Code: 0103*

Strictly as per the compliance and regulations of:

Ref

1. M. Corcione, Effects of the thermal boundary conditions at the sidewalls upon natural convection in rectangular enclosures heated from below and cooled from above, International Journal of Thermal Sciences, 42 (2): 199-208, (2003).

Effects of Heat Transfer of Natural Convection Laminar Flow of Fluid's Velocity Components and Viscous Dissipation through a Circular Heated Plate on a Square Body

Md Saiful Islam [α], Md. Mahmud Alam [σ] & Somaiya Islam Shuchy [ρ]

Abstract- This research is on natural convection flow of a fluid through a circular plate on a two-dimensional square body. In this paper heat transfer of laminar flow will be investigated based on mesh modeling and graphs under heat flux and viscosity. Temperatures, thermal conductivity, velocity components, viscous dissipation of fluid with time are used as variables. A flat square surface with circle is considered in the investigation by using COMSOLMULTIPHYSICS. Here square body and circular plate are insulated. As material we have used water. The parameters of the fluid have inlet velocity, outlet velocity, heat flux and thermal conductivity. Using boundary condition and velocity components with temperature, we can solve governing equations numerically. The solution has been expressed as meshes and graphs in the results. Different sectors of science like magneto hydrodynamic, fluid flow, natural convection, Radiation, solar, engineering and medical science will be benefitted by this research. We construct many mesh models through our studied technique which is shown in Figure 'a' to 'o'. We will also discuss 2D graphical representation through the proposed technique which is shown in Figures 'p' to 't'. In future, we may try to apply our results in space heating, domestic hot or cold water processes, heat from electricity and many other related fields.

Keywords: viscous dissipation, viscosity, laminar flow, heat transfer, heat flux, thermal conductivity, insulation and mesh.

I. Introduction

Natural convection laminar wave takes place in multiple scientific and industrial treatments in nature. Heat transfers take place in a low-velocity, solar receivers exposed breeze currents, nuclear reactors to be cooled during emergency shutdown; electronic devices cooled by fans and many more [1-20]. Fluid mechanics is the study of effects of force on a fluid. If the fluid is at rest, it remains static. When fluid is in motion, the studies are known as fluid dynamics. Fluid is any substance which can flow and relative change of position of particles with respect to time. Fluid has no certain shape which occupies the shape of vessels can flow under its own weight. There are no voids between the molecules of fluid. Different forces act upon fluid as surface force, body force (gravitational force). This force acts on the mass of the fluid that is the quantity of the component of the particular fluid. Due to the conservation of mass, it can't create or

Author α: Principal, Pabna Cadet College, Pabna, Bangladesh, PhD Fellow, Department of Mathematics, Dhaka University of Engineering & Technology, Gazipur, Bangladesh. e-mail: saifulislam_1965@yahoo.com
Author σ: Professor, Department of Mathematics, Dhaka University of Engineering & Technology, Gazipur, Bangladesh.
Author ρ: Lecturer, Department of Nutrition and Food Engineering, Daffodil International University, Dhaka-1216, Bangladesh.

destroy. Viscous dissipation occurs by the continuous force on the fluid where the molecule deformation happens. Fluid flows are of two types: laminar flows and turbulent flows. When the fluid flows in infinitesimal parallel layers with no disruption between them. Laminar flow occurs fluid layers flow in parallel and current is normal to the flow itself. This type of flow is also referred to as streamline flow because nature this flow does not cross the streamlines flow. Laminar flow in a straight pipe may be considered as the relative motion of a set of concentric cylinders of fluid, the outside one fixed at the pipe wall. Others movement is approached increasing speed at the center of the pipe. Regarding the smolder rising in a straight path from a burned material is creating laminar flow. With the rising of a small distance, the smoke usually changes to turbulent flow. For the laminar flow channel is relatively small, that is why the fluid moves slowly. So, its viscosity is relatively high. Oil flow through a thin pipe like blood flow is laminar. Turbulent flows are flowing near solid boundaries, where the flow is often laminar, in a thin layer just adjacent to the surface is noticeable. Laminar and turbulent flow takes place depend the velocity of the fluid flow. But viscosity of the fluid when a fluid flow spreads around the heated circular plate. At the lower velocity laminar flow takes place under an edge when the flow turns turbulent. Heat transfer is the phenomena; it conveys energy and entropy from one location to another. If any heated circular plate is placed on a square solid body than heat transfer will occur. Heat transfer has gained considerable attention due to its numerous applications in the area of energy conservations. The applications also include refrigeration of electrical devices, electronic components, design of solar collectors and heat exchangers. The determination of the velocity profile greatly influences the heat transfer process creates the main difficulty in solving natural convection problems. Heat flux is the amount of heat transferred per unit area and per unit time to or from a surface. The heat transfer results from the resulting quantity of the heat flux per unit time and per unit area to or from the surface. Viscosity is the inter frictional force between adjacent layers of fluid relative in motion. The resistance of a fluid from deformation at a definite rate is viscosity- for example syrup and water.

The continuous process like electromagnetic fields. Fluid dynamics and deformable bodies cause Navier-Stokes equations for the fluids flow. The nature of the fluid can be described by using Navier-Stoke's partial differential equation based on continuity, momentum and conservation of energy. Effects of viscous dissipation have an important role to play in free convection in various gadgets subjected to large deceleration that operates at high rotational speeds. It is also a strong gravitational field processes on large scales in geological processes. MHD is the fluid dynamics of conducting fluid. A typical feature of MHD leads to forming a singular current density. Current sheets are associated with the breaking and reconnecting magnetic field lines due to presence of viscous dissipation. Study of heat transfer, temperature, MHD fluid flow, natural convection and viscous dissipation are very important because these are used in many branches of science. Some researchers have also showed that both the flows are affected by geometrical parameters.

Circular cavity with various heated isothermal wall has received the devotion because of it has wide application, cooling of electric devices, fire control in a building, extraction of oil from container, the construction of stars, planet and blood flow. The density and viscous dissipation effect play a vital role in free convection flow over a circular heated plate or various gadgets in geological system. Dissipation is the output for formulating a viscous material that is converted into energy. The discussion of process of free convection flows, viscosity, density and viscous dissipation on the heated

Notes

circular plate on a rectangular body are usually ignored. In this work, it is considered the effect of heat transfer through a circular heated plate on natural convection flow with time. To solve extraction one or more components of a liquid mixture are extracted by solution in a selective solvent. In humidification water is transferred from the liquid to air. The biological applications include oxygenation of blood, food and drug and respiration mechanism. In the process of analogous or digital heat diffusion resulting from temperature gradient when the fluid is in motion heat transfer take in place by molecular diffusion. Hence velocity component field is needed to the heat transfer process. So, the heat transfers equation and the coefficient of thermal conductivity obtained by COMSOL MULTIPHYSICS.

Meanwhile we considered heat transferred in forced convection in which the fluid flow imposed external by surface pressure, fan, blower and pump. Convection flow is set up within the fluid without a force where velocity 'u' is the free convection.

The wave velocity in free convection is much smaller than in forced convection. So, heat transfer by free convection is much smaller than forced convection.

Meshing means mesh generation which is the procedure of two-dimensional and three-dimensional grid. These divided compound geometries into elements then it can be used to discretize a domain. The most important steps of meshing procedure in performance model using finite element analysis. A mesh is the last presentation of elements. The coordinate sites in space which can vary on element nature belong to nodes that represent the contour of the geometry. The performance of mesh is to divide the model into cells in sequence behavior a simulation analysis or concentrate a digital model. Meshing is the collective of coordinates, boundaries, and faces which describe the form of aim in two or three dimensions.

Steady laminar free convection in air-filled 2D rectangular enclosures heated from below and cooled from above is studied numerically for a wide variety of thermal boundary conditions at the walls [1]. Another study shows that, complex interactions between Nano fluids and the walls of cavity. This complexity may increase with a change of geometry or orientation of the cavity. So, study of natural convection fluid flow and heat transfer in a trapezoidal geometry is more difficult than that of square or rectangular enclosures due to the presence of sloping walls [3]. A good number of investigations on natural convection fluid flows and heat transfers in trapezoidal cavities have already been published. A complete study of the laminar solution of the problem (Ra up to l05) was given by MacGregor and Emery' together with experimental results covering a wide range of Prandtl numbers. Many correlations of Nusselt number (Nu) and Ra concerning experimental results can also be found in this paper. Jaluria and Gebhart have worked on vertical natural convection flows. Their studies expressed the process occurring during the transition from laminar to turbulent flow near a vertical flat plate when the surface heat flux is uniform. The interaction of the velocity and temperature fields during the transition and the effect on Nu were also investigated. Mallinson and De Vahl Davis presented detailed three-dimensional calculations for laminar flow. The calculations were performed for different values of Prandtl number and aspect ratio of cavity dimensions. Natural convection flow in a square cavity revisited: laminar and turbulent models with wall functions [2, 9]. They found that the flow and temperature fields are obstructed by the presence of the body. Oztop et al. analyzed the effects of the surface of the insulated body for partially heated enclosure [10].Natural convection in porous triangular enclosure with a circular obstacle in presence of heat generation [11]. Compared to other study, this study includes the velocity profile of fluid particles and temperature with respect to time. The purpose of

Ref

1. M. Corcione, Effects of the thermal boundary conditions at the sidewalls upon natural convection in rectangular enclosures heated from below and cooled from above, International Journal of Thermal Sciences, 42 (2): 199-208, (2003).

this research is the exploration of the effect of heat transfer of natural convection laminar flow of a fluid's velocity components and viscous dissipation through a circular heated plate on a square body with time.

II. Aim, Findings, Future Scope

a) Aim

The aim of the study is to investigate the effects of heat transfer on natural convection laminar flow of fluid's velocity components and viscous dissipation through a circular heated plate on a square body. Specifically, the study aims to understand how the velocity and temperature fields of the fluid are affected by the presence of the circular heated plate and how the viscous dissipation affects the heat transfer characteristics. This information is important for understanding, optimizing heat transfer processes in several industrial and engineering uses, such as in heat exchangers, cooling process, and energy production policy. The study also aims to contribute to the scientific understanding of natural convection laminar flow and its relationship with heat transfer in flat media.

1. Investigation of the heat transfer characteristics of natural convection laminar flow over a circular heated plate on a square body.
2. To analysis the effects of fluid's velocity components and viscous dissipation on the heat transfer rate.
3. To determination the optimal conditions for improving the heat transfer rate in the method, such as the best area for the circular heated plate, the optimal size of the square body and the optimal properties of the fluid flow.
4. Comparing the results in the present study compare to earlier studies on related systems and authenticate the requirement.

Overall, the research improve our understanding of the natural convection laminar flow over a circular heated plate on a square body, which has practical uses in many fields like energy, environmental science and engineering.

b) Findings

1. Mesh Type Impact: The choice of mesh type significantly affects the simulation results, particularly the temperature distribution and fluid flow patterns. Different types of meshes, such as triangular and free triangular meshes, yield distinct temperature profiles and flow characteristics. This highlights the importance of mesh selection in accurately capturing the physics of the problem and obtaining reliable results.
2. Shaded Area: The simulation identifies certain shaded area in the domain where fluid particles exhibit more frequent movement. These regions high correspond to regions of high turbulences, vortices, or areas with intense convective heat transfer. Understanding these shaded areas is crucial as they can indicate regions of higher thermal efficiency or potential challenges in the design of heat transfer systems.
3. Fluid Particle Heating and Interactions: The incorporation of fluid particles on the square body, particularly when heated by the circular plate, leads to an increase in their intermolecular distance due to heat transfer. This phenomenon is related to the expansion of the fluid as it absorbs heat. Such thermal expansion indicates the fluid flow dynamics and heat transfer rate in the process.
4. Formation of Triangular Shape: The simulation results indicate the heat is generated in the square body at the same time a triangular mesh is used then the

Notes

particles of water tend to arrange themselves in a triangular pattern. This organization might be influenced by the boundary conditions and the convective flow patterns arising cause of the heated circular plate.

5. Non-Uniform Heating: If the circular heated plate does not cover the entire square body, the area near the circular plate experiences more intense heating related to the distant regions. This non-uniform heating distribution introduces complexities in the flow field, leading to variations in temperature and velocity profiles.

Notes

Overall, the study focuses on the complex interaction among heat transfer, fluid dynamics and mesh characteristics in conduct of a heated circular plate on a square body. The findings provide valuable insights for optimizing heat transfer producing and learning flow behavior. However, it is no table that these findings are based on simulation results and further experimental confirmation is necessary for the accuracy and applicability of the observed phenomena in real-world scenarios. Additionally, conducting sensitivity analyses on various parameters would allow for a more comprehensive understanding of the system's behavior.

c) Future Scope

1. Thermal energy storage involves technologies for collecting and storing energy for later use. It can be used for the stability of energy demand between day and night. Temperature above or below that of the ambient environment is required to maintain the thermal reservoir. Its usage takes in space heating, domestic or processing warm water process.
2. Efficient energy use is the goal to decrease the quantity of energy necessary for heating or cooling. In architecture, condensation and air wind can cause of cosmetic or structural deformation. A power audit may help to measure the implementation of recommended correct procedure. For example, insulation improvements, air shutting of mechanical leaks, the addition of energy-efficient windows and doors can be learned.
3. Smart meter is used to record electric energy consumption during intervals.
4. Thermal transition is the rate of transfer of heat through a model divided by the difference in temperature across the structure. It is measured in watts per square meter per kelvin, or W/ (m2K).
5. Thermostat is a device that automatically regulates the temperature.
6. Greenhouse effect is the representation of the exchanges of energy among the heat source, Earth's surface, Earth's atmosphere and the eventual sink outer space. The greenhouse is the natural rule to give benefits the animal and the human by recycle temperature. Heat emitted by the Earth surface is the defining characteristic of the greenhouse effect. This process ensures the thermal radiation from a planetary surface is absorbed by atmospheric greenhouse gases and clouds are re-radiated in all directions. As a results reduction in the amount of thermal radiation reaching space relative to reach space in the absence of absorbing materials. This reduction of outgoing radiation leads to rise in the temperature of the surface and troposphere. The rate of outgoing radiation again and again equals to the rate of heat arrives from the sun.
7. To find out how the body transfers heat? Principles of heat transfer in engineering systems can be applied to the human body. Continuous metabolism of nutrients which provides energy of the body produces heat in the animal and human body. A consistent internal temperature in human body is necessary for smooth physical

functions. Therefore, excess heat must be removed from the body. At the same time keep the internal temperature at a healthy level.

Finally, heat transfer by convection is driven by the movement of fluids over the surface body. This convective fluid can be either a liquid or a gas. For heat transfer from the outer surface of the body, the convection mechanism is dependent on the surface area of the body, the velocity of the fluid flow, temperature gradient between the surface of the skin and the ambient fluid. Significantly less temperature of the surroundings than the normal body temperature leads to heat transfer. For that reason ones feel cold when there is no enough worn cloth with him then he is exposed to a cold weather. Here clothing as insulator acts thermal resistance to heat flow over the covered portion of the body. On the other hand, abnormally excessive heat in the body may even cause death. To ensure that one portion of the body is not significantly hotter than another portion. Heat must flow evenly through the bodily tissues. Blood flowing through blood vessels acts as a convective fluid flow and helps to protect any grownup of excess heat inside the tissues of the body. The heat carried by the blood is resolute the temperature of the surrounding tissue, the diameter of the blood vessel, the thickness of the fluid, velocity of the flow and the heat transfer coefficient of the blood.

Notes

III. Physical Model

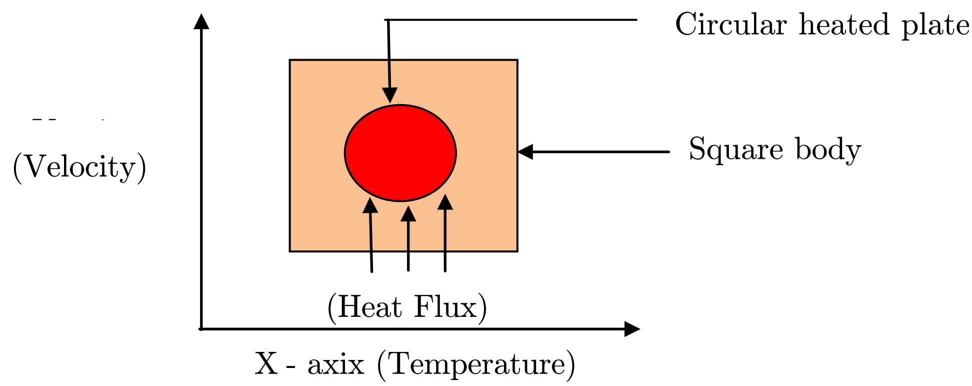


Figure 1: Two-dimensional heat transfer in laminar fluid

The mentioned problem describes the two-dimensional circular heated plate on square body horizontally. The figure indicates y axis as velocity and x axis as temperature which defines the velocity profile of the fluid flow and temperature with respect to time when the fluid will run through the heated circular plate. The circular plate is on the square body. Through this model, viscous dissipation of the fluid by heat transfer with time is determined.

a) *Mathematical equation*

Laminar flow and fluid properties of water find the results by using Stokes equation with the help of COMSOL MULTIPHYSICS.

Continuity equation,

$$\nabla \cdot (\rho u) = 0. \tag{1}$$

Stokes energy equation with surface pressure is

$$\rho(u\cdot\nabla)u=\nabla\cdot[-\rho I+\mu(\nabla u+(\nabla u)^{T})-\frac{2}{3}\mu(\nabla\cdot u)I]+F. \tag{2}$$

Initial value on the wall, $u=0$.

Where, rho (ρ) is the density, l is the length of square body, u is the initial velocity, Mu (μ) is the Viscosity and F is the surface force.

Heat transfer in water surface

Notes

$$\rho C_p u\cdot\nabla T=\nabla\cdot(k\nabla T)+Q+Q_{vh}+W_p. \tag{3}$$

Where, C_p the specific heat at constant is pressure and W_p is the constant pressure in water.

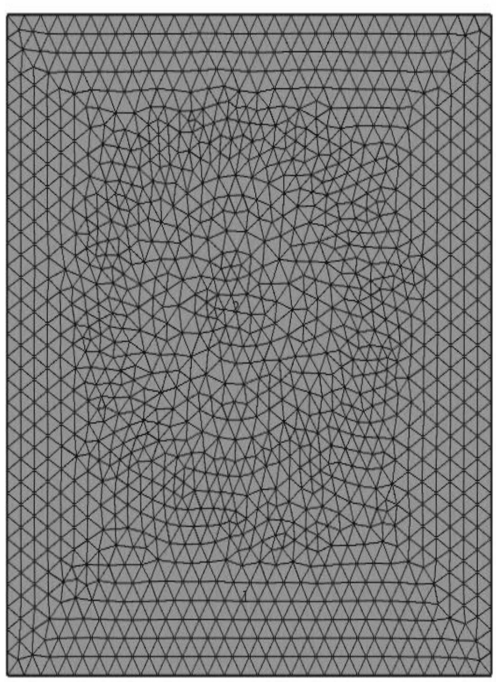
Thermal insulation

$$-n(-k\nabla T)=0. \tag{4}$$

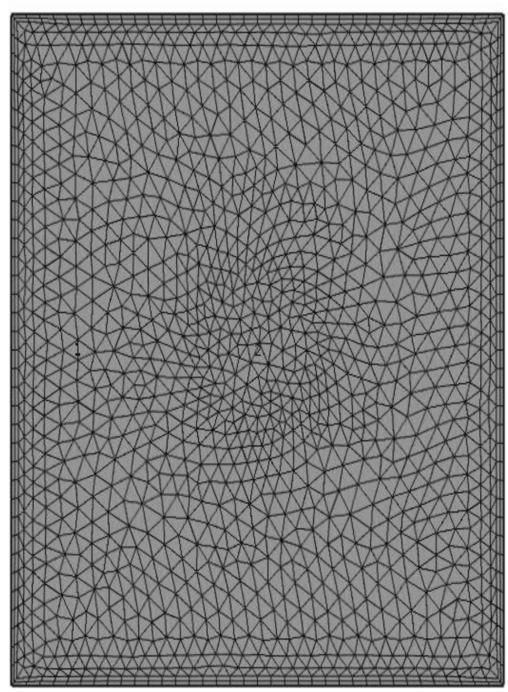
Where, k is the Thermal conductivity, T is the Temperature.

b) Analysis of Mesh modeling

This model includes borderline layer, heat transfer free triangular mesh, heat transfer in fluid, laminar flow in Isothermal process, contour in extra fine and fine mesh are given as model. Isothermal mesh size is fine (Fig-a) and Isothermal mesh size is normal (Fig-b).



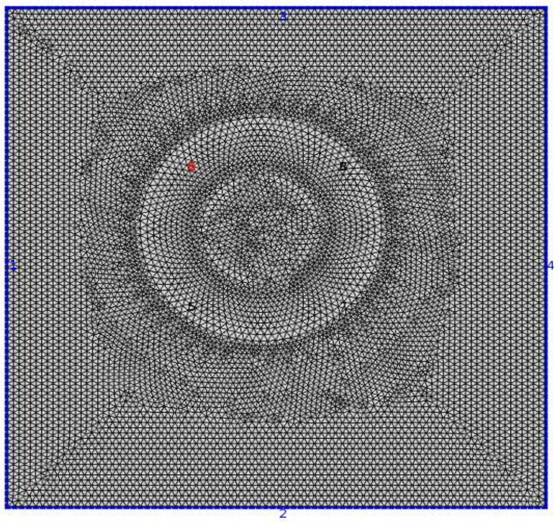
(Fig-a)



(Fig-b)

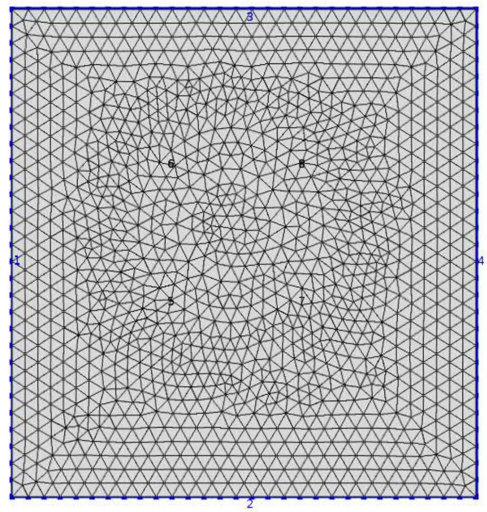
This mesh describes the molecular movement of water is driven on the heated circular plate on a square body. In this mesh, material is taken as water and the properties are Minimum element quality (0.7615), Average element quality (0.9789), Triangular elements (16832), Edge elements (404), and Vertex elements (8). The calibrated parameters are Maximum element size (0.0091), Minimum element size (0.00105), Resolution of curvature (0.25), Maximum element growth rate (1.08) and Extra fine size.

Isothermal Boundary layer mesh size is extra fine (Fig-c)



(Fig-c)

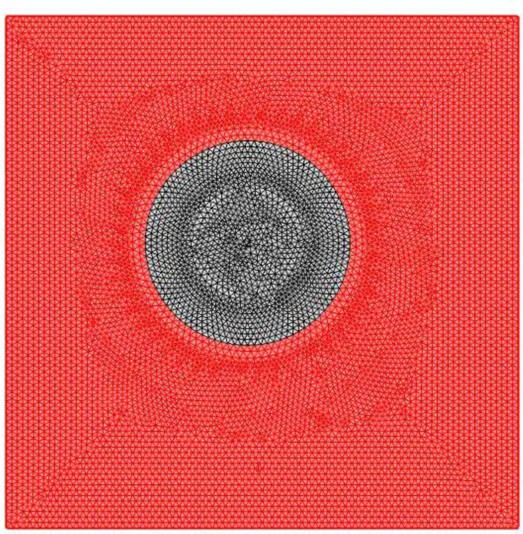
Isothermal Boundary layer mesh size is fine (Fig-d).



(Fig-d)

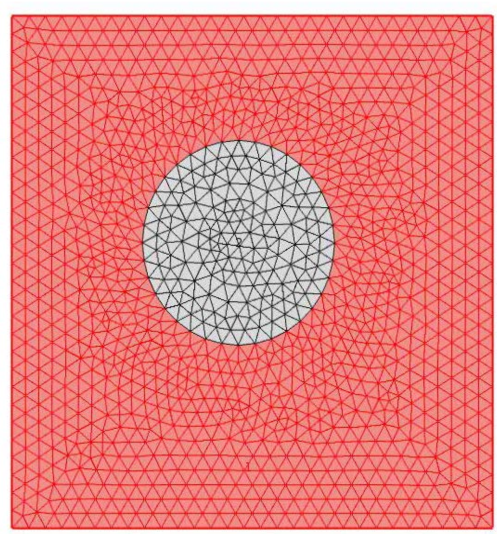
The boundary layer is displayed in the mesh by blue color lining which keep the water particles inside the square body. After heat generation, the water particles become dispersed and create dense circular shape seems like a hole. (Fig-c) and (Fig-d) are Heat transfer of free triangular mesh.

Isothermal Mesh Model size is extra fine (Fig-e).



(Fig-e)

Isothermal Mesh Model size is fine (Fig-f).



(Fig-f)

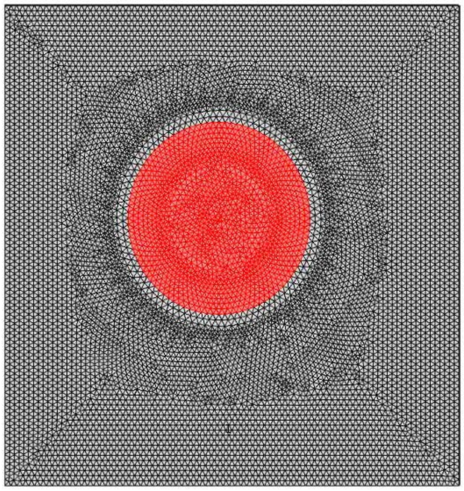
In this model (Fig-e and Fig-f) a simulation is taken as, heat transfer of free triangular mesh is shown red and gray color where due to heated circular plate the heat is generated to square body and a triangular shape will be formed by the particles of water.

Notes

Notes

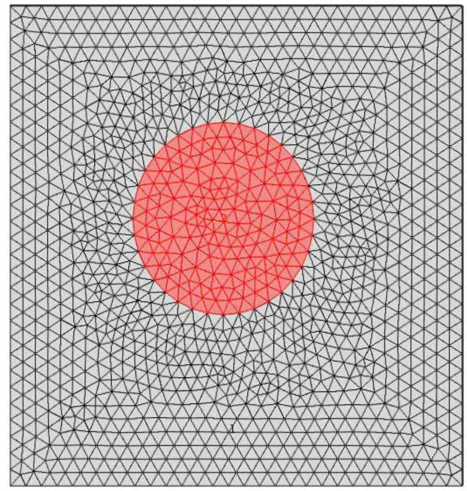
.

Heat transfer free triangular mesh size is extra fine (Fig-g)



(Fig-g)

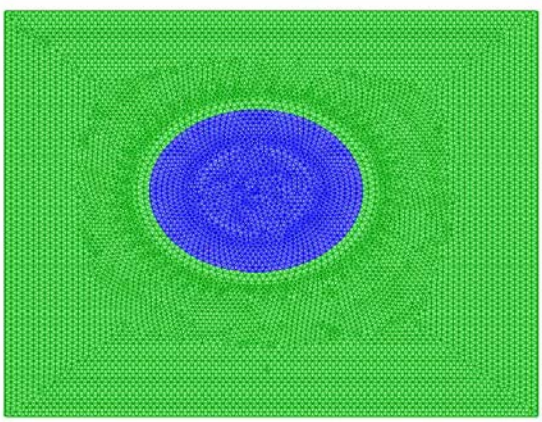
Heat transfer free triangular mesh size is fine (Fig-h).



(Fig-h)

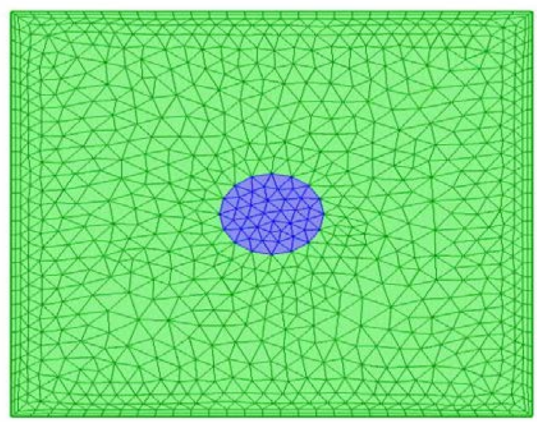
This figure represents the circular heated plate which is not generated to the whole square body but the area nearer to the circular plate is heated and shows a densely circular area comparative to the distant area.

Heat transfer in fluid mesh size is extra fine (Fig-i).



(Fig-i)

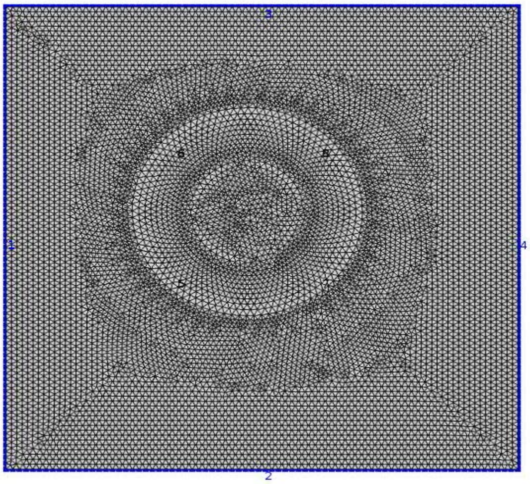
Heat transfer in fluid mesh size is fine (Fig-j).



(Fig-j)

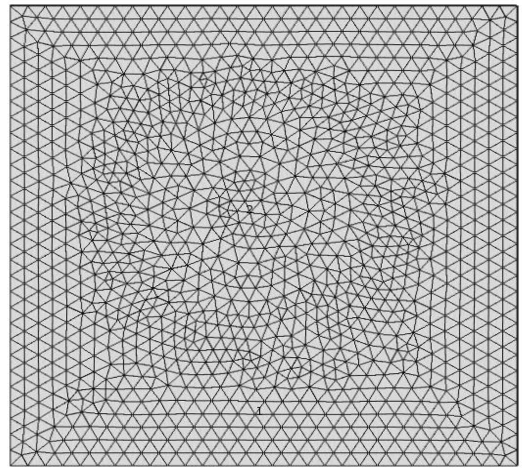
In this Fig-i & j, fluid is incorporated on the square body along with heated circular plate. Here the water particles are heated and the intermolecular distance increases. Some dense area represents the movements of the particles through which it can be found that the water particles are moving discretely due to temperature.Color of the meshes is green and blue.

Laminar flow in Isothermal process size is extra fine (Fig-k).



(Fig-k)

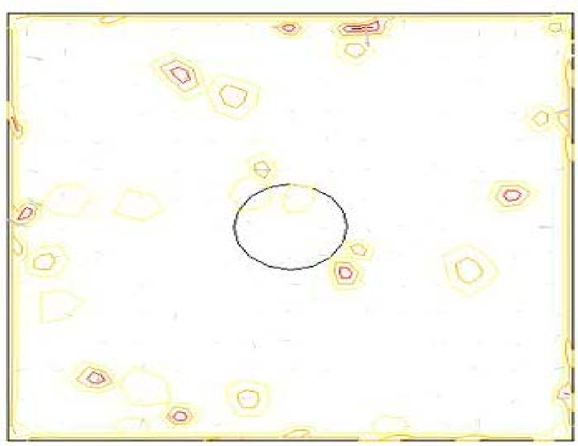
Laminar flow in Isothermal process size is fine (Fig-l).



(Fig-l)

In these Fig-k & l, Laminar flow is considered by fluid particles the smooth paths on layers, fluid particles moving smoothly past the adjacent layers with small or no partying. The fluid particles tend to flow at low velocities without lateral mixing, and adjacent layers meshes past one another.

Contour of the mesh (Fig-m).

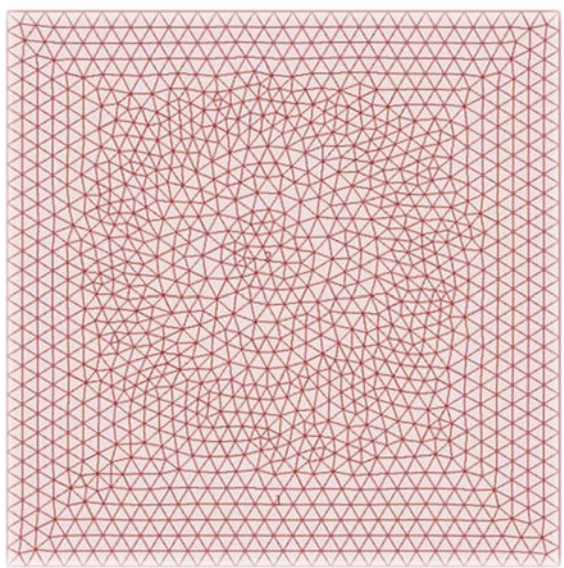


(Fig-m)

This (Fig-m) contour represents the identical curved outlines of the movement of the fluid particles when heat is generated by the circular plate on the square body. Different shapes of the water particles are visible in the mesh by contour linings which represents the condition of the particles. The interpolation data can round structured and defined on a general point rain cloud.

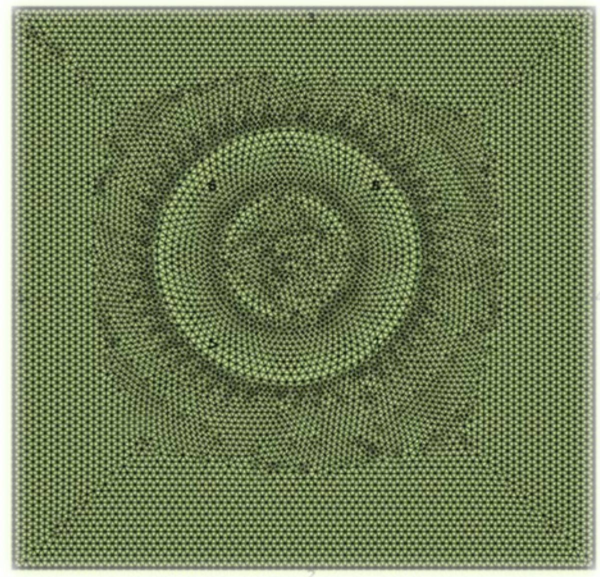
Notes

Thermal conduction area(Fig-0).



(Fig-o)

Laminar flow wall (Fig-n).



(Fig-n)

In these Fig-n & o, boundary layer meshing simulations creates isotropic meshes close to walls without having to use swept meshes or specially designated domains. These fluid particles are required due to the boundary layer that typically forms at no-slip walls. Boundary layer meshes are added after the domain has been meshed. A triangular path of particles is pushed into the computational domain. Observe that the qualities of elements are good in spite of isotropy. Due to boundary layer mesh built from triangles with high quality, and that results in high-quality prismatic elements as well.

IV. Methodology

The continuity equations, stokes energy equation, heat transfer equation and thermal insulation and natural conservation equations are transformed into a system of integral equations by using the Galerkin weighted residual method of finite-element formulation. The nonlinear algebraic equations so obtained are modified by imposition of boundary conditions. These modified nonlinear equations are transferred into linear algebraic equations with the help of Newton's method. Lastly, these linear equations are solved by applying Triangular factorization. For numerical computation and post processing, the software COMSOL MULTIPHYSICS is used.

V. Results and Discussion

In different type of meshes Model is change for their temperature changes. There are some shaded area in which the frequent movement of fluid particle is more. If the fluid is incorporated on the square body along with heated circular plate then the water particles are heated and the intermolecular distance increases due to heat transfer. A Simulation is taken as, heat transfer of free triangular mesh due to heated circular plate, if the heat is generated to square body and a triangular shape will be formed by the particles of water. If the circular heated plate is not generated to the whole square body, the area nearer to the circular plate is heated then the densely circular area comparatively to the distant area. Results are presented with the help of graph.

Notes

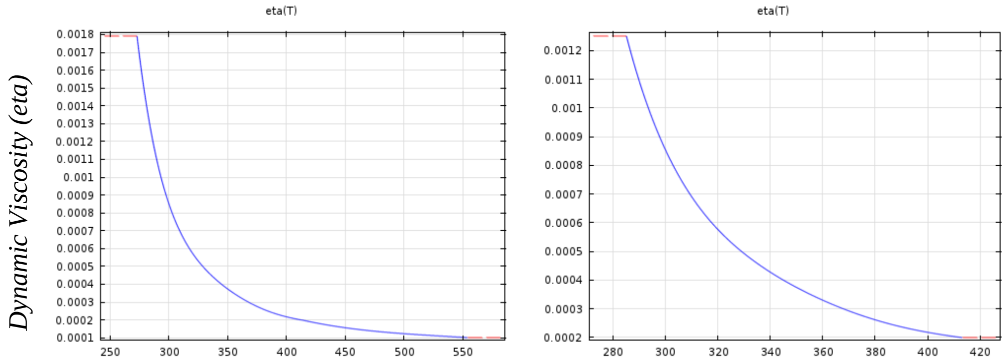


Fig-p: Temperature (T)

This graph (Fig-p) shows the change of viscosity with respect to temperature. For increasing the temperature viscosity decreased.

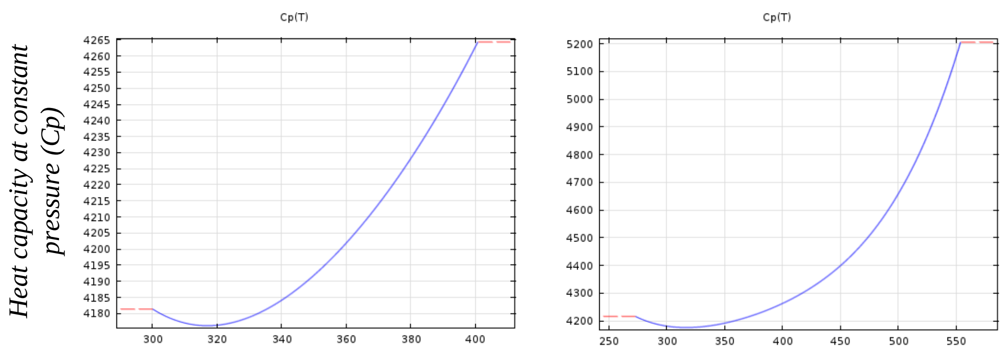


Fig-q: Temperature (T)

This graph (Fig-q) shows the change of Heat capacity at constant pressure (Cp) with respect to temperature. For increasing the temperature Heat capacity at constant pressure (Cp) also increases.

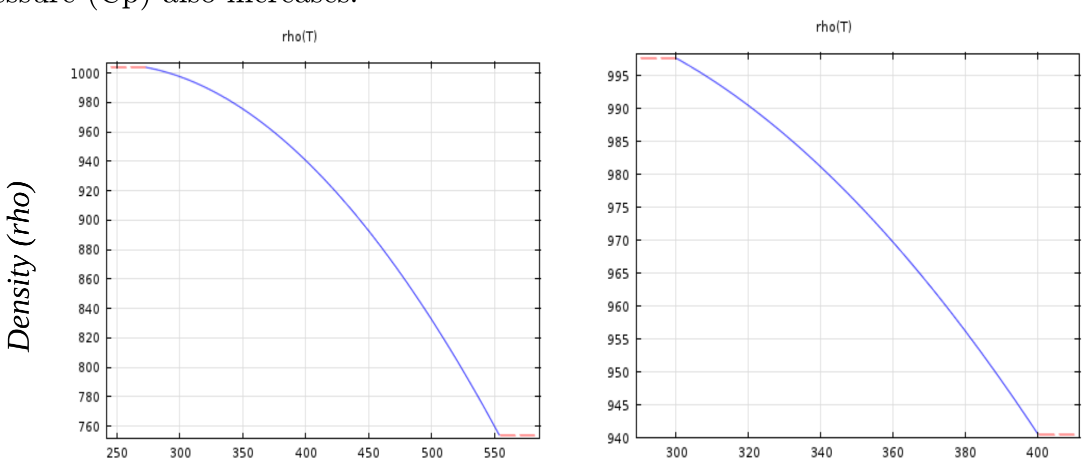


Fig-r: Temperature (T)

This graph (Fig-r) shows the change of Density (rho) with respect to temperature. For increasing the temperature Density (ρ) gradually decreases and minimum at maximum temperature.

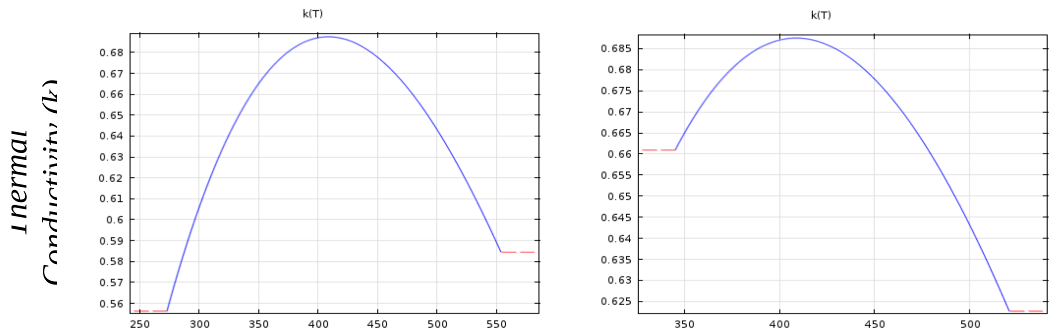


Fig-s: Temperature (T)

This graph (Fig-s) shows the change of Thermal Conductivity (k)) with respect to temperature. For increasing the temperature Thermal Conductivity (k) increases and at a certain temperature becomes maximum after that for increasing temperature Thermal Conductivity (k) gradually decreases.

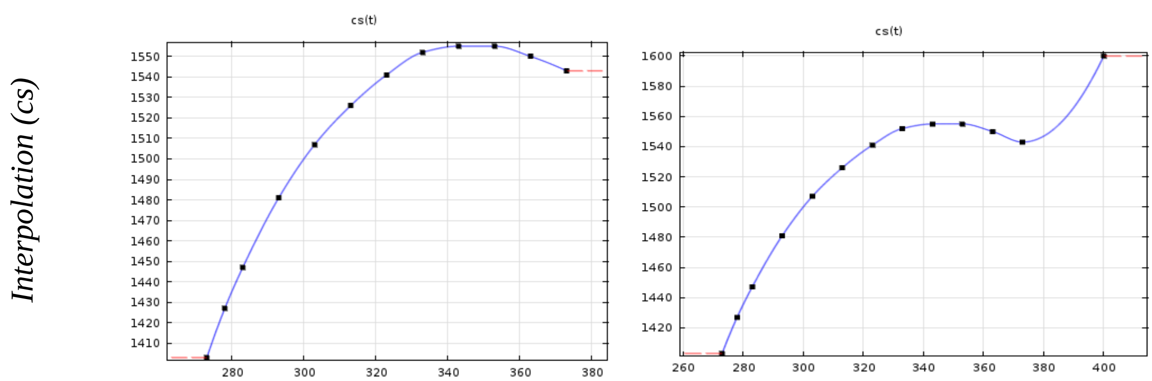


Fig-t: Temperature (T)

This graph (Fig-05) shows the velocity profile of laminar flow fluid has a definite viscosity. Interpolation function is defined by a graph containing the values of the function in discrete points. Variation of given temperature, the discrete points of interpolation are varied according to the different value of temperature. The interpolation is a process of predicting unknown data points such as fluid that is the speed of rainfall.

VI. Conclusion

Solving the numerical model equations for multiphase flow may be a very challenging task, even with access to supercomputers. In reality, these models are limited to Nano fluids and for the study of natural surfaces of viscous liquids. The disseminated multiphase flow phenomena allow for the studying of systems with millions and billions of bubbles, droplets, or particles. But even the simplest dispersed flow models can lead to the generation of very complicated and daunting model equations. The development of these models into variations that are well adapted to describe specific mixtures has allowed for engineers and scientists to study multiphase flow with a relatively good accuracy and with reasonable computational costs. We have successfully constructed many mesh models through our studied techniques which are shown in Figures a to o and we have also discussed 2D graphical representation through the proposed techniques which are shown in Figures p to t. In future, we will try to apply our results in fluid flow, natural convection, Radiation, solar energy, engineering sector, medical science and many others fields.

References Références Referencias

1. M. Corcione, Effects of the thermal boundary conditions at the sidewalls upon natural convection in rectangular enclosures heated from below and cooled from above, International Journal of Thermal Sciences, 42 (2): 199-208, (2003).
2. M.R. Haque, M.M. Ali, M.M. Alam and M.A. Alim, Effects of viscous dissipation on natural convection flow over a sphere with temperature dependent thermal conductivity, J. Comp. Math. Sci.,5 (1): 5-14, (2014).
3. M.H. Esfe, A.A.A. Arani, M. Rezaie, W.M. Yan and A. Karimipour, Experimental determination of thermal conductivity and dynamic viscosity of Ag–MgO/water hybrid nanofluid, International Communications in Heat and Mass Transfer, 66: 189-195, (2015).
4. M.M. Alam, M.A. Alim and M.M.K. Chowdhury, Effect of pressure stress work and viscous dissipation in natural convection flow along a vertical flat plate with heat conduction, Journal of Naval Architecture and Marine Engineering, 3: 69-76, (2006).
5. M.M. Alim, M.M. Alamand A. Al-Momun, Joule heating effects on the coupling of conduction with magnetohydrodynamic free convection flow from a vertical flat plate, Nonlinear analysis: Modeling and Control, 12(3): 307-316, (2007).
6. M.M. Alam, M.M. Alimand M.K. Chowdhury, Stress work effect on natural convection flow along a vertical flat plate with joule heating and heat conduction, Journal Mechanical Engineering, 38: 18-24, (2007).
7. M.M. Alim, M.M.Alam, A. Al-Momun and M.B. Hossain, Combined effect of viscous dissipation and Joule heating on the coupling of conduction and free convection along a vertical flat plate, International communication of heat and mass transfer, 35: 338-346, (2008).
8. M.A. Hossainand M.A. Alim, Natural convection-radiation interaction on boundary layer flow along a thin cylinder, J. Heat and Mass Transfer, 32: 515-520, (1997).
9. G. Barakos, E. Mitsoulis and D. Assimacopoulos, Natural convection flow in a square cavity revisited: laminar and turbulent models with wall functions, International Journal For Numerical Methods In Fluids, 18: 695-719, (1994).
10. A. Pozziand M. Lupo, The coupling of conduction with laminar natural convection along a flat plate, Int. J. Heat Mass Transfer, 31(9): 1807-1814, (1988).
11. T. Cebeci and P. Bradshaw, Physical and computational aspects of convective heat transfer, Springer, N. Y. (1984).
12. Y. Joshi and B. Gebhart, Effect of pressure stress work and viscous dissipation in some natural convection flows, Int. J. Heat Mass Transfer, 24(10): 1377-1388, (1981).
13. P.S. Mahapatra, N.K. Manna and K. Ghosh, Effect of active wall location in a partially heated enclosure, International Communications in Heat and Mass Transfer, 61: 69-77, (2015).
14. R. Chowdhury, M.A.H. Khan and M.N.AA. Siddiki, Natural Convection in Porous Triangular Enclosure with a Circular Obstacle in Presence of Heat Generation, American Journal of Applied Mathematics, 3(2): 51-58, (2015).

Notes

Notes

15. N. Sarwar, M.I. Asjad, S. Hussain, M.N. Alam and M. Inc, Inclined magnetic field and variable viscosity effects on bioconvection of Cassonnanofluid slip flow over non linearly stretching sheet, Propulsion and Power Research, 11(4): 565-574, (2022).
16. B. Ullah, U. Khan, H.A. Wahab, I. Khan and M.N. Alam, Entropy Generation Analysis for MHD Flow of Hybrid Nanofluids over a Curved Stretching Surface with Shape Effects, Journal of Nanomaterials, 2022: 8929985, 10 pages, (2022).
17. M.F.A. Asad, M. Yavuz, M.N. Alam, M.M.A. Sarker and O. Bazighifan, Influence of Fin Length on Magneto-Combined Convection Heat Transfer Performance in a Lid-Driven Wavy Cavity, Fractal and Fractional, 5: 107, (2021).
18. M.F.A Asad, M.N. Alam, A.M. Rashad and M.M.A. Sarkar, Impact of undulation on magneto-free convective heat transport in an enclosure having vertical wavy sides, International Communications in Heat and Mass Transfer, 127: 105579, (2021).
19. T. Islam, M.N. Alam, N. Parveen, Y.M. Chu and M.I. Asjad, Heatline Visualization of MHD Natural Convection Heat Transfer of Nanofluid in a Prismatic Enclosure, Scientific Reports, 11: 10972 (2021).
20. M.F.A. Asad, M.N.Alam, H. Ahmad, M.M.A. Sarker, M.D. Alsulami and K.A.Gepreel, Impact of a Closed Space Rectangular Heat Source on Natural Convective Flow through Triangular Cavity, Results in Physics, 23: 104011, (2021).

This page is intentionally left blank

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Distinguished Couple of Integer Right Triangles and Canada Numbers

By Janaki G & Gowri Shankari A
Bharathidasan University

Abstract- We propose a couple of integer right triangles whose perimeter differences are each equal to four times the Canada number. We also provide the number of couples containing primitive and non-primitive integer right triangles.

Keywords: couple of integer right triangles, canada number, primitive and non-primitive integer right triangles.

GJSFR-F Classification: MSC: 11-XX

Strictly as per the compliance and regulations of:

Distinguished Couple of Integer Right Triangles and Canada Numbers

Janaki G [α] & Gowri Shankari A [σ]

Abstract- We propose a couple of integer right triangles whose perimeter differences are each equal to four times the Canada number. We also provide the number of couples containing primitive and non-primitive integer right triangles.

Keywords: couple of integer right triangles, canada number, primitive and non-primitive integer right triangles.

I. Introduction

The theory of numbers is a fascinating area of mathematics. Right integer triangles have attracted the attention of many mathematicians and math enthusiasts because it is a treasure house where finding many hidden connections is like going on a treasure hunt. Refer to [1]-[3] for various fascinating challenges. In addition to polygonal numbers, we also have the Jarasandha numbers, Nasty numbers, Dhuruva numbers, and Canada Numbers, which are all intriguing patterns of numbers. These figures are displayed in [4]-[9]. Special Pythagorean triangles linked to Nasty and polygonal numbers are derived in [10]–[15].

In this writing, we look for a distinguished couple of right integer triangles where the difference in their perimeters in each pair is four times the Canada numbers.

II. Basic Definitions

Definition: 1

The ternary quadratic Diophantine equation given by $s^2+t^2=r^2$ is known as Integer right equation, where s, t and r are natural numbers and denotes it by $\Delta(s,t,r)$. Also, in $\Delta(s,t,r): s^2+t^2=r^2$, s and t are called its legs and r its hypotenuse.

Definition: 2

The most cited solution of the Integer right equation is

$$s=a^2-b^2,\ t=2ab,\ r=a^2+b^2,$$

where $a>b>0$. If a and b have opposing parities and $\gcd(a,b)=1$, then this solution is referred to as primitive.

Author α: Associate Professor, Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University), Trichy – 18, India. e-mail: janakikarun@rediffmail.com

Author σ: Assistant Professor, Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University), Trichy – 18, India. e-mail: gowrirajinikanth@gmail.com

Ref

1. Sierpinski W, Pythagorean triangles, Dover publications, INC, New York, 2003.

Definition: 3

Canada numbers are those n such that the sum of the squares of the digits of n is equal to the sum of the non- trivial divisors of n, i.e., $\sigma(n)-n-1$.
The Canada numbers are 125, 581, 8549 and 16999.

The name of these numbers is due to the fact they were defined by some mathematicians from Manitoba University to celebrate the 125^{th} anniversary of Canada.

III. Materials and Methods

Let Δ_1 and Δ_2 be two distinct right integer triangles with generators $a, c\ (a > c > 0)$, $b, c\ (b > c > 0)$ respectively. Let P_1 and P_2 be the perimeters and Λ_1 *and* Λ_2 be the areas of Δ_1 and Δ_2 such that $P_1 - P_2 = 4$ times the 1^{st} Canada number 125.
The equation derived from the relationship above is

$$(2a+c)^2-(2b+c)^2=1000 \tag{1}$$

It is observed after completing numerical calculations that there are 20 different values for $a, c,$ and b satisfied (1) provided $a+b+c=$ Canada number
The values of a, c, b, P_1 and P_2 are shown in Table I below for clarity and simplicity.

Table I

S. No.	a	c	b	P_1	P_2	$\frac{P_1 - P_2}{4}$
1.	44	39	42	7304	6804	125
2.	45	37	43	7380	6880	125
3.	46	35	44	7452	6952	125
4.	47	33	45	7520	7020	125
5.	48	31	46	7584	7084	125
6.	49	29	47	7644	7144	125
7.	50	27	48	7700	7200	125
8.	51	25	49	7752	7252	125
9.	52	23	50	7800	7300	125
10.	53	21	51	7844	7344	125
11.	54	19	52	7884	7384	125
12.	55	17	53	7920	7420	125
13.	56	15	54	7952	7452	125
14.	57	13	55	7980	7480	125
15.	58	11	56	8004	7504	125
16.	59	9	57	8024	7524	125
17.	60	7	58	8040	7540	125
18.	61	5	59	8052	7552	125
19.	62	3	60	8060	7560	125
20.	63	1	61	8064	7564	125

Thus, it can be observed that there are 20 couples of right integer triangles, where each couple's difference in perimeters equals four times the first Canada number (125). Of these 20 couples, ten are non-primitive, six are primitive, and four are couples, where one is a primitive triangle and the other is non-primitive.

The following Table II illustrates a similar observation of other Canada numbers:

Notes

Notes

Table II

Canada Number	Couples of Right integer Triangles	Couples of primitive Right integer Triangles	Couples of non-primitive Right integer Triangles	Couples of primitive and non-primitive Right integer Triangles
581	96	28	50	18
8549	1424	337	733	354
16999	2833	903	1438	492

IV. Conclusion

In this article, we propose a couple of integer right triangles whose perimeter differences are each equal to four times the Canada number. We also provide the number of couples containing primitive and non-primitive integer right triangles. In conclusion, one can look for relationships between distinguished couples of integer right triangles and other unique numbers and number patterns.

References Références Referencias

1. Sierpinski W, Pythagorean triangles, Dover publications, INC, New York, 2003.
2. Gopalan M. A, Gnanam A, and Janaki G, A Remarkable Pythagorean problem, Acta Ciencia Indica, Volume XXXIII M, No 4, 2007, Pages 1429-1434.
3. Gopalan M. A and Janaki G, Pythagorean triangle with perimeter as Pentagonal number, Antartica J. Math., Volume 5(2), 2008, Pages 15-18.
4. Bert Miller, Nasty numbers, The mathematics teacher, Volume 73, No.9, 649, 1980.
5. Sastry P. S. N, Jarasandha numbers, The mathematics teacher, No.9, Volume 37, issues 3 and 4, 2001.
6. Gopalan M. A and Janaki G, Pythagorean triangle with nasty number as a leg, Journal of applied Mathematical Analysis and Applications, Volume, No 1-2, 2008, Pages 13-17.
7. Gopalan M. A, Sangeetha V. and Manju somanath, Pythagorean triangle and Polygonal number, Cayley J. Math., Volume 2(2), 2013, Pages 151-156.
8. Janaki G and Gowri Shankari A, Integer Right triangle with Area/Perimeter as a Canada Numbers, Asian Journal of Science and Technology, Volume 14, Issue 02, 2023, Pages 12399-12402.
9. Janaki G and Gowri Shankari A , Connection between Distinguished Integer Right Triangle and Canada Numbers, Journal For Basic Sciences, Volume 23, No. 7, 2023, Pages 391-396.
10. Gopalan M. A , Vidhyalaksmi S, Premalatha E and Presenna R, Special Pairs of Pythagorean triangles and Dhuruva numbers, Global Journal of Science Frontier Research (F), Volume XV, Issue I, 2015.
11. Janaki G, Saranya C, Special Pairs Of Pythagorean Triangles And Jarasandha Numbers, American International Journal of Research in Science, Technology, Engineering and Mathematics, Issue 13, 2016, Pages 118-120.
12. Janaki G and Saranya P, Special Pairs of Pythagorean Triangles and Narcissistic Number, International Journal of Multidisciplinary Research and Development, Volume 3, Issue 4 , 2016 , Pages 106-108.
13. Janaki G and Vidhya S, Special pairs of Pythagorean triangles and 2- digit Sphenic numbers, American International Journal of Research in Science, Technology, Engineering & Mathematics, Volume 15, Issue 1, 2016, Pages 19-22.

14. Janaki G and Vidhya S, Special pairs of Pythagorean triangles and 3- digit consecutive Sphenic numbers, International Journal of Academic Research and Development, Volume 1, Issue 11, 2016, Pages 29-31.
15. Janaki G and Radha R, Special Pairs Of Pythagorean Triangles And Harshad Numbers, Asian Journal of Science and Technology, Volume 07, Issue, 08, 2016, Pages 3397-3399.

Notes

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Development of Mathematical Simulation of Hydrodynamic Oscillation Generators

By A. S. Korneev

Abstract- The unsteady turbulent swirled water flow in a channel in the presence of cavitation is calculated. The comparison of two forms of corrections to the k-ε-model of turbulence, taking into account the swirl of the flow, is performed as applied to the problem of calculating hydrodynamic oscillation generators. It is shown that both considered corrections it made possible to achieve agreement between the calculated and experimental data on the pressure distribution along the wall of the generator channel and on the form of the amplitude-frequency characteristics of oscillations. At the same time, the linear correction it made possible to improve the stability of the calculation procedure and prevent the appearance of zones with non-physical negative pressures, which in some cases were obtained using a quadratic correction. The results obtained can be used in mathematical modeling of hydrodynamic oscillation generators for various purposes, particularly for chemical technologies, oil production and medicine.

Keywords: hydrodynamic generators of oscillations, turbulence, swirl, mathematical modeling, pressure distribution, oscillation characteristics, fluid dynamics.

GJSFR-F Classification: LCC Code: TA1-2040

Strictly as per the compliance and regulations of:

Development of Mathematical Simulation of Hydrodynamic Oscillation Generators

A. S. Korneev

Abstract- The unsteady turbulent swirled water flow in a channel in the presence of cavitation is calculated. The comparison of two forms of corrections to the k-ε-model of turbulence, taking into account the swirl of the flow, is performed as applied to the problem of calculating hydrodynamic oscillation generators. It is shown that both considered corrections it made possible to achieve agreement between the calculated and experimental data on the pressure distribution along the wall of the generator channel and on the form of the amplitude-frequency characteristics of oscillations. At the same time, the linear correction it made possible to improve the stability of the calculation procedure and prevent the appearance of zones with non-physical negative pressures, which in some cases were obtained using a quadratic correction. The results obtained can be used in mathematical modeling of hydrodynamic oscillation generators for various purposes, particularly for chemical technologies, oil production and medicine.

Keywords: hydrodynamic generators of oscillations, turbulence, swirl, mathematical modeling, pressure distribution, oscillation characteristics, fluid dynamics.

I. Introduction

Hydrodynamic oscillation generators [1, 2] produce pressure waves when a fluid flows through channels of specific shapes and dimensions. These generators have no movable elements, which provides their reliability and long service life. Such devices can be used to disperse gas in liquid in many fields of chemical technologies [3], in systems for the biological purification of waste water and the chlorination and ozonization of drinking water, etc [2]. Generators of this type for medical purposes work like hydromassages [4]. Auspicious is the use such devices in oil production [5] to intensify oil production processes and enhance oil recovery. The results of wave technology processing of more than 1000 oil wells showed that the oil well productivity increased by an average of 30–40%, and the oil recovery increased by 5–10%.

It is necessary to develop methods for mathematical modeling to improve the performances of hydrodynamic generators. This method was presented in [6]. The system of continuity equation and Reynolds-averaged Navier–Stokes equations [7] with the "standard" k-ε turbulence model [8, 9] and the full cavitation model [10] was solved. The amplitude-frequency characteristics of the oscillations were calculated. The calculated positions of the amplitude maxima agreed with the available experimental data.

However, the experience of using the model [6] showed that for some values of the operating parameters of the generators, instabilities of the calculation procedure arose, leading to the appearance of the negative pressure in the channel, which increased infinitely in absolute value. In addition, for those cases when these

Author: Federal state budgetary science establishment of Mechanical Engineering Institute of A. A. Blagonravov of the Russian Academy of Sciences, Moscow, Russia. e-mail: korneev47@gmail.com

Ref

2. R .F. Ganiev and L. E. Ukrainski, Nonlinear Wave Mechanics and Generatory Phenomena on the Basis of High Technologies, Begell House, 2012.

instabilities did not arise, the calculated pressure distributions along the channel wall differed markedly from the experimental values. It was suggested that these problems are related to the flow swirl, which is not considered in the "standard" *k*-ε turbulence model [8, 9]. There are several experimental works that have shown that the swirl of the flow can lead to the weakening of turbulent pulsations.

In particular, Murakami and Kikuyama [12] studied the turbulent flow of water in a rotating tube. The water was supplied into a long fixed pipe with a diameter of D = 32 mm and a length of 60D, forming a steady turbulent longitudinal velocity profile. The unswirled flow entered the rotating pipe and was involved in the rotation due to friction against the walls. The rotating tube had segments of various lengths: 30D, 50D, 70D, 120D, 140D and 160D. Between these segments there were receivers of total pressure, which could move along the radius. The velocity profile was determined from the value of the dynamic pressure. The liquid then entered a fixed outlet pipe 200D long. The measurements showed that when moving along a rotating pipe, the profile of the axial velocity component changed and transformed from a steady turbulent to a parabolic one, which is typical for a laminar flow.

A. I. Borisenko, O. N. Kostikov, and V. I. Chumachenko [13] used a hot-wire anemometer to measure the intensity of turbulent pulsations in a rotating pipe with a diameter of D = 52 mm, through which an airflow passed. It was found that the intensity of pulsations decreases in the rotating channel. This process began at the wall, and as it moved away from the inlet, it reached the central region of the pipe. Thereby, under conditions typical of [12] and [13], the swirl led to flow laminarization. This must be taken into account in the mathematical model.

A correction to the *k*-ε turbulence model for the flow swirl was proposed in [11]. The accounting for this amendment it made possible to achieve agreement between the calculated and experimental data on the pressure distribution along the channel wall of the hydrodynamic generator and to calculate amplitude-frequency characteristics. However, even with this correction, the instabilities of the calculation procedure as mentioned earlier arose in a number of cases. To overcome them, this paper proposes another form of correction for flow swirl.

II. Mathematical Model

The system of continuity equation (1) and Reynolds-averaged Navier–Stokes equations (2), (3), (4) for an axisymmetric flow [7] and the two-equation (5), (6) turbulence model [8, 9] was solved:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial z}+\frac{1}{r}\frac{\partial(r\rho \upsilon)}{\partial r}=0; \tag{1}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial z}+\rho\upsilon\frac{\partial u}{\partial r}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left(\mu_s\frac{\partial u}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_s\frac{\partial u}{\partial r}\right); \tag{2}$$

$$\rho\frac{\partial \upsilon}{\partial t}+\rho u\frac{\partial \upsilon}{\partial z}+\rho\upsilon\frac{\partial \upsilon}{\partial r}=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial z}\left(\mu_s\frac{\partial \upsilon}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_s\frac{\partial \upsilon}{\partial r}\right)-\mu_s\frac{\upsilon}{r^2}+\rho\frac{w^2}{r}; \tag{3}$$

Ref

12. M. Murakami, and K. Kikuyama, *Trans. A SME. J. Fluids Eng.* 102, 97–103 (1980).

$$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial z}+\rho\upsilon\frac{\partial w}{\partial r}=\frac{\partial}{\partial z}\left(\mu_s\frac{\partial w}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_s\frac{\partial w}{\partial r}\right)-\mu_s\frac{w}{r^2}-\rho\frac{\upsilon w}{r}; \quad (4)$$

$$\rho\frac{\partial k}{\partial t}+\rho u\frac{\partial k}{\partial z}+\rho\upsilon\frac{\partial k}{\partial r}=\frac{\partial}{\partial z}\left[\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial r}\right]+G-\rho\varepsilon; \quad (5)$$

$$\rho\frac{\partial \varepsilon}{\partial t}+\rho u\frac{\partial \varepsilon}{\partial z}+\rho\upsilon\frac{\partial \varepsilon}{\partial r}=\frac{\partial}{\partial z}\left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial r}\right]+C_1\frac{\varepsilon}{k}G-C_2\rho\frac{\varepsilon^2}{k}. \quad (6)$$

Here

$$G=G_{u,v}+G_w;$$

$$G_{u,v}=\mu_t\left\{2\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial \upsilon}{\partial r}\right)^2+\left(\frac{\upsilon}{r}\right)^2\right]+\left(\frac{\partial u}{\partial r}+\frac{\partial \upsilon}{\partial z}\right)^2\right\};$$

$$G_w=\mu_t\left[\left(\frac{\partial w}{\partial z}\right)^2+\left(r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right)^2-\frac{\partial}{\partial r}\left(\frac{w^2}{r}\right)\right];$$

$$C_1=1.44;\ C_2=1.92;\ C_\mu=0.09;\ \sigma_k=1.0;\ \sigma_\varepsilon=1.3.$$

$$\mu_s=\mu+\mu_t;$$

Turbulent viscosity `$_t$ was defined as

$$\mu_t=C_\mu\rho f_\mu\frac{k^2}{\varepsilon}; \quad (7)$$

Two forms of corrections f_μ to turbulent viscosity, taking into account the flow swirl, were studied: quadratic (8) and linear (9):

$$f_\mu=1-\frac{w^2}{u^2+\upsilon^2+w^2}. \quad (8)$$

$$f_\mu=1-\frac{|w|}{|u|+|\upsilon|+|w|}. \quad (9)$$

Here u, υ, and w are the axial, radial, and tangential velocity components, ρ is the fluid density, k is the turbulence kinetic energy, and ε is the turbulence energy dissipation rate. In the absence of a swirl ($w=0$), the presented turbulence model with each of the corrections (8) and (9) goes into the standard one [8].

Ref

8. B. E. Launder, and D. B. Spalding, *Comput. Methods Appl. Mech. Eng.* 3, 269–289 (1974).

The cavitation was taken into account using the equation of transfer of the vapor mass fraction [10]

$$\rho\frac{\partial f_\upsilon}{\partial t}+\rho u\frac{\partial f_\upsilon}{\partial z}+\rho\upsilon\frac{\partial f_\upsilon}{\partial r}=R_{ce.}\cdot \tag{10}$$

$$p\le p_\upsilon:\quad R_{ce}=C_e\frac{\rho_l\rho_\upsilon}{\sigma}\left(1-f_\upsilon\right)\sqrt{\frac{2\left(p_\upsilon-p\right)k}{3\rho_l}}; \tag{11}$$

$$p>p_\upsilon:\quad R_{ce}=-C_c\frac{\rho_l\rho_\upsilon}{\sigma}\left(1-f_\upsilon\right)\sqrt{\frac{2\left(p-p_\upsilon\right)k}{3\rho_l}}; \tag{12}$$

$$\rho=\left(\frac{f_\upsilon}{\rho_\upsilon}+\frac{1-f_\upsilon}{\rho_l}\right)^{-1};\ \mu=\left(\frac{f_\upsilon}{\mu_\upsilon}+\frac{1-f_\upsilon}{\mu_l}\right)^{-1}. \tag{13}$$

$$C_e=0.02,\ C_c=0.01.$$

Here, f_υ is the mass fraction of vapor, ρ_l is the density of the liquid, $\rho_\upsilon=p_{sat}M_\upsilon/(RT)$ is the density of saturated vapor, p_{sat} is the pressure of saturated vapor of the liquid at temperature T, M_υ is the molar mass of vapor, $p_\upsilon=p_{sat}+p_{turb}/2$ is the phase-change threshold pressure, $p_{turb}=0.39\rho k$ is the turbulent pressure fluctuations, R_{ce} is the rate of evaporation of the liquid (at $R_{ce}>0$) or steam condensation (at $R_{ce}<0$), σ is the surface tension.

The system of equations (1) - (10) was solved by the pressure correction method [14] using the SIMPLE algorithm. In this method, instead of the continuity equation, the equation derived from it for corrections to pressure p' is solved. As the steady state solution is reached in the cycle of pressure iterations, the corrections p' approach zero.

Near the channel walls, in a laminar sublayer the near-wall functions proposed in [9] were used instead of Eq. (6).

In the calculation scheme of the generator (Fig. 1), the segments AB and EF are the face walls, BE is the cylindrical wall of the channel, FG is the cylindrical wall of the operation chamber, and GH is the output cross-section. The supply orifices were placed in segment CD. The cross-section I–I was set through the axes of symmetry of these orifices.

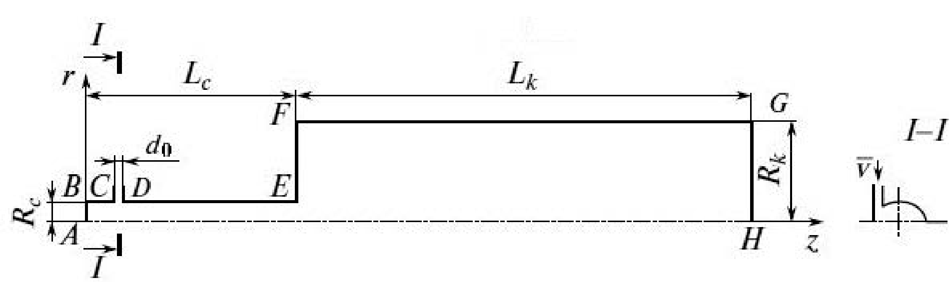


Figure 1: The scheme of the calculation region

10. A. K. Singhal, M. M. Athavale, H. Li, and Y. Jiang, *J. Fluids Eng.* 124 (3), 617–624 (2002).

Notes

As boundary conditions on solid walls AB, BC, DE, EF, and FG (Fig. 1), the no-slip conditions were set: $u = \upsilon = w = 0$, $p' = 0$, $\partial p / \partial n = 0$, $k = 0$, $\partial \varepsilon / \partial n = 0$, $f_\upsilon = 0$. where n is the coordinate along the normal to the wall.

On the axis of symmetry AH: $\partial u / \partial r = 0$, $\upsilon = w = 0$, $\partial p' / \partial r = 0$, $\partial p / \partial r = 0$, $\partial k / \partial r = 0$, $\partial \varepsilon / \partial r = 0$, $\partial f_\upsilon / \partial r = 0$.

In the region of fluid inlet CD: $u = 0$, $\upsilon = \upsilon_0$, $w = w_0$, $k = k_0$, $\varepsilon = \varepsilon_0$, $f_\upsilon = 0$. Here are $\upsilon_0 = Q/(2\pi R_c d_0)$, $w_0 = 4Q/(\pi d_0^2 n_0)$, $k_0 = (k_{in}\upsilon_0)^2/2$, $\varepsilon_0 = C_\mu k_0 \sqrt{k_0}/(0.1 d_0)$, Q is the volumetric flow rate of the liquid, n_0 is the number of supply orifices. In this work it was taken $n_0 = 2$, $k_{in} = 0.05$.

At the output of GH: $p = p_{out}$, and “soft” boundary conditions $\partial F / \partial z = 0$, where F is one of the variables: u, υ, w, p', k, ε, and f_υ.

III. Object of Investigation

An especial test generator (Fig. 2,*a*) was developed [16] to verify the mathematical model.

This generator made it possible to obtain the pressure distribution along the channel wall in order to compare calculated and experimental data.

The generator was of a cylindrical channel with a flaring part (operation chamber). The working fluid (tap water) was fed through two tangential orifices ensuring swirl of the flow. Six orifices in planes xz and yz have been made on a channel wall for static pressure measurement. Using of tubes the orifices were connected to manometers.

Tap water was chosen due to its availability. These generators can also operate on other liquids, particularly mineral oil and special drilling fluids.

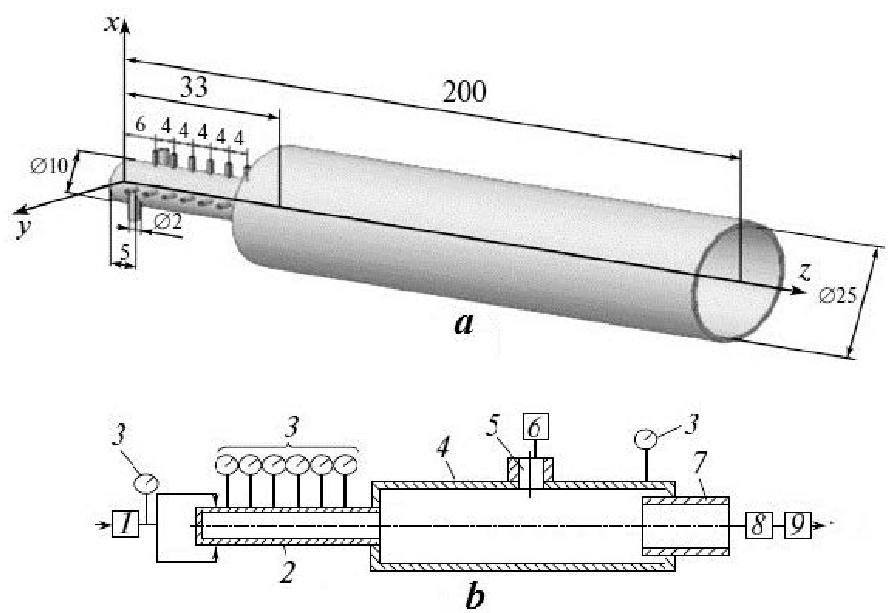


Figure 2: The test generator: (*a*) three-dimensional model, (*b*) scheme of experimental facility: (*1*) pump, (*2*) hydrodynamic generator, (*3*) manometers, (*4*) operation chamber, (*5*) pressure sensor, (*6*) oscilloscope, (*7*) throttle, (*8*) flowmeter, and (*9*) adjusting ventil.

The experimental results of the study of this test generator were given in [16]. Experiments were performed on the facility, the scheme of which is presented in Fig. 2,*b*. The tap water was fed into the input of pump *1* and directed into the generator *2* under the pressure p_{in}. The water pressure was measured by manometers *3* of the accuracy class 1. After the generator, the water entered operation chamber *4*. The piezoelectric pressure sensor *5* of type 701A produced by Kistler was mounted to measure the pressure pulsations. The oscilloscope LeCroy Wave Surfer[ff] MXs-B *6* was used to record and proceed the spectra. The used water went to the drain through throttle *7*, flowmeter *8*, and adjusting ventil *9*. The required pressure p_{out} in the chamber was set using ventil *9*.

IV. Results and Discussions

The results presented below were obtained at the value of absolute pressure of water p_{in} = 5.1 MPa at the generator input and p_{out} = 0.24 MPa at the generator output. The flow rate was equal to Q = 23.3 dm^3/min. The Reynolds number computed by the averaged velocity in the channel was Re ≈ 50000. These parameters corresponded to values typical for generators used in chemical technologies.

a) Time dependences of the pressure

The calculated dependences of pressure p on time t on the axis of symmetry (r = 0 mm) in the cross-section of the supply orifices (z = 5 mm) are shown in Fig. 3.

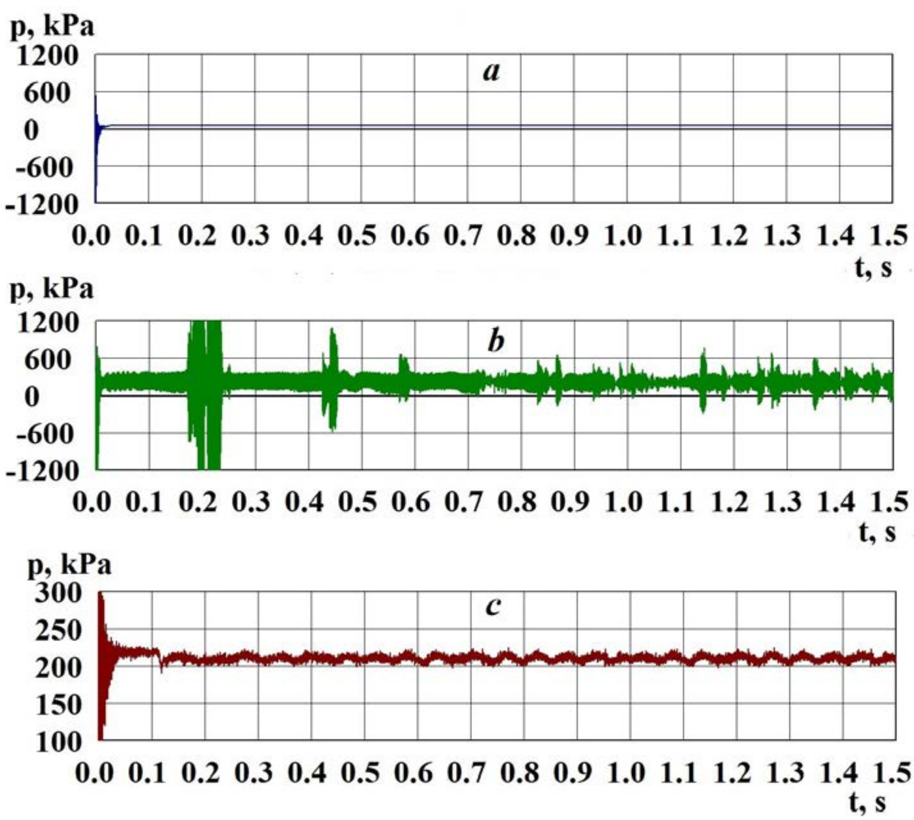


Figure 3: The time dependences of the pressure at the axis of symmetry at the plane of input orifices (r = 0 mm, z = 5 mm): (a) without correction for the swirl ($f_\mu = 1$), (b) with a quadratic correction f_μ according to (8), (c) with a linear correction f_μ according to (9).

Ref

16. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 48 (5), 401–407 (2019).

The calculation without correction for the swirl, $f_{\mu}=1$ (Fig. 3,*a*), led to the damping of pressure fluctuations with time. In contrast, in experiments [15, 16] these fluctuations existed constantly. The calculations with a quadratic correction for swirl according to expression (8) (Fig. 3,*b*) gave undamped oscillations in time. However, at some time intervals, instabilities of the calculation procedure were observed with a sharp increase in pressure in absolute value, which disappeared as the calculation continued. This significantly increased the calculation time required to achieve a steady state of oscillations. In addition, non-physical negative pressure values appeared at some channel points. The calculations with a linear correction for the swirl according to expression (9) (Fig. 3,*c*) gave stable fluctuations in time. In this case, there were no negative pressures.

Similar calculated data at the location of the pressure sensor in the experiments [15, 16] (r = 14 mm, z = 100 mm) are shown in Fig. 4.

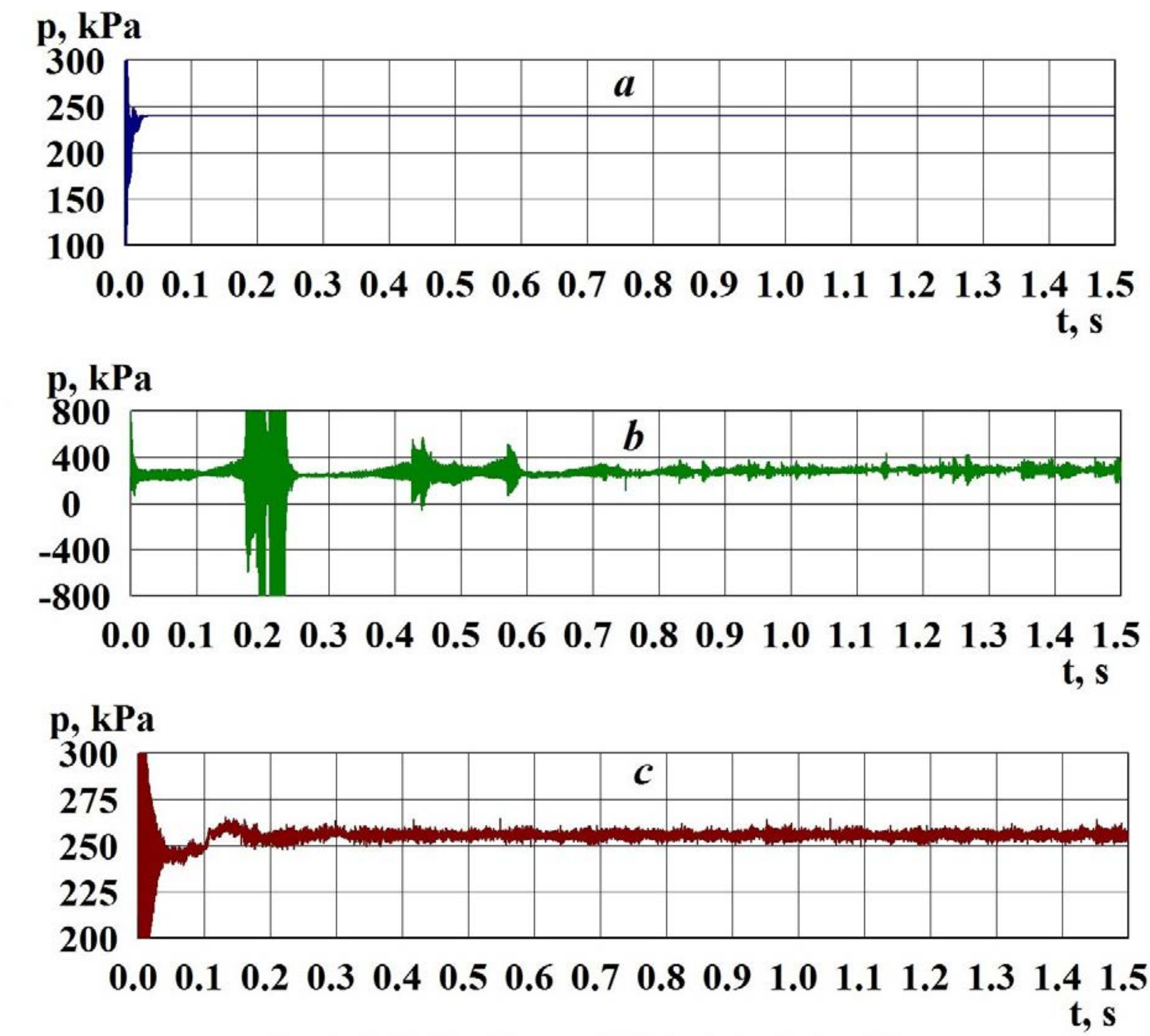


Figure 4: The time dependences of the pressure at the location of the pressure sensor (r = 14 mm, z = 100 mm): (*a*) without correction for the swirl ($f_{\mu}=1$), (*b*) with a quadratic correction f_{μ} according (8), (*c*) with a linear correction f_{μ} according (9).

Ref

15. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 46 (4), 356–363 (2017).

b) Amplitude-frequency characteristics of the oscillations

By Fourier transformations of the data presented in Fig.4,*b* and 4,*c*, for time intervals of steady oscillations t = 0.6 ... 1.5 s, the calculated amplitude-frequency characteristics of the oscillations were obtained (Fig. 5,*a* and 5,*b*). The corresponding experimental frequency response according to [15] is shown in Fig. 5, *c*.

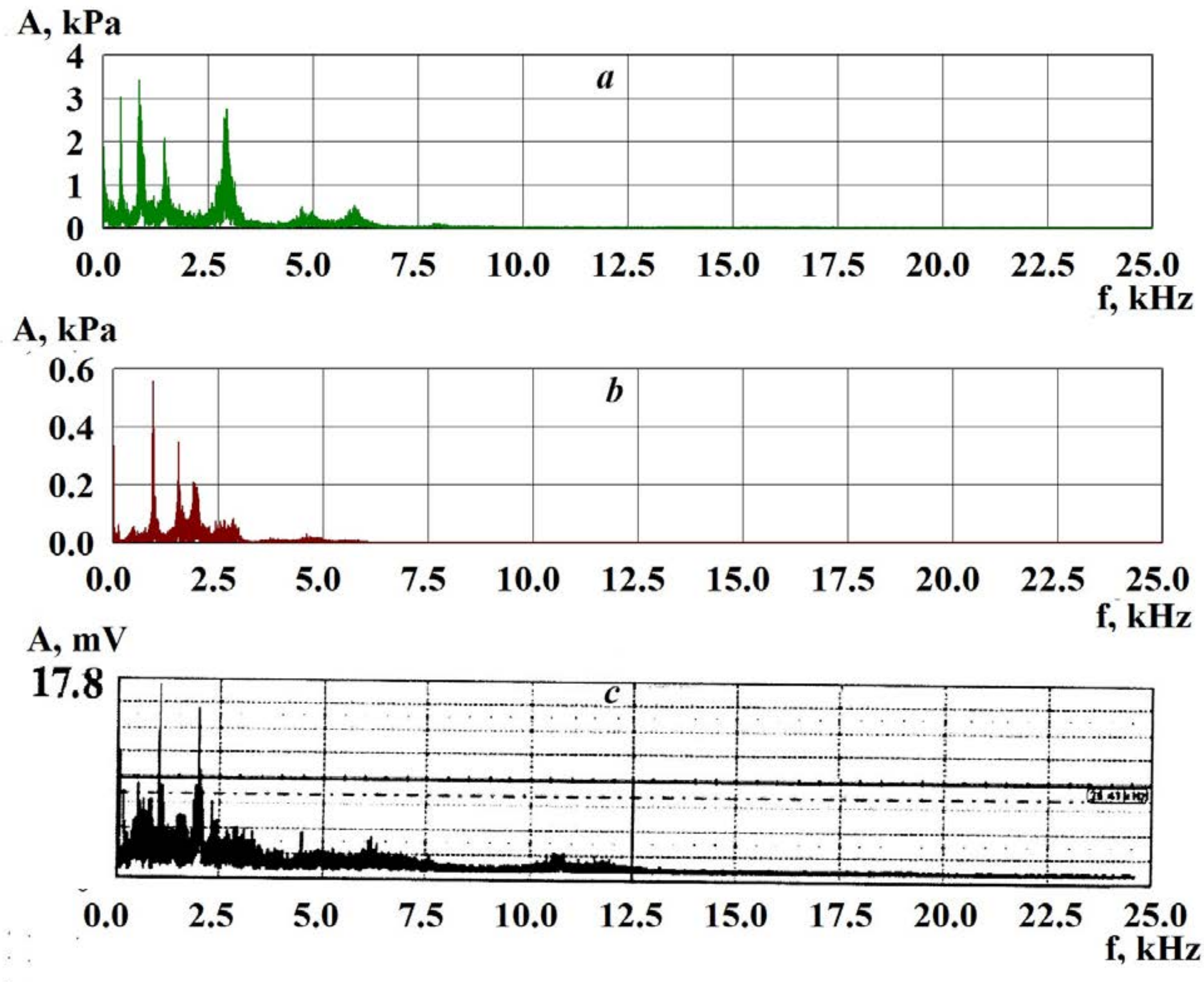


Figure 5: The amplitude-frequency characteristics of the oscillations at the location of the pressure sensor (r = 14 mm, z = 100 mm): (*a*) calculations with a quadratic correction f_{μ} according to expression (8), (*b*) with a linear correction f_{μ} according to expression (9), (*c*) experiment [15].

The experimental values of the amplitudes in Fig. 5,*c* are presented in units of the oscilloscope scale (millivolts). The experimental frequency characteristic exhibits two principal maxima at frequencies f = 1.04 kHz and f = 1.98 kHz. The calculated frequency characteristic with a quadratic correction according to expression (8) yielded four maxima at f = 0.42, 0.83, 1.48, and 2.90 kHz. The calculation with a linear correction using expression (9) gave the values of three principal maxima corresponding to the frequencies f = 0.94, 1.55, and 1.98 kHz.

Ref

15. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 46 (4), 356–363 (2017).

c) The pressure distribution along a wall of the channel

Experimental pressure distributions [16] along the wall of the generator channel (r = 5 mm) in a plane parallel to the axes of the supply orifices are shown in Fig. 6, points 1, and in the plane perpendicular to these axes in Fig. 6, points 2. It can be seen that deviations from axisymmetry manifest themselves up to distances z = 10 ...12 mm from the left end wall of the channel.

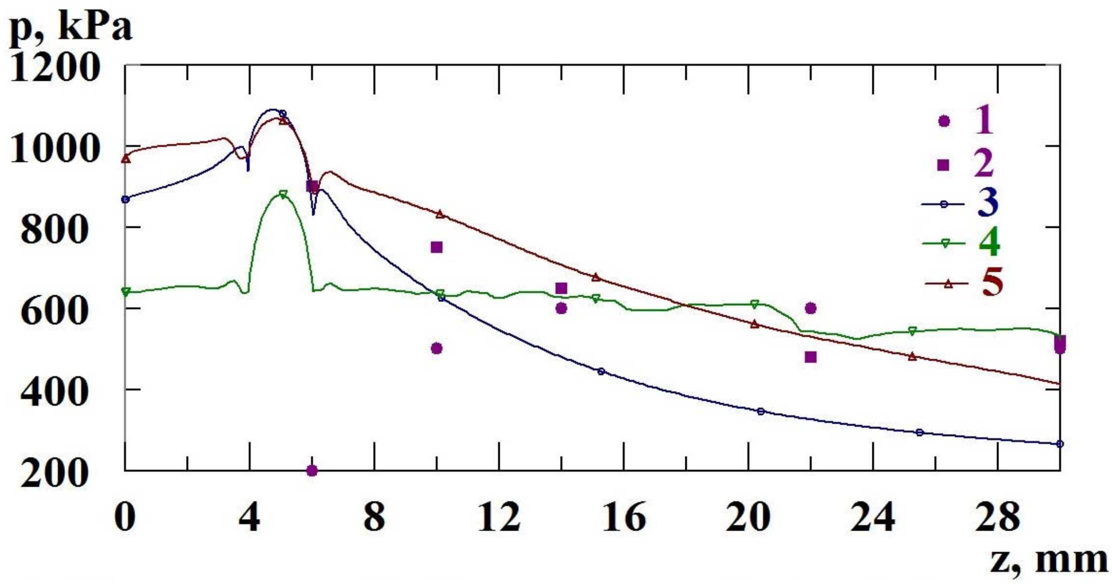


Figure 6: The distribution of the time-averaged pressure along the wall of the generator channel: points (1) experiment in a plane *xz* parallel to the axes of the supply orifices, points (2) experiment in a plane *yz* perpendicular to the axes of the supply orifices, line (3) calculation without correction for the swirl ($f_\mu = 1$), line (4) calculation with quadratic correction f_μ according to expression (8), line (5) calculation with a linear correction f_μ according to expression (9)

The calculations without considering the correction for the flow swirl (Fig. 6, line 3) showed a noticeable deviation from the experimental values. The calculated data obtained in the axisymmetric approximation with a quadratic correction f_μ (Fig. 6, line 4) according to expression (8) are between the experimental points, and with a linear correction f_μ (Fig. 6, line 5) according to expression (9) deviate slightly from experimental points.

V. Conclusions

1. In conditions, which are typical for hydrodynamic generators, the swirl of a stream leads to laminarization of the current. It occurs owing to the stabilising influence of a field of centrifugal forces. Besides, because of centrifugal effects the liquid is rejected on stream periphery where there is a damping of turbulent pulsations of speed due to a friction about a wall.
2. A comparison of two forms of corrections for the flow swirl in the k-ε-model of turbulence is carried out. The accounting for these corrections it made possible to achieve consistency between the calculated and experimental data on the pressure distribution along the channel wall of the hydrodynamic generator and it made

Ref

16. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 48 (5), 401–407 (2019).

possible to calculate the amplitude-frequency characteristics of oscillations. If there is no swirl, both models shown are automatically converted to the standard *k*-ε-model of turbulence.

3. The results obtained can be used in mathematical modeling of hydrodynamic oscillation generators for various purposes, particularly for chemical technologies, oil production and medicine.

Acknowledgments

The author is grateful to the head of the Nonlinear Wave Mechanics and Technology Center of the RAS - the Branch of the Mechanical Engineering Institute of A. A. Blagonravov of the Russian Academy of Sciences, academician R. F. Ganiev, as well as the Center team for the help and useful work discussions. The author is especially grateful to O. V. Shmyrkov for the provided experimental data.

The study was carried out using the computational resources of the Joint Supercomputer Center of the Russian Academy of Sciences (JSCC of RAS).

Notes

References Références Referencias

1. S. Avduevskii, R. F. Ganiev, G. A. Kalashnikov, *et al.,* Russian Federation Patent No. 2015749 (12 July 1994).
2. R .F. Ganiev and L. E. Ukrainski, Nonlinear Wave Mechanics and Generatory Phenomena on the Basis of High Technologies, Begell House, 2012.
3. R. F. Ganiev, D. A. Zhebynev, A. S. Korneev, L. E. Ukrainsky, Fluid Dynamics, 43 (2), 297 - 302 (2008).
4. E. I. Veliev, R. F. Ganiev, A.S. Korneev, et al., Dokl. Phys, 66, 353–357 (2021).
5. R. F. Ganiev, ed., Wave Technology and Engineering, Logos, Moscow, 55–57 (1993).
6. A. S. Korneev, Fluid Dynamics 48, 471–476 (2013).
7. L. G. Loitsyanskii, Mechanics of Liquids and Gases, Elsevier (1966).
8. B. E. Launder, and D. B. Spalding, *Comput. Methods Appl. Mech. Eng.* 3, 269–289 (1974).
9. T. J. Craft, A. V. Gerasimov, H. Iacovides, B. E. Launder, *Int. J. Heat Fluid Flow* 23, 148–160 (2002).
10. A. K. Singhal, M. M. Athavale, H. Li, and Y. Jiang, *J. Fluids Eng.* 124 (3), 617–624 (2002).
11. A. S. Korneev, Int. J. of Research in Eng. and Science (IJRES) 7 (1), Ser. II , 29–34 (2019).
12. M. Murakami, and K. Kikuyama, *Trans. A SME. J. Fluids Eng.* 102, 97–103 (1980).
13. A. I. Borisenko, O. N. Kostikov, and V. I. Chumachenko, J. Eng. Physics 24, 770–773 (1975).
14. S. V. Patankar, Numerical Heat Transfer and Fluid Flow, McGraw Hill, (1980).
15. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 46 (4), 356–363 (2017).
16. A. S. Korneev and O. V. Shmyrkov, J. Machinery Manufacture and Reliability 48 (5), 401–407 (2019).

Global Journals Guidelines Handbook 2023

www.GlobalJournals.org

Memberships

FELLOWS/ASSOCIATES OF SCIENCE FRONTIER RESEARCH COUNCIL

FSFRC/ASFRC MEMBERSHIPS

Introduction

FSFRC/ASFRC is the most prestigious membership of Global Journals accredited by Open Association of Research Society, U.S.A (OARS). The credentials of Fellow and Associate designations signify that the researcher has gained the knowledge of the fundamental and high-level concepts, and is a subject matter expert, proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice. The credentials are designated only to the researchers, scientists, and professionals that have been selected by a rigorous process by our Editorial Board and Management Board.

Associates of FSFRC/ASFRC are scientists and researchers from around the world are working on projects/researches that have huge potentials. Members support Global Journals' mission to advance technology for humanity and the profession.

FSFRC

FELLOW OF SCIENCE FRONTIER RESEARCH COUNCIL

FELLOW OF SCIENCE FRONTIER RESEARCH COUNCIL is the most prestigious membership of Global Journals. It is an award and membership granted to individuals that the Open Association of Research Society judges to have made a 'substantial contribution to the improvement of computer science, technology, and electronics engineering.

The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Fellows are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Fellow Members.

Benefit

To the institution

Get letter of appreciation

Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

Exclusive Network

Get access to a closed network

A FSFRC member gets access to a closed network of Tier 1 researchers and scientists with direct communication channel through our website. Fellows can reach out to other members or researchers directly. They should also be open to reaching out by other.

Career | Credibility | Exclusive | Reputation

Certificate

Receive a print ed copy of a certificate

Fellows receive a printed copy of a certificate signed by our Chief Author that may be used for academic purposes and a personal recommendation letter to the dean of member's university.

Career | Credibility | Exclusive | Reputation

Designation

Get honored title of membership

Fellows can use the honored title of membership. The “FSFRC” is an honored title which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., FSFRC or William Walldroff, M.S., FSFRC.

Career | Credibility | Exclusive | Reputation

Recognition on the Platform

Better visibility and citation

All the Fellow members of FSFRC get a badge of "Leading Member of Global Journals" on the Research Community that distinguishes them from others. Additionally, the profile is also partially maintained by our team for better visibility and citation. All fellows get a dedicated page on the website with their biography.

Career | Credibility | Reputation

Future Work

Get discounts on the future publications

Fellows receive discounts on future publications with Global Journals up to 60%. Through our recommendation programs, members also receive discounts on publications made with OARS affiliated organizations.

Career Financial

GJ Internal Account

Unlimited forward of Emails

Fellows get secure and fast GJ work emails with unlimited forward of emails that they may use them as their primary email. For example, john [AT] globaljournals [DOT] org.

Career Credibility Reputation

Premium Tools

Access to all the premium tools

To take future researches to the zenith, fellows and associates receive access to all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

Conferences & Events

Organize seminar/conference

Fellows are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career Credibility Financial

Early Invitations

Early invitations to all the symposiums, seminars, conferences

All fellows receive the early invitations to all the symposiums, seminars, conferences and webinars hosted by Global Journals in their subject.

Exclusive

Publishing Articles & Books

Earn 60% of sales proceeds

Fellows can publish articles (limited) without any fees. Also, they can earn up to 60% of sales proceeds from the sale of reference/review books/literature/ publishing of research paper. The FSFRC member can decide its price and we can help in making the right decision.

Exclusive Financial

Reviewers

Get a remuneration of 15% of author fees

Fellow members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

Access to Editorial Board

Become a member of the Editorial Board

Fellows may join as a member of the Editorial Board of Global Journals Incorporation (USA) after successful completion of three years as Fellow and as Peer Reviewer. Additionally, Fellows get a chance to nominate other members for Editorial Board.

Career Credibility Exclusive Reputation

And Much More

Get access to scientific museums and observatories across the globe

All members get access to 5 selected scientific museums and observatories across the globe. All researches published with Global Journals will be kept under deep archival facilities across regions for future protections and disaster recovery. They get 10 GB free secure cloud access for storing research files.

ASFRC

ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL

ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL is the membership of Global Journals awarded to individuals that the Open Association of Research Society judges to have made a 'substantial contribution to the improvement of computer science, technology, and electronics engineering.

> The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Associate membership can later be promoted to Fellow Membership. Associates are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Associate Members.

Benefit

To the Institution

Get letter of appreciation

Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

Exclusive Network

Get access to a closed network

A ASFRC member gets access to a closed network of Tier 1 researchers and scientists with direct communication channel through our website. Associates can reach out to other members or researchers directly. They should also be open to reaching out by other.

Career | Credibility | Exclusive | Reputation

Certificate

Receive a print ed copy of a certificate

Associates receive a printed copy of a certificate signed by our Chief Author that may be used for academic purposes and a personal recommendation letter to the dean of member's university.

Career | Credibility | Exclusive | Reputation

Designation

Get honored title of membership

Associates can use the honored title of membership. The “ASFRC” is an honored title which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., ASFRC or William Walldroff, M.S., ASFRC.

Career | Credibility | Exclusive | Reputation

Recognition on the Platform

Better visibility and citation

All the Associate members of ASFRC get a badge of "Leading Member of Global Journals" on the Research Community that distinguishes them from others. Additionally, the profile is also partially maintained by our team for better visibility and citation. All associates get a dedicated page on the website with their biography.

Career | Credibility | Reputation

Future Work

Get discounts on the future publications

Associates receive discounts on the future publications with Global Journals up to 60%. Through our recommendation programs, members also receive discounts on publications made with OARS affiliated organizations.

Career Financial

GJ Internal Account

Unlimited forward of Emails

Associates get secure and fast GJ work emails with unlimited forward of emails that they may use them as their primary email. For example, john [AT] globaljournals [DOT] org.

Career Credibility Reputation

Premium Tools

Access to all the premium tools

To take future researches to the zenith, fellows receive access to almost all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

Conferences & Events

Organize seminar/conference

Associates are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career Credibility Financial

Early Invitations

Early invitations to all the symposiums, seminars, conferences

All associates receive the early invitations to all the symposiums, seminars, conferences and webinars hosted by Global Journals in their subject.

Exclusive

Publishing Articles & Books

Earn 30-40% of sales proceeds

Associates can publish articles (limited) without any fees. Also, they can earn up to 30-40% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.

Exclusive Financial

Reviewers

Get a remuneration of 15% of author fees

Associate members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

And Much More

Get access to scientific museums and observatories across the globe

All members get access to 2 selected scientific museums and observatories across the globe. All researches published with Global Journals will be kept under deep archival facilities across regions for future protections and disaster recovery. They get 5 GB free secure cloud access for storing research files.

Associate	Fellow	Research Group	Basic
$4800 **lifetime designation**	$6800 **lifetime designation**	$12500.00 **organizational**	APC **per article**
Certificate, LoR and Momento 2 discounted publishing/year **Gradation** of Research 10 research contacts/day 1 **GB** Cloud Storage **GJ** Community Access	**Certificate,** LoR and Momento **Unlimited** discounted publishing/year **Gradation** of Research **Unlimited** research contacts/day 5 **GB** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**Certificates,** LoRs and Momentos **Unlimited** free publishing/year **Gradation** of Research **Unlimited** research contacts/day **Unlimited** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**GJ** Community Access

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

During submission of the manuscript, the author is confirming an exclusive license agreement with Global Journals which gives Global Journals the authority to reproduce, reuse, and republish authors' research. We also believe in flexible copyright terms where copyright may remain with authors/employers/institutions as well. Contact your editor after acceptance to choose your copyright policy. You may follow this form for copyright transfers.

Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

save our planet

Global Journal of Science Frontier Research

Visit us on the Web at www.GlobalJournals.org | www.JournalofScience.org
or email us at helpdesk@globaljournals.org

ISSN 9755896