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About Stability of Solutions to Systems of Differential Equations

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Abstract- The stability conditions for solutions of systems of ordinary differential equations are considered. The conditions and criteria for the use of partial and external derivatives are proposed. This allows us to investigate the behavior of a function of several variables, without requiring its differentiability, but using only information on partial derivatives. This reduces the restrictions on the degree of smoothness of the studied functions. The use of the apparatus of external derived numbers makes it possible to reduce the restrictions on the degree of smoothness of manifolds when studying the question of the integrability of the field of hyperplanes. Using the apparatus of partial and external derived numbers, it can be shown that the investigation of the stability of solutions of a system of differential equations can be reduced to an investigation of the solvability of a system of equations of a special form.

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About Stability of Solutions to Systems of Differential Equations

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Abstract- The stability conditions for solutions of systems of ordinary differential equations are considered. The conditions and criteria for the use of partial and external derivatives are proposed. This allows us to investigate the behavior of a function of several variables, without requiring its differentiability, but using only information on partial derivatives. This reduces the restrictions on the degree of smoothness of the studied functions. The use of the apparatus of external derived numbers makes it possible to reduce the restrictions on the degree of smoothness of manifolds when studying the question of the integrability of the field of hyperplanes. Using the apparatus of partial and external derived numbers, it can be shown that the investigation of the stability of solutions of a system of differential equations can be reduced to an investigation of the solvability of a system of equations of a special form.

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I. Introduction

Many sciences are engaged in the creation of mathematical models of various processes. The problems in the study of dynamic processes lead to complex system of differential equations [1-22]. The concepts of stability of solutions or asymptotic stability are often used in studies of solutions of equations and the ability to control the behavior in the presence of perturbations [4-7]. For their solution or successive approximations to the exact solution necessary to check the conditions and criteria that must be met. The study of control problems and the stability of solutions of systems of differential equations to describe processes that are defined as linear operations makes it possible to divide all tasks into classes and identify important properties inherent in systems of differential equations of the same class. In the study of the problems of controlling the motion of mechanical systems [9-11] in the transition from a general formal description to the construction of mathematical models take into account:

- Content and properties of functions in the system of equations of dynamics,
- Structure of control functions, restrictions or boundary conditions,
- Type of functional or quality criterion of solutions,
- Stability conditions for solutions for admissible controls.

The concepts of partial derivatives of numbers and external derivatives of numbers are considered in order to use them to study the stability of solutions of a system of differential equations through the study of the solvability of a system of

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equations of a special form. The proposed method can be used to obtain necessary or sufficient stability conditions for solutions of systems of differential equations.

II. Features of Stability Conditions

Let the change of parameters$x$ or the object's behavior be described by a system of ordinary differential equations of the form

$$\dot{x} = Ax \tag{1}$$

Notes

From equation (1) for a linear stationary system follows the validity of the following equation

$$\frac{d}{dt} x^* x = (A^* + A)\, x \tag{2}$$

Here, an asterisk means a transpose operation. Let $(A^* + A)$ be a nonsingular matrix symmetric with respect to the diagonal. Then applying the Lagrange method to equation (2) reduction of quadratic forms to the sum of squares [3], it is easy to verify that there is a linear transformation$x = Ly$, reducing equation (2) to the form

$$\frac{d}{dt} y^* L^* L y = y^* B y$$

where $B = L^* (A^* + A) L$ is the diagonal matrix. If the matrix B is negative definite, i.e. all its elements are negative, then system (1) is asymptotically stable. In general we can talk about the stability of solutions under additional conditions.

a) The partial derivatives numbers

Using the apparatus of private and external derivatives of numbers, show that the study of the stability of solutions systems of differential equations can be reduced to the study of the solvability of systems of equations of a special kind. The present studies are based on [1-3,8].

Let the function f be given in some open region of space R^n , and let it go $x = (x_1, \dots, x_n)^*$ — an arbitrary point of this areas as $\Delta x = (\Delta x_1, \dots, \Delta x_n)^*$—arbitrary increment of function f arguments

$$\psi_i[f](x; \Delta x) = \frac{\omega_i}{2^{n-1} \Delta x_i}, \qquad i = 1,2, \dots, n.$$

$$\omega_i = \sum_{\mu \in \nu_i} [\, f(x_1 + \mu_1 \Delta x_1, \dots, x_n + \mu_n \Delta x_n) -$$

$$- f(x_1 + \mu_1 \Delta x_1, \dots, x_{n-1} + \mu_{i-1} \Delta x_{i-1}, x_{i+1} +$$

$$+ \mu_{i+1} \Delta x_{i+1}, \dots, x_n + \mu_n \Delta x_n)],$$

where $\mu = (\mu_1, \dots, \mu_n)$, ν_i, $i = 1,2, \dots, n$, marked a bunch of n-dimensional vectors consisting of zeros and ones and having unit at the i–th place.

Definition 1: The number λ is called the partial derivative of the function *f* in point x in the variable x_i if there is a sequence $\triangle x^k$ such that for any $\Delta x_j^k \to 0, j \in (1, \dots, n)$, at $\Delta x_j^k \to 0, k \to \infty$, and

$$\lim_{k\to\infty} \psi_i[f]\,(x;\triangle x^k) = \lambda.$$

Notes

The fact that λ is a partial derivative functions f at the point x with respect to the variable x_i , we will write this:

$$\lambda = \lambda_{x_i}[f](x).$$

Perform a study of the stability of solutions of systems of ordinary differential equations.

b) The external derivatives numbers

The definition of the external derivative number allows us to find the conditions for the complete integrability of continuous fields of hyperplanes. Let *M* be a Hausdorff space with a countable base, and let *p* be an arbitrary point of *M*. If a point *p* has a neighborhood *U* that is homeomorphic to an open subspace of an *n*-dimensional Euclidean space R^n, then *M* is called an *n*-dimensional topological manifold. Let M^n be an *n*-dimensional topological manifold. Let *V* be an *n*-dimensional vector space over a field of real numbers. Every linear mapping $f : V \in R$, i.e. display at which

$$f(av + bw) = af(v) + bf(w), \qquad v, w \in V, \qquad a, b \in R.$$

Definition 2: The form $\lambda[\omega](p)$ is called the external derivative of the external differential *q*-form of the class C^r , $r \geq 0$, on variety M^n at the point $p \in M^n$, if in R^n there is a sequence converging to zero $-\triangle x^k$", such that

$$(\Phi_\kappa^*)^{-1}\lambda[\omega](p) = \lim_{k\to\infty}\left\{\sum_{j_1,\dots,j_q}\left(\sum_{i=1}^{n}\psi_{x_i}\left[a_{j_1,\dots j_q},\right](x{:}\,\Delta x^k)dx_i\right)\wedge dx_{j_1}\wedge\dots\wedge dx_{j_q}\right\} =$$

$$= \sum_{j_1,\dots,j_q}\left(\sum_{i=1}^{n}\lambda_{x_i}\left[a_{j_1,\dots j_q},\right](\phi_k(p))dx_i\right)\wedge dx_{j_1}\wedge\dots\wedge dx_{j_q}$$

c) Investigation of the stability of solutions

Let the behavior of an object be described by a system of ordinary differential equations of the form

$$\dot{x} = F(t, x), \;\; F(t, 0) \equiv 0, \tag{3}$$

where $= (x_1, \dots, x_n)^*$, $F(t, x) = (F_1(t, x), \dots, F_n(t, x))^*$.

We say that the solution $x = 0$ of system (3) is Lyapunov stable if, for any ε ¿ 0 and $t_0 \geq 0$ can find $\delta(\varepsilon, t_0) > 0$ such that from $\| x_0 \| < \delta$ it follows $\| x(t; t_0, x_0) \| < \varepsilon$ for all $t \geq t_0$.

We introduce the class of functions H, assuming that the function $l(r)$ belongs to this class $(l(r) \in H)$, if $l(r)$ is continuous, strictly increasing for $r \in [0,H]$, $H = const > 0$, or for $r \in [0,\infty)$, the function is $l(0) = 0$.

The function is H, which means that $l(r)$ is optional this class $(l(r) \in H)$, if $l(r)$ — continuous strictly increasing at$r \in [0,H]$, $H = const > 0$, or at$r \in [0,\infty)$ function, moreover $l(0) = 0$.

Definition 3: The function $V(t,x)$, $V(t,0) \equiv 0$, $t \geq 0$, will call definitely positive if there is a function $(r) \in H$, such that in

$$t \geq 0, \quad \| x \| \leq H$$

inequality holds

$$V(t, x) \geq l(\|x\|).$$

This definition is equivalent to the generally accepted definition of positive definiteness of a function $V(t, x)$.

In the future, we will adhere to the following notation:

$$K_r(x_0) = \{x : \| c - x_0 \| \leq r\},$$

$$S_r(x_0) = \{x : \| x - x_0 \| = r\}, \; r = const > 0. \tag{4}$$

For brevity we put $S_1(0) = S$.

Theorem 1: Suppose that in region (4) there exist continuous partial derivatives

$$\frac{\partial F_i}{\partial x_j}, \; i,j = 1,2,\ldots,n.$$

Then, in order for the solution $x = 0$ of system (3) was stable according to Lyapunov, it is necessary and sufficient that in the region

$$t \geq 0, \quad \| x \| \leq h, \; 0 < h = const < H, \tag{5}$$

system

$$a_0(t,x) + a(t,x) \cdot F(t,x) \leq 0, \; a(t,x) = (a_1(t,x),\ldots,a_n(t,x)), \tag{6}$$

$$\omega \wedge \lambda[\omega] \equiv 0, \; \omega = a_0 dt + a_1 dx_1 + \ldots + a_n dx_n, \tag{7}$$

had a continuous solution $a_0(t,x) + a(t,x)$ satisfying the following requirements:

1) in the region of

$$t \geq 0, \; x \in K_h(0) \setminus \{0\},$$

$$\sum_{i=0}^{n} a_i^2 (t,x) > 0, \tag{8}$$

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2) in the region of

$$t \geq 0, \ \mu \in [0,h], \ x \in S,$$

$$\int_0^{\mu} a(t,\mu' x) \cdot x d\mu' \geq l(\mu), l(\mu) \in H \ . \qquad (9)$$

Notes

The solution $x = 0$ of system (3) will be called uniformly sustainable if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that from $t_0 \geq 0, \| x_0 \| < \delta$ should

$$\| x(t; t_0, x_0) \| < \varepsilon, \quad t \geq t_0.$$

We will say that the solution $x = 0$ of system (3) is evenly attractive if exists $\Delta_0 = const > 0$ such that the condition

$$\lim_{t \to \infty} \| x(t; t_0, x_0) \| = 0$$

performed uniformly by t_0, x_0 from area

$$t_0 \geq 0, \qquad \| x_0 \| < \Delta_0.$$

If the solution $x = 0$ of system (3) is simultaneously uniformly stable and evenly attractive, then we will call uniformly asymptotically stable.

d) Stability conditions

Theorem 2. Suppose that in region (10) there exist continuous partial derivatives. Then, in order for the solution $x = 0$ of system (3) to be Lyapunov stable, it is necessary and sufficient: system had a continuous solution $(a_0(t,x), a(t,x))$, satisfying the following requirements in the region of $t \geq 0$.

A solution $x = 0$ of system (3) will be called uniformly stable if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\| x_0 \| < \delta$ follows

$$\| x(t, t_0, x_0) \| < \delta$$

for all $t \geq t_0$,

The proposed method allows one to obtain statements that give necessary or sufficient conditions for uniform stability or asymptotic stability for solutions of systems of differential equations.

Theorem 3. Suppose that in region (4) the functions $F_i$and their partial derivatives $\frac{\partial F_i(t,x)}{\partial x_j}$ are continuous and bounded:

$$|F_i(t,x)| \leq B, \ B = const, \quad \left|\frac{\partial F_i(t,x)}{\partial x_j}\right| \leq A, \ A = const, \ i,j = 1,2,\dots,n. \qquad (10)$$

Then, for the solution $x = 0$ of system (3) to be uniformly asymptotically stable, it is necessary and sufficient that in region (5), where h is a sufficiently small constant, system (6)–(7) has a continuous solution $(a_0(t,x), a(t,x))$, satisfying in the area (8) or (9) the following constraints:

1) $\sum_{i-0}^{n} a_i^2\,(t,x) > 0$;

2) $l_1(\mu) \leq \int_0^{\mu} a(t,\mu' x) \cdot x d\mu' \leq l_2(\mu)$;

3) $\max_{t\geq 0, \|x\|=1}[a_0(t,\mu x) + a(t,\mu x) \cdot F(t,\mu x)] \leq -l_3(\mu),\ l_k(r) \in H$.

The proposed method allows to obtain the necessary or sufficient conditions for the stability of solutions of systems of differential equations.

Notes

e) Stability of Almost Periodic Solutions

On the basis of the previous theorems, the authors obtain the conditions to determine the maximum possible number of almost periodic solutions in first-order differential equation. Now the problem of the existence of almost periodic solutions for the equation is under consideration, since this allows for the determination of the minimum possible number of almost periodic solutions for the differential equation considered.

So, consider the first-order differential equation

$$\dot{x} = f(t,x), \tag{11}$$

where f is a function continuous on R^2 that is almost periodic in t uniformly in x in every compact set and such that equation (11) has the property of existence and uniqueness of its solutions.

To prove the existence of almost periodic solution for equation (11), the result obtained should be used. Let it be formulated in the form of the following theorem.

This study allows to determine the minimum possible number of almost periodic solutions for the considered differential equation. Consider the first-order differential equation (1), where f is a function continuous on R^2 almost periodic in t uniformly in x on each compact set and such that equation (1) has the property of existence and uniqueness of solutions. In proving the existence of an almost periodic solution of equation (1), the results obtained in [9] are used.

Consider now stability of the solutions of equation (11) [6-10, 18-22].

Theorem 4: If the right-hand side of equation (11) is a function decreasing with respect to x for each fixed t, then all solutions of this equation are uniformly stable.

Proof. Let $u(t)$ be an arbitrary solution of equation (11). Suppose $y = x - u$. The equation for y is of the following form:

$$\dot{y} = f(t,u+y) - f(t,u) = g(t,y). \tag{12}$$

Let the following function be the Lyapunov function:

$$v(y) = \frac{1}{2} y^2. \tag{13}$$

Since $f(t,x)$ decreases with respect to x at each fixed t, the derivative of the function (13) on the solutions of Equation (12) satisfies the inequality

$$\left.\frac{dv}{dt}\right|_{(16)} = yg(t,y) \le 0,$$

which implies the uniform stability of solution $y = 0$ of equation (12), and hence, solution $u(t)$ of equation (11). Taking into account the fact that $u(t)$ is an arbitrary solution of equation (11), it is clear the theorem is proven.

Notes

Note that the theorem implies in the conditions of Theorem 14 that all n almost periodic solutions of equation (11) are stable, either as $t \to +\infty$ or with $t \to -\infty$.

Let $\lambda_x[f](t,x)$ denote an arbitrary derived number of the function $f(t,x)$ at the point x for a fixed t .

Theorem 5: If there exists a constant α > 0 such that for any fixed t and each derived number $\lambda_x[f](t,x)$ performed inequality

$$\lambda_x[f](t,x) \le -\alpha,$$

then all the solutions of equation (11) are uniformly asymptotically stable in general. If it is additionally known that equation (11) has an almost periodic solution, then all the solutions of equation (11) are asymptotically almost periodic.

Proof: Let $u(t)$ be an arbitrary solution of equation (11). Let a function y be introduced, setting that

$$y = x - u\cdot$$

It is clear that if x is a solution of equation (11), then y is a solution of equation (12). Let us obtain a derivative of equation (13) on solutions of equation (12).

Repeating the proof of Theorem 12 [21], it is easy to show that there exist derived numbers for which the following relation holds:

$$f(t, y+u) - f(t,u) \le y\lambda_{u+\theta y}[f](t, u+\theta y),$$

$$\theta \in (0,1).$$

Taking into account that by the condition of the theorem

$$\lambda_{u+\theta y}[f](t, u+\theta y) \le -\alpha,$$

the following estimation is obtained:

$$\left.\frac{dv}{dt}\right|_{(16)} \le -\alpha y^2.$$

It follows from this inequality that the solution $y = 0$ of equation (12) is uniformly asymptotically stable, as well as the solution $u(t)$ of equation (11). Since $u(t)$ is an arbitrary solution of equation (11), all the solutions of equation (11) are asymptotically stable.

If equation (11) has an almost periodic solution, then all the solutions of equation (11) are asymptotically almost periodic in the view of its uniform asymptotic.

Theorem 6: If the function $f(t,x)$ from the right-hand side of equation (11) decreases with respect to x at each fixed t, and on each compact set

$$\{(y,u): |u| \le u_0, d_1 \le |y| \le d_2, d_1 > 0\}$$

as $t \to \infty$

$$sign(y)\int_0^t [f(\tau, y+u) - f(\tau, u)]d\tau \to -\infty$$

uniformly, then the solution $y = 0$ of equation (12) is uniformly asymptotically stable.

Proof: Let $u(t)$ be an arbitrary bounded solution of equation (11).
Suppose that

$$y = x - u .$$

It follows from Theorem 12 that the solution $y = 0$ of equation (12) is uniformly stable. Let us prove that all the solutions of equation (12) tend to zero as $t \to \infty$.

Suppose the contrary. Then for some solution $y(t;0,y_0)$ of equation (12), there exists $d > 0$, such that

$$y(t;0,y_0) > d.$$

Here it is assumed that $y_0 > 0$, for definiteness. In the proof of Theorem 12, it is shown that the inequality

$$y\dot{y} \le 0,$$

which implies that —y— does not increase on the solutions of the equation (12).
Therefore, in the considered case for $t \ge 0$,

$$d \le y(t) \le y_0.$$

Suppose that

$$u_0 = \sup_t |u(t)|.$$

It follows from equation (12) that

$$\frac{\dot{y}}{y} = \frac{g(t,y)}{y} \le \frac{g(t,y)}{y_0}.$$

Hence, by virtue of the conditions of the assertion, we obtain

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$$\lim_{t\to\infty} y(t) \le \lim_{t\to\infty} y_0 e^{1/y_0 \int_0^t g(\tau,y)d\tau} = 0,$$

which contradicts the introduced assumption. The case when $y_0 < 0$ is treated in a similar way. Thus, the solution $y = 0$ of equation (12) is uniformly asymptotically stable.

III. Conclusion

The proposed apparatus of partial and external derived numbers allows us to investigate the behavior of a function of several variables, without requiring its differentiability, but using only information about partial derived numbers. This reduces the limitations imposed on the degree of smoothness of the functions studied.

The use of the apparatus of external derived numbers also makes it possible to reduce the restrictions on the degree of smoothness of manifolds when studying the question of the integrability of the hyperplanes field.

Theorems of the derived numbers method to estimate the number of periodic solutions of first-order ordinary differential equations are formulated and proved.

Using the apparatus of derived numbers allows to weaken the constraints imposed on the right-hand sides of the differential equations analized in this paper, and thereby increase the generality degree of the results. The upper and lower bounds for the numbers of periodic and almost periodic solutions of ordinary first-order dofferential equations are carried out. Conditions for the existence of periodic and almost periodic solutions are established. Using the apparatus of derived numbers allowed us to expand the scope of the results obtained. The application of the method of derivative numbers in problems of estimating the number of almost periodic solutions of first-order differential equations is shown. Conditions are found for determining upper and lower bounds for almost periodic solutions of ordinary differential equations of the first order. The questions of existence and stability of these solutions are investigated.

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