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Application of Laplace Transform for Solving Improper Integrals Containing Bessel's Function as Integrand, In the Form of Hypergeometric Function

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Abstract- In this article we use Laplace Transform for solving Improper integrals whose Integrand contains Bessel's function in the form of hypergeometric function. The theory of Bessel's function is a rich subject due to its significant role in providing solutions for differential equations associated with many applications Their applications span across disciplines like heat conduction, electromagnetism, signal processing, and more.

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Application of Laplace Transform for Solving Improper Integrals Containing Bessel's Function as Integrand, In the Form of Hypergeometric Function

Ranjana Shrivastava [α], Usha Gill [σ] & Kaleem Quaraishi [ρ]

Abstract- In this article we use Laplace Transform for solving Improper integrals whose Integrand contains Bessel's function in the form of hypergeometric function. The theory of Bessel's function is a rich subject due to its significant role in providing solutions for differential equations associated with many applications Their applications span across disciplines like heat conduction, electromagnetism, signal processing, and more.

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I. Introduction

Special functions are applicable in solving variety of problems of mathematical physics, statistics mechanics, dynamics, functional analysis etc. In recent few decades, many researchers are working on this field, especially with Bessel's function .Many researchers applied different Integral transforms include the Fourier Transform, Laplace Transform and the Mellin Transform for solving many problems of science and engineering The solution of many Engineering problems like acoustics, electromagnetism, quantum mechanics and other areas frequently lead to integrals involving Bessel's functions. When we solve these types of problems by using Integral Transform it is necessary to know the Integral Transform of Bessel's Function.

Mathematically, Bessel's function is defined by

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}$$

And is known as Bessel's function of the first kind of order 'n'

The Laplace transform of a real-valued function of real-valued function $f(t) of\ t$ when $t > 0$ is denoted by $f(s)\ or\ L\big(f(t)\big)$ where s is a real or complex parameter and it is denoted by improper integration given by $f(s) = \int_0^{\infty} e^{-st} f(t) dt$

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The aim of present article is to determine the value of improper integral containing Bessel's function as Integrand by using Laplace transform.

a) Some properties of Laplace Transform

Linearity Property:

The linearity property of the Laplace transform is a fundamental property that states the transform of a linear combination of functions is equal to the linear combination of the individual transforms. Mathematically, this property can be expressed as follows:

$$L(a(f(t)) + L(b(f(t)) = aL(f(t)) + bL(f(t))$$

Change of scale Property: If Laplace transform of function $f(t)\ is\ F(s)$ then, Laplace transform of $e^{at} f(at)$ is given by $\frac{1}{a} f\left(\frac{s}{a}\right)$

Shifting Property: If Laplace transform of a function$f(t)\ is\ (f(s)$then, Laplace transform of $e^{at} f(t)$ is given by $f(s-a)$.

Laplace transform of the derivative of the function: The Laplace transform of the derivative of a function $f(t)$ is given by $sf(s) - f(0)$ where $f(s)$ is the Laplace Transform of $f(t)$.

Laplace transform of Integral of the function: If Laplace transform of function $f(t)\ is\ f(s)$ then $L\int_0^t f(t)dt = \frac{1}{s} f(s)$

Laplace transform of Function, (Multiplication by 't' theorem): If Laplace transform of function $f(t)\ is\ f(s)$then,$L(t(f(t)) = (-1)\frac{d}{ds} f(s)$

Laplace transform of Bessel's Function:

The Bessel's Function is defined as

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}$$

$$LJ_n(t) = L\left(\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}\right)$$

$$\int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n}\ k!\,\Gamma(n+k+1)} \int_0^{\infty} e^{-st} t^{2k+n} dt$$

$$\frac{1}{s^{n+1}} \cdot \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k}\ k!\,\Gamma(n+k+1)} \frac{\Gamma(2k+n+1)}{\Gamma(n+1)\,.\,s^{2k}}$$

Notes

$$\frac{1}{s^{n+1}} \cdot \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k (1+n)_{2k}}{2^{2k}\ k!\ (1+n)_k} \frac{1}{.\, s^{2k}}$$

$$\frac{1}{s^{n+1}} \cdot \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} \left(\frac{1+n}{2}\right)_k \left(1+\frac{n}{2}\right)_k}{2^{2k}\ k!\ (1+n)_k} \frac{1}{.\, s^{2k}}$$

$$\frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1 \left[\begin{matrix} \frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) \frac{1}{s^2} \\ & 1+n\,; & \end{matrix} \right]$$

Hypergeometric Function: Hypergeometric function denoted as $2F_1(\mathrm{a,b;c;z})$ is a special function that arises in various areas of mathematics including analysis and mathematical physics. It is defined by the series

$$2F_1(\mathrm{a,b;c;z}) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\ (c)_k} z^k$$

Where $(a)_k$ denotes the Pochhammer's symbol

$$(a)_k = a\,(a+1)(a+2) \ldots \ldots \ldots \ldots (a+n-1) \text{ and } (a)_0 = 1$$

The hypergeometric function is a generalization of several elementary function such as the binomial series and it satisfies various differential equations including the hypergeometric differential equation. It has applications in solving differential equation, evaluating definite integrals and expressing solutions to certain problems. The hypergeometric function has many special cases and identities making it a powerful tool in mathematical analysis. It is extensively studied and applied in different branches of mathematics and physics.

II. Applications

In this section, some applications are given in order to explain the advantage of Laplace transform of Bessel's function for evaluating the improper Integral containing Bessel's function as Integrand.

1. Evaluation of $I = \int_0^{\infty} e^{-t} J_n(t) dt$

we have, $LJ_n(t) = \int_0^{\infty} e^{-st} J_n(t) dt = \frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1 \left[\begin{matrix} \frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) \frac{1}{s^2} \\ & 1+n\,; & \end{matrix} \right]$

here $s \to 1$, therefore,

Notes

$$\int_0^{\infty} e^{-t} J_n(t)dt = \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) & 1 \\ & 1+n\,; & & \end{matrix}\right]$$

2. Evaluation of $I = \int_0^{\infty} te^{-2t} J_n(t)dt$

$$LJ_n(t) = \frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

$$L(tJ_n(t)) = \frac{1}{s^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

here s → 2,

$$\int_0^{\infty} te^{-2t} J_n(t)dt = \frac{1}{2^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{2^2} \\ & 1+n\,; & \end{matrix}\right]$$

3. Evaluation of $I = \int_0^{\infty} e^{-t} \left(\int_0^t J_n(u)\, du\right)dt$

$$LJ_n(t) = \frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

By property of Laplace Transform of integral,

$$L\int_0^t J_n(u)\, du = \frac{1}{s} L(J_n(u)) = \frac{1}{s^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

here s → 1,

$$L\int_0^t J_n(u)\, du = \frac{1}{s} L\left(J_n(u)\right) = . \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) & 1 \\ & 1+n\,; & & \end{matrix}\right]$$

Notes

Notes

4. Evaluation of $I = \int_0^\infty e^{-2t}\left[\frac{d}{dt}J_n(t)\right]dt$

$$L(J_n(t)) = \frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(n+1)}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{\Gamma(n+1)\,.\,s^{2k}}$$

Now, by using the property of Laplace transform of derivative of function, we have

$$\mathrm{L}\left[\frac{d}{dt}J_n(t)\right] = s.\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.s^{2k}}$$

$$=.\frac{1}{s^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.s^{2k}} \qquad (1)$$

Then by the differentiation of Laplace transform we have

$$\mathrm{L}\left[\frac{d}{dt}J_n(t)\right] = \int_0^\infty e^{-st}\left[\frac{d}{dt}J_n(t)\right]dt \qquad (2)$$

$$\int_0^\infty e^{-st}\left[\frac{d}{dt}J_n(t)\right]dt =$$

$$.\frac{1}{s^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.\,s^{2k}}$$

Taking s → 2,

$$\int_0^\infty e^{-2t}\left[\frac{d}{dt}J_n(t)\right]dt =$$

$$.\frac{1}{2^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.\,2^{2k}}$$

$$= \frac{1}{2^{2n+2k}}\,2F_1\begin{bmatrix}\frac{1+n}{2} & , \left(1+\frac{n}{2}\right) & 1\\ & 1+n\,; & \end{bmatrix}$$

III. Conclusion

In this article, we have successfully discussed the application of Laplace transform for solving improper integrals whose Integrand contains Bessel's Function. The given numerical applications show the advantage of Laplace transform for evaluating improper integral whose Integrand contains Bessel's function.

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