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## of Science Frontier Research: F

# Mathematics and Decision Science

Highlights

Separability and a Lax Pair

Generalized Modal Syllogisms

Interactive Sliding Controls

Application of Laplace Transform

Discovering Thoughts, Inventing Future

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# Separability and a Lax Representation for the $C_2^{(1)}$ Toda Lattice

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*Abstract-* We consider the Toda lattice associated to the twisted affine Lie algebra $C_2^{(1)}$. It is well known that this system is a two-dimensional algebraic completely integrable system. By using algebraic geometric methods, we give a linearisation of the system by determining the linearizing variables. This allows us to explain a morphism between this system and the Mumford system. Finally, a Lax representation in terms of 2 x 2 matrices is constructed for this system.

*Keywords: toda lattice, integrable system, linearisation, lax representation.*

*GJSFR-F Classification: MSC 2010: 34G20, 34M55, 37J35*

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*Strictly as per the compliance and regulations of:*

[8] Chaplygin, C. A., *New case of rotation of rigid body with a fixed point.* Gostekhizdat, Moscow, 1954, pp. 472-476.

# Separability and a Lax Representation for the $C_2^{(1)}$ Toda Lattice

Djagwa Dehainsala [α], J. Moussounda Mouanda [σ] & G. F. Wankap Nono [ρ]

*Abstract-* We consider the Toda lattice associated to the twisted affine Lie algebra $\boldsymbol{C_2^{(1)}}$. It is well known that this system is a two-dimensional algebraic completely integrable system. By using algebraic geometric methods, we give a linearisation of the system by determining the linearizing variables. This allows us to explain a morphism between this system and the Mumford system. Finally, a Lax representation in terms of 2 x 2 matrices is constructed for this system.

*Keywords:* *toda lattice, integrable system, linearisation, lax representation.*

## I. Introduction

Completely integrable systems have been largely investigated during the past years. Some of them possess much richer structures that are the subject of extensive research and are called algebraic completely integrable system. This concept was introduced by Adler and van Moerbeke in [6]. An integrable polynomial system is algebraic completely integrable (a.c.i.) if the complexified system linearizes on an appropriate Abelian variety.

Many algebraic completely integrable systems possess matrix Lax representations whose spectral curves admit symmetries; in particular, involutions. The Jacobians of these curves contain Abelian subvarieties whose subsets are identified with the complex invariant manifolds of the system. The list of such systems includes the well known integrable cases of the Henon-Heiles systems [20], the integrable cases of quartic potentials [21], the Chaplygin top [8] and [14, 17], etc. A Lax representation for these systems can be constructed in terms of a direct product of Lax operators [20]. The literature on Lax equations is immense. The original references are [18, 15, 16, 4, 5]. Indeed, in order to simplify quantum problems it would be more convenient to use Lax representations in terms of $2 \times 2$ matrices.

In this paper, we consider the $\mathfrak{c}_2^{(1)}$ Toda lattice. It is a algebraic completely integrable (a.c.i.) system in the sense of Adler-van Moerbeke [2, 3] which means it can be linearized on a complex algebraic torus $\mathbb{C}^r/\wedge$ ($\wedge$ a lattice in $\mathbb{C}^r$) i.e. an Abelian variety $\mathbb{T}^r$. The aim is to find the separating variables and to show how to construct for this system a Lax representation. To this end, we use the algebraic structure of the problem. The separating variables give us a simple way of constructing a Lax equation. These separating variables can be found by inspecting the Painleve expansions of the solutions near some special divisor on the compactified invariant manifolds of the problem.

In two dimensional, $\mathbb{T}^2$ is a Abelian surface. If $\mathbb{T}^2$ is Jacobian surface i.e. contains a smooth curve of genus two, then there exists a general procedure, due to Pol Vanhaecke [23] for finding the separating variables. It turns out, however, that the Abelian surface in the case of the $\mathfrak{c}_2^{(1)}$ Toda

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lattice is not a Jacobian surface [12]. Furthemore, if one of the componentd is a 2:1 unramified cover of a smooth curve of genus, the procedure still applies. We shall see this is the case for our two-dimensional a.c.i. system $\mathfrak{c}_2^{(1)}$ Toda lattice.

## II. Linearisation of Two-Dimensional Algebraic Completely Integrable Systems

In this section, we recall some basic tools which will allow us to study algebraic completely integrable (a.c.i.) systems [6]. Consider the Hamiltonian system

$$\dot{x} = J\frac{\partial H}{\partial x} \equiv f(x), \quad x \in \mathbb{R}^m, \tag{2.1}$$

where $H$ is the Hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in $x$, for which the corresponding Poisson bracket $\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial x}, J\frac{\partial H_j}{\partial x} \right\rangle$ satisfies the Jacobi identity. The system (2.1) is integrable if it possesses $n+k$ independent polynomial invariants $H_1, \ldots, H_{n+k}$ of which $k$ invariants are Casimirs, the $n$ remaining ones are in involution and $m = 2n + k$. The intersection

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m \mid H_i(x) = c_i\}$$

is invariant by Poisson-commutativity for the flows of all $\mathcal{X}_H$, and is smooth for generic values of $c = (c_1, \ldots, c_m)$. By the well-known Arnold-Liouville theorem, the compact connected components of these invariant manifolds are diffeomorphic to real tori. Moreover, the flows of vector field $\mathcal{X}_H$ are linear, when they are seen as flows on the tori using the diffeomorphism. The integer $n$ is called the dimension of the system.

The Poisson structure and the vector field are easily complexified, giving a Poisson-commuting family of functions on $\mathbb{C}^m$ and for generic $c = (c_1, \ldots, c_n)$ in $\mathbb{C}^{n+k}$, the invariant manifolds

$$\mathcal{A}_c = \bigcap_{i=1}^{n+k} \{x \in \mathbb{C}^m \mid H_i(x) = c_i\}$$

are smooth affine (algebraic) varieties. In this case, the integrable system will be called *algebraic completely integrable* if these generic invariant manifolds $\mathcal{A}_c$ are smooth affine parts of an Abelian variety $\mathbb{T}_c^n$ and the flows of integrable vector fields are linear. This means that $\mathcal{A}_c = \mathbb{T}_c^n \setminus \mathcal{D}_c$, where $\mathcal{D}_c$ is the minimal divisor with the coordinate functions $x(t)$, restricted to the invariant manifolds, blow up for some value of $t \in \mathbb{C}$ and if the (complex) flow of the vector fields on $\mathbb{T}_c$ is linear [23].

In the two-dimensional case, that is $n = 2$, the invariant manifolds complete into Abelian surfaces by adding one (or several) curves to the affine surfaces $\mathcal{A}_c$. In this case, Vanhaecke proposed in [23] a method which leads to an explicit linearization of the vector field of the a.c.i. system. The computation of the first few terms of the Laurent solutions to the differential equations enables us to construct an embedding of the invariant manifolds in the projective space $\mathbb{P}^N$. From this embedding, one deduces the structure of the divisors $\mathcal{D}_c$ to be adjoined to the generic affine $\mathcal{A}_c$ in order to complete them into Abelian surfaces $\mathbb{T}_c$. Thus, the system is a.c.i.. The different steps of the algorithm of Vanhaecke are given by:

Ref

[12] Dehainsala, D., *Algebraic completely integrability of the $\mathfrak{c}_2^{(1)}$ Toda lattice*, Journal of Geometry and Physics, 135 (2019), pp. 80-97.

1. (a) If one of the components of $\mathcal{D}_c$ is a smooth curve $\Gamma_c$ of genus two, compute the image of the rational map $\phi_{[2\Gamma_c]} : \mathbb{T}_c^2 \to \mathbb{P}^3$ which is a singular surface in $\mathbb{P}^3$, the Kummer surface $\mathcal{K}_c$ of the the jacobian $\mathrm{Jac}(\Gamma_c)$ of the curve $\Gamma_c$.

   (b) Otherwise, if one of the components of $\mathcal{D}_c$ is a $d : 1$ unramified cover $\mathcal{C}_c$ of a smooth curve $\Gamma_c$ of genus two, the map $p : \mathcal{C}_c \to \Gamma_c$ extends to the map $\widetilde{p} : \mathbb{T}_c^2 \to \mathrm{Jac}(\Gamma_c)$. In this case, let $\mathcal{C}_c$ denote the (non complete) linear system $\widetilde{p}^* [2\Gamma_c] \subset [2\mathcal{C}_c]$ which corresponds to the complete linear system $[2\mathcal{C}_c]$ and compute now the Kummer surface $\mathcal{C}_c$ of $\mathrm{Jac}(\Gamma_c)$ as image of $\phi_{\mathcal{E}_c} : \mathbb{T}_c^2 \to \mathbb{P}^3$.

   (c) Otherwise, change the divisor at infinity so as to arrive in cas $(a)$ or $(b)$. This can always be done for any irreducible Abelian surface.

2. Choose a Weierstrass point $W$ on the curve $\Gamma_c$ and coordinates $(z_0 : z_1 : z_2 : z_3)$ for $\mathbb{P}^3$ such $\phi_{[2\Gamma_c]}(W) = (0 : 0 : 0 : 1)$ in case 1.$(a)$ and $\phi_{\mathcal{E}_c}(W) = (0 : 0 : 0 : 1)$ in case 1.$(b)$. Then this point will be a singular point (node) for the Kummer surface $\mathcal{K}_c$ whose equation is

$$p_2(z_o, z_1, z_2)z_3^2 + p_3(z_o, z_1, z_2)z_3 + p_4(z_o, z_1, z_2) = 0,$$

   where the $p_i$ are polynomials of degree $i$. After a projective transformation which fixes $(0 : 0 : 0 : 1)$, we may assume that $p_2(z_o, z_1, z_2) = z_1^2 - 4z_0z_2$.

3. Finally, let $x_1$ and $x_2$ be the roots of the quadractic equation $z_0x^2 + z_1x + z_2 = 0$, whose discriminant is $p_2(z_o, z_1, z_2)$, with the $z_i$ expressed in terms of the original variables. Then the differential equations describing the vector field of the system are rewritten by direct computation in the classical Weierstrass form

$$\begin{aligned} \frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} &= \alpha_1 dt, \\ \frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} &= \alpha_2 dt, \end{aligned} \tag{2.2}$$

   where $\alpha_1$ and $\alpha_2$ depend on $c$ (i.e., on the torus). From it, the symmetric functions $x_1 + x_2 (= -z_1/z_0)$, $x_1x_2 (= z_2/z_0)$ and the original variables can be written in terms of the Riemann theta function associated to the curve $y^2 = f(x)$.

## III. The $\mathfrak{c}_2^{(1)}$ -Toda System: Algebraic Completely Integrability

In this section, we recall some results relating the two-dimensional $\mathfrak{c}_2^{(1)}$ toda system. It is well known that this system is a.c.i. (see [12]).

The Toda lattice associated to the twisted affine Lie algebra $\mathfrak{c}_2^{(1)}$ consists of three particles interconnected by means of exponential springs and constrained to move on a circle. The motion is determined by the following equations

$$\begin{aligned} \dot{x}_0 &= x_0x_3, & \dot{x}_3 &= 2x_0 - 2x_1, \\ \dot{x}_1 &= x_1x_4, & \dot{x}_4 &= -x_0 + 2x_1 - x_2, \\ \dot{x}_2 &= x_2x_5, & \dot{x}_5 &= 2x_2 - 2x_1. \end{aligned} \tag{3.1}$$

on the hyperplane $\mathcal{H} = \left\{(x_0, x_1, \ldots, x_5) \in \mathbb{C}^6 \mid x_3 + 2x_4 + x_5 = 0\right\}$. We denote by $\mathcal{V}$ the vector field defined by the above differentials equations (3.1).

[12] Dehainsala, D., *Algebraic completely integrability of the $\mathfrak{c}_2^{(1)}$ Toda lattice*, Journal of Geometry and Physics, 135 (2019), pp. 80-97.

There are three independent constants of motion, namely

$$\begin{aligned} F_1 &= x_0 x_1^2 x_2, \\ F_2 &= x_3^2 + x_5^2 - 4x_0 - 8x_1 - 4x_2, \\ F_3 &= (x_3^2 - 4x_0)(x_5^2 - 4x_2) - 8x_1(x_3x_5 - 2x_1). \end{aligned} \tag{3.2}$$

The field $\mathcal{V}$ is the Hamiltonian vector field with the function $F_2$, with respect to the Poisson structure defined by the following skew-symmetric matrix

$$J := \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 2\,x_0 & -x_0 & 0 \\ 0 & 0 & 0 & -x_1 & x_1 & -x_1 \\ 0 & 0 & 0 & 0 & -x_2 & 2\,x_2 \\ -2\,x_0 & x_1 & 0 & 0 & 0 & 0 \\ x_0 & -x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & -2\,x_2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

If we assign $x_0, x_1$ and $x_2$ weight 2 and $x_3, x_4$ and $x_5$ weight 1, then the invariants are all homogeneous with weights $8, 2$ and 4 respectively. If we give time weight $-1$, the vector field $\mathcal{V}$ also becomes weight homogeneous. It is shown in [6] that, for such vector field, it is easy to find the weight homogeneous Laurent solutions to the differential equations. On $\mathbb{C}^6$ there are two involutions $\sigma$ and $\tau$ which preserve the constants of motion $F_1, F_2$ and $F_3$. These involutions, restrict to the hyperplane $\mathcal{H}$, are given by

$$\begin{aligned} \sigma(x_0, x_1, x_2, x_3, x_4, x_5) &= (x_2, x_1, x_0, x_5, x_4, x_3), \\ \tau(x_0, x_1, x_2, x_3, x_4, x_5) &= (x_0, x_1, x_2, -x_3, -x_4, -x_5). \end{aligned} \tag{3.4}$$

The involution $\sigma$ preserves the vector field $\mathcal{V}$ (3.1) while the involution $\tau$ changes its sign. Both involutions will have strong implications on the geometry of the integrable system [12]. We have shown [12] that the set of regular values of the momentum map $\mathbb{F}$ is the Zariski open in $\mathbb{C}^3$ given by

$$\Omega = \{(c_1, c_2, c_3) \in \mathbb{C}^3 \mid c_1 \neq 0,\ c_3^2 - 1024c_1 \neq 0 \text{ and } (c_2^2 - 4c_3)^2 - 16384c_1 \neq 0\}. \tag{3.5}$$

Throughout the rest, a generic point $c = (c_1, c_2, c_3)$ in $\mathbb{C}^3$ will be an element of the set $\Omega$.

The involution $\sigma$ simplifies the Painlevé analysis to the system. We show that the system of differential equations (3.1) possesses three families of Laurent solutions depending on the maximal number free parameters (4 in this case). Such families are called *principal balances*. The first principal balance $x(t, m_0)$ is given by

$$\begin{aligned} x_0(t; m_0) &= \frac{1}{t^2} + d + et + O(t^2), \\ x_1(t; m_0) &= -2et + O(t^2), \\ x_2(t; m_0) &= c + act + O(t^2), \\ x_3(t; m_0) &= -\frac{2}{t} + 2dt + 3et^2 + O(t^3), \\ x_4(t; m_0) &= \frac{1}{t} - \frac{a}{2} - (c+d)t - \frac{1}{2}(ac - 5e)t^2 + O(t^3), \\ x_5(t; m_0) &= a + 2ct + (2e + ac)t^2 + O(t^3), \end{aligned} \tag{3.6}$$

Ref

[6] Adler, M. and van Moerbeke, P. and Vanhaecke, P., *Algebraic integrability, Painlevé geometry and Lie algebras*. Ergeb. Math. Grenzgeb. (3), vol. 47, Berlin-Heidelberg: Springer, 2004.

where the four free parameters have been denoted by $a, c, d$ and $e$. The second principal balance $x(t; m_1)$ is given by

$$
\begin{aligned}
x_0(t;m_1) &= \beta t^2 + O(t^3),\\
x_1(t;m_1) &= \frac{1}{t^2} + \gamma + \frac{1}{10}(6\gamma^2 - \beta - \delta)t^2 + O(t^3),\\
x_2(t;m_1) &= \delta t^2 + O(t^3)\\
x_3(t;m_1) &= \frac{2}{t} + \alpha - 2\gamma t - \frac{1}{15}(6\gamma^2 - 11\beta - \delta)t^3 + O(t^4),\\
x_4(t;m_1) &= -\frac{2}{t} + 2\gamma t - \frac{2}{5}(\gamma^2 - \beta - \delta)t^3 + O(t^4),\\
x_5(t;m_1) &= \frac{2}{t} - \alpha - 2\gamma t - \frac{1}{15}(6\gamma^2 - \beta - 11\delta)t^3 + O(t^4).
\end{aligned}
\tag{3.7}
$$

where the four free parameters are denoted by $\alpha, \beta, \gamma$ and $\delta$. The last principal balance $x(t; m_2)$ is obtained from the above formulas for $x(t; m_0)$ by applying the involution $\sigma$. Using the majorant method [6], one shows that these series are convergent for small $|t| \neq 0$.

Substituting the Laurent solution (3.6) into (3.2): $F_1 = c_1$, $F_2 = c_2$ and $F_3 = c_3$, and equaling the $t^0$-terms yieds $c_1 = 4ce^2$, $c_2 = a^2 - 4c - 12d$, $c_3 = 48cd - 12a^2d - 32ae$. Eliminating $c$ and $d$ from these equations leads to an equation connecting the two remaining parameters $a$ and $e$. Namely,

$$
\Gamma_c^0 : a^4e^4 - (2c_1 + c_2e^2)a^2e^2 + 32ae^5 + c_3e^4 + c_1c_2e^2 + c_1^2 = 0. \tag{3.8}
$$

It is shown in [9, 12] that this curve can be compactified into a Riemann surface, denoted by $\overline{\Gamma}_c^0$, by just adding six points at infinity and the genus of $\overline{\Gamma}_c^0$ is two. Upon computing the abstract Painlevé divisor $\Gamma_c^2$, which corresponds to the Laurent solution $x(t, m_2)$, we obtain the same equation (3.8), since the involution $\sigma$ preserves the constants of motion, so that the Riemann surface $\overline{\Gamma}_c^2$ is isomorphic to $\overline{\Gamma}_c^0$.

At last, a direct substitution of the Laurent solution $x(t; m_1)$ (3.7) in the three equations $F_i = c_i$, $i = 1, 2, 3$; leads to the algebraic equations in terms of the four parameters $\alpha, \beta, \gamma$ and $\delta$, to wit

$$
c_1 = \beta\delta,\ c_2 = 2\alpha^2 - 24\gamma,\ c_3 = \alpha^4 + 24\alpha^2\gamma + 144\gamma^2 - 16\beta - 16\delta.
$$

Since $c_1 \neq 0$, by eliminating the parameters $\gamma$ and $\delta$ in these equations, we find a curve $\Gamma_c^1$ whose an equation is given, in the two remaining parameters $\alpha$ and $\beta$, by

$$
64\beta^2 + (4c_3 - (4\alpha^2 - c_2)^2)\beta + 64c_1 = 0.
$$

For $c$ generic, the affine curve $\Gamma_c^1$ is smooth and can be compactified into a Riemann surface, denoted $\overline{\Gamma}_c^1$, by adding two points at infinity $\infty'$ and $\infty''$. A local parametrization of neighborhood of these points is given by

$$
\infty' : \qquad \alpha = \frac{1}{\varsigma}, \qquad \beta = \frac{1}{8\varsigma^4}\left(2 - c_2\varsigma^2 + \frac{1}{8}(c_2^2 - 4c_3)\varsigma^4 + O(\varsigma^6)\right), \tag{3.9}
$$

$$
\infty'' : \qquad \alpha = \frac{1}{\varsigma}, \qquad \beta = 4c_1\varsigma^4 + 2c_1c_2\varsigma^6 + O(\varsigma^6). \tag{3.10}
$$

Ref

[6] Adler, M. and van Moerbeke, P. and Vanhaecke, P., *Algebraic integrability, Painlevé geometry and Lie algebras.* Ergeb. Math. Grenzgeb. (3), vol. 47, Berlin-Heidelberg: Springer, 2004.

The genus of the Riemann surface $\overline{\Gamma}_c^1$ is three. Indeed, by making the change of the variable $\xi = 128\beta + (4c_3 - (4\alpha^2 - c_2)^2)$, we can see that the curve $\overline{\Gamma}_c^1$ is isomorphic to the smooth genus three hyperelliptic Riemann surface $\mathcal{C}_c^1 : \xi^2 = h(\alpha) = ((4\alpha^2 - c_2)^2 - 4c_3)^2 - 16384c_1$.

The affine invariant surface

$$\mathbb{F}_c := \mathbb{F}^{-1}(c) = \bigcap_{i=1}^{3} \{x \in \mathcal{H} : F_i(x) = c_i\}$$

defined by the three constants of motion can be embedded in the projective space $\mathbb{P}^{17}$ means of eighteen functions

$$\begin{aligned}
&z_0 = 1, && z_1 = x_3, && z_2 = x_3 + x_5,\\
&z_3 = 4x_1 - x_3x_5, && z_4 = 4(x_0 - x_2) + x_5^2 - x_3^2, && z_5 = x_3z_3 + 4x_0x_5,\\
&z_6 = x_5z_3 + 4x_2x_3, && z_7 = x_1x_0, && z_8 = x_1x_2,\\
&z_9 = x_3x_5z_4 + 4x_1(x_3^2 - x_5^2), && z_{10} = x_1x_2x_3, && z_{11} = x_1x_0x_5,\\
&z_{12} = x_0x_1x_2, && z_{13} = x_1x_0(x_5^2 - 4x_1), && z_{14} = x_1x_2(x_3^2 - 4x_1),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
z_{15} &= x_1x_0(4x_1(x_3 - 2x_5) + x_5(x_5^2 - 4x_2)),\\
z_{16} &= x_1x_2(4x_1(x_5 - 2x_3) + x_3(x_3^2 - 4x_0)),\\
z_{17} &= x_0x_1x_2(x_3^2 + x_5^2 - 4(x_0 + x_2)),
\end{aligned}$$

which behave like $t^{-1}$ at worst when the three principal balances are substituted into them. Using the embedding

$$\begin{array}{cccc} \varphi_c : & \mathbb{F}_c & \to & \mathbb{P}^{17} \\ & (x_0, x_1, x_2, x_3, x_4, x_5) & \mapsto & (1 : z_1 : \cdots : z_{17}), \end{array} \tag{3.12}$$

and the three principal balances, it is possible to show that the closure of the image of this affine surface $\mathbb{F}_c$ is an Abelian surface $\mathbb{T}_c^2$ for generic values $c = (c_1, c_2, c_3) \in \Omega$ of constants of motion. Indeed, the map $\varphi_c$ induces three injective maps $\varphi_c^i : \Gamma_c^i \to \mathbb{P}^{17}$ $(i = 0, 1, 2)$ that define the divisor $\mathcal{D}_c$ to be added to $\mathbb{F}_c$ for its completion into the Abelian suface $\mathbb{T}_c^2$. The closure of the image of each of these maps, $\overline{\varphi_c^i(\Gamma_c^i)}$ will be denoted by $\mathcal{D}_c^i$. We have the following result:

*Theorem 3.1. [12]*

1. For generic points of $\mathbb{C}^3$, the invariant surface $\mathbb{F}_c$ is the affine part of an Abelian surface $\mathbb{T}_c^2$. The divisor at infinity $\mathcal{D}_c$ on $\mathbb{T}_c^2$ consists of three irreducible components $\mathcal{D}_c^0$, $\mathcal{D}_c^1$ and $\mathcal{D}_c^2$ where

   (a) $\mathcal{D}_c^0$ and $\mathcal{D}_c^2$ are both singular curves isomorphic to $\overline{\Gamma}_c^0$ defined by

   $$e^4a^4 - (2c_1 + c_2e^2)e^2a^2 + 32e^5a + c_3e^4 + c_1c_2e^2 + c_1^2 = 0, \tag{3.13}$$

   (b) $\mathcal{D}_c^1$ is isomorphic to the smooth hyperelliptic curve of genus three $\overline{\Gamma}_c^1$ defined by

   $$64\beta^2 + (4c_3 - (4\alpha^2 - c_2)^2)\beta + 64c_1 = 0. \tag{3.14}$$

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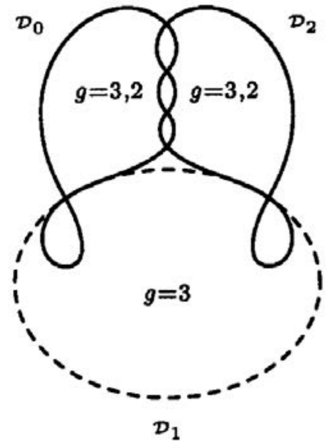
2. The system of differential equations (3.1) is algebraically completely integrable and the flows of integrable vector fields are linear on the Abelian surfaces $\mathbb{T}_c^2$.




*Remark 3.2.* The divisor that completes the invariant surface $\mathbb{F}_c$ into Abelian surface is made up by three curves $\mathcal{D}_i := \mathcal{D}_c^i$. $\mathcal{D}_1$ intersects the over curves at one point each. The letter intersect each other at four points.

## IV. Separation of the Variables

This section is entirely devoted to the linearization of the system. As we have seen in the previous section, a two-dimensional algebraic completely integrable system is linearizable if one of the components of the divisor $\mathcal{D}_c$ (to be adjoined to $\mathbb{F}_c$ in order to complete $\mathbb{F}_c$ into an Abelian surface) is a smooth curve of genus two; which is not the case for our system. Indeed, $\mathbb{T}_c^2$ is not a Jacobian surface because one of the components of $\mathcal{D}_c$ is not a smooth curve of genus two. We show in [12] that $\mathbb{T}_c^2$ is Prym variety of polarization of type $(1, 2)$. In this situation, according to Vanhaecke, the system is linearizable if one of the components of $\mathcal{D}_c$ is a $d : 1$ unramified $\mathcal{C}_c$ of a smooth curve of genus two. In order to check this condition, we consider the curve $\overline{\Gamma}_c^1$ defined by

$$\overline{\Gamma}_c^1 : 64\beta^2 + (4c_3 - (4\alpha^2 - c_2)^2)\beta + 64c_1 = 0, \tag{4.1}$$

which is a component of the divisor $\mathcal{D}_c$. For $c \in \Omega$, the curve $\overline{\Gamma}_c^1$ is a smooth hyperelliptic curve of genus three. The involution $\sigma : (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_1, x_0, x_5, x_4, x_3)$ acts on the parameters $\alpha, \beta, \gamma$ and $\delta$ in the following way:

$$\sigma(\alpha, \beta, \gamma, \delta) = (-\alpha, \delta, \gamma, \beta).$$

For generic $c$, $\beta\delta = c_1 \neq 0$. It follows that the map

$$\sigma : (\alpha, \beta) \mapsto \left(-\alpha, \frac{c_1}{\beta}\right) \tag{4.2}$$

is an involution for the curve $\overline{\Gamma}_c^1$. Indeed, let $\sigma$ be the involution which acts on the curve $\overline{\Gamma}_c^1$, the equation (4.1) becomes

$$64\left(\frac{c_1}{\beta}\right)^2 + (4c_3 - (4\alpha^2 - c_2)^2)\frac{c_1}{\beta} + 64c_1 = 0.$$

Ref

[12] Dehainsala, D., *Algebraic completely integrability of the* $\mathcal{C}_2^{(1)}$ *Toda lattice*, Journal of Geometry and Physics, 135 (2019), pp. 80-97.

Simplifying by $\frac{c_1}{\beta}$, this leads to

$$64\left(\frac{c_1}{\beta}\right)+(4c_3-(4\alpha^2-c_2)^2)+64\beta=0,$$

which can be written as

$$\frac{1}{\beta}\left(64\beta^2+(4c_3-(4\alpha^2-c_2)^2)\beta+64c_1\right)=0.$$

Since $\beta\neq 0$, we find the same initial equation of the curve $\Gamma_c^1$. Namely,

$$64\beta^2+(4c_3-(4\alpha^2-c_2)^2)\beta+64c_1=0.$$

*Remark 4.1.* The invariants of the involution $\sigma$ are

$$Y=\alpha^2,\quad X=\beta+\frac{c_1}{\beta}\quad\text{and}\quad Z=\alpha\left(\beta-\frac{c_1}{\beta}\right). \tag{4.3}$$

Indeed, we have $\sigma(Z)=-\alpha\left(\frac{c_1}{\beta}-\beta\right)=\alpha\left(\beta-\frac{c_1}{\beta}\right)=Z$; cleary we have $\sigma(Y)=Y$ and $\sigma(X)=X$.

*Proposition 4.2.* Let $\mathcal{K}_c$ be the quotient of the curve $\overline{\Gamma}_c^1$ by the involution $\sigma$. For generic $c$, the quotient curve $\mathcal{K}_c$ is a smooth curve of genus two and the map $\overline{\Gamma}_c^1\to\mathcal{K}_c$ is an unramified $2:1$ map.

*Proof.* We determine the genus of the curve $\mathcal{K}_c:=\overline{\Gamma}_c^1/\sigma$. We observe that the equation of $\overline{\Gamma}_c^1$ can be written in the following form:

$$64\left(\beta+\frac{c_1}{\beta}\right)+(4c_3-(4\alpha^2-c_2)^2)=0,$$

such that

$$64X-(4Y-c_2)^2+4c_3=0.$$

We deduce that

$$X=\frac{1}{64}\left[(4Y-c_2)^2-4c_3\right]. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}Z^2&=\alpha^2\left(\beta-\frac{c_1}{\beta}\right)^2\\&=Y\left[\left(\beta+\frac{c_1}{\beta}\right)-\frac{2c_1}{\beta}\right]^2\\&=Y\left[\left(\beta+\frac{c_1}{\beta}\right)^2-4\frac{c_1}{\beta}\left(\beta+\frac{c_1}{\beta}\right)+\frac{4c_1^2}{\beta^2}\right]\end{aligned} \tag{4.5}$$

$$Z^2=Y\left(X^2-4c_1\right)$$



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Substituting (4.4) in (4.5), one obtains the equation of the curve $\mathcal{K}_c$. Namely

$$\mathcal{K}_c : Z^2 = Y\left(\frac{1}{4096}\left((4Y-c_2)^2-4c_3\right)^2-4c_1\right).$$

Thus, the curve $\mathcal{K}_c$ is isomorphic to the hyperelliptic of genus two whose the equation is

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$$z^2 = h(y) = y\left(\left((y-c_2)^2-4c_3\right)^2-16384c_1\right). \tag{4.6}$$

For $c \in \Omega$, this curve is smooth because the polynomial $h(y)$ is without multiple roots; indeed, its discriminant is equals , up to a constant, to

$$c_1^2(c_3^2-1024c_1)^2((c_2^2-4c_3)^2-16384c_1),$$

which does not vanish for $c \in \Omega$. Finally let us show that the involution $\sigma$ has no fixed point for $c \in \Omega$. A point $(x_0, x_1, x_2, x_3, x_4, x_5)$ de $\mathbb{F}_c$ is a fixed point for involution $\sigma$ if and only if $x_0 = x_2$ and $x_3 = x_5$. By substituting the coordinates $x_2$ and $x_5$ respectively by $x_0$ and $x_3$ in the functions $F_i = c_i$ (for $i = 1, 2, 3$), one obtains the system

$$\begin{cases} c_1 = x_0^2 x_1^2, \\ c_2 = 2x_3^2 - 8(x_0+x_1), \\ c_3 = (x_3^2-4x_0)^2 - 8x_1(x_3^2-2x_1). \end{cases}$$

By a direct computation, we find the relation $c_2^2 - 4c_3 = 128x_0x_1$. This leads to the equality

$$(c_2^2-4c_3)^2-16384c_1 = 0,$$

which is impossible for a generic point $c \in \mathbb{C}^3$. Thus the involution $\sigma$ has no fixed point in $\mathbb{F}_c$. Using (4.2), it is easy to verify that the points at infinity $\infty'$ and $\infty''$ also aren't fixed points for $\sigma$. We have

$$g\left(\overline{\Gamma}_c^1\right) = 2g\left(\mathcal{K}_c\right) - 1.$$

We can conclude that the map $\pi : \overline{\Gamma}_c^1 \longrightarrow \mathcal{K}_c$ is an unramified double cover. $\square$

*Theorem 4.3.* The vector field $\mathcal{V}$ (3.1) extends to a linear vector field on the Abelian surface $\mathbb{T}_c^2$ and the Jacobi form for the differentials equation can be written as

$$\frac{d\mu_1}{\sqrt{f(\mu_1)}} + \frac{d\mu_2}{\sqrt{f(\mu_2)}} = 0,$$

$$\frac{\mu_1 d\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 d\mu_2}{\sqrt{f(\mu_2)}} = \frac{1}{i\sqrt{2}}dt,$$

where $f(\mu) = \left(\mu^4 - 2c_3\mu^2 - 1024c_1 + c_3^2\right)\left(\mu - \frac{1}{2}c_2\right)$; and the curve $v^2 = f(\mu)$ is birational equivalent to the hyperelliptic curve of genus two $\mathcal{K}_c$ (4.6).

*Proof.* The demonstration is based on the Vanhaecke's procedure described above. We first construct, following [7], an explicit map from the generic fiber $\mathbb{F}_c$ into the Jacobian of the Riemann surface $\overline{\Gamma}_c^1$.

We consider the functions which have at worst a double pole along the component $\mathcal{D}_c^1$ of the divisor $\mathcal{D}_c$ on $\mathrm{Jac}(\overline{\Gamma}_c^1)$, and no others poles. These functions are obtained by constructing those polynomials on $\mathcal{H}$ which have at worst a double pole in $t$ when the principal balance $x(t,m_1)$ is substituted into them and no poles when the other principal balances are substituted.

From (3.7), we easily show that the space of such polynomials is spanned by

$$\begin{aligned}
s_0 &= 1\\
s_1 &= x_1\\
s_2 &= x_5^2 - x_3^2 - 4(x_2 - x_0)\\
s_3 &= x_1(x_3 - x_5)\\
s_4 &= x_1(4x_1 - x_3x_5)\\
s_5 &= x_1(x_3^3 - x_5^3 + x_0x_3 - 4x_2x_5 + 12x_1x_3 - 12x_1x_5)\\
s_6 &= x_1^2x_0\\
s_7 &= x_1^2(x_0 + x_2)
\end{aligned}$$

The leading terms are given by

$$(s_0, s_1, \ldots, s_7) = \left(1, \frac{1}{t^2}, -\frac{2\alpha}{t}, \frac{\alpha}{t^2}, \frac{\alpha^2 + 12\gamma}{t^2}, \frac{2\alpha(-\alpha^2 + 36\gamma)}{t^2}, \frac{\beta}{t^2}, \frac{\beta + \delta}{t^2}\right).$$

Among these functions, only the following are invariants by the involution $\sigma$:

$$\begin{aligned}
\theta_0 &:= 1, & \theta_2 &:= x_1(4x_1 - 4x_3x_5),\\
\theta_1 &:= x_1, & \theta_3 &:= x_1^2(x_0 + x_2).
\end{aligned} \tag{4.7}$$

The above functions allow us to embed the Kummer surface of $\mathrm{Jac}(\mathcal{K}_c)$ in the projective space $\mathbb{P}^3$. Consider now the Koidara map which correspond to these functions.

$$\begin{array}{cccc}
\varphi_c & : & \mathrm{Jac}(\mathcal{K}_c) \setminus \mathcal{D}_c^1 & \to & \mathbb{P}^3\\
& & p = (x_0, x_1, x_2, x_3, x_5) & \mapsto & (\theta_0(p) : \theta_1(p) : \theta_2(p) : \theta_3(p)).
\end{array} \tag{4.8}$$

Since the functions $\theta_i$ correspond to the sections of the line bundle $\left[2\mathcal{D}_c^1\right]$, the map $\varphi_c$ maps the $\mathrm{Jac}(\mathcal{K}_c)$ into its Kummer surface, which is a singular quartic in the projective space $\mathbb{P}^3$.

An equation for this quartic surface, in terms of $\theta_i$, can be computed by eliminating the variables $x_0, x_1, x_2, x_3, x_5$ from (4.7) and from the equations

$$\begin{aligned}
c_1 &= x_0x_1^2x_2,\\
c_2 &= x_3^2 + x_5^2 - 4x_0 - 8x_1 - 4x_2,\\
c_3 &= (x_3^2 - 4x_0)(x_5^2 - 4x_2) - 8x_1(x_3x_5 - 2x_1).
\end{aligned} \tag{4.9}$$

Ref
[7] Bueken, P. and Vanhaecke, P., *The moduli problem for integrable systems: the example of a geodesic flow on SO(4)*. J. London Math. Soc. (2), 62 (2), pp. 357-369, 2000.

Notes

The result is a quartic equation of the Kummer surface of Jac$(\mathcal{K}_c)$ which it can be put in the following form :

$$((8\theta_1)^2 - 4(8\theta_2 - c_3))\theta_3^2 + P_3(\theta_1, \theta_2)\theta_3 + \frac{1}{4}P_4(\theta_1, \theta_2) = 0, \tag{4.10}$$

where $P_3$ (respectively $P_4$) is a polynomial of degree three (respectively four) in $\theta_1$ and $\theta_2$, given by

$$P_3(\theta_1, \theta_2) = (8\theta_1 + c_2)(c_3\theta_1^2 - \theta_2^2 + 16c_1^2),$$
$$P_4(\theta_1, \theta_2) = c_3^2\theta_1^4 + 256c_1c_2\theta_1^3 + (16c_1(32\theta_2 + c_2^2 - 2c_3) - 2c_3\theta_2^2)\theta_1^2 + (\theta_2^2 - 16c_1)^2.$$

The coefficient of $\theta_3$ in (4.10) can be written, in terms of the initial variables $x_0, x_1, x_2, x_3$ and $x_5$, as follows:

$$\theta_3 = (8x_1)^2 - 4(8x_1(4x_1 - x_3x_5) - c_3).$$

Let $\mu_1$ and $\mu_2$ be roots of the polynomial

$$f(\mu) = \mu^2 + 8x_1\mu + 8x_1(4x_1 - x_3x_5) - c_3.$$

By [23, Theorem 9], the vector field $\mathcal{V}$ defining the Toda lattice linearizes upon the setting

$$\begin{aligned} \mu_1 + \mu_2 &= -8x_1, \\ \mu_1\mu_2 &= 8x_1(4x_1 - x_3x_5) - c_3, \end{aligned} \tag{4.11}$$

which implies, with respect to the vector field $\mathcal{V}$, that

$$\begin{aligned} \dot{\mu}_1 + \dot{\mu}_2 &= 4x_1(x_3 + x_5), \\ \dot{\mu}_1\mu_2 + \mu_1\dot{\mu}_2 &= -4x_1((x_3 + x_5)(x_3x_5 - 4x_1) + 4(x_0x_5 + x_2x_3)). \end{aligned} \tag{4.12}$$

Substituting (4.11) and (4.12) in the invariants (4.9) and eliminating the variables $x_0, x_1, x_2, x_3$ and $x_5$, two quadratic polynomials in $\dot{\mu}_1^2$ and $\dot{\mu}_2^2$ are found. Solving them for in $\dot{\mu}_1^2$ and $\dot{\mu}_2^2$ yields

$$\dot{\mu}_i^2 = \frac{\left(\mu_i^4 - 2c_3\mu_i^2 - 1024c_1 + c_3^2\right)(\mu_i - c_2/2)}{4(\mu_1 - \mu_2)^2}, \quad i = 1, 2.$$

It follows that

$$\begin{aligned} \frac{d\mu_1}{\sqrt{f(\mu_1)}} + \frac{d\mu_2}{\sqrt{f(\mu_2)}} &= 0, \\ \frac{\mu_1 d\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 d\mu_2}{\sqrt{f(\mu_2)}} &= \frac{1}{i\sqrt{2}}dt, \end{aligned} \tag{4.13}$$

where $f$ is the polynomial

$$f(\mu) = \left(\mu^4 - 2c_3\mu^2 - 1024c_1 + c_3^2\right)\left(\mu - \frac{1}{2}c_2\right).$$

Integrating (4.13) we see that the field $\mathcal{V}$ is a linear vector field on $\mathbb{F}_c$ which obviously extends to linear vector field on the Jacobian variety of the curve $\mathcal{K}_c$ and the factor of its generic complex invariant manifold $\mathbb{F}_c$ by $\sigma$ is an open subset of Jac$(\mathcal{K}_c)$.

By using [19, Theorem 5.3], we show that the symmetric functions $\mu_1$, $\mu_2$ and the original phase variables can be written in terms of theta functions.

In this connection, a natural question is how the curve $\mathcal{K}_c$ and the curve $v^2 = f(\mu)$ are related. The answer comes out immediately when we observe that the curves $v^2 = f(\mu)$ and $\mathcal{K}_c$ are birationally equivalent. Indeed, setting $u = \mu - c_2/2$, we obtain the following:

$$v^2 = u\left(\frac{1}{16}\left((2u+c_2)^2 - 4c_3\right)^2 - 1024c_1\right). \tag{4.14}$$

Next, putting $y = -2u$, and $z = 4i\sqrt{2}v$, we obtain the equation

$$z^2 = y\left(\left((y-c_2)^2 - 4c_3\right)^2 - 16384c_1\right)$$

of the curve $\mathcal{K}_c$ whose the jacobian is canonically associated to the Abelian surface $\mathbb{T}_c^2$. $\square$

## V. Lax Pairs

In this Section, we shall find a new Lax pair for the $c_2^{(1)}$ Toda lattice.

Let $u, v$ and $w$ be functions in $t$ whith the property $uw + v^2 = c$, $c$ is a constant. Then the following obvious identity holds

$$\frac{d}{dt}X = [X, Y], \qquad [X, Y] = XY - YX, \tag{5.1}$$

where

$$X = \begin{pmatrix} v & u \\ w & -v \end{pmatrix}, \qquad Y = \frac{1}{2v} = \begin{pmatrix} 0 & \frac{d}{dt}u \\ -\frac{d}{dt}w & 0 \end{pmatrix}.$$

Suppose that a completely integrable Hamiltonian system is given which linearizes on a Jacobian variety Jac$(\Gamma)$ of a hyperelliptic curve $\Gamma : y^2 = f(\lambda)$, where $f(\lambda)$ is a polynomial with coefficients depending upon the constant of motion. Then, as it has been first by Fairbanks [13] and also Pol Vanhaecke [23], we may take $c = f(\lambda)$, and define $u, v, w$ to be the Jacobi polynomials on Jac$(\Gamma)$. Thus we obtain a Lax pair (5.1) depending on a spectral parameter $\lambda$, and the coefficients of $f(\lambda)$ (and hence the first integrals) are reconstruted from the identies

$$det\,(X - yI) = y^2 - v^2 - uv = y^2 - f(\lambda) = constant.$$

Suppose first that the curve $\Gamma$ is a genus 2 curve. Let $p_1$ and $p_2$ be points on $\Gamma$ and denote $\mu_1 = \lambda(p_1)$, $\mu_2 = \lambda(p_2)$. Then the Jacobi polynomials associated with $\Gamma$ read

$$u(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2), \qquad v(\lambda) = \frac{\sqrt{f(\mu_1)}(\lambda - \mu_2) - \sqrt{f(\mu_2)}(\lambda - \mu_1)}{\mu_1 - \mu_2},$$

$$\omega(\lambda) = \frac{f(\lambda) - v(\lambda)^2}{u(\lambda)}.$$

Note that $w(\lambda)$ is in fact a polynomial in $\lambda$.

Ref

[23] Vanhaecke, P., *Linearising two-dimensional integrable systems and the construction of action-angle variables*. Math. Z., 1992, vol. 211, no. 2, pp. 265-313.

Let us find now a new Lax pair for the $c_2^{(1)}$ Toda lattice. According to Mumford's description of hyperelliptic Jacobians (see [19, Section 3.1]), if $\Gamma$ a hyperelliptic curve of genus two then the Riemann surface $\overline{\Gamma}$ is embedded in its jacobian in a such way that $\mathrm{Jac}(\overline{\Gamma}) \setminus \Gamma$ is isomorphic to the space of pairs of polynomials $(u(\lambda), v(\lambda))$ such that $u(\lambda)$ is a monic of degree two, $v(\lambda)$ is of degree less than two and $f(\lambda) - v^2(\lambda)$ is divisible by $u(\lambda)$. Let us describe the map from $\mathbb{F}_c$ into $\mathrm{Jac}(\overline{\Gamma}_c^1)$ in terms of these polynomoials. Let $\mu_1$ and $\mu_2$ be the roots of the polynomial $u(\lambda)$. From (4.11), we can conclude that

$$u(\lambda) = \lambda^2 + 8x_1\lambda + 16(x_1^2 - x_0x_2) + 4(x_0x_5^2 + x_2x_3^2) - x_3^2x_5^2.$$

The polynomial $v(\lambda)$ is defined as the derivative, suitable normalised, of $u(\lambda)$ in the direction of the vector field $\mathcal{V}$ (3.1), we find out that

$$v(\lambda) = 4i\sqrt{2}\left[-x_1(x_3 + x_5)\lambda + x_1((x_3 + x_5)(x_3x_5 - 4x_1) - 4x_1(x_0x_5 + x_2x_3)\right].$$

It is easy to check that the expression $f(\lambda) - v^2(\lambda)$ is divisible by $u(\lambda)$ such that the above formulas define a point of $\mathrm{Jac}(\overline{\Gamma}_c^1) \setminus \Gamma_c^1$. Let's put

$$w(\lambda) = \frac{f(\lambda) - v^2(\lambda)}{u(\lambda)}.$$

$w(\lambda)$ is a polynomial in $\lambda$ of degree $3 = \deg u + 1$. By direct calculation, we find

$$w(\lambda) = \lambda^3 + w_2\lambda^2 + w_1\lambda + w_0,$$

where

$$w_2 = -\frac{1}{2}(x_3^2 + x_5^2) + 2x_0 - 4x_1 + 2x_2,$$

$$w_1 = 4x_1(x_3^2 + x_5^2 - 4x_0 - 4x_1 - 4x_2 + 4x_3x_5) - (x_3^2 - 4x_0)(x_5^2 - 4x_2)$$

and

$$w_0 = \frac{1}{2}(x_3^2 + x_5^2 - 4x_0 - 8x_1 - 4x_2)(x_3^2 - 4x_0)(x_5^2 - 4x_2) + 8x_1\left(8x_1^2 + 4(x_0 + x_2)(x_1 + x_3x_5) + (x_3^2 + x_5^2)(3x_1 - x_3x_5)\right).$$

Based on above and [22, Chapter VII.2], the linearizing variables (4.11) and (4.12) suggest a morphism $\phi$ from the $\mathfrak{c}_2^{(1)}$ Toda lattice to genus 2 odd Mumford system:

$$\mathcal{M} = \left\{ \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix} \in M_2(\mathbb{C}[\lambda]) \mid \begin{array}{l} deg(u) = 2 = deg(w) - 1, \\ deg(v) < 2;\ u, w \text{ monic} \end{array} \right\} \cong \mathbb{C}^7.$$

It is well known that the Mumford system $\mathcal{M}$ is algebraic completely integrable. The morphism $\phi : \mathcal{H} \to \mathbb{C}^7$ is given by

$$(x_0, x_1, x_2, x_3, x_5) \mapsto \begin{cases} u(\lambda) = \lambda^2 + u_1\lambda + u_0, \\ v(\lambda) = v_1\lambda + v_0, \\ w(\lambda) = \lambda^3 + w_2\lambda^2 + w_1\lambda + w_0. \end{cases}$$

The form of the Lax pair then follows from [23]. We have:

Ref

[23] Vanhaecke, P., *Linearising two-dimensional integrable systems and the construction of action-angle variables.* Math. Z., 1992, vol. 211, no. 2, pp. 265-313.

*Theorem 5.1.* The Lax equation for the Hamiltonian vector field $\mathcal{V}$ is given by

$$\dot{X}(\lambda) = [X(\lambda), Y(\lambda)]$$

by taking

$$X(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix} \quad \text{and} \quad Y(\lambda) = \begin{pmatrix} 0 & 1 \\ b(\lambda) & 0 \end{pmatrix},$$

where $u(\lambda)$, $v(\lambda)$ and $w(\lambda)$ are the polynomials defined above. The coefficient $b(\lambda)$ of the matrix $Y(\lambda)$ is the polynomial part of the rational function $w(\lambda)/u(\lambda)$.

By direct computation, one finds

$$b(\lambda) = \lambda - \frac{1}{2}(x_3^2 + x_5^2) + 2x_0 - 12x_1 + 2x_2.$$

And we can show that the characteristic polynomial of the matrix $X(\lambda)$ is precisely the polynomial which defines the curve $\mathcal{K}_c$.

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Notes

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# Dynamics of Triangle Similarity: Exploring Similitude Ratios through Interactive Sliding Controls

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*Keywords: dynamic geometry environment, online interactions, teacher training, VMTwG, slider control.*

*GJSFR-F Classification: UDC Code: 51*

DYNAMICSOFTRIANGLESIMILARITYEXPLORINGSIMILITUDERATIOSTHROUGHINTERACTIVESLIDINGCONTROLS

*Strictly as per the compliance and regulations of:*

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# Dynamics of Triangle Similarity: Exploring Similitude Ratios through Interactive Sliding Controls

Marcelo Bairral [α] & Cristiano de Souza Brito [σ]

*Abstract-* Similarity is crucial in mathematics and other fields. It relates to different curricular mathematics content, for instance, proportionality, measures, and shapes. This article discusses the interactions of prospective teachers when performing similarity tasks in a multiuser, online, virtual, and synchronous environment with Geo Gebra, the VMTwG. The analyzed task focused on the similarity ratio from a slider. The focus here is to illustrate how the subjects interact when dragging free points of two triangles created with dependence on each other and using a checkered mesh. Data production came from registers written and Geo Gebra screen construction by participants and VMT replayer. The use of the slider may represent a more potent form of thought process, which implies both the global observation of geometric properties, the validation of conjectures and checking detailed results.

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## I. Introduction

This study was motivated by the need for new paths in the teaching and learning geometry, precisely triangle similarity, to understand what contributions and synchronous interactions in a Virtual Math Teams (VMT) environment can be observed in the learning of (prospective) teachers. VMT allows for the creation of workspaces using Desmos or GeoGebra, and the latter is called Virtual Math teams with GeoGebra (VMTwG)[1].

VMT platform adds the Dynamic Geometry Environment (DGE) GeoGebra. It makes possible to drag points, free or linked to a figure, allowing to produce hypotheses on the observed properties (Arzarello, Olivero, Paola & Robutti, 2002), as well as for the verification of relations among objects and the dynamics of relations (Alqahtani & Powell, 2016). As it is a multiuser environment, the VMT constitutes collaborating groups that explore, question, and learn together (Stahl, Koschmann & Suthers, 2008), involving the interaction between the individual and the device, and among the subjects. We consider not only the importance of an individual's learning, one's mind with itself, but also the possibility of collective learning and group cognition (Stahl, 2015).

[1] This environment is designed by Drexel University (Philadelphia, USA). We would acknowledge Drexel University and Dr. Arthur Powell (Rutgers University, Newark, USA) for join us in this research project.

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We understand that tasks in a DGE open the way to producing mathematical meanings (conceptual, procedurals, etc.), although not automatically (Bussi & Mariotti, 2008). It becomes necessary, therefore, to have a didactic project that supports the students in these environments. In this way, we seek to understand the role of the checkered grid, the task script, and the chat in a task of geometry, virtual and synchronous, in the configuration of thinking among teachers and prospective teachers. We would like to stress that a checkered grid/mesh is a resource available at the graphic zone of VMTwG, the chat is one of the spaces of VMTwG designed to exchange posting, and the script for the task was made by the authors (Brito, 2022).

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In our research group[2] we are designing, implementing, and analyzing tasks in the VMTcG concerning similarity (Brito & Bairral, 2023) and isometry (Bairral & Silvano, 2023). This article[3] illustrates and analyzes synchronous interactions of (prospective) teachers[4] in online tasks on triangle similarities and establish the role of the checkered mesh and sliding control in the configuration of geometric thinking among the participants. It summarizes results from Brito & Bairral (2023) concerning reasoning and geometric processes involved in using sliding control in the provided task. Since in Brazil similarity instruction is mainly focused on static geometric shapes, on numerical procedures and applying the rule of three, the study's contribution is reflecting about another possibility of teaching and learning similarity with online DGE.



## II. Teaching and Learning Similarity

The similarity of plane figures is usually related to the notion of measure, amplification, and reduction, keeping the proportion of the corresponding sides and angles. The similarity of figures is related to homothety, the right triangle metric ratios, the invariance of trigonometric ratios (Jaconiano, Barbosa, Concordido & Costa, 2019; González et al., 1990), the demonstration of mathematic theorems, as the fundamental theorem of proportionality and the Thales theorem (Pereira, 2017).

Similarity is applied in different fields of knowledge such as engineering, architecture, the study of optical phenomena, videogame programming and digital image processing. Such fields of study have the professional aim to control the dimensions and proportions of the shapes involved (Maciel & Almouloud, 2004; Powell & Alqahtani, 2021). One example of the application of similarity of plane figures is when you slide your fingers over the screen of a smartphone, tablet, or any other touch screen devices, to visualize an image without deforming it. It is also usual in computer programs when you alter the length and height of figures, keeping the proportion of their corresponding sides when you drag the image diagonally (Bairral, 2020; Brito & Bairral, 2023).

The concept of similitude, mainly of triangles, is exceptionally prominent (Lima, 2011). Books usually define similar triangles as those that have equal angles and proportional homologous sides, and this definition is extended to polygons. Triangles are the only group of non-deformable polygons, and the two conditions that guarantee similitude always occur together [...]: if the sides are proportional, the corresponding angles will automatically be equal, and vice-versa (Machado, 2000). The triangles show specific characteristics of similitude, which constitute the basis for understanding the similitude of other polygons.

[2] www.gepeticem.ufrrj.br
[3] It comes from a research project granted by CNPq and Faperj, Brazil.
[4] We use the expression (prospective) teachers to indicate the inclusion of graduate and undergraduate teachers in our working team.

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The criteria that establish the similitude of triangles are not well established for the learning subjects (González et al. 1990). According to authors, there is some language interference of mathematical language into everyday language, because the word *similar* has different meanings depending on the context of the language being used as pointed out by Bairral (1998) and Machado (2000). We have specified the way mathematics defines similar figures. Nevertheless, in everyday language, similar objects can be either equal or only resembling each other. We observe that from a mathematics point of view, we are dealing with something clearly more defined: figures that have the same shape, with equal or different sizes (Pereira, 2017).

González et al. (1990) stress two criteria – equal corresponding angles and proportional homologous sides – to identify similarity. The first one is probably the easiest to identify. Still, the second criterion, the similarity ratio between the figures, is a concept that is closely related to the proportionality between quantities. It needs to be worked upon with plenty of activities significant enough to ensure their acquisition, including tasks using floor plans and the scale concept (Bairral, 1998).

Galvão, Souza and Miashiro (2016) developed work on trigonometric functions among undergraduate students of mathematics. They observed that the students' lack of mastery of trigonometry when dealing with right triangles, Pythagoras theorem, measuring angles (in degrees and radians), and in the definition of the sine function was all a result of the little understanding they had about triangle similitude.

Other studies involving mathematics teachers stress the abusive use of the rule of three in solving geometry problems without considering concepts of proportionality and similarity (Costa & Allevato, 2015; Tinoco, 1996). Along these lines, it is argued that the teaching of similarity should not be restricted to how to solve something, but rather stress on the reason why one method is used, to avoid prejudice when learning (Jaconiano et al., 2019; Menduni-Bortoloti & Barbosa, 2018).

We understand that this scenario demands new practices in the teaching of triangle similarity to enable the development of geometric and proportional thinking among learners. The concept of proportionality is essential to make connections, as it integrates content of different branches of mathematics (Tinoco, 1996). And, more than the content to be taught, proportionality works in building cognitive structures that are necessary for the comprehension of other mathematics concepts, whether in the numeric or the geometric field (Costa & Allevato, 2015).

González et al. (1990) propose working the concept of similitude from figure enlarging or reduction and observing variant and invariant characteristics. Since similarity is a relevant concept in mathematics, exploring geometrically – and not only numerically or applying the rule of three – these processes of enlarging, reducing, and verifying congruence sounds powerful in mathematical thinking and using DGE to enhance learning.

*a) Learning similarity using DGE and VMTwG*

Mathematic activity takes place in the dialog of a mind with itself over observations of objects, relations among objects, and relations among relations or dynamics (Gattegno, 1987). We consider that mathematic activity of (prospective) teachers regarding triangle similitude takes place from dialogs with themselves over observations of objects (angles, sides, and their measurements), relations among objects (corresponding angles, sides, and points), and relations among relations or dynamics (congruence of corresponding angles, ratio, proportion, and proportionality of homologous sides, as well as similitude among triangles).

We consider essential to establish a new approach on the similitude of triangles from the use of DGE with the audience of prospective mathematics teachers. The logic involved in DGE emphasizes the development of mathematic activity through moving or deformable figures as a resource to promote exploration, discovery, and research of mathematic objects, whether through software or not (Arceo, 2009 apud Bairral & Barreira, 2017). It is crucial to highlight that those processes do not occur automatically. They require the teacher's mediation while the actions are taking place in the DGE (Arzarello et al., 2002).

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Among the particularities of a DGE through software, we are stressing the possibility to drag free points and transforming figures, keeping, or not keeping Euclidian properties (Bairral & Barreira, 2017). The dragging action helps the individual to produce conjectures through the observation and exploration of the movements of a figure and through the discovery of invariant properties (Azarello et al, 2002).

Those features can be enhanced by making groups of interaction in virtual environments, working on the resolution of mathematic tasks in a collaborative way (Powell, 2014). It was with this aim that VTM was developed, so that participants could interact with one another through a chat, build figures and drag points in the graphic area of GeoGebra.

In this way, with GeoGebra added to VMT, besides the interaction among individuals and DGE, other elements are related, like the interaction among the participants in the group that has come into play. We use the term interaction, from discourse analysis studies, as a synonym for interlocution and meaning exchange, reflection, and negotiation of meanings, with the possibility to take place among two or more interlocutors subjected to shared norms, orally or in writing, with or without the use of technology (Maingueneau, 2004; Oliveira & Bairral, 2020).

Interaction related to the use of VMT focuses on the collaboration among individuals, so it is a collaborative interaction in which the group works for on joint task and learning (Oliveira & Bairral, 2020), where the students learn through their questions, pursuing joint lines of reasoning, teaching one another, and watching how others are learning (Stahl et al., 2008). Collaborative interaction implies that the task is carried out jointly - even the discourse that issues from it does not belong to one individual but to the collective that has issued from the joint work.

DGE and its functions comprise a microworld in which the logic of Euclidean geometry is embedded (Mariotti, 2000), and it is possible to produce graphic strokes where the properties of the object are preserved thanks to the working of the tools that operate in this micro world. We shall call this micro world an artifact (Bussi & Mariotti, 2008), and the functions of dragging are classified as direct or indirect dragging. In direct dragging it is possible to drag a point without interfering with another. In contrast, in whereas in indirect dragging there is some interference in the movement of other points and elements built from the original point.It is worth noting that the meaning of artifact is not restricted to the DGE, but is a broader notion, including every type of human creation that has a practical nature, everything to which an individual attributes a use and which, at the same time, moves from the sphere of intellect to the sphere of practical doings and vice-versa (Bussi & Mariotti, 2008).

In the next section we shall discuss about the importance of dragging, artifacts, signs in each task and in mathematical learning in DGE.

## III. Theoretical Framework

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Online and synchronous virtual environments can provide a favorable space for learning both individually and collectively. Collaborative interactions and the role of VMT can be analyzed through discursive aspects, through interactions among individuals with DGE, through the dragging action, and through the meanings that individuals apply to each form of dragging. For instance, Alqahtani & Powell (2016, 2017b) organized a training course for teachers of mathematics, online and synchronously on VMTwG, and they noticed that the dragging[5] resource was used more knowingly, to the extent that teachers didn't only check the features contained in the construction, but they also checked the validity of the construction.

The tasks proposed by Alqahtani & Powell (2016, 2017b) involved triangles that were dependent on one another, where indirect dragging was necessary for the resolution of the problem that was proposed. The teachers participating in the research used the dragging resource together with the algebra window, aiming to understand the dependence of measures of the angles and the sides. Besides, the teachers also "dragged a triangle on top of another, or whatever they called overlapping triangles… so that in almost every task after this one, they discussed some form of congruence and similitude of objects" (Alqahtani & Powell, 2017b, p.32).

The authors stress the importance of educators' understanding how their students take over virtual collaborating environments, and how this appropriation brings a new configuration to the geometric thinking of the learners. These authors highlight two aspects of tasks with DGE: the manipulation of geometric objects by dragging and the discourse based on the relations and dependencies observed when dragging the objects. The resource of dragging is related to the external context of the student, that is, to the DGE itself and to the icons that allow the manipulation of objects. The dragging action is interlinked with the students' discourse, as it opens the way to actions that can become part of the mathematic discourse, thinking, and communication of geometric ideas (Alqahtani & Powell, 2017b).

It is in this sense that Bussi & Mariotti (2008) discuss the role of artifacts and signs in a given mathematic task, comprising the external context (the artifact) and the internal one (the signs). Artifacts act externally and are related to their practical perspective in the task, that is, how individuals attribute them a use. For instance, there are situations that come into play in the use of hammer. Depending on what it is aimed for, it can be used to hammera nail into the wall or to take it out. It will be necessary to consider the amount of strength, the position etc. Situations like the ones regarding the use of the hammer make it an artifact. We shall now move on to deal with the internal context.

Besides the physical device, an artifact such as the VMT, through its dragging function, can internally guide the behavior of an individual and it can affect their cognitive activity. Despite differences, the sign and the artifact have in common the function of mediation in the resolution of a task. Still, they differ in the way they orient human behavior. Vigotski calls "internalization" the process "in which individuals transform external activities linked to artifacts into internal activities which are linked to signs" (Alqahtani & Powell (2017b, p.3), involving "the inner reconstruction of an external operation" (Bussi & Mariotti, 2008, p. 751).

[5] In Brito & Bairral (2023) we detail the seven modalities of dragging according Arzarello et al. (2002).

Internalization is led by semiotic processes. In other words, this means that the process involves a semiotic mediation that consists not only in encouraging the relation of mathematic knowledge with the student, but also comprises the links between signs and the content being studied. According to Bussi & Mariotti (2008), there is a system of signs of the artifact that relate the artifact to the specific task that is aimed. These signs have a solid link to the procedures that are carried out, and the semiotic means through which these signs are produced go from gestures, words, figures, etc.

The second system of signs - called mathematic signs - is parallel to the one mentioned above, and it consists of the relation between an artifact and mathematic knowledge. This relationship is expressed through signs that reveal properties embedded in the artifact. These two systems do not relate spontaneously, and so it is vital for the teacher to understand the evolution of signs, from artifact to mathematics. Usually, the teacher can explore the signs elaborated socially, aiming to guide the evolution of signs towards what is known as mathematics, relating personal meanings generated using the artifact to mathematic meanings (Bussi & Mariotti, 2008).

The teacher who masters both personal and mathematic meanings can orient the evolution of the signs in the context of the artifact, relating to the artifact and to experience for its use, for mathematic signs, as definitions, conjectures to be proven or mathematic proof. There is a category of signs called pivot signs (Bussi & Mariotti (2008). They work as pivots or hinges, and they allow for the signs of the context of the artifact to evolve towards the mathematical realm. These signs belong to the mathematic field as much as to the artifact. As what is at play are signs, the teacher can use the pivot signs to relate the personal meanings of their students, leading them to mathematic meanings. This double relation is called semiotic potential and does more than relate mathematic knowledge to students. It demands a system of transformations of signs into other signs. It is therefore not just a mediation process; it is semiotic mediation.

The artifact also plays a role in semiotic mediation, although not in an automatic way. The teacher's guiding is necessary for the evolution of signs with a stress on the use into signs in the mathematic context. It is worth noting that any artifact will be referred to as a tool for semiotic mediation, if it is used (or meant to be used) intentionally by the teacher to mediate a mathematic content through a planned didactic intervention (Bussi & Mariotti, 2008).

## IV. Methodology

The pedagogical intervention occurred in 2020 with prospective mathematics teachers and MA students. Both groups were following online courses in Subject *Teaching and Learning Mathematics in Virtual Environments.* This was their first experience with VMTwG and the proposed tasks.

The VMT allows the user to create independent rooms or courses organized in groups of rooms. The working space can be created for the use of Desmos or GeoGebra. The tasks were planned to take place in an average time of two hours and with the participation of three to four users per room. We chose this organization mode to favor understanding among participants during a high flow of data generated in the chat. We looked for ways to understand in which ways learners construct their knowledge both individually and collectively, paying particular attention to dragging and group interaction.

The VMTwG interface comprises: GeoGebra area of tools which we shall refer to as icons, GeoGebra graphic zone, algebraic zone, chat area, button to ask for the

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control, button to refer previous postings in the chat and area for the presentation of instructions to the task. Figure 1 presents the interface of a VMTwG room and the subareas that belong in it.


*Source: Our own elaboration from VMT environment*

*Figure 1:* Subareas in VMTwG Interface

The button to request the control allows one user at a time to use the GeoGebra icons to make constructions on the graphic area, drag points and manipulate an embedded sliding control. While this occurs, the other participants can observe and share their ideas on the chat. If another participant has already selected the button 'take control' the posting 'Can I take control' is generated automatically on the chat. The candidates must wait for the control to get free.

All the data generated by the functions of a VMT room are saved in a sort of cloud on the site, which can be accessed later, and the data can be reviewed in chronological order. This resource - called *Replayer* - allowed for the analysis of the data of the implementations carried out in the research done by Brito (2022), even with data in charts, graphs, and filters generated by the platform itself. The editing functions of the room are available only to the teacher/monitor of the room, and they can insert the statement of a problem in the area meant for the instructions for the task.

The three tasks and it is learning aims (Appendix):

- *Task 1*– research the concept of congruence of triangles by the overlapping of two triangles and identifying the corresponding angles and sides.
- *Task 2*– make out variation and covariation factors of two similar triangles, by comparing the ratio of their corresponding sides.
- *Task 3* – check necessary and sufficient conditions for two triangles to be similar, from the exploration of the case of similitude of triangles Angle-Angle (AA).

The tasks are focused on the procedures of construction, the involved properties in the construction, and the concepts that are already known by the participants (Barreira & Bairral, 2017). In the tasks, triangles were created dependent

(Task 2 and 3) on each other, but there are free points in the original triangle that can have changed position. We summarize here the analysis based on the second task, on the ratio of triangle similitude to understand the subjects' aims when dragging values on a slider that linked two triangles. The first triangle had free points and the second one reacted to the dragging of the values of the slider and the points of the original triangle. As task 2 was designed with the use of the slider, there is a dependence of one triangle on the other. Although the participants used the checkerboard mesh to verify their conjectures the triangles continued to depend on each other.

Data production came from registers written and GeoGebra screen construction by participants and VMT replayer. From these registers which were taken out of interaction in the chat, we recovered the moments of the use of the resource, and we associated that to the registers in the chat. We looked at the postings before and after the dragging action and, finally, we crossed the discursive data and graphs to interpret that action.

## V. Results

As the task involved the construction and manipulation of triangles with free points, during the carrying out the participants stressed the primary function of a DGE - the movement - with the possibility to build similarity triangles. As the movement of a triangle implied the alteration of the other one, the role of dragging was to point out the properties that justify the dependence between the two triangles: congruence of the corresponding angles and proportionality of the homologous sides. The chat was used to share the personal meanings, interpret what the participants were saying and doing, and inform of the observations and properties. Together with the task script, the dragging resource was used to construct some types of congruent triangles, using the checkered mesh, which worked as a type of comparison measure between the sides of the triangles.

In the episode illustrated below – focused on Task 2 – we analyze the role of sliding control of semiotic mediation to identify the pivot sign and the relation between signs of the artifact and mathematic signs. To do this, we sought the interpretation of the aim of the participant's dragging through the crossing of the written registers on the chat, both before and after the dragging movement of the sliding control.

### a) *Episode: sliding control and the ratio of similitude*

In this episode, three (prospective) teachers used the sliding control to explore the ratio of similitude between two triangles. The second task aimed to build any triangle, a sliding control, and a second triangle dependent on the first, through the function of "homothety" of GeoGebra. The script proposal was to direct the participants to explore the angular properties and the ratio between the length of the sides of the similar triangles.

Nicole and Maria, who participated in the previous task, were also involved in the task with sliding control. The team consisted only of undergraduate students, and their fictional names are: Júlio, Suzi, Nicole, and Maria. Júlio came in one hour earlier than planned and wrote down his initial remarks in the chat. Some of them were considered by the group during the proposed task. After the construction of the sliding control and depending on triangles, the group members started to alter the value of the sliding control and, later, shared their remarks in the chat. Chart 2 presents the data from Replayer during the exploration of sliding control by Nicole.

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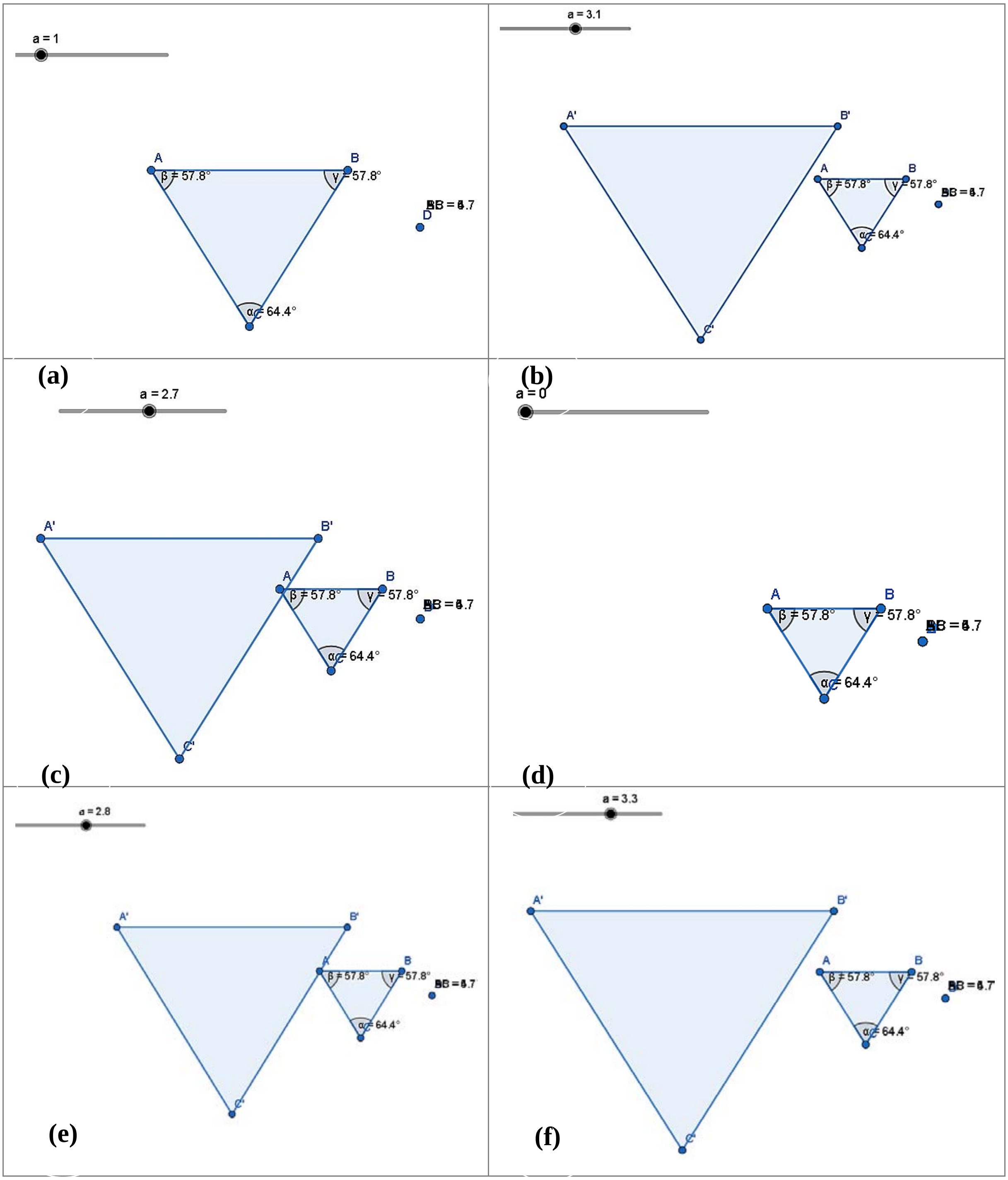
*Chart 2:* Process of alteration of values of the sliding control by Nicole (19:44:05 – 19:44:45)



*Source: Our own elaboration, based on data from the Replayer(Brito & Bairral, 2023)*

Notes

The values explored by Nicole belong to the interval between zero and five (Chart 2, figures a to d) and then concentrate on values higher than one (Chart 2, figures e to f). The same pattern in the variation of values repeated itself at the later moment when Maria dragged first between intervals from one to five and then altered to the value of zero and values over five. Both students explored on a broader way the sliding control between the initial value and the last. Therefore, the remarks in the chat relate to more general aspects. Maria observed the existence of the relation of the movement of the sliding control with the enlarging and reduction of triangle A'B'C'. Nicole conjectured about the distancing or approaching of the triangle to D, the point where the homothety function from GeoGebra was applied to the extent that it altered the values of the sliding control.

Maria and Nicole generated different signs, although they were using the same artifact, as they were sharing their meanings with the group, resulting from their own experience with the sliding control. Later the participants explored more specific values of the sliding control. They noted cases as a=1 and a = 0, and they observed that the triangles are congruent and that in the second case, the distance from the points in the triangle to point D is zero. Concerning the first moment, there was an advancement of the most generic aspects to the more specific ones to the extent that the personal meanings began to converge.

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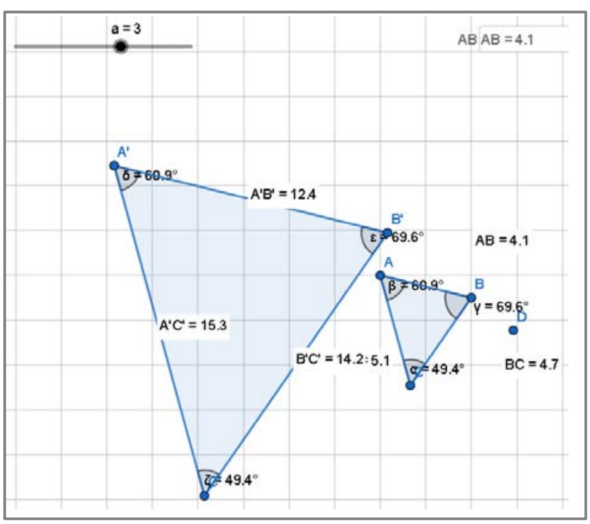
The participants still had not interacted about the interval between one and zero, neither had they measured the length of the sides, only the internal angles. They used the random dragging to discover the regularities of the figure, and by the reaction of the environment, they identified the corresponding angles and observed that they were equal. The group members measured the length of the sides of the triangles.

When we reached step 8 in the task script, we asked the participants: "Compare the length of the sides of triangle A'B'C' with the corresponding sides of triangle ABC. To do so, select the entrance field with the icon a=1 In this field, calculate the ratio of segment AB by A'B'". The participants tried to insert the length ratio of the sides in the entrance box of the commands. They were not successful because of a limitation of VMT which didn't make the insertion of formulas available.

The participants questioned the mediating teacher (second author) about a problem with the entrance box for the GeoGebra commands and were oriented to use the calculator in their computers to obtain the ratio between the corresponding sides. The final configuration in the Graphic zone at that instant in the task is shown in Figure 1.




*Source: Replayer Data*

*Figure 1:* Configuration of Graphic zone, the data of which were extracted to use with the calculator (146-167)

*Chart 3:* Presents the interaction segment among participants and second author when they were using their computer or smart phone calculators and sharing the results in the chat. Artifact signs and mathematic signs were identified. We are representing those signs by the symbols "<" and ">". The pivot sign, the sliding control <a>, is represented between the columns of Chart 3 in the text boxes.

*Chart 3:* Synthesis of the evolution from artifact to mathematic signs

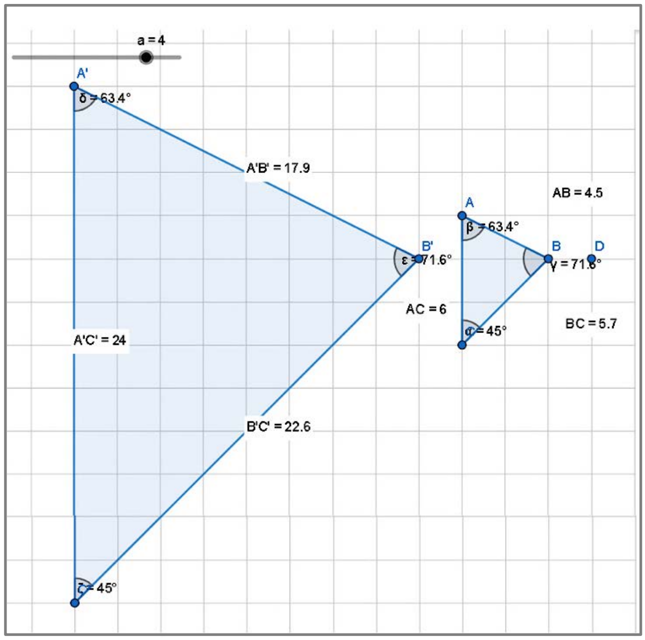
| Index | Participant | Posting in *chat* | Artifact sign | Mathematic sign |
|---|---|---|---|---|
| 148 | Author | The other groups also had a problem with this tool. You can use your computer calculator in case you cannot use this tool. | **<a>** | |
| 149 | Nicole | Because the ratio would be A'B'=a*AB | <A'B'>, <a> | <A'B'=a*AB> |
| 150 | Nicole | And so on, for each side | <side> | |
| 151 | Maria | OK | | |
| 152 | Maria | 8) ratio AB/A'B'= 0,3307 | | <ratio> |
| 153 | Maria | Approximately that value | | |
| 154 | Nicole | 8) the ratio for a=3, A'B'/AB, is 3 | <a=3> | <A'B'/AB, is 3> |
| 155 | Suzi | It's AB/A'B' | | |
| 156 | Maria | I did the ratio for a=3, but for AB/A'B'=0,3307 approximately | | <AB/A'B'=0,3307> |
| 157 | Nicole | Doing AB/A'B' it also was 0,3306 | | <AB/A'B' also was 0, 3306> |
| 158 | Maria | 9) a=3, ratio BC/B'C'= 0,3309 | <a=3> | |
| 159 | Nicole | BC/B'C'= 0,3309 | | <BC/B'C'=0,3309> |
| 160 | Maria | 9) a=3, ratio AC/A'C'= 0,33333 | <a=3> | |
| 161 | Nicole | AC/A'C'= 0,333... | | <AC/A'C'=0,33333> |
| 162 | Nicole | For me, the ratio of AC and A'C' is the most precise, because it would be 1/3 which is exactly 0,333... | **<a>** | <1/3> |
| 163 | Author | does this result have anything to do with the value of "a"? | <a> | |
| 164 | Nicole | I think that a is the ratio | <a> **<a>** | tio> |
| 165 | Maria | Yes, altering the value of a, the triangle A'B'C' changes its size, therefore the ratio changes | <triangle A'B'C'> | |
| 166 | Nicole | That's why it says that the ratio from ABC to A'B'C' is 1/3 | | <ratio from ABC to AB'C' é 1/3> |
| 167 | Nicole | Now you can see from the side AC=6, as a=4, then A'C'=24 | **<a>** | |

*Source: Brito & Bairral (2023)*

Various artifact signs were generated by the participants, regarding the constructions manipulated in the graphic zone. In the same way, mathematic signs were generated regarding the value obtained by the ratio of corresponding sides, represented by the number in decimal modality. The sign <a> of the sliding control was often referred to and played the role of connecting the two contexts to promote the passage from personal meanings to mathematic meanings represented by the respective ratios of the triangle sides.

Nicole realized that, performing the ratio of side AB to A'B', the values were close to de decimal representation 0,333 or 1/3 in fractional form (Chart 3, index 162). As she noticed that the sliding control was set at a value of three, Nicole conjectured that the value of the inverted ratio of 1/3 was represented by the value of the sliding control <a=3> (Chart 3, index 154). So, she proposed to her mates they calculate the ratio of A'B' for AB.

We also registered that Nicole manipulated the sliding control again and, more intentionally this time, she modified the sliding control to the whole value <a=4> and she obtained the length AC = 6 e A'C = 24, as shown in Figure 4.




*Source: Replayer data*

*Figure 1:* Configuration of the Graphic zone that comprises the descending process

Figure 4 and Chart 3 (indexes 149, 167) illustrate how Nicole concluded that A'C is equal to the product of < a > by the length of segment AC. Therefore, Nicole used a process with a descending flow - from the theory to the construction - thus checking the validity of the property and that it was kept for other measures of the sides of the triangles. In this process, the calculator was usedto construct a conjecture about the ratio of the length of the sides of A'B'C' by ABC, and the process of altering the values of the sliding control played the role to help in the production of conjectures and the validation of an observed property.

The use of grid was restricted to the exploration of more global geometric properties while those related to the use of the slider may represent a more robust form of thought process. Remarkably, regarding the use of sliding control:

- It was often referred to and played the role of connecting the two contexts to promote the passage from personal meanings to mathematic meanings represented by the respective ratios of the triangle sides.
- It played the role to help in the production of conjectures and the validation (or refutation) of an observed property and checking result.

## VI. Conclusions

In this article,we illustrated (prospective) teachers interacting in VMTwG in activities concerning triangle similitude. In the illustrated episode – from the measures of the sides and the calculus of the ratios with the calculator – the participants constructed the conjecture of the proportion of the corresponding sides. Using of the sliding control enabled us to explore approaches, inferences on measures and the notion of ratio as fraction, the analysis of global and cases (a = 1 and a = 0). The processes involved in the use of sliding control can assume a more potent form, as they allow not only global observation, but also the checking. The types of flow of thought identified were ascending and descending and encouraged exploration, building conjectures, and intentionally checking their suppositions. Further research needs to be done using sliding control with different numerical intervals.

This intertwined movement of observation, exploration and checking were provided by the design of the task integrating slider control and the interactions within the dynamic environment. For instance, when subjects perform congruence tasks using different triangles cut out of paper they only compare – the angles and the sides – the shape by superposition. When dealing with the task using slider control the subject can explore a variety of shapes and mathematic objects. This process makes it possible for a different geometric property that emerges to be analyzed and verified whether it is true or false.

Notes

We are not saying the use of shapes cut out of paper is not essential. Still, the development of mathematical thinking with deep understanding and improving different forms of justification (writing, construction, and motion on screen, graphical, etc.) about some property or mathematical routine is more potent than merely checking measurements of angles or sides as we often observe in static resources. The semiotic mediation provided by VMTwG and other DGE can orient the evolution of the different signs enriched by interaction.

In the analytical episode summarized here it was possible to have a notion of the semiotic mediation process of thought involved in dragging, eitherby free points or sliding control. We hope that this article can contribute to other mathematics teachers' exploration of the semiotic potential of geometric tasks with DGE and that they can help their students learn triangle similitude and other geometry subjects paying less attention on calculus or applying mechanically the rule of three. We would like to stress the importance of VMTwG as an environment of interrelated spaces that enables a constant interactive process, where the students themselves explore, share, interpret, and check their thoughts and discovers.

We must warn that, as it is a robust system involving various functions, a good connection is required, so that all tasks may be carried out without obstacles. The VMT environment is in constant construction and relies on a technical support team who listens to suggestions to the problems reported by the users. Some suggestions were made through our research, including the availability of the box to the access of VMTwG commands, following the problem that took place in episode 2. This was duly solved, and it is already possible to insert commands for the side ratios directly in the environment.

This research had its educational outcome[6] - Sequence of activities on triangle similitude in Virtual Math Teams environment - where we present, besides the tasks mentioned above, other tasks, discussing the semiotic process involved in each task, and give suggestions to the teachers (Brito, 2022). We welcome the reader to learn, develop, and adapt the tasks with VMT among their students. As a closing remark, let us highlight that VMTwG is one of the scenarios that can occur in mathematic instruction, and we would not boast about it being the only one. On the contrary, let's rejoice in the fact that it is there as one more field for the exploration of mathematics concepts and to awaken students' interest and pleasure in learning mathematics online and/or face-to-face.

[6] Available at: https://gepeticem.ufrrj.br/sequencia-didatica-sobre-semelhanca-de-triangulos-no-ambiente-virtual-math-teams, access April 24. 2022.

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## Appendix

### *Task 1:* Congruence of triangles

*Estimated time:* 2 hours.
*Objective:* To investigate the concept of congruence of triangles by overlapping two triangles and identifying the corresponding angles and sides.

1. Enable the GeoGebra mesh using the icon . Select the "Display Mesh" option and then "Main Mesh". Construct any triangle **ABC** using the icon .
2. Measure the internal angles of the triangle **ABC** by selecting .
3. Select to find the measure of each side of the triangle.

a) With the tool selected, move the points on the vertices of the triangle freely and comment on what you see.

4. Construct any triangle **DEF** by repeating the previous steps.

b) What did you notice when you compared the lengths of the sides and angles of triangles **ABC** and **DEF**?
c) With the help of the grid, move the vertices of triangle ABC with , so that triangle **ABC** coincides exactly with triangle **DEF**.

5. Tap the center of the triangle and drag it to the side.

d) Comparing the measure of angles and sides: what can you say about the characteristics of overlapping triangles?
e) Observe and comment in the chat, what are the corresponding angles of triangles **ABC** and **DEF**?
f) What are the corresponding sides of triangles **ABC** and **DEF**?
g) Based on the construction you have made, comment on whether (or not) triangles **ABC** and **DEF** are congruent and explain the reasons that led you to this conclusion.

### *Task 2:* Similarity ratio

*Estimated time:* 2 hours.

*Objective:* Distinguish components of variation and covariation of two similar triangles by comparing the ratios of their corresponding sides.

1. Let's create a numerical parameter. Select the tool and click in the top corner of the GeoGebra screen. When a new tab opens, put the value 0 (zero) in the "min" box and then press "OK".
2. Construct a triangle **ABC** with and measure its internal angles and the lengths of its sides.
3. Create a point **D** outside triangle **ABC** using .
4. Selecting , press on the center of triangle **ABC** and then tap on point **D**, which is outside triangle **ABC**. When the Homothety box appears under "factor", type in "a". Then press "ok".
5. Select and then increase and decrease the values of "a" and observe.

a) What happened to triangle **A'B'C'** when you changed the values of "a"?

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6. After constructing triangle **A'B'C'**, measure its interior angles.
7. With the option , move the vertices of triangle **ABC** freely and observe.
b) Compare the corresponding angles of the two triangles, what do you see?
8. Compare the lengths of the sides of triangle **A'B'C'** with the corresponding sides of triangle **ABC**. What do you notice?
9. Calculate the ratio of the other corresponding sides. To do this, enter the ratio of the segment **A'B' to AB** in the input box.
c) What is the relationship between the calculated ratios?
d) If you change the value of "a", what do you notice when you compare it to the ratios you calculated earlier?
e) What is the relationship between the lengths of the sides of triangles **A'B'C'** and **ABC**?
10. Finally, change the value of "a" to 1 and move the vertices of the triangle by changing their angles and side lengths.
f) How does this activity differ from or enrich the activity you experienced in the previous section?

*Task 3:* Cases of similarity of triangles

*Estimated time:* 1 hour.

*Objective:* Check the necessary and sufficient situations for two triangles to be similar by exploring the case of triangle similarity Angle-Angle (AA).

1. Enable the GeoGebra mesh using the icon . Select the "Display Mesh" option and then the **"Isometric"** option.
2. Construct a triangle **ABC** with . Left click inside the triangle and choose a color by selecting .
3. Measure **just two internal angles** of triangle **ABC**.
4. Next, construct any triangle **DEF** and measure **just two internal angles**.
5. Position points **D**, **E** and **F** so that triangle **DEF** is similar to **ABC**.
a) Look at the construction of triangles **ABC** and **DEF** and explain why you think they are similar.
6. Measure the angles and sides that have not yet been measured in the two triangles.
b) Now that you know the measures of the internal angles and sides, what can you say about all the triangles that have two equal internal angles?
c) How does this activity differ from or enrich the activities you experienced in the previous sections?

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# The Validity of Generalized Modal Syllogisms with the Generalized Quantifiers in Square{*most*}

By Siyi Yu & Xiaojun Zhang

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*Abstract-* Due to the large number of generalized quantifiers in the English language, this paper only studies the fragment of generalized modal syllogistic that contains the quantifiers in Square{all} and Square{most}. On the basis of generalized quantifier theory, possible-world semantics, and set theory, this paper shows that there are reducible relations between/among the generalized modal syllogism £EM◇O-3 and at least the other 29 valid generalized modal syllogisms. This method can also be used to study syllogisms with other generalized quantifiers. The results obtained by means of formal deductive method have not only consistency, but also theoretical value for the development of inference theory in artificial intelligence.

*Keywords:* *generalized modal syllogisms; reducibility; modality; validity.*

*GJSFR-F Classification:* *LCC: QA9.64*

THEVALIDITYOFGENERALIZEDMODALSYLLOGISMSWITHTHEGENERALIZEDQUANTIFIERSINSQUAREMOST

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# The Validity of Generalized Modal Syllogisms with the Generalized Quantifiers in Square {*most*}

Siyi Yu [α] & Xiaojun Zhang [σ]

*Abstract-* Due to the large number of generalized quantifiers in the English language, this paper only studies the fragment of generalized modal syllogistic that contains the quantifiers in Square{all} and Square{most}. On the basis of generalized quantifier theory, possible-world semantics, and set theory, this paper shows that there are reducible relations between/among the generalized modal syllogism £EM◇O-3 and at least the other 29 valid generalized modal syllogisms. This method can also be used to study syllogisms with other generalized quantifiers. The results obtained by means of formal deductive method have not only consistency, but also theoretical value for the development of inference theory in artificial intelligence.

*Keywords:* *generalized modal syllogisms; reducibility; modality; validity.*

## I. Introduction

Syllogism is one of the significant forms of reasoning in natural language and human thinking. There are various kinds of syllogisms, such as Aristotelian syllogisms (Patzig, 1969; Long, 2023; Hui, 2023), Aristotelian modal syllogisms (Johnson, 2004; Łukasiewicz, 1957; Cheng and Xiaojun, 2023), generalized syllogisms (Murinová and Novák, 2012; Xiaojun and Baoxiang, 2021; Endrullis and Moss, 2015), and generalized modal syllogisms (Jing and Xiaojun, 2023).

Although many generalized modal syllogisms exist in natural language, there is little literature on their reducibilities. Therefore, this paper mainly focuses on them. The four Aristotelian quantifiers (that is, *not all*, *all*, *some* and *no*) constitute Square{*all*}. And '*most*' and its three negative (i.e. inner, outer and dual), *fewer than half of the*, *at most half of the*, *and at least half of the*, form Square{*most*}. The generalized modal syllogisms studied in this paper only involve the quantifiers in Square{*all*} and Square{*most*}.

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## II. Preliminaries

In this paper, let $w$, $v$ and $z$ be the lexical variables, which are elements in the set $W$, $V$ and $Z$ respectively, $D$ be the domain of lexical variables, $|W|$ the cardinality of the set $W$, and $m$, $n$, $s$ and $t$ propositional variables. $Q$ stands for any generalized quantifiers, $\neg Q$ and $Q\neg$ for the outer and inner negative quantifier of $Q$ respectively. The generalized modal syllogisms discussed in this paper comprise the following sentences as follows: 'all $w$s are $v$s', 'no $w$s are $v$s', 'some $w$s are $v$s', 'not all $w$s are $v$s', 'most $w$s are $v$s', 'fewer than half of the $w$s are $v$s', 'at most half of the $w$s are $v$s', and 'at least half of the $w$s are $v$s'. They can be denoted as: *all*($w$, $v$), *no*($w$, $v$), *some*($w$, $v$), *not all*($w$, $v$), *most*($w$, $v$), *fewer than half of the*($w$, $v$), *at most half of the*($w$, $v$), *at least half of the*($w$, $v$), and are respectively abbreviated as Proposition $A$, $E$, $I$, $O$, $M$, $F$, $H$ and $S$.

A non-trivial generalized modal syllogism includes at least one and at most three non- overlapping modalities (possible modality ($\Diamond$) or necessary modality ($\Box$)) and non-trivial generalized quantifiers, such as the quantifiers in Square–*most*".

*Example 1:*

Major premise: No grapes are necessarily blueberries.

Minor premise: Most grapes are purple fruits.

Conclusion: Not all purple fruits are possibly blueberries.

Let $w$ be the lexical variable for a blueberry in the domain, $v$ be the lexical variable for a grape in the domain, and $z$ be the lexical variable for a purple fruit in the domain. Then the syllogism in example 1 can be formalized as: $\Box$*no*($v$, $w$)$\wedge$*most* ($v$, $z$)$\rightarrow\Diamond$*not all*($z$, $w$), which abbreviated as $\Box$EM$\Diamond$O-3.

According to generalized quantifier theory, set theory (Halmos, 1974) and possible world semantics (Chellas, 1980), the truth value definitions of sentences with quantification, relevant facts and rules used in the paper are as follows:

*Definition 1* (truth value definitions):

(1.1) *all*($w$, $v$) is true when and only when $W\subseteq V$ is true in all real worlds.

(1.2) *no*($w$, $v$) is true when and only when $W\cap V=\varnothing$ is true in all real worlds.

(1.3) *some*($w$, $v$) is true when and only when $W\cap V\neq\varnothing$ is true in all real worlds.

(1.4) *not all*($w$, $v$) is true when and only when $W\nsubseteq V$ is true in all real worlds.

(1.5) *most*($w$, $v$) is true when and only when $|W\cap V|>0.5|W|$ is true in all real worlds.

(1.6) $\Box$*all*($w$, $v$) is true when and only when $W\subseteq V$ is true in all possible worlds.

(1.7) $\Diamond$*all*($w$, $v$) is true when and only when $W\subseteq V$ is true in some possible worlds.

(1.8) $\Box$*no*($w$, $v$) is true when and only when $W\cap V=\varnothing$ is true in all possible worlds.

(1.9) $\Diamond$*no*($w$, $v$) is true when and only when $W\cap V=\varnothing$ is true in some possible worlds.

(1.10) $\Box$*some*($w$, $v$) is true when and only when $W\cap V\neq\varnothing$ is true in all possible worlds.

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(1.11) $\Diamond$*some*$(w, v)$ is true when and only when $W\cap V\neq\varnothing$ is true in some possible worlds.

(1.12) $\Box$*not all*$(w, v)$ is true when and only when $W\nsubseteq V$ is true in all possible worlds.

(1.13) $\Diamond$*not all*$(w, v)$ is true when and only when $W\nsubseteq V$ is true in some possible worlds.

(1.14) $\Box$*most*$(w, v)$ is true when and only when $|W\cap V|>0.5|W|$ is true in all possible worlds.

(1.15) $\Diamond$*most*$(w, v)$ is true when and only when $|W\cap V|>0.5|W|$ is true in some possible worlds.

*Definition 2* (inner negation): $Q\neg(w, v)=_{\text{def}}Q(w, D-v)$.

*Definition 3* (outer negation): $\neg Q(w, v)=_{\text{def}}$It is not that $Q(w, v)$.

*Fact 1* (inner negation):

(1.1) $\vdash$*all*$(w, v)\leftrightarrow$*no*$\neg(w, v)$;

(1.2) $\vdash$*no*$(w, v)\leftrightarrow$*all*$\neg(w, v)$;

(1.3) $\vdash$*some*$(w, v)\leftrightarrow$*not all*$\neg(w, v)$;

(1.4) $\vdash$*not all*$(w, v)\leftrightarrow$*some*$\neg(w, v)$;

(1.5) $\vdash$*fewer than half of the*$(w, v)\leftrightarrow$*most*$\neg(w, v)$;

(1.6) $\vdash$*most*$(w, v)\leftrightarrow$*fewer than half of the*$\neg(w, v)$;

(1.7) $\vdash$*at most half of the*$(w, v)\leftrightarrow$*at least half of the*$\neg(w, v)$;

(1.8) $\vdash$*at least half of the*$(w, v)\leftrightarrow$*at most half of the*$\neg(w, v)$.

*Fact 2* (outer negation):

(2.1) $\vdash\neg$*not all*$(w, v)\leftrightarrow$*all*$(w, v)$;

(2.2) $\vdash\neg$*all*$(w, v)\leftrightarrow$*not all*$(w, v)$;

(2.3) $\vdash\neg$*no*$(w, v)\leftrightarrow$*some*$(w, v)$;

(2.4) $\vdash\neg$*some*$(w, v)\leftrightarrow$*no*$(w, v)$;

(2.5) $\vdash\neg$*most*$(w, v)\leftrightarrow$*at most half of the*$(w, v)$;

(2.6) $\vdash\neg$*at most half of the*$(w, v)\leftrightarrow$*most*$(w, v)$;

(2.7) $\vdash\neg$*fewer than half of the*$(w, v)\leftrightarrow$*at least half of the*$(w, v)$;

(2.8) $\vdash\neg$*at least half of the*$(w, v)\leftrightarrow$*fewer than half of the*$(w, v)$.

*Fact 3* (dual):

(3.1) $\vdash\neg\Box Q(w, v)\leftrightarrow\Diamond\neg Q(w, v)$;

(3.2) $\vdash\neg\Diamond Q(w, v)\leftrightarrow\Box\neg Q(w, v)$.

*Fact 4* (symmetry):

(4.1) $\vdash$*some*$(w, v)\leftrightarrow$*some*$(v, w)$;

(4.2) $\vdash$*no*$(w, v)\leftrightarrow$*no*$(v, w)$.

*Fact 5* (subordination):

(5.1) $\vdash\Box Q(w, v)\rightarrow Q(w, v)$;

(5.2) $\vdash\Box Q(w, v)\rightarrow\Diamond Q(w, v)$;

(5.3) $\vdash Q(w, v)\rightarrow\Diamond Q(w, v)$;

(5.4) $\vdash all(w, v)\rightarrow some(w, v)$;

(5.5) $\vdash no(w, v)\rightarrow not\ all(w, v)$.

Notes

*Rule 1* (subsequent weakening): If $\vdash(m\wedge n\rightarrow s)$ and $\vdash(s\rightarrow t)$, then $\vdash(m\wedge n\rightarrow t)$.

*Rule 2* (anti-syllogism): If $\vdash(m\wedge n\rightarrow s)$, then $\vdash(\neg s\wedge m\rightarrow\neg n)$ or $\vdash(\neg s\wedge n\rightarrow\neg m)$.



## III. The Validity of the Syllogism □EM◇O-3

In order to discuss the reducibility of generalized modal syllogisms based on the syllogism □EM◇O-3, it is necessary to prove the validity of the syllogism □EM◇O-3.

*Theorem 1 (□EM◇O-3):* The generalized modal syllogism $\Box no(v, w)\wedge most(v, z)\rightarrow\Diamond not\ all(z, w)$ is valid.

*Proof:* According to Example 1, □EM◇O-3 is the abbreviation of the syllogism $\Box no(v, w)\wedge most(v, z)\rightarrow\Diamond not\ all(z, w)$. Suppose that $\Box no(v, w)$ and $most(v, z)$ are true, then in virtue of Definition (1.8), $\Box no(v, w)$ is true when and only when $V\cap W=\emptyset$ is true in all possible worlds. Similarly, in line with Definition (1.5), $most(v, z)$ is true when and only when $|V\cap Z|>0.5|V|$ is true in all real worlds. Real worlds are elements in the set of all possible worlds. Thus, it is easily seen that $V\cap W=\emptyset$ and $|V\cap Z|>0.5|V|$ are true in some possible worlds. Then, it is clear that $Z\nsubseteq W$ is true in some possible worlds. $\Diamond not\ all(z, w)$ is true in terms of Definition (1.13). The above proves that the syllogism $\Box no(v, w)\wedge most(v, z)\rightarrow\Diamond not\ all(z, w)$ is valid.

## IV. The Other Generalized Modal Syllogisms Derived From □EM◇O-3

Theorem 1 states that □EM◇O-3 is valid, and '□EM◇O-3→□EM◇O-4' in Theorem 2(1) expresses that the validity of syllogism □EM◇O-4 is deduced from that of syllogism □EM◇O-3. That is to show that there are reducible relations between these two syllogisms, and the others are similar.

*Theorem 2:* There are at least the following 29 valid generalized modal syllogisms obtained from □EM◇O-3:

(1) □EM◇O-3→□EM◇O-4

(2) □EM◇O-3→□A□EH-2

(3) □EM◇O-3→□AM◇I-1

(4) □EM◇O-3→□AM◇I-3

Notes

(5) □EM◇O-3→□EM◇O-4→□A□EH-4

(6) □EM◇O-3→□EM◇O-4→M□A◇I-4

(7) □EM◇O-3→□A□EH-2→□E□AH-2

(8) □EM◇O-3→□AM◇I-1→□EM◇O-1

(9) □EM◇O-3→□AM◇I-3→M□A◇I-3

(10) □EM◇O-3→□A□EH-2→□E□AH-2→□E□AH-1

(11) □EM◇O-3→□AM◇I-1→□EM◇O-1→□EM◇O-2

(12) □EM◇O-3→□AM◇I-3→M□A◇I-3→F□A◇O-3

(13) □EM◇O-3→□A□EH-2→□E□AH-2→□E□AH-1→□A□AS-1

(14) □EM◇O-3→□AM◇I-1→□EM◇O-1→□EM◇O-2→□AF◇O-2

(15) □EM◇O-3→□A□EH-2→□A□E◇H-2

(16) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□E◇H-4

(17) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□M◇I-1

(18) □EM◇O-3→□A□EH-2→□A□E◇H-2→□E□M◇O-3

(19) □EM◇O-3→□A□EH-2→□A□E◇H-2→□E□A◇H-2

(20) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□E◇H-4→□M□A◇I-4

(21) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□E◇H-4→□E□M◇O-4

(22) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□M◇I-1→□E□M◇O-1

(23) □EM◇O-3→□A□EH-2→□A□E◇H-2→□E□M◇O-3→□A□M◇I-3

(24) □EM◇O-3→□A□EH-2→□A□E◇H-2→□E□A◇H-2→□E□A◇H-1

(25) □EM◇O-3→□A□EH-2→□A□E◇H-2→□A□M◇I-1→□E□M◇O-1
→ □E□M◇O-2

(26) □EM◇O-3→□A□EH-2→□A□E◇H-2→□E□M◇O-3→□A□M◇I-3
→□M□A◇I-3

(27) □EM◇O-3→□A□EH-2→□E□AH-2→□E□AH-1→□A□AS-1→□A□A◇S-1

(28) □EM◇O-3→□A□EH-2→□E□AH-2→□E□AH-1→□A□AS-1→□A□A◇S-1
→□A□F◇O-2

(29) □EM◇O-3→□A□EH-2→□E□AH-2→□E□AH-1→□A□AS-1→□A□A◇S-1
→□F□A◇O-3

*Proof:*

[1] $\vdash\Box no(v, w)\wedge most(v, z)\rightarrow\Diamond not\ all(z, w)$ (i.e. □EM◇O-3, Theorem 1)

[2] $\vdash\Box no(w, v)\wedge most(v, z)\rightarrow\Diamond not\ all(z, w)$ (i.e. □EM◇O-4, by [1] and Fact (4.2))

[3] $\vdash\neg\Diamond not\ all(z, w)\wedge\Box no(v, w)\rightarrow\neg most(v, z)$ (by [1] and Rule 2)

[4] $\vdash\Box\neg not\ all(z, w)\wedge\Box no(v, w)\rightarrow\neg most(v, z)$ (by [3] and Fact (3.2))

[5] $\vdash\Box all(z, w)\wedge\Box no(v, w)\rightarrow at\ most\ half\ of\ the(v, z)$
(i.e. □A□EH-2, by [4], Fact (2.1) and Fact (2.5))

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[6] $\vdash\neg\Diamond$*not all*$(z, w)\wedge$*most*$(v, z)\rightarrow\neg\Box$*no*$(v, w)$ (by [1] and Rule 2)

[7] $\vdash\Box\neg$*not all*$(z, w)\wedge$*most*$(v, z)\rightarrow\Diamond\neg$*no*$(v, w)$ (by [6], Fact (3.1) and Fact (3.2))

[8] $\vdash\Box$*all*$(z, w)\wedge$*most*$(v, z)\rightarrow\Diamond$*some*$(v, w)$

(i.e. □AM◇I-1, by [7], Fact (2.1) and Fact (2.3))

[9] $\vdash\Box$*all*$\neg(v, w)\wedge$*most*$(v, z)\rightarrow\Diamond$*some*$\neg(z, w)$ (by [1], Fact (1.2) and Fact (1.4))

[10] $\vdash\Box$*all*$(v, D\text{-}w)\wedge$*most*$(v, z)\rightarrow\Diamond$*some*$(z, D\text{-}w)$(i.e. □AM◇I-3, by [9] and Definition 2)

[11] $\vdash\neg\Diamond$*not all*$(z, w)\wedge\Box$*no*$(w, v)\rightarrow\neg$*most*$(v, z)$ (by [2] and Rule 2)

[12] $\vdash\Box\neg$*not all*$(z, w)\wedge\Box$*no*$(w, v)\rightarrow\neg$*most*$(v, z)$ (by [11] and Fact (3.2))

[13] $\vdash\Box$*all*$(z, w)\wedge\Box$*no*$(w, v)\rightarrow$*at most half of the*$(v, z)$

(i.e. □A□EH-4, by [12], Fact (2.1) and Fact (2.5))

[14] $\vdash\neg\Diamond$*not all*$(z, w)\wedge$*most*$(v, z)\rightarrow\neg\Box$*no*$(w, v)$ (by [2] and Rule 2)

[15] $\vdash\Box\neg$*not all*$(z, w)\wedge$*most*$(v, z)\rightarrow\Diamond\neg$*no*$(w, v)$ (by [14], Fact (3.1) and Fact (3.2))

[16] $\vdash\Box$*all*$(z, w)\wedge$*most*$(v, z)\rightarrow\Diamond$*some*$(w, v)$

(i.e. M□A◇I-4, by [15], Fact (2.1) and Fact (2.3))

[17] $\vdash\Box$*no*$\neg(z, w)\wedge\Box$*all*$\neg(v, w)\rightarrow$*at most half of the*$(v, z)$

(by [5], Fact (1.1) and Fact (1.2))

[18] $\vdash\Box$*no*$(z, D\text{-}w)\wedge\Box$*all*$(v, D\text{-}w)\rightarrow$*at most half of the*$(v, z)$

(i.e. □E□AH-2, by [17] and Definition 2)

[19] $\vdash\Box$*no*$\neg(z, w)\wedge$*most*$(v, z)\rightarrow\Diamond$*not all*$\neg(v, w)$ (by [8], Fact (1.1) and Fact (1.3))

[20] $\vdash\Box$*no*$(z, D\text{-}w)\wedge$*most*$(v, z)\rightarrow\Diamond$*not all*$(v, D\text{-}w)$

(i.e. □EM◇O-1, by [19] and Definition 2)

[21] $\vdash\Box$*all*$(v, D\text{-}w)\wedge$*most*$(v, z)\rightarrow\Diamond$*some*$(D\text{-}w, z)$ (i.e. M□A◇I-3, by [10] and Fact (4.1))

[22] $\vdash\Box$*no*$(D\text{-}w, z)\wedge\Box$*all*$(v, D\text{-}w)\rightarrow$*at most half of the*$(v, z)$

(i.e. □E□AH-1, by [18] and Fact (4.2))

[23] $\vdash\Box$*no*$(D\text{-}w, z)\wedge$*most*$(v, z)\rightarrow\Diamond$*not all*$(v, D\text{-}w)$

(i.e. □EM◇O-2, by [20] and Fact (4.2))

[24] $\vdash\Box$*all*$(v, D\text{-}w)\wedge$*fewer than half of the*$\neg(v, z)\rightarrow\Diamond$*not all*$\neg(D\text{-}w, z)$

(by [21], Fact (1.6) and Fact (1.3))

[25] $\vdash\Box$*all*$(v, D\text{-}w)\wedge$*fewer than half of the*$(v, D\text{-}z)\rightarrow\Diamond$*not all*$(D\text{-}w, D\text{-}z)$

(i.e. F□A◇O-3, by [24] and Definition 2)

[26] $\vdash\Box$*all*$\neg(D\text{-}w, z)\wedge\Box$*all*$(v, D\text{-}w)\rightarrow$*at least half of the*$\neg(v, z)$

(by [22], Fact (1.2) and Fact (1.7))

Notes

[27] $\vdash\Box all(D\text{-}w, D\text{-}z)\wedge\Box all(v, D\text{-}w)\rightarrow at\ least\ half\ of\ the(v, D\text{-}z)$

(i.e. □A□AS-1, by [26] and Definition 2)

[28] $\vdash\Box all\neg(D\text{-}w, z)\wedge fewer\ than\ half\ of\ the\neg(v, z)\rightarrow\Diamond not\ all(v, D\text{-}w)$

(by [23], Fact (1.2) and Fact (1.6))

Notes

[29] $\vdash\Box all(D\text{-}w, D\text{-}z)\wedge fewer\ than\ half\ of\ the(v, D\text{-}z)\rightarrow\Diamond not\ all(v, D\text{-}w)$

(i.e. □AF◇O-2, by [28] and Definition 2)

[30] $\vdash\Box all(z, w)\wedge\Box no(v, w)\rightarrow\Diamond at\ most\ half\ of\ the(v, z)$

(i.e. □A□E◇H-2, by [5], Fact (5.3) and Rule 1)

[31] $\vdash\Box all(z, w)\wedge\Box no(w, v)\rightarrow\Diamond at\ most\ half\ of\ the(v, z)$

(i.e. □A□E◇H-4, by [30] and Fact (4.2))

[32] $\vdash\neg\Diamond at\ most\ half\ of\ the(v, z)\wedge\Box all(z, w)\rightarrow\neg\Box no(v, w)$ (by [30] and Rule 2)

[33] $\vdash\Box\neg at\ most\ half\ of\ the(v, z)\wedge\Box all(z, w)\rightarrow\Diamond\neg no(v, w)$

(by [32], Fact (3.1) and Fact (3.2))

[34] $\vdash\Box most(v, z)\wedge\Box all(z, w)\rightarrow\Diamond some(v, w)$

(i.e. □A□M◇I-1, by [33], Fact (2.6) and Fact (2.3))

[35] $\vdash\neg\Diamond at\ most\ half\ of\ the(v, z)\wedge\Box no(v, w)\rightarrow\neg\Box all(z, w)$ (by [30] and Rule 2)

[36] $\vdash\Box\neg at\ most\ half\ of\ the(v, z)\wedge\Box no(v, w)\rightarrow\Diamond\neg all(z, w)$

(by [35], Fact (3.1) and Fact (3.2))

[37] $\vdash\Box most(v, z)\wedge\Box no(v, w)\rightarrow\Diamond not\ all(z, w)$

(i.e. □E□M◇O-3, by [36], Fact (2.6) and Fact (2.2))

[38] $\vdash\Box no\neg(z, w)\wedge\Box all\neg(v, w)\rightarrow\Diamond at\ most\ half\ of\ the(v, z)$

(by [30], Fact (1.1) and Fact (1.2))

[39] $\vdash\Box no(z, D\text{-}w)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond at\ most\ half\ of\ the(v, z)$

(i.e. □E□A◇H-2, by [38] and Definition 2)

[40] $\vdash\neg\Diamond at\ most\ half\ of\ the(v, z)\wedge\Box all(z, w)\rightarrow\neg\Box no(w, v)$ (by [31] and Rule 2)

[41] $\vdash\Box\neg at\ most\ half\ of\ the(v, z)\wedge\Box all(z, w)\rightarrow\Diamond\neg no(w, v)$

(by [40], Fact (3.1) and Fact (3.2))

[42] $\vdash\Box most(v, z)\wedge\Box all(z, w)\rightarrow\Diamond some(w, v)$

(i.e. □M□A◇I-4, by [41], Fact (2.6) and Fact (2.3))

[43] $\vdash\neg\Diamond at\ most\ half\ of\ the(v, z)\wedge\Box no(w, v)\rightarrow\neg\Box all(z, w)$ (by [31] and Rule 2)

[44] $\vdash\Box\neg at\ most\ half\ of\ the(v, z)\wedge\Box no(w, v)\rightarrow\Diamond\neg all(z, w)$ (by [43], Fact (3.1) and Fact (3.2))

[45] $\vdash\Box most(v, z)\wedge\Box no(w, v)\rightarrow\Diamond not\ all(z, w)$

(i.e. □E□M◇O-4, by [44], Fact (2.6) and Fact (2.2))

[46] $\vdash\Box most(v, z)\wedge\Box no\neg(z, w)\rightarrow\Diamond not\ all\neg(v, w)$ (by [34], Fact (1.1) and Fact (1.3))

[47] $\vdash\Box most(v, z)\wedge\Box no(z, D\text{-}w)\rightarrow\Diamond not\ all(v, D\text{-}w)$

(i.e.□E□M◇O-1, by [46] and Definition 2)

[48] $\vdash\Box most(v, z)\wedge\Box all\neg(v, w)\rightarrow\Diamond some\neg(z, w)$ (by [37], Fact (1.2) and Fact (1.4))

[49] $\vdash\Box most(v, z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond some(z, D\text{-}w)$

(i.e.□A□M◇I-3, by [48] and Definition 2)

[50] $\vdash\Box no(D\text{-}w, z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond at\ most\ half\ of\ the(v, z)$

(i.e. □E□A◇H-1, by [39] and Fact (4.2))

[51] $\vdash\Box most(v, z)\wedge\Box no(D\text{-}w, z)\rightarrow\Diamond not\ all(v, D\text{-}w)$

(i.e. □E□M◇O-2, by [47] and Fact (4.2))

[52] $\vdash\Box most(v, z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond some(D\text{-}w, z)$

(i.e. □M□A◇I-3, by [49] and Fact (4.1))

[53] $\vdash\Box all(D\text{-}w, D\text{-}z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond at\ least\ half\ of\ the(v, D\text{-}z)$

(i.e. □A□A◇S-1, by [27], Fact (5.3) and Rule 1)

[54] $\vdash\neg\Diamond at\ least\ half\ of\ the(v, D\text{-}z)\wedge\Box all(D\text{-}w, D\text{-}z)\rightarrow\neg\Box all(v, D\text{-}w)$

(by [53] and Rule 2)

[55] $\vdash\Box\neg at\ least\ half\ of\ the(v, D\text{-}z)\wedge\Box all(D\text{-}w, D\text{-}z)\rightarrow\Diamond\neg all(v, D\text{-}w)$

(by [54], Fact (3.1) and Fact (3.2))

[56] $\vdash\Box fewer\ than\ half\ of\ the(v, D\text{-}z)\wedge\Box all(D\text{-}w, D\text{-}z)\rightarrow\Diamond not\ all(v, D\text{-}w)$

(i.e. □A□F◇O-2, by [55], Fact (2.8) and Fact (2.2))

[57] $\vdash\neg\Diamond at\ least\ half\ of\ the(v, D\text{-}z)\wedge\Box all(v, D\text{-}w)\rightarrow\neg\Box all(D\text{-}w, D\text{-}z)$

(by [53] and Rule 2)

[58] $\vdash\Box\neg at\ least\ half\ of\ the(v, D\text{-}z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond\neg all(D\text{-}w, D\text{-}z)$

(by [57], Fact (3.1) and Fact (3.2))

[59] $\vdash\Box fewer\ than\ half\ of\ the(v, D\text{-}z)\wedge\Box all(v, D\text{-}w)\rightarrow\Diamond not\ all(D\text{-}w, D\text{-}z)$

(i.e. □F□A◇O-3, by [58], Fact (2.8) and Fact (2.2))

Now, the other 29 generalized modal syllogisms have been deduced from the validity of □EM◇O-3. Similarly, more valid syllogisms can be inferred from it. This indicates that there are reducible relations between/among these syllogisms. Their validity can be proven similar to Theorem 1.

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## V. Conclusion

Due to the large number of generalized quantifiers in the English language, this paper only studies the fragment of generalized modal syllogistic that contains the quantifiers in Square–*all*" and Square–*most*". This paper proves that there are reducible relations between/ among the generalized modal syllogism □EM◇O-3 and at least the above 29 valid generalized modal syllogisms. To be specific, this paper firstly proves the validity of □EM◇O-3 on the basis of generalized quantifier theory, possible-world semantics, and set theory. Then, according to some facts and inference rules, the above 29 valid generalized modal syllogisms are derived from □EM◇O-3.

This method can also be used to study syllogisms with other generalized quantifiers, such as *at most* 1/3 *of the*, *more than* 1/3 *of the*, *at least* 2/3 *of the*, *fewer than* 2/3 *of the.* It is obvious that the above results obtained by deduction have not only consistency, but also theoretical value for the development of inference theory in artificial intelligence.

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# Application of Laplace Transform for Solving Improper Integrals Containing Bessel's Function as Integrand, In the Form of Hypergeometric Function

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*Abstract-* In this article we use Laplace Transform for solving Improper integrals whose Integrand contains Bessel's function in the form of hypergeometric function. The theory of Bessel's function is a rich subject due to its significant role in providing solutions for differential equations associated with many applications Their applications span across disciplines like heat conduction, electromagnetism, signal processing, and more.

*Keywords:* *laplace transform, bessel's function, hypergeometric function, improper integral.*

*GJSFR-F Classification:* *MSC Code: 33-XX*

*Strictly as per the compliance and regulations of:*

Notes

# Application of Laplace Transform for Solving Improper Integrals Containing Bessel's Function as Integrand, In the Form of Hypergeometric Function

Ranjana Shrivastava [α], Usha Gill [σ] & Kaleem Quaraishi [ρ]

*Abstract-* In this article we use Laplace Transform for solving Improper integrals whose Integrand contains Bessel's function in the form of hypergeometric function. The theory of Bessel's function is a rich subject due to its significant role in providing solutions for differential equations associated with many applications Their applications span across disciplines like heat conduction, electromagnetism, signal processing, and more.

*Keywords: laplace transform, bessel's function, hypergeometric function, improper integral.*

## I. Introduction

Special functions are applicable in solving variety of problems of mathematical physics, statistics mechanics, dynamics, functional analysis etc. In recent few decades, many researchers are working on this field, especially with Bessel's function .Many researchers applied different Integral transforms include the Fourier Transform, Laplace Transform and the Mellin Transform for solving many problems of science and engineering The solution of many Engineering problems like acoustics, electromagnetism, quantum mechanics and other areas frequently lead to integrals involving Bessel's functions. When we solve these types of problems by using Integral Transform it is necessary to know the Integral Transform of Bessel's Function.

Mathematically, Bessel's function is defined by

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(\mathrm{n+k+1})} (t/2)^{n+2k}$$

And is known as Bessel's function of the first kind of order 'n'

The Laplace transform of a real-valued function of real-valued function $f(t) of\ t$ when $t > 0$ is denoted by $f(s)\ or\ L\big(f(t)\big)$ where s is a real or complex parameter and it is denoted by improper integration given by $f(s) = \int_0^\infty e^{-st} f(t)dt$

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The aim of present article is to determine the value of improper integral containing Bessel's function as Integrand by using Laplace transform.

*a) Some properties of Laplace Transform*

*Linearity Property:*

The linearity property of the Laplace transform is a fundamental property that states the transform of a linear combination of functions is equal to the linear combination of the individual transforms. Mathematically, this property can be expressed as follows:

Notes

$$L(a(f(t)) + L(b(f(t)) = aL(f(t)) + bL(f(t))$$

*Change of scale Property:* If Laplace transform of function $f(t)$ *is* $F(s)$ then, Laplace transform of $e^{at} f(at)$ is given by $\frac{1}{a} f\left(\frac{s}{a}\right)$

*Shifting Property:* If Laplace transform of a function$f(t)$ *is* $(f(s)$then, Laplace transform of $e^{at} f(t)$ is given by $f(s-a)$.

*Laplace transform of the derivative of the function:* The Laplace transform of the derivative of a function $f(t)$ is given by $sf(s) - f(0)$ where $f(s)$ is the Laplace Transform of $f(t)$.

*Laplace transform of Integral of the function:* If Laplace transform of function $f(t)$ *is* $f(s)$ then $L\int_0^t f(t)dt = \frac{1}{s} f(s)$

*Laplace transform of Function, (Multiplication by 't' theorem):* If Laplace transform of function $f(t)$ *is* $f(s)$then,$L(t(f(t)) = (-1)\frac{d}{ds} f(s)$

Laplace transform of Bessel's Function:

The Bessel's Function is defined as

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}$$

$$LJ_n(t) = L\left(\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}\right)$$

$$\int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} (t/2)^{n+2k}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n}\, k!\,\Gamma(n+k+1)} \int_0^{\infty} e^{-st} t^{2k+n} dt$$

$$\frac{1}{s^{n+1}} \cdot \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k}\, k!\,\Gamma(n+k+1)} \frac{\Gamma(2k+n+1)}{\Gamma(n+1)\,.\,s^{2k}}$$

$$\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k(1+n)_{2k}}{2^{2k}\ k!\ (1+n)_k}\frac{1}{.s^{2k}}$$

$$\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}\left(\frac{1+n}{2}\right)_k\left(1+\frac{n}{2}\right)_k}{2^{2k}\ k!\ (1+n)_k}\frac{1}{.s^{2k}}$$

$$\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}2F_1\begin{bmatrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{bmatrix}$$

Notes

*Hypergeometric Function:* Hypergeometric function denoted as $2F_1(\mathrm{a,b;c;z})$ is a special function that arises in various areas of mathematics including analysis and mathematical physics. It is defined by the series

$$2F_1(\mathrm{a,b;c;z})=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!\ (c)_k}z^k$$

Where $(a)_k$ denotes the Pochhammer's symbol

$$(a)_k=a\,(a+1)(a+2)\ldots\ldots\ldots\ldots(a+n-1)\text{ and }\ (a)_0=1$$

The hypergeometric function is a generalization of several elementary function such as the binomial series and it satisfies various differential equations including the hypergeometric differential equation. It has applications in solving differential equation, evaluating definite integrals and expressing solutions to certain problems. The hypergeometric function has many special cases and identities making it a powerful tool in mathematical analysis. It is extensively studied and applied in different branches of mathematics and physics.

## II. Applications

In this section, some applications are given in order to explain the advantage of Laplace transform of Bessel's function for evaluating the improper Integral containing Bessel's function as Integrand.

1. Evaluation of $I=\int_0^{\infty}e^{-t}J_n(t)dt$

we have, $LJ_n(t)=\int_0^{\infty}e^{-st}J_n(t)dt=\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}2F_1\begin{bmatrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{bmatrix}$

here $\mathrm{s}\rightarrow 1$, therefore,

$$\int_0^\infty e^{-t} J_n(t)dt = \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) & 1 \\ & 1+n\,; & & \end{matrix}\right]$$

2. Evaluation of $I = \int_0^\infty te^{-2t} J_n(t)dt$

$$LJ_n(t) = \frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

$$L(tJ_n(t)) = \frac{1}{s^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

here s → 2,

$$\int_0^\infty te^{-2t} J_n(t)dt = \frac{1}{2^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{2^2} \\ & 1+n\,; & \end{matrix}\right]$$

3. Evaluation of $I = \int_0^\infty e^{-t} \left(\int_0^t J_n(u)\, du\right)dt$

$$LJ_n(t) = \frac{1}{s^{n+1}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

By property of Laplace Transform of integral,

$$L\int_0^t J_n(u)\, du = \frac{1}{s} L(J_n(u)) = \frac{1}{s^{n+2}} \cdot \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right)\frac{1}{s^2} \\ & 1+n\,; & \end{matrix}\right]$$

here s → 1,

$$L\int_0^t J_n(u)\, du = \frac{1}{s} L\left(J_n(u)\right) =. \frac{1}{2^n} 2F_1\left[\begin{matrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) & 1 \\ & 1+n\,; & & \end{matrix}\right]$$

Notes

Notes

4. Evaluation of $I = \int_0^\infty e^{-2t}\left[\frac{d}{dt}J_n(t)\right]dt$

$$L(J_n(t)) = \frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(n+1)}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{\Gamma(n+1)\,.s^{2k}}$$

Now, by using the property of Laplace transform of derivative of function, we have

$$\mathrm{L}\left[\frac{d}{dt}J_n(t)\right] = s.\frac{1}{s^{n+1}}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.s^{2k}}$$

$$=.\frac{1}{s^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.s^{2k}} \quad (1)$$

Then by the differentiation of Laplace transform we have

$$\mathrm{L}\left[\frac{d}{dt}J_n(t)\right] = \int_0^\infty e^{-st}\left[\frac{d}{dt}J_n(t)\right]dt \quad (2)$$

$$\int_0^\infty e^{-st}\left[\frac{d}{dt}J_n(t)\right]dt =$$

$$.\frac{1}{s^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.s^{2k}}$$

Taking s → 2,

$$\int_0^\infty e^{-2t}\left[\frac{d}{dt}J_n(t)\right]dt =$$

$$.\frac{1}{2^n}\cdot\frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\ k!\,\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{.2^{2k}}$$

$$= \frac{1}{2^{2n+2k}}\,2F_1\begin{bmatrix}\frac{1+n}{2} & , & \left(1+\frac{n}{2}\right) & & 1\\ & & 1+n\,; & & \end{bmatrix}$$

## III. Conclusion

In this article, we have successfully discussed the application of Laplace transform for solving improper integrals whose Integrand contains Bessel's Function. The given numerical applications show the advantage of Laplace transform for evaluating improper integral whose Integrand contains Bessel's function.

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# Entropy-Based Stability of Fractional Self-Organizing Maps with Different Time Scales

By C. A Peña Fernández

*Abstract-* The behavior of self-organizing neural maps, which develop through a combination of long and short-term memory, involves different time scales. Such a neural network's activity is characterized by a neural activity equation representing the fast phenomenon and a synaptic information efficiency equation representing the slow part of the neural system. The work reported here proposes a new method to analyze the dynamics of self-organizing maps based on the flow-invariance principle, considering the performance of the system's different time scales. In this approach, the equilibrium point is determined based on the estimate for the entropy at each iteration of the learning rule, which is generally sufficient to analyze existence and uniqueness. In this sense, the viewpoint reported here proves the existence and uniqueness of the equilibrium point on a fractional approach by using a Lyapunov method extension for Caputo derivatives when $0 < \alpha < 1$. Furthermore, the global exponential stability of the equilibrium point is proven with a strict Lyapunov function for the flow of the system with different time scales and some numerical simulations.

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# Entropy-Based Stability of Fractional Self-Organizing Maps with Different Time Scales

C. A Peña Fernández

*Abstract-* The behavior of self-organizing neural maps, which develop through a combination of long and short-term memory, involves different time scales. Such a neural network's activity is characterized by a neural activity equation representing the fast phenomenon and a synaptic information efficiency equation representing the slow part of the neural system. The work reported here proposes a new method to analyze the dynamics of self-organizing maps based on the flow-invariance principle, considering the performance of the system's different time scales. In this approach, the equilibrium point is determined based on the estimate for the entropy at each iteration of the learning rule, which is generally sufficient to analyze existence and uniqueness. In this sense, the viewpoint reported here proves the existence and uniqueness of the equilibrium point on a fractional approach by using a Lyapunov method extension for Caputo derivatives when $0 < \alpha < 1$. Furthermore, the global exponential stability of the equilibrium point is proven with a strict Lyapunov function for the flow of the system with different time scales and some numerical simulations.

## *Highlights*

- Self-organizing neural maps develop through long and short-term memory and have fast and slow activity equations.
- Lyapunov method extended for Caputo derivatives helps estimate and examine entropy behavior during learning.
- Self-organizing neural maps based on competitive differential equations lack entropy-based synaptic efficiency.
- The final stage of training is not affected by external stimuli.
- Pattern learning has lower entropy-rate without external stimuli during learning process.

## I. Introduction

Self-organizing neural maps, also known as competitive neural networks, are an important class of neural networks. These networks focus on two key aspects: the ability to store desired patterns as stable equilibrium points and the mutual interference between neuron and learning dynamics. This research examines how cortical cognitive maps work by using a self-organizing map with differential equations for neural activity levels, short-term memory (STM), and synaptic information efficiency, long-term memory (LTM).

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STM and LTM models are usually based on classical Grossberg's approach or Amari's model for primitive neural competition [1, 2]. These models often involve mutually inhibitory neurons with fixed synaptic connections [3, 4, 1]. Researchers have studied competitive neural systems using flow invariance theory and singular perturbation theory on large-scale networks. These networks have two types of state variables that describe the slow unsupervised dynamics of synapses and the fast dynamics of neural activity. The fast dynamics are usually represented by the goal of the self-organizing map, such as clustering or recognition. One example is the Willshaw-Malsburg model [1, 5] of topographic formation, which uses solutions of equations of synaptic self-organization coupled with the field equation of neural excitations to improve the understanding of the dynamics of cortical cognitive maps [4, 6]. However, the design using classical competitive differential equations is not broad enough because the synaptic efficiency is not dependent on entropy. Therefore, this paper extends these approaches to incorporate systems where external stimuli can modify the synaptic information efficiency. This is done by estimating the entropy of information transfer between neurons. In other words, this research focuses on LTM in terms of how information transfer evolves. Fractional differential equations are used to model synaptic efficiency based on entropy estimation, which accurately improves the models based on STM and LTM approaches [7, 8, 9, 10, 11].

The study proposes an alternate model for information transfer between neurons based on entropy. It applies McMillan-Shannon's approach to fractional competitive differential equations to determine the mathematical conditions for when STM and the estimation of entropy related to LTM have bounded trajectories. The study uses an alternative version of Lyapunov functions to examine exponential stability in fractional-order systems [7]. This proposal is more comprehensive than previous studies, such as those conducted by [3, 4, 1, 8]. It presents a strict Lyapunov function for the neural multi-time scale system, which demonstrates the existence and uniqueness of the equilibrium point. Additionally, this proposal provides some conditions for global exponential stability based on singular perturbation theory and variable entropy-dependent synaptic efficiency.

The paper is structured as follows: In Section 2, the mathematical background related to self-organizing maps modeled by Caputo derivatives is presented. This allows for the inclusion of fractional order in the differential equations associated with STM and LTM dynamics. Section 3 analyzes the equilibrium point from the perspective of synaptic efficiency, and covers the existence and uniqueness of the equilibrium point. Section 4 presents numerical simulations that provide findings about SOM's equilibrium point and entropy when exposed to external stimulus. Finally, Section 5 offers closing remarks and final comments.

## II. Mathematical Background

The synaptic information efficiency (SIE) can be defined according to the existence of an ergodic process related to binned input and output spike

Ref

[7] N. Aguila-Camacho, M. A. Duarte-Mermoud, J. A. Gallegos, Lyapunov functions for fractional order systems, Communications in Nonlinear Science and Numerical Simulation 19 (9) (2014) 2951–2957. doi:10.1016/j.cnsns.2014.01.022.

trains whose length is $N$ bins [4]. In computational experiments, $N$ is associated with iterations during the learning procedure on time domain. In this way, by assuming $m_{ij}$ the synaptic efficiency between $i$-th and the $j$-th neuron then let $S_{in}, S_{out}$ the binned input and output spike trains, respectively, both defined by $\{\sigma_1, \ldots, \sigma_N\}$, where $\sigma_i \in \{0, 1\}$ represents $i$-th bin, and assume that this string is the realization of a stationary and ergodic stochastic process. By using the Shannon-MacMillan-Brieman's theorem [12, 13] $-\frac{1}{N}\log_2 p(\sigma_1, \ldots, \sigma_N) \to H$, where $H$ is the entropy rate of $X$ (events set) and $p(\sigma, \ldots, \sigma_N)$ is the probability of obtaining the string $(\sigma_1, \ldots, \sigma_N)$ as a realization of $X$. In that sense, the estimate for the entropy at $N$-th iteration will be represented by $\hat{H}_N = -\frac{1}{N}\log_2 \hat{p}(\sigma_1, \ldots, \sigma_N)$. So, for $i$-th neuron,

$$\begin{aligned}\mathrm{SIE}_i &= \hat{H}_N(\sigma_1, \ldots, \sigma_N)_i - \hat{H}_N(\sigma_1, \ldots, \sigma_N|S_{in})_i \\ &= \frac{1}{N}\log_2 \hat{p}_i(\sigma_1, \ldots, \sigma_N|S_{in}) - \frac{1}{N}\log_2 \hat{p}_i(\sigma_1, \ldots, \sigma_N),\end{aligned}$$

where $\hat{H}_N(\sigma_1, \ldots, \sigma_N|S_{in})$ is the estimated output spike train entropy given the input spike train $S_{in}$, also known as conditional entropy (see [4]). In this way, for $N$ iterations,

$$m_{ij} = \frac{1}{N}\log_2 \beta_{ij} \triangleq \frac{1}{N}\log_2 \frac{\hat{p}_i(\sigma_1, \ldots, \sigma_N|S_{in})\hat{p}_j(\sigma_1, \ldots, \sigma_N)}{\hat{p}_j(\sigma_1, \ldots, \sigma_N|S_{in})\hat{p}_i(\sigma_1, \ldots, \sigma_N)}. \tag{1}$$

Nevertheless, on the time domain, $\beta_{ij}$ can be represented as $\beta_{ij}(t)$, for $t \geq 0$, since at $N$-th iteration there is some time related to the learning procedure. Sometimes, to guarantee coherence with symbolic representation for $N$ iterations, the notation $\beta_{ij}$ is retained.

Since the nature of the transfer process related to synaptic information becomes more precise using fractional differential equations, the Caputo definition for the fractional derivative is more suitable because it incorporates initial conditions and its integer order derivatives. Although there are several definitions regarding the fractional derivative of order $\alpha \geq 0$, in the time domain the general network equations describing the temporal evolution of the STM and LTM states for the $j$-th neuron of $M$ neurons are

$$\frac{\varepsilon}{\Gamma(1-\alpha)}\int_0^t \frac{x_j^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda = -a_j x_j + \sum_{i=1}^{M} D_{ij} f(x_i) + \frac{B_j}{N}\sum_{i=1}^{P}\log_2 \beta_{ij} y_i$$

$$\frac{1}{N\Gamma(1-\alpha)}\int_0^t \frac{[\log_2 \beta_{ij}(\lambda)]^{(m+1)}}{(t-\lambda)^\alpha}d\lambda = -\frac{1}{N}\log_2 \beta_{ij} + y_i f(x_j),$$

where $\alpha$ is the Caputo's fractional order defined by $\alpha = m + \gamma$, $m \in \mathbb{Z}^+$, $0 < \gamma \leq 1$; $\Gamma(\cdot)$ is the Gamma function, $x_j$ is the current activity level (STM),

Ref

[4] M. London, A. Schreibman, M. Häusser, M. E. Larkum, I. Segev, The information efficacy of a synapse, Nature Neuroscience 5 (4) (2002) 332–340. doi:10.1038/nn826.

$a_j$ is the time constant of the neuron, $B_j$ is the contribution of the external stimulus term, $f(x_i)$ is the neuron's output, $y_i$ is the external stimulus, and $\varepsilon$ is the fast time-scale associated with the STM state. $D_{ij}$ represents a synaptic connection parameter between the $i$-th neuron and $j$-th neuron.

According to the definition of $m_{ij}$ in (1) and unlike [1, 9, 6], the self-organizing map is implicitly modeled by a network of sources emitting input signals with a prescribed probability distribution and external stimulus $\boldsymbol{y} \triangleq [y_i]$. By using the dynamic transform $w_j = \langle \boldsymbol{y}, \log_2 \boldsymbol{\beta}_j \rangle$ the model gets as follows:

$$\frac{\varepsilon}{\Gamma(1-\alpha)} \int_0^t \frac{x_j^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda = -a_j x_j + \sum_{i=1}^{M} D_{ij} f(x_i) + \frac{B_j}{N} w_j \tag{2}$$

$$\frac{1}{N\Gamma(1-\alpha)} \int_0^t \frac{w_j^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda = -\frac{1}{N} w_j + \|\boldsymbol{y}\|^2 f(x_j), \tag{3}$$

where the external stimuli are assumed to be normalized vectors of unit magnitude $\|\boldsymbol{y}\|^2 = 1$.

As each string is the realization of a stationary and ergodic stochastic process, it will be assumed a stochastic column matrix for each iteration of $\boldsymbol{\beta}_j$, such that there exists $P \triangleq [\bar{p}_{ij}] \in \mathbb{R}^{M \times M}$, with $\sum_{i=1}^{M} \bar{p}_{ik} = 1$ and $\boldsymbol{\beta}_j^{\sigma+1} = P \boldsymbol{\beta}_j^{\sigma}$ for $(\sigma+1)$-th iteration. In this way, it can be noted that $\dot{\boldsymbol{\beta}}_j = \lim_{\Delta t \to 0} \frac{1}{\Delta^k t} (P - I)^k \boldsymbol{\beta}_j$, such that the sum of elements in each column of $(P-I)^k \triangleq [p_{ij}]$ is equal to zero, i.e.,

$$\beta_{ij}^{(k)} = \lim_{\Delta t \to 0} \frac{1}{\Delta^k t} \sum_{s=1}^{M} p_{is} \beta_{sj}, \qquad \sum_{i=1}^{M} p_{ik} = 0, \quad \text{for } \forall\, k. \tag{4}$$

## III. Equilibrium and Global Asymptotic Stability

The existence and uniqueness of the equilibrium point are given based on flow-invariance while the global exponential stability will be based on a strict Lyapunov function for fractional-order approaches. It is well-known that the flow-invariance theory provides a qualitative viewpoint about the dynamics of a system. To begin with, it will be defined the following theorem before presenting the main results of this paper.

*Theorem 1. Consider the system of fractional differential equations:*

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda = -a_i x_i + \sum_{j=1}^{M} D_{ij} f(x_j) + \frac{B_i}{N} w_i, \tag{5}$$

Notes

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda = -\frac{1}{N}w_i + f(x_i), , \tag{6}$$

*for $i = 1, \ldots, M$, where $a_i > 0$ for all $i = 1, \ldots, M$, and $f$ is locally Lipschitz and bounded, that is, there exists a constant $C > 0$ such that $-C \leq f(x) \leq C$ for all $x \in \mathbb{R}$. Then for any $\varepsilon > 0$ and for any initial condition $\{x(0), w(0)\} \in \mathbb{R}^{2M}$ there exists a $T \geq 0$ such that*

$$w_i(t) \in [-C-\varepsilon, C+\varepsilon], \qquad x_i(t) \in [-l_i-\varepsilon, l_i+\varepsilon],$$

*for all $i = 1, \ldots, M$, with equilibrium point $e = [\bar{x}_i \quad \bar{w}_i] = [\bar{x}_i \ \langle \boldsymbol{y}, \log_2 \bar{\boldsymbol{\beta}}_i \rangle]$, where $\forall\, \bar{\beta}_{ij} \in \bar{\boldsymbol{\beta}}_i$ satisfies*

$$\int_0^t\int_0^\lambda \frac{y_j}{\bar{\beta}_{ij}^2 \ln 2}\sum_{r=1}^M\sum_{q=1}^M\sum_{s=1}^M p_{is}p_{sq}p_{ir}\bar{\beta}_{qj}\bar{\beta}_{rj}\cdot d\lambda^2 = 0 \tag{7}$$

$$\int_0^t \frac{y_j}{\bar{\beta}_{ij}^2 \ln 2}\sum_{r=1}^M\sum_{q=1}^M\sum_{s=1}^M p_{is}p_{sq}p_{ir}\bar{\beta}_{qj}\bar{\beta}_{rj}\cdot d\lambda = 0 \tag{8}$$

$$S_m\sum_{s=1}^M p_{is}\bar{\beta}_{sj}y_j = \frac{y_j}{\bar{\beta}_{ij}\ln 2}\sum_{r=1}^M\sum_{q=1}^M\sum_{s=1}^M p_{is}p_{sq}p_{ir}\bar{\beta}_{qj}\bar{\beta}_{rj}, \tag{9}$$

*for $m = 0, 1$ and $2$, respectively, and $S_m \in \mathbb{Z}^+$ for all $t \geq T$.*

*Proof.* Since $f$ is Lipschitz, system (5)-(6) has local existence and uniqueness of solutions. Furthermore, since $f$ is uniformly bounded, there exist constants $K_1, \ldots, K_5$ such that

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda\right| \leq K_1 + K_2\|x_i(t)\| + K_3\|w_i(t)\|$$

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda\right| \leq K_4 + K_5\|w_i(t)\|,$$

thus all solutions are defined globally (for all $t \geq 0$).

Given $\varepsilon > 0$, let $\delta_i > 0$ be defined as

$$\delta_i = \begin{cases} \min\left\{\frac{a_i\varepsilon}{2|B_i|}, \varepsilon\right\} & B_i \neq 0 \\ \varepsilon & B_i = 0 \end{cases},$$

such that $-|B_i|\delta_i + a_i\varepsilon \geq a_i\varepsilon/2$, for all $i = 1, \ldots, M$. Then for $t \geq 0$ and for $w_i(t) \leq -C - \delta_i$ the following inequality holds:

Notes

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda \geq -\frac{1}{N}(-C-\delta_i)+f(x_i)=\frac{1}{N}\delta_i+$$

$$\left[f(x_i)+\frac{1}{N}C\right]\geq \delta_i>0. \tag{10}$$

Similarly, for $t \geq 0$ and for $w_i(t) \geq C+\delta_i$, the following inequality holds:

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha}d\lambda \leq -\frac{1}{N}(C+\delta_i)+f(x_i)=-\frac{1}{N}\delta_i+$$

$$\left[f(x_i)-\frac{1}{N}C\right]\leq -\delta_i<0. \tag{11}$$

Since $\alpha \leq 1$ then $\Gamma(1-\alpha)>0$ and $(t-\lambda)^\alpha>0$. In that sense, from (10)-(11), both inequalities are guaranteed if and only if $w_i^{(m+1)}(\lambda)>0$ and $w_i^{(m+1)}(\lambda)<0$, respectively.

By mathematical induction, it can be noted that

$$w_i^{(m+1)}=\sum_{j=1}^{M}\left[\sum_{k=0}^{m}\frac{A_k}{\beta_{ij}^{m+1-k}}(\dot{\beta}_{ij})^{m+1-k}y_j^{(k)}+\frac{S_k}{\beta_{ij}^{m+1-k}}\beta_{ij}^{(k+1)}y_j^{(m-k)}\right]$$

$$+\frac{d^{m-2}}{d\lambda^{m-2}}\quad\left.\frac{\dot{\beta}_{ij}\ddot{\beta}_{ij}y_j}{\beta_{ij}^2\ln 2}\right),$$

where $A_k, B_k \in \mathbb{Z}^+$. So, for $w_i(t) \leq -C-\delta_i$ and $w_i(t) \geq C+\delta_i$,

$$\sum_{j=1}^{M}\left[\sum_{k=0}^{m}\frac{A_k}{\beta_{ij}^{m+1-k}}(\dot{\beta}_{ij})^{m+1-k}y_j^{(k)}+\frac{S_k}{\beta_{ij}^{m+1-k}}\beta_{ij}^{(k+1)}y_j^{(m-k)}\right]>$$

$$-\frac{d^{m-2}}{d\lambda^{m-2}}\quad\left.\frac{\dot{\beta}_{ij}\ddot{\beta}_{ij}y_j}{\beta_{ij}^2\ln 2}\right), \tag{12}$$

$$\sum_{j=1}^{M}\left[\sum_{k=0}^{m}\frac{A_k}{\beta_{ij}^{m+1-k}}(\dot{\beta}_{ij})^{m+1-k}y_j^{(k)}+\frac{S_k}{\beta_{ij}^{m+1-k}}\beta_{ij}^{(k+1)}y_j^{(m-k)}\right]<$$

$$-\frac{d^{m-2}}{d\lambda^{m-2}}\quad\left.\frac{\dot{\beta}_{ij}\ddot{\beta}_{ij}y_j}{\beta_{ij}^2\ln 2}\right), \tag{13}$$

respectively.

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By using (4) in (12), it can be obtained

$$\lim_{\Delta t\to 0}\sum_{j=1}^{M}\left[\sum_{k=0}^{m}\frac{A_k}{\Delta^{m-k-2}t\beta_{ij}^{m+1-k}}\sum_{s=1}^{M}p_{is}\beta_{sj}y_j^{(k)}\right.$$

$$\left.+\frac{S_k}{\Delta^{k-2}t\beta_{ij}^{m+1-k}}\sum_{s=1}^{M}p_{is}\beta_{sj}y_j^{(m-k)}\right]>$$

$$-\lim_{\Delta t\to 0}\frac{d^{m-2}}{d\lambda^{m-2}}\quad\frac{y_j}{\beta_{ij}^2\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}\beta_{qj}p_{ir}\beta_{rj}\Bigg).$$

For $k<m$, the evaluation of limits above yields,

$$\infty>-\frac{d^{m-2}}{d\lambda^{m-2}}\quad\frac{y_j}{\beta_{ij}^2\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}\beta_{qj}p_{ir}\beta_{rj}\Bigg),\tag{14}$$

which represents an obvious condition. For $k=m$,

$$\lim_{\Delta t\to 0}\sum_{j=1}^{M}\left[\frac{A_m\Delta^2 t}{\beta_{ij}}\sum_{s=1}^{M}p_{is}\beta_{sj}y_j^{(m)}+\frac{S_m}{\Delta^{m-2}t\beta_{ij}}\sum_{s=0}^{M}p_{is}\beta_{sj}y_j\right]$$

$$>-\lim_{\Delta t\to 0}\frac{d^{m-2}}{d\lambda^{m-2}}\quad\frac{y_j}{\beta_{ij}^2\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}\beta_{qj}p_{ir}\beta_{rj}\Bigg).$$

Therefore, by evaluating the limits for $m>2$, the inequality (14) is obtained again. However, for $0\le m\le 2$ (i.e., $m=0,1$ and 2),

$$\int_0^t\int_0^\lambda\quad\frac{y_j}{\beta_{ij}^2\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}p_{ir}\beta_{qj}\beta_{rj}\Bigg)\,d\lambda^2>0,\tag{15}$$

$$\int_0^t\quad\frac{y_j}{\beta_{ij}^2\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}p_{ir}\beta_{qj}\beta_{rj}\Bigg)\,d\lambda>0,\tag{16}$$

$$S_m\sum_{s=1}^{M}p_{is}\beta_{sj}y_j<\frac{y_j}{\beta_{ij}\ln 2}\sum_{r=1}^{M}\sum_{q=1}^{M}\sum_{s=1}^{M}p_{is}p_{sq}p_{ir}\beta_{qj}\beta_{rj},\tag{17}$$

respectively. Since the operator $d^{m-2}/d\lambda^{m-2}$ becomes an integral for $m<2$.

So, for $w_i(t) \leq -C-\delta_i$, the inequalities (15)-(17) guarantee $w^{(m+1)} > 0$. The same treatment applies to (13), it is only to change the less than symbol in (15)-(17) by greater than symbol. Therefore, for any $i \in \{1, \ldots, M\}$ there exists a $T_i > 0$ such that

$$w_i(t) \in [-C-\delta_i, C+\delta_i] \subseteq [-C-\varepsilon, C+\varepsilon], \tag{18}$$

for all $t \geq T_i$. So, the equilibrium point $\bar{w}_i = \langle \boldsymbol{y}, \log_2 \bar{\boldsymbol{\beta}}_i \rangle \in [-C-\varepsilon, C+\varepsilon]$, where $\forall\ \bar{\beta}_{ij} \in \bar{\boldsymbol{\beta}}_i$ satisfies

$$\int_0^t \int_0^\lambda \quad \frac{y_j}{\bar{\beta}_{ij}^2 \ln 2} \sum_{r=1}^M \sum_{q=1}^M \sum_{s=1}^M p_{is} p_{sq} p_{ir} \bar{\beta}_{qj} \bar{\beta}_{rj} \Bigg) \, d\lambda^2 = 0, \quad (m=0)$$

$$\int_0^t \quad \frac{y_j}{\bar{\beta}_{ij}^2 \ln 2} \sum_{r=1}^M \sum_{q=1}^M \sum_{s=1}^M p_{is} p_{sq} p_{ir} \bar{\beta}_{qj} \bar{\beta}_{rj} \Bigg) \, d\lambda = 0, \quad (m=1)$$

$$S_m \sum_{s=1}^M p_{is} \bar{\beta}_{sj} y_j = \frac{y_j}{\bar{\beta}_{ij} \ln 2} \sum_{r=1}^M \sum_{q=1}^M \sum_{s=1}^M p_{is} p_{sq} p_{ir} \bar{\beta}_{qj} \bar{\beta}_{rj} \quad (m=2).$$

Defining $T_S = \max T_i$ then $w_i(t) \in [-C-\varepsilon, C+\varepsilon]$ holds for all $i \in \{1, \ldots, M\}$ and for all $t \geq T_S$.

Now, let $t \geq T_S$. For $x_i(t) \leq -l_i - \varepsilon$, (5) and (18) imply that

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda \geq a_i(l_i+\varepsilon) + \sum_{j=1}^M D_{ij} f(x_j)$$

$$+ \frac{B_i}{N}(-C-\delta_i) \geq a_i l_i + a_i \varepsilon - C \quad \sum_{j=1}^M |D_{ij}| + \frac{|B_i|}{N} \Bigg) - |B_i| \delta_i.$$

By defining $l_i = \frac{C}{a_i} \left( \sum_{j=1}^M |D_{ij}| + \frac{1}{N} |B_i| \right) > 0$, for $i = 1, \ldots, M$ then

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda \geq -|B_i|\delta_i + a_i \varepsilon \geq \frac{a_i \varepsilon}{2} > 0.$$

Similarly, for $t \geq T_S$ and for $x_j(t) \geq l_i + \varepsilon$, (5) and (18) imply that

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda \leq |B_i|\delta_i - a_i \varepsilon \leq -\frac{a_i \varepsilon}{2} < 0.$$

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Since $\alpha \leq 1$ then $\Gamma(1-\alpha) > 0$ and $(t-\lambda)^\alpha > 0$. In that sense, both before inequalities are guaranteed if and only if $x_i^{(m+1)}(\lambda) > 0$ and $x_i^{(m+1)}(\lambda) < 0$, respectively. Therefore, for any $i \in \{1, \ldots, M\}$ there exists a $T_i > 0$ such that

$$x_i(t) \in [-l_i - \varepsilon, l_i + \varepsilon]$$

for all $t \geq T_i$. Defining $T_X = \max T_i$ then $x_i(t) \in [-l_i - \varepsilon, l_i + \varepsilon]$ holds for all $i \in \{1, \ldots, M\}$ and for all $t \geq T_X$. $\square$

The system (5)-(6) is dissipative in $\mathbb{R}^{2M}$ and therefore, it has a compact global attractor

$$\mathcal{A} \subseteq D = \prod_{i=1}^{M} [-l_i, l_i] \times \prod_{i=1}^{M} [-C, C].$$

It follows from the proof of Theorem 1 that the set $D$ is flow invariant under (5)-(6). In other words, $D$ is positively invariant set of (5)-(6), i.e., any solution starting in $D$ at $t = 0$ remains in $D$ for all $t \geq 0$. Furthermore, from the proof of Theorem 1 the set $H$ that contains $D$ can be contracted to a point, and since $D$ is flow-invariant with respect to (5)-(6) then by the Brower fixed point theorem implies that there exists a point $e \in D$ is an equilibrium point of (5)-(6).

*Theorem 2. Suppose that $f(x)$ is $C^1$ with $|\dot{f}(x)| < k$ for all $x$ and*

$$a_i > k \left. \sum_{j=1}^{M} |D_{ij}| + |B_i| \right), \qquad i = 1, \ldots, M, \tag{19}$$

*then the equilibrium $e$ is unique.*

*Proof.* At the equilibrium point, $f(x_i) = \frac{1}{N} w_i$ from (6). Substituting these expressions in (5), it is obtained

$$0 = -a_i x_i + \sum_{j=1}^{M} D_{ij} f(x_j) + B_i f(x_i), \quad \text{for } i = 1, \ldots, M.$$

Since $a_i > 0$, $x_i$ can be expressed as

$$x_i = \frac{1}{a_i} \left. \sum_{j=1}^{M} D_{ij} f(x_j) + B_i f(x_i) \right) = G_i(x_1, \ldots, x_M).$$

The inequality (19) implies that

$$|G_i(x_1', \ldots, x_M') - G_i(x_1'', \ldots, x_M'')| < k |(x_1', \ldots, x_M') - (x_1'', \ldots, x_M'')|,$$

i.e., $G$ is a contracting map in $\mathbb{R}^M$. Consequently, there exists a unique fixed point of $G$. The $x_i$-coordinates of this fixed point uniquely determine the $w_i$-coordinates via $f(x_i) = \frac{1}{N}w_i$. Therefore, the equilibrium point is unique. $\square$

Now, let $e = [\bar{x}_i \;\; \bar{w}_i] = [\bar{x}_i \; \langle \boldsymbol{y}, \log_2 \bar{\boldsymbol{\beta}}_i \rangle]$ be the equilibrium point of (5)-(6) and introduce the change of variables $\phi_i = x_i - \bar{x}_i$, $\varphi_i = w_i - \bar{w}_i$ which shifts $e$ to the origin. Specifically, if $f_i(\phi_i) = f(\phi_i + \bar{x}_i) - \frac{1}{N}\bar{w}_i$, then $f_i(0) = 0$ and (5)-(6) may be rewritten as

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\phi_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda = -a_i\phi_i + \sum_{j=1}^{M} D_{ij} f(\phi_j) + \frac{B_i}{N}\varphi_i, \tag{20}$$

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\varphi_i^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda = -\frac{1}{N}\varphi_i + f(\phi_i) \;\; \text{for } i = 1, \ldots, M. \tag{21}$$

By assuming that there exists a Lyapunov function $V(t, \phi_i, \varphi_i)$ and class-$K$ functions $\gamma_i$ (for $i = 1, 2, 3$) satisfying

$$\gamma_1(|\phi_i|, |\varphi_i|) \le V(t, \phi_i, \varphi_i) \le \gamma_1(|\phi_i|, |\varphi_i|),$$

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{V^{(m+1)}(\lambda, \phi_i, \varphi_i)(\lambda)}{(t-\lambda)^\alpha} d\lambda \le -\gamma_3(|\phi_i|, |\varphi_i|),$$

then the system (20)-(21) becomes asymptotically stable at equilibrium point $e$ [7].

*Lemma 1. [see [7]] Let $r(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time $t \ge t_0$*

$$\frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{[r^2(\lambda)]^{(m+1)}}{(t-\lambda)^\alpha} d\lambda \le \frac{r(t)}{2\Gamma(1-\alpha)} \int_0^t \frac{r(\lambda)^{(m+1)}(\lambda)}{(t-\lambda)^\alpha} d\lambda.$$

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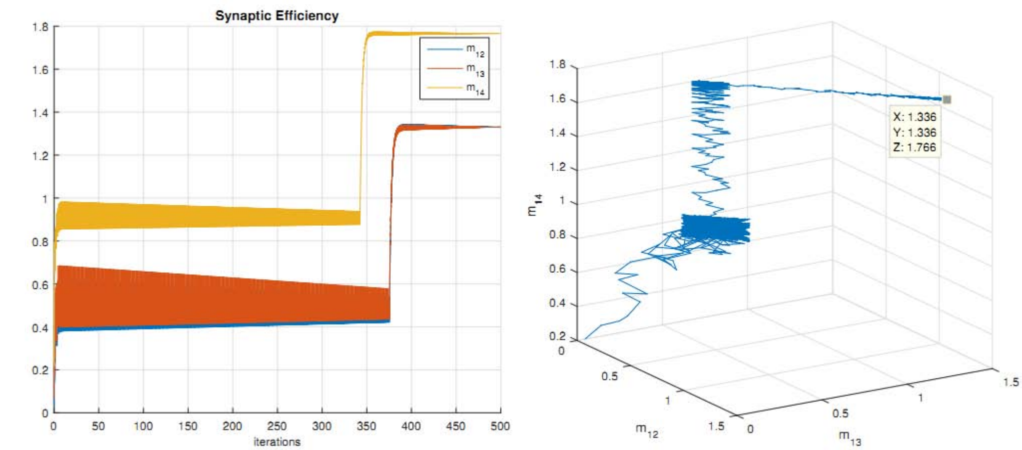


*Figure 1:* Evolution of synaptic efficiency $m_{ij}$ (for $i = 1, j = 2, 3, 4$) toward equilibrium point $(1.336, 1.336, 1.768)$ after $N = 500$ iterations when each neuron receives an external stimulus based on neighborhood's winner.

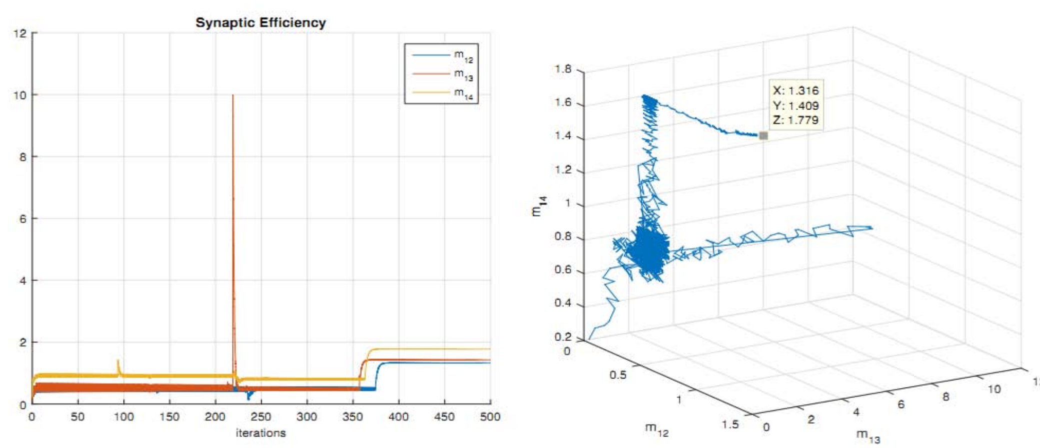


*Figure 2:* Evolution of synaptic efficiency $m_{ij}$ (for $i = 1, j = 2, 3, 4$) toward equilibrium point $(1.316, 1.409, 1.779)$ after $N = 500$ iterations when each neuron updates its weights without external stimulus.

*Theorem 3. Suppose that $f(x)$ is $\mathcal{C}^1$ with $|\dot{f}(x)| \leq k$ for all $x$ and $a_i > 0$.*

*If*

$$\max_i \left\{ \frac{1}{2}\left(\frac{|B_i|}{a_i} + k\right) + \sum_{j=1}^{M} \frac{1}{2} k \left(\frac{|D_{ij}|}{a_i} + \frac{|D_{ji}|}{a_j}\right) \right\} < 1 \qquad (22)$$

*then $e$ is a global attractor for the system* (20)-(21)*. Moreover, all solutions of* (20)-(21) *converge to $e$ exponentially fast as $t \to \infty$.*

*Proof.* The global convergence will be proved by using a Lyapunov function for (20)-(21). Let

$$V(\phi_i, \varphi_i) = \frac{1}{2}\sum_{i=1}^{M}\left(\frac{\phi_i^2}{a_i} + \varphi_i^2\right),$$

then the fractional derivative for $V(\phi_i, \varphi_i)$ is defined by

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V^{(m+1)}(\phi_i, \varphi_i)}{(t-\lambda)^{\xi}} d\lambda =$$

$$\frac{1}{2\Gamma(1-\alpha)}\sum_{i=1}^{M}\left(\frac{1}{a_i}\int_0^t \frac{[\phi_i^2(\lambda)]^{(m+1)}}{(t-\lambda)^{\alpha}} d\lambda + \int_0^t \frac{[\varphi_i^2(\lambda)]^{(m+1)}}{(t-\lambda)^{\alpha}} d\lambda\right),$$

and by using the Lemma 1 together with (20)-(21) yields

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V^{(m+1)}(\phi_i, \varphi_i)}{(t-\lambda)^{\alpha}} d\lambda \leq -\sum_{i=1}^{M}\left(\phi_i^2 + \varphi_i^2\right) \tag{23}$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{M} D_{ij}\frac{\phi_i}{a_i} f_j(\phi_j) + \sum_{i=1}^{M}\varphi_i\left(\frac{\phi_i}{a_i}B_i + f_i(\phi_i)\right).$$

Since $f_i(0) = 0$ and $|\dot{f}_i(x)| = |\dot{f}(x + \bar{x}_i)| < k$, it is possible to have $|f_i(\phi_i)| < k|\phi_i|$. Consequently, with this last fact and the Minkowski inequality, (23) can be replaced by the inequality

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V^{(m+1)}(\phi_i, \varphi_i)}{(t-\lambda)^{\alpha}} d\lambda < -\sum_{i=1}^{M}\left(\phi_i^2 + \varphi_i^2\right)$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{M}|D_{ij}|k\frac{1}{a_i}|\phi_i||\phi_j| + \sum_{i=1}^{M}\left(\frac{|B_i|}{a_i} + k\right)|\varphi_i||\phi_i|.$$

The right-hand side of this inequality is given by the quadratic form with the matrix $-Q$ where $Q$ has the following structure:

$$Q = \begin{bmatrix} 1 - \frac{1}{2}k\left(\frac{|D_{11}|}{a_1} + \frac{|D_{11}|}{a_1}\right) & -\frac{1}{2}\left(\frac{|B_1|}{a_1} + k\right) & -\frac{1}{2}k\left(\frac{|D_{12}|}{a_1} + \frac{|D_{21}|}{a_2}\right) & 0 & \cdots \\ -\frac{1}{2}\left(\frac{|B_1|}{a_1} + k\right) & 1 & 0 & 0 & \cdots \\ -\frac{1}{2}k\left(\frac{|D_{21}|}{a_2} + \frac{|D_{12}|}{a_1}\right) & 0 & 1 - \frac{1}{2}k\left(\frac{|D_{22}|}{a_2} + \frac{|D_{22}|}{a_2}\right) & -\frac{1}{2}\left(\frac{|B_2|}{a_2} + k\right) & \cdots \\ 0 & 0 & -\frac{1}{2}\left(\frac{|B_2|}{a_2} + k\right) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$



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According to Gerschgorin's theorem applied to $-Q$, there are $M$ disks centered at $z = -1$ (in $\mathbb{C}$-plane) with radius $\frac{1}{2}\left(\frac{|B_i|}{a_i} + k\right)$. In the same way, there are $M$ disks centered at $z = \frac{1}{2}k\left(\frac{|D_{ii}|}{a_i} + \frac{|D_{ii}|}{a_i}\right) - 1$ and radius $\frac{1}{2}\left(\frac{|B_i|}{a_i} + k\right) + \sum_{j=1}^{M} \frac{1}{2}k\left(\frac{|D_{ij}|}{a_i} + \frac{|D_{ji}|}{a_j}\right)$. If condition (22) is valid then all eigenvalues $q$ of $-Q$ satisfy $q < 1$ or

$$\frac{1}{2}k\left(\frac{|D_{ii}|}{a_i} + \frac{|D_{ii}|}{a_i}\right) - 2 < q < \frac{1}{2}k\left(\frac{|D_{ii}|}{a_i} + \frac{|D_{ii}|}{a_i}\right).$$

Since $\frac{1}{2}k\left(\frac{|D_{ii}|}{a_i} + \frac{|D_{ii}|}{a_i}\right) > 0$ for $\forall\, i$ then $-Q$ is positive definite and $Q$ is negative definite. Therefore, let $\xi > 0$ the smallest eigenvalue of $-Q$ such that

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V^{(m+1)}(\phi_i,\varphi_i)}{(t-\lambda)^\alpha}d\lambda < -\xi\sum_{i=1}^{M}\left(\phi_i^2 + \varphi_i^2\right),$$

and consequently, $V$ is a strict Lyapunov function for (20)-(21). Moreover, $2V \leq \sum_{i=1}^{M}\left(\phi_i^2 + \varphi_i^2\right)$ thus

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V^{(m+1)}(\phi_i,\varphi_i)}{(t-\lambda)^\alpha}d\lambda < -2\xi V.$$

This result implies that $V$ converges to zero exponentially fast, and the solutions $(\phi_i, \varphi_i)$ converge to the origin also exponentially fast, i.e., solutions $(x, w)$ in system (5)-(6) converges exponentially fast to equilibrium $e = [\bar{x}\ \langle \boldsymbol{y}, \log_2 \bar{\boldsymbol{\beta}}_1\rangle \ldots \langle \boldsymbol{y}, \log_2 \bar{\boldsymbol{\beta}}_M\rangle]$.

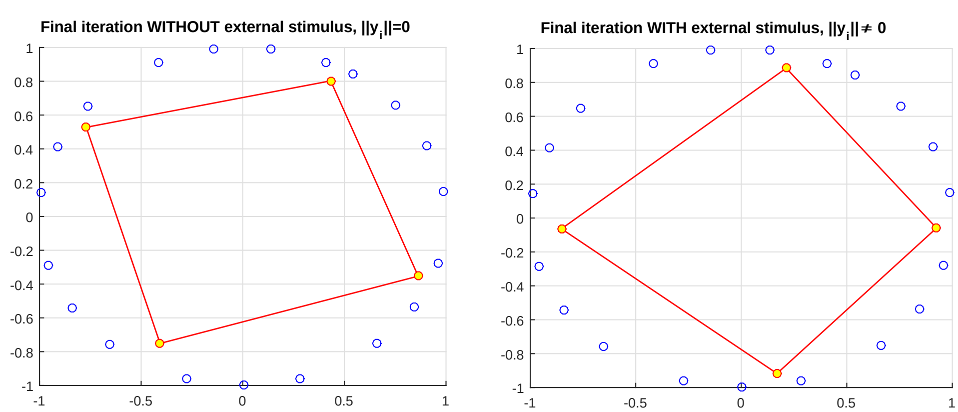


*Figure 3:* Comparing last iteration ($N = 500$ iterations) related to SOM for circle pattern composed by 20 points.

## IV. Accessing Numerical Simulations

To validate the proposed model, a self-organizing map with four neurons ($M = 4$) is utilized in the following example. The aim is to find the best updating of synaptic weights to adjust a circle shape that consists of 20 patterns and has a radius of 0.5. This will be achieved using unsupervised Hebb's learning rule with $N = 500$ and a learning rate of 0.05. Let's assume that each neuron in a neural network has an activation value of $a_j = 0.5$. Two parameters, $D_{ij}$ and $B_j$, were set randomly for each neuron within the range of 0 to 1. Additionally, $\alpha$ was set to 0.5, $m$ to 0, and $\varepsilon$ to $10^{-4}$. To analyze the behavior of the equilibrium point in this model, the algorithm ran for 500 iterations and observed how the equilibrium point behaves. Theorem 1 was used to determine the stability of this point. Our results show that the model is locally stable, which means that it is capable of reaching an equilibrium point and maintaining it over time.

In order to illustrate this point, Fig. 1 illustrates the equilibrium point $(1.336, 1.336, 1.768)$ after 500 iterations with no external stimulus, meaning that $\|y_i\| = 0$ and $m_{ij} = \frac{1}{N}\log_2 \beta_{ij}$ for $i = 1$ and $j = 2, 3, 4$. This means that all sequences $(\sigma_1, \ldots, \sigma_{500})$ for events set $X = \emptyset$ between two neurons have a probability of one, and the entropy $H_i$ $(i = 1, 2, 3, 4)$ approaches zero, which is the ideal situation.

The following text describes the behavior of neurons when an external stimulus is applied. So, let's compare two figures - Fig. 1 and Fig. 2. The former shows the neurons' behavior when no external stimuli are applied, while the latter shows their behavior when stimuli are applied. The amplitude of the impulses generated by the stimuli is set at 0.001, and the stimuli are defined by one impulse over the threshold. The sequences $(\sigma_1, \ldots, \sigma_{500})$ associated with each neuron are considered as a result of updating synaptic weights according to the training rule at each iteration. In a neural network, when a neuron is declared a winner and its weights are updated, the weights of the neighboring neuron also need to be updated. Therefore, for each iteration in the process, every neuron must have a term called $\sigma_i$ in its sequence. This term should be equal to one when the neuron is updated and zero when it doesn't receive any updates in its weights. Figure 3 illustrates that neurons form a circular pattern consisting of 20 points; however, the final position of neurons changes due to external stimuli during the last iteration.

For both cases in Figures 1 and 2, the equilibrium point $(\bar{\beta}_{12}, \bar{\beta}_{13}, \bar{\beta}_{14})$ is represented by $(1.336, 1.336, 1.768)$ and $(1.316, 1.409, 1.779)$. By assuming only the synaptic efficiencies $m_{12}, m_{13}, m_{14}$, condition (7) becomes the following three conditions:

$$\frac{p_{11}}{\ln 2} \sum_{s=1}^{M} p_{1s} p_{s1} \int_0^t \int_0^\lambda y_4 \cdot d\lambda^2 = 0,$$

$$\frac{p_{11}}{\ln 2} \sum_{s=1}^{M} p_{1s} p_{s1} \int_0^t \int_0^\lambda y_3 \cdot d\lambda^2 = 0,$$

$$\frac{p_{11}}{\ln 2} \sum_{s=1}^{M} p_{1s} p_{s1} \int_0^t \int_0^\lambda y_2 \cdot d\lambda^2 = 0,$$

Notes

i.e., the external stimulus has no contribution in the final stage of training for neurons connected to neuron 1.

On another side, it can be observed from Figs. 1 and 2 that the evolution of $m_{1j}$ $(j = 2, 3, 4)$ ensures the condition $SIE_1 > SIE_j$, more specifically,

$$H_i(\sigma_1, \ldots, \sigma_{500}|S_{in}) - H_j(\sigma_1, \ldots, \sigma_{500}|S_{in}) \\ > H_i(\sigma_1, \ldots, \sigma_{500}) - H_j(\sigma_1, \ldots, \sigma_{500}).$$

This result indicates that the motion of neurons without an external stimulus is less disorganized than the motion of neurons with an external stimulus, as expected.

## V. Final Remarks

The following paper establishes the global exponential stability of SOMs. To this end, this proposal has used fractional order derivatives to describe cognitive cortical maps that result from self-organization, with both LTM and STM approaches. It applied McMillan-Shannon's approach to fractional competitive differential equations, which helps to understand the entropy behavior of SOM in response to external stimuli. The proposal not only proves the existence and uniqueness of the equilibrium point but also shows how fractional order derivative operators can improve our understanding of entropy associated with synaptic efficiency. Synaptic efficiency was modeled by using sequences of updating impulses at each iteration, all of them as a result of applying external stimuli, not only for the winner neuron but also for neighbor neurons. In this way, the concept of conditional entropy plays an important role in the proposal as it shows that it must be greater than entropy to ensure convergence to the equilibrium point.

In future studies, it is hoped to investigate the correlation between the development of information transfer and the division of fast dynamics into distinct time scales. This will involve integrating the Dynamic Confined Space of Velocities criterion (DCSV) [14] into restricted self-organizing maps, which are extensively utilized to address constrained multi-objective optimization issues (CMOPs) [15].

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Notes

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# Memberships

## FELLOWS/ASSOCIATES OF SCIENCE FRONTIER RESEARCH COUNCIL

### FSFRC/ASFRC MEMBERSHIPS

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***15. Never start at the last minute:*** Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

***16. Multitasking in research is not good:*** Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

***17. Never copy others' work:*** Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

***18. Go to seminars:*** Attend seminars if the topic is relevant to your research area. Utilize all your resources.

***19. Refresh your mind after intervals:*** Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

***20. Think technically:*** Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

***21. Adding unnecessary information:*** Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

***22. Report concluded results:*** Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

***23. Upon conclusion:*** Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

## Informal Guidelines of Research Paper Writing

***Key points to remember:***

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

**Final points:**

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

*The introduction:* This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

**The discussion section:**

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

**General style:**

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

**To make a paper clear:** Adhere to recommended page limits.

*Mistakes to avoid:*

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

**Title page:**

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

**Abstract:** This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

*Reason for writing the article—theory, overall issue, purpose.*

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

**Approach:**

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

**Introduction:**

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

*The following approach can create a valuable beginning:*

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

**Approach:**

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

**Procedures (methods and materials):**

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

**Materials:**

*Materials may be reported in part of a section or else they may be recognized along with your measures.*

**Methods:**

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

**Approach:**

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

**What to keep away from:**

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

**Results:**

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

**Content:**

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

**What to stay away from:**

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

**Approach:**

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

**Figures and tables:**

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

**Discussion:**

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

**Approach:**

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

## The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

*Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection*.

*Segment draft and final research paper:* You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

*Written material:* You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

## Criterion for Grading a Research Paper (Compilation) by Global Journals

**Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.**

| Topics | Grades | | |
|---|---|---|---|
| | **A-B** | **C-D** | **E-F** |
| ***Abstract*** | Clear and concise with appropriate content, Correct format. 200 words or below | Unclear summary and no specific data, Incorrect form<br><br>Above 200 words | No specific data with ambiguous information<br><br>Above 250 words |
| ***Introduction*** | Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited | Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter | Out of place depth and content, hazy format |
| ***Methods and Procedures*** | Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads | Difficult to comprehend with embarrassed text, too much explanation but completed | Incorrect and unorganized structure with hazy meaning |
| ***Result*** | Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake | Complete and embarrassed text, difficult to comprehend | Irregular format with wrong facts and figures |
| ***Discussion*** | Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited | Wordy, unclear conclusion, spurious | Conclusion is not cited, unorganized, difficult to comprehend |
| ***References*** | Complete and correct format, well organized | Beside the point, Incomplete | Wrong format and structuring |

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## S

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