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The Iterative Technique for a Fourth-Order Three-Point Nonlinear BVP with Changing Sign Green's Function

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which has the sign-changing Green's function. where $\alpha \in [0,2)$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ and $\eta \in [\frac{1}{2}, 1)$. The point is that although the corresponding Green is changing the sign, by applying iterative methods, We can still obtain the existence of a monotonic positive solution under certain suitable conditions of f.

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GJSFR-F Classification: *MSC 2010: 34B15; 34B18; 34B27; 34C23*

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The Iterative Technique for a Fourth-Order Three-Point Nonlinear BVP with Changing Sign Green's Function

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I. Introduction

In this article, we aim to study the existence of a monotonic positive solution for fourth-order three-point Nonlinear BVP with changing sign Green's function

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u'''(0) = u(1) = 0, \\ \alpha u(0) + u''(\eta) = 0 \end{cases} \tag{1.1}$$

where $\alpha \in [0,2)$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ and $\eta \in [\frac{1}{2}, 1)$. By using iterative methods, We can still obtain the existence of a monotonic positive solution under certain suitable conditions of f.

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In recent decades, The differential equations come from various fields of a mathematical applied and physics, for example, in the deflection of curved beams with constant or varying cross-sections, triple-layer beams, electromagnetic waves or gravity-driven currents, etc. [1].

In the recent years, existence of single or multiple positive solutions for some third-order three-point (BVP) has attracted the attention of many authors. Please refer to [2-7] and its references. When the corresponding Green's function is non-negative, the paper can be completed, This is the condition that is an important . A natural question is whether we can get it? When the corresponding Green's function performs sign conversion, there are some positive solutions for the third-order three-point BVP.

Recently, when the corresponding Green's function is undergoing sign conversion, there has been some work on the positive solution of the second and third-order BVP. For example,in [8] the existence of at least one positive solution of the following second-order periodic BVP with positive and negative transformation Green's function studied by Zhong and An

$$\begin{cases} u''(t)+\rho^2 u = f(u), & ,0<t<T, \\ u(0)=u(T), \\ u'(0)=u'(T), \end{cases}$$

where $\eta \in (\frac{17}{24},1)$,Palamide and Smirlis $[9]$ discussed the existence of at least one positive solution. Their technique is a combination of GuoKrasnosel'sski fixed point theory and the corresponding vector field characteristics. In 2012, Sun and Zhao $[10], [11]$ obtained single or multiple positive solutions with three-point positive and negative BVP by applying the fixed point theory of Guo-Krasnosel'skii .

$$\begin{cases} u'''(t) = f(t,u(t)), & t\in[0,1], \\ u(0)=u(1)=u''(\eta)=0, \end{cases}$$

Motivated. Through the above work, this article will study BVP (1.1)Through an iterative method. Throughout this article, we always assume $\alpha \in [0,2)$ and $\eta \in [\frac{1}{2},1)$. Although the corresponding Green function is changing its sign, under certain suitable conditions, we can still obtain the existence of the monotonic positive solution of BVP (1.1) on f. Moreover, our iterative scheme starts with a zero function, This means that iterative scheme is feasible.

$$\begin{cases} u^{(4)}(t) = \lambda f(t,u(t)), & t\in[0,1], \\ u'(0)=u'''(0)=u(1)=0, \\ \alpha u(0)+u''(\eta)=0 \end{cases} \tag{1.1}$$

In this article, by applying an iterative approach, we always assume that $\alpha \in [0,2)$ and $\eta \in [\frac{1}{2},1)$.

Obviously,the BVP (3.1)is a special case of the BVP (1.1). Although the corresponding Green's function is changing the sign, under certain suitable conditions, for f, we still obtain the existence of a monotonic positive solution of BVP (1.1). furthermore, our iterative scheme starts with a zero function, which means iterating The program is feasible.

Ref
1. G, Cheng., Li, X., Zhang, F. Eigenvalues of discrete Sturm CLiouville problems with nonlinear eigenparameter dependent boundary conditions. Quaest. Math. 41(2018), 773C797.

Notes

We first recall the following fixed point of Krasnoselskii's type.

Theorem 1.1. Let E be a Banach space and K be a cone in E. Assume that Ω_1 and Ω_1 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $A : K \cap (\Omega_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either
(1) $\|Au\| \leq \|Au\|$ *foru* $\in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*
(2) $\|Au\| \geq \|Au\|$*for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

* *A. has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

II. Preliminaries

For the BVP

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u'''(0) = u(1) = 0, \\ \alpha u(0) + u''(\eta) = 0 \end{cases} \tag{2.1}$$

we have the following lemma

Lemma 2.1. The BVP (2.1) *has only trivial solution.*

Proof. It is simple to check. for any $y \in C[0,1]$, we consider the boundary value problems

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u'''(0) = u(1) = 0, \\ \alpha u(0) + u''(\eta) = 0 \end{cases} \tag{2.2}$$

After a direct computation, one may obtain the expression of Greens function $G(t,s)$ of the BVP (2.2) as following:

Proof. Integrating four times the linear problem gives us that

$$u'''(t) = u'''(0) + \int_0^t y(s)ds,$$

$$u''(t) = u''(0) + tu'''(0) + \int_0^t (t-s)y(s)ds,$$

$$u'(t) = u'(0) + tu''(0) + \frac{t^2}{2}u'''(0) + \frac{1}{2}\int_0^t (t-s)^2 y(s)ds,$$

$$u(t) = u(0) + tu'(0) + \frac{t^2}{2}u''(0) + \frac{t^3}{6}u'''(0) + \frac{1}{6}\int_0^t (t-s)^3 y(s)ds.$$

The conditions $u'(0) = u'''(0) = 0$ implies that

$$u(t) = u(0) + \frac{t^2}{2}u''(0) + \frac{1}{6}\int_0^t (t-s)^3 y(s)ds,$$

and the conditions $u(1) = 0$ this means

$$u(1) = u(0) + \frac{1}{2}u''(0) + \frac{1}{6}\int_0^1 (1-s)^3 y(s)ds = 0,$$

Next, $\alpha u(0) + u''(\eta) = 0$ is rewritten Such as

$$u''(0) + \int_0^\eta (\eta - s)y(s)ds - \frac{\alpha}{2}u''(0) - \frac{\alpha}{6}\int_0^1 (1-s)^3 y(s)ds = 0,$$

whence

$$u''(0) = \frac{\alpha}{3(2-\alpha)}\int_0^1 (1-s)^3 y(s)ds - \frac{2}{2-\alpha}\int_0^\eta (\eta - s)y(s)ds. \qquad (2.3)$$

form The conditions $u(1) = 0$ we have

$$u(0) = -\frac{1}{2}u''(0) - \frac{1}{6}\int_0^1 (1-s)^3 y(s)ds$$

If we substitute(1.3) with the expression from above and simplify, we get that

$$u(0) = \frac{1}{2-\alpha}\int_0^\eta (\eta - s)y(s)ds - \frac{1}{3(2-\alpha)}\int_0^1 (1-s)^3 y(s)ds. \qquad (2.4)$$

Finally, we obtain that

$$\begin{aligned} u(t) &= \frac{1}{2-\alpha}\int_0^\eta (\eta - s)y(s)ds - \frac{1}{3(2-\alpha)}\int_0^1 (1-s)^3 y(s)ds \\ &+ \frac{\alpha t^2}{6(2-\alpha)}\int_0^1 (1-s)^3 y(s)ds - \frac{t^2}{2-\alpha}\int_0^\eta (\eta - s)y(s)ds \\ &+ \frac{1}{6}\int_0^t (t-s)^3 y(s)ds. \end{aligned}$$

As a result, we have that

For $s \geq \eta$

$$G(t,s) = \begin{cases} \frac{-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)} & 0 \leq t \leq s, \\ \frac{(t-s)^3}{6} - \frac{(2-\alpha t^3)(1-s)^3}{6(2-\alpha)} & s \leq t \leq 1 \end{cases}$$

and $s < \eta$

$$G(t,s) = \begin{cases} \frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)} & 0 \leq t \leq s, \\ \frac{(t-s)^3}{6} + \frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)} & s \leq t \leq 1 \end{cases}$$

Notes

Remark 2.1. It is not difficult to verify that $G(t,s)$ has the following characteristics:

$$G(t,s) \geq 0 \text{ for } 0 \leq s \leq \eta \text{ and } G(t,s) \leq 0 \text{ for } \eta \leq s \leq 1.$$

Moreover, if $s \geq \eta$, then

$$\max G(t,s) : t \in [0,1] = G(1,s) = 0,$$

$$\min G(t,s) : t \in [0,1] = G(0,s) = \frac{-(1-s)^3}{3(2-\alpha)} \geq \frac{-(1-\eta)^3}{3(2-\alpha)}$$

if $s < \eta$, then

$$\begin{aligned} \max G(t,s) : t \in [0,1] &= G(0,s) = \tfrac{(s^3-3s^2)+(3\eta-1)}{3(2-\alpha)} \leq \tfrac{(\eta^3-3\eta^2)+(3\eta-1)}{3(2-\alpha)}, \\ \min G(t,s) : t \in [0,1] &= G(1,s) = 0 \end{aligned}$$

therefore,if we let $\delta = \max |G(t,s)| : t,s \in [0,1]$ then

$$\delta = \max\left\{\frac{-(1-\eta)^3}{3(2-\alpha)}, \frac{(\eta^3-3\eta^2)+(3\eta-1)}{3(2-\alpha)}\right\} < \frac{\eta - s}{(2-\alpha)}$$

Now, let Banach space $E = C[0,1]$ is equipped with the $\|u\| = \max_{t\in[0,1]} |u(t)|$.

$K = \{y \in C[0,1] : y(t)\}$ is nonnegative and decreasing on $[0,1]$. Then K is a cone in $C[0,1]$.

Note that this order relationship is inducesan in E by defining uv if and only if $u - v \in K$.

In the remainder of this paper, we always assume that $f : C[0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and satisfies the following conditions:

$(F1)$ For each $u \in [0,+\infty)$, the mapping $t \mapsto f(t,u)$ is decreasing;

$(F2)$ For each $t \in [0,1]$, the mapping $u \mapsto f(t,u)$ is increasing.

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds, t \in [0,1] \tag{2.5}$$

Obviously, if$u$ is a fixed point of A in K, then u is a nonnegative and decreasing solution of the BVP (1.1).

Lemma 2.2. Let $A : K \to K$. is completely continuous.

Proof. let $u \in K$. Then, for $0 \leq t \leq \eta$, we have

$$\begin{aligned} (Au)(t) &= \int_0^t \left[\frac{(t-s)^3}{6} + \frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right] y(s)ds \\ &+ \int_t^\eta \left[\frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right] y(s)ds \\ &+ \int_\eta^1 \frac{-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)} y(s)ds \end{aligned}$$

Notes

which together with $(F1)$ and $(F2)$ implies that

$$\begin{aligned}
(Au)'(t) &= \int_0^\eta \frac{3t^2(2-\alpha)+2\alpha t(1-s)^3-12t(\eta-s)}{6(2-\alpha)}y(s)ds \\
&+\frac{1}{2}\int_0^t (s^2-2ts)y(s)ds+\frac{t^2}{2}\int_t^\eta y(s)ds \\
&+\int_\eta^1 \frac{\alpha t(1-s)^3}{3(2-\alpha)}y(s)ds \\
&\le y(\eta)\left[\int_0^\eta \frac{3t^2(2-\alpha)+2\alpha t(1-s)^3-12t(\eta-s)}{6(2-\alpha)}\right. \\
&\left.+\frac{1}{2}\int_0^t (s^2-2ts)+\frac{t^2}{2}\int_t^\eta+\int_\eta^1 \frac{\alpha t(1-s)^3}{3(2-\alpha)}\right]ds \\
&\le ty(\eta)\left[\frac{4\eta t(2-\alpha)+(\alpha-8\eta)}{12(2-\alpha)}-\frac{5t^2}{6}+\frac{t\eta}{2}\right] \\
&\le bty(\eta)\left[\frac{4\eta t(2-\alpha)+(\alpha-8\eta)}{12(2-\alpha)}-\frac{\eta^8}{6}\right] \\
&\le 0
\end{aligned}$$

At the same time, $\eta > \frac{1}{2}$ shows that

$$\begin{aligned}
(Au)''(t) &= \int_0^\eta \frac{6t(2-\alpha)+2\alpha(1-s)^3-12(\eta-s)}{6(2-\alpha)}y(s)ds \\
&-\int_0^t sy(s)ds+t\int_t^\eta y(s)ds \\
&+\int_\eta^1 \frac{\alpha(1-s)^3}{3(2-\alpha)}y(s)ds \\
&\le y(\eta)\left[\int_0^\eta \frac{6t(2-\alpha)+2\alpha(1-s)^3-12(\eta-s)}{6(2-\alpha)}-\int_0^t sds\right. \\
&\left.+t\int_t^\eta+\int_\eta^1 \frac{\alpha(1-s)^3}{3(2-\alpha)}\right]ds \\
&\le y(\eta)\left[\frac{\alpha(3t-2\eta)+6(\eta-t)}{2(2-\alpha)}-2\eta^2+\alpha\right] \\
&\le y(\eta)\left[\frac{\eta(\alpha-2\eta)}{2(2-\alpha)}+\alpha\right] \\
&\le 0 \quad t\in(0,\eta)
\end{aligned}$$

Notes

For $t \in [\eta, 1]$, we have

$$
\begin{aligned}
(Au)(t) \quad &= \int_0^{\eta} \left[\frac{(t-s)^3}{6} - \frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right] y(s)ds \\
&+ \int_{\eta}^{t} \left[\frac{(t-s)^3}{6} - \frac{(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right] y(s)ds \\
&+ \int_{t}^{1} \left[\frac{-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right] y(s)ds
\end{aligned}
$$

which together with $(F1)$ and $(F2)$ implies that

$$
\begin{aligned}
(Au)'(t) \quad &= \int_0^{\eta} \frac{3t^2(2-\alpha)+2\alpha t(1-s)^3-12t(\eta-s)}{6(2-\alpha)} ds \\
&+ \frac{1}{2}\int_0^{\eta}(s^2-2ts)y(s)ds + \int_{\eta}^{t}\frac{(t-s)^2}{2} \\
&+ \int_{\eta}^{1}\frac{2\alpha t(1-s)^3}{6(2-\alpha)} y(s)ds \\
&\le y(\eta)\left[\int_0^{\eta}\frac{3t^2(2-\alpha)+2\alpha t(1-s)^3-12t(\eta-s)}{6(2-\alpha)}\right. \\
&\left. + \frac{1}{2}\int_0^{\eta}(s^2-2ts) + \int_{\eta}^{t}\frac{(t-s)^2}{2} + \int_{\eta}^{1}\frac{\alpha t(1-s)^3}{6(2-\alpha)}\right] ds \\
&= ty(\eta)\left[\frac{(\alpha-12\eta)}{12(2-\alpha)} + t\eta - \frac{\eta^2}{2} + 1 + \frac{\eta^3}{6} - \eta\right] \\
&\le ty(\eta)\left[\frac{(\alpha-12\eta)}{12(2-\alpha)} + \frac{\eta^3}{6} + \frac{\eta^2}{2} - \eta + 1\right] \\
&\le 0 \quad t \in (\eta, 1)
\end{aligned}
$$

So,$(Au)(t)$ is decreasing on $[0,1]$.At the same time,since $(Au)(1) = 0$, we know that $(Au)(t)$ is nonnegative on $[0,1]$. This indicates that $(Au)(t) \in K$. Furthermore, although $G(t,s)$ is not continuous, it follows from known text book results, for example, see [12], that $A : K \to K$ is completely continuous

Theorem 2.3. Assume that $f(t,0) \not\equiv 0$ for $t \in [0,1]$ and there exist two positive constants a and b such that the following conditions are satisfied:

$(H1)$ $f(0,a) \le 6(2-\alpha)a$,

$(H2)$ $b(u_2-u_1) \le f(t,u_2) - f(t,u_1) \ge 2b(u_2-u_1), \quad t \in [0,1]$,

$$0 \le u_1 \le u_2 \le a.$$

If we construct a iterative sequence $v_{n+1} = Av_n$ and $n = 0,1,2,\cdots$, where $v_0(t) \equiv 0$ for $t \in [0,1]$, then $v_{n_{n=1}^{\infty}}$ converges to $v^{\dagger}$ in E and $v^{\dagger}$ is a decreasing positive solution of the BVP (1.1)

Ref

12. R. H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, John Wiley Sons, New York, NY, USA, 1976.

Proof. Let $K_a = u \in K : \|u\| \leq a$. Then we may assert that $A : K_a \to K_a$. In fact, if $u \in K_a$, then it follows from $(H1)$ that

$$0 \leq (Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds$$

$$\leq \int_0^1 |G(t,s)|f(0,a)ds$$

$$\leq 6(2-\alpha)a\delta$$

$$< a, \quad t \in [0,1],$$

which indicate that $\|Au\| \leq a$

so $A : K_a \to K_a$.

Now, we prove that $v_{n}{}_{n=1}^{\infty}$ converges to $v^{\dagger}$ in E and $v^{\dagger}$ is a decreasing positive solution of (1.1). Indeed, in view of $v_0 \in K_a$ and $A : K_a \to K_a$, we have $v_n \in K_a$, $n = 1, 2, \cdots$ Since the set $v_{n}{}_{n=0}^{\infty}$ is bounded and A is completely continuous, we know that the set $v_{n}{}_{n=1}^{\infty}$ is relatively compact. In what follows, we prove that $v_{n}{}_{n=0}^{\infty}$ is monotone by induction. First, it is explicit that $v_1 - v_0 = v_1 \in K$, Which indicates this $v_1 v_0$ Subsequently, we suppose that $v_{k-1} v_k$. Then $v_k - v_{k-1}$ is decreasing and $0 \leq v_{k-1}(t) \leq v_k(t) \leq a$,$0 \leq t \leq 1$. So, it follows from $(H2)$ that for $0 \leq t \leq \eta$

$$v'_{k+1}(t) - v'_k(t)$$

$$= \frac{1}{6(2-\alpha)} \int_0^{\eta} 3t^2(2-\alpha) + 2\alpha t(1-s)^3 - 12t(\eta - s)[f(s, v_k(s)) - f(s, v_{k-1}(s))]ds$$

$$+ \frac{1}{2} \int_0^{\eta} (s^2 - 2ts)[f(s, v_k(s)) - f(s, v_{k-1}(s))]ds$$

$$+ \frac{1}{2} \int_{\eta}^{t} (t-s)^2 [f(s, v_k(s)) - f(s, v_{k-1}(s))]ds$$

$$+ \frac{\alpha t}{3(2-\alpha)} \int_{\eta}^{1} (1-s)^3 [f(s, v_k(s)) - f(s, v_{k-1}(s))]ds$$

$$\leq \frac{b}{6(2-\alpha)} \int_0^{\eta} 3t^2(2-\alpha) + 2\alpha t(1-s)^3 - 12t(\eta - s)[v_k(s) - v_{k-1}(s)]ds$$

$$+ \frac{b}{2} \int_0^{\eta} (s^2 - 2ts)[v_k(s) - v_{k-1}(s)]ds$$

$$+ \frac{b}{2} \int_{\eta}^{t} (t-s)^2 [v_k(s) - v_{k-1}(s)]ds$$

$$+ \frac{\alpha t}{6(2-\alpha)} \int_{\eta}^{1} (1-s)^3 [v_k(s) - v_{k-1}(s)]ds$$

Notes

$$\leq b[v_k(\eta) - v_{k-1}(\eta)]$$

$$\times [\frac{1}{6(2-\alpha)} \int_0^\eta 3t^2(2-\alpha) + 2\alpha t(1-s)^3 - 12t(\eta - s)$$

$$+ \frac{1}{2}\int_0^\eta (s^2 - 2ts) + \frac{1}{2}\int_\eta^t (t-s)^2 + \frac{\alpha t}{3(2-\alpha)} \int_\eta^1 (1-s)^3]ds$$

$$= b[v_k(\eta) - v_{k-1}(\eta)]t\left[\frac{(\alpha - 12\eta)}{12(2-\alpha)} + t\eta - \frac{\eta^2}{2} + 1 + \frac{\eta^3}{6} - \eta\right]$$

$$\leq b[v_k(\eta) - v_{k-1}(\eta)]t\left[\frac{(\alpha - 12\eta)}{12(2-\alpha)} + \frac{\eta^3}{6} + \frac{\eta^2}{2} - \eta + 1\right]$$

$$\leq 0 \quad t \in (\eta, 1)$$

And therefore,

$$v'_{k+1}(t) - v'_k(t) \leq 0, \quad v''_{k+1}(t) - v''_k(t) \leq 0 \quad t \in [0,1] \tag{2.6}$$

This together with

$$v'_{k+1}(t) - v'_k(t) = \int_0^1 G(1,s)[f(s, v_k(s)) - f(s, v_{k-1}(s))]ds, t \in [0,1]. \tag{2.7}$$

$v_{k+1}(t) - v_k(t) \geq 0, \quad t \in [0,1]$ Subsequently, given the above (1.7) and (1.8) that $v_{k+1} - v_k \in K$, Which shows $v_{k+1}v_k \in K$.

Thus, we have shown that $v_{k+1}v_k \in K$, $n = 0, 1, 2....$ Since $v_{n}{}_{n=1}^{\infty}$ Relatively compact and monotonous, there exists a $v^\dagger \in K_a$ such that $\lim_{n\to\infty} v_n = v^\dagger$, which together with the continuity of A and the fact that $v_{n+1} = Av_n$ It means that $v^\dagger = Av^\dagger$. This indicate that $v^\dagger$ is a decreasing non-negative solution of (1.1). Moreover, in view of $f(t,0) \not\equiv 0$, $t \in [0,1]$, we know that zero function is not a solution of (1.1), which indicates that $v^\dagger$ is a positive solution of (1.1).

III. An Example

Consider the boundary value problem:

$$\begin{cases} u^{(4)}(t) = \frac{1}{8}u^2(t) + u(t) + (1-t),, \quad t \in [0,1], \\ u'(0) = u'''(0) = u(1) = 0, \\ \alpha u(0) + u''(\frac{1}{2}) = 0 \end{cases} \tag{3.1}$$

If we let $\eta = \frac{1}{2}$,$\alpha = 1$ and $f(t,u) = \frac{1}{8}u^2(t) + u(t) + (1-t)$, $(t,u) \in [0,1] \times [0, +\infty)$, Then all the assumptions of Theorem 2.2 $a = 2$ and $b = 1$. It follows from Theorem 2.2 that (3.1) has a decreasing positive solution $v^\dagger$. Moreover, the iterative scheme is $v_0(t) \equiv 0$ for $t \in [0,1]$

Notes

$$v_{n+1}(t)=\begin{cases}\int_0^t\left[\frac{(t-s)^3}{6}+\frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right][\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ \quad+\int_t^\eta\left[\frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right][\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ +\int_\eta^1\frac{-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}[\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ if \quad t\in[0,\frac{1}{2}] \quad n=0,1,2,3,4,\cdots\\ \int_0^\eta\left[\frac{(t-s)^3}{6}-\frac{6(1-t^2)(\eta-s)-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right][\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ \quad+\int_\eta^t\left[\frac{(t-s)^3}{6}-\frac{(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right][\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ \quad+\int_t^1\left[\frac{-(2-\alpha t^2)(1-s)^3}{6(2-\alpha)}\right][\frac{1}{8}u_n^2(s)+u_n(s)+(1-t)]ds\\ if \quad t\in[\frac{1}{2},1] \quad n=0,1,2,3,\cdots\end{cases}$$

IV. Conclusion

in this paper, when $\alpha\in[0,2)$ and $\eta\in[\frac{1}{2},1)$, we have successfully constructed an animation sequence whose limit is just the ideal monotonic positive solution of boundary value problem (1.1).

In addition, a zero function started with the iterative scheme , which shows that the iterative scheme is feasible.

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