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Positivity and the Michael's Problem

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Positivity and the Michael's Problem

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Abstract

We define an order on $\mathbb{C}$ that makes it a Banach lattice. Having established properties of this order, we shall use them to provide a brief proof of the solution to the problem posed by E. Michael in 1952 [8], which was resolved in [6] in 2024 using theorems of functional analysis.

Keywords: *Banach lattice, Fréchet lattice, Michael's problem, Complex cone, Gelfand transform, Automatic continuity*

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1. Preliminary

Although the work on lattices is varied and interesting, it mainly concerns normed vector spaces over the field of real numbers. In functional analysis, and particularly in spectral analysis, spaces over the field of complex numbers are often studied. In order to avoid extension through complexification, we study in this paper properties of the complex lattice $C(X, \mathbb{C})$, the space of continuous complex functions on a hemicompact space (containing an exhaustive sequence of compact sets K_n) by considering an adequate cone on the field of complex numbers that strictly contains $\mathbb{R}_+$.

A Fréchet space is a topological vector space E whose topology is complete metrizable and defined by a countable family of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ that separates points (if $\|x\|_n = 0$ for all n, then $x = 0$). We shall always assume that for all $x \in E$, $\|x\|_n \leqslant \|x\|_{n+1}$. A basic device in the study of a Fréchet space is to represent E as the inverse limit of a sequence of Banach spaces $\overline{E_n}$, where $\overline{E_n}$ is the completion of $E_n = E/I_n$ with norm $\|x + I_n\|_n = \|x\|_n$ and I_n is the subspace of all $x \in E$ such that $\|x\|_n = 0$. The subspaces I_n satisfy $I_{n+1} \subset I_n$ and $\bigcap_n I_n = \{0\}$.

The homomorphism $\pi_{mn} : \overline{E_n} \to \overline{E_m}$ for $m \leqslant n$ is defined as the completion of the mapping $x + I_n \mapsto x + I_m$. This representation enables one to construct an element in E by constructing a sequence x_n such that for each n, $x_n \in \overline{E_n}$ and $\pi_{n(n+1)}(x_{n+1}) = x_n$, according to the projective system:

$$\ldots \to \overline{E_{n+1}} \xrightarrow{\pi_{n(n+1)}} \overline{E_n} \to \ldots$$

The homomorphism $\pi_n : E \to E_n$ is defined as $x \mapsto x + I_n$.

We will show that E is a Fréchet lattice.

2. A Lattice Cone of $\mathbb{C}$

Definition 2.1 A set C is a cone in a complex Fréchet space E if it is closed, nonvoid, the sum of two members of C is a member of C, and non-negative real number scalar multiples of members of C are members of C.

The principal method of attack is via an ordering of the algebra, the positive cone being the closure of the set of sums of squares.

Now on $\mathbb{C}$ we consider the cone K consisting of complex numbers $\alpha + i\beta$ such that

$$0 \leqslant \alpha \text{ and } |\beta| \leqslant \alpha$$

so that K is in the half-plane of complexes with positive real part and limited by the lines of equations $y = x$ and $y = -x$ (see Figure 1).

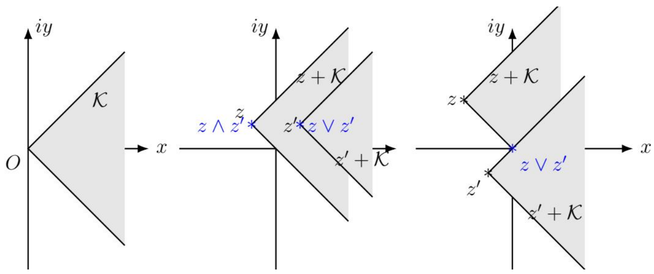

Figure 1. The cone K in the complex plane.

Also, it is obvious that K is a closed pointed convex cone of the real vector space $\mathbb{C}$. Moreover, we have the following.

Lemma 2.1 K is a lattice cone of the real vector space $\mathbb{C}$.

Preuve. By Theorem 1.16 of [2], it suffices to show that for any $z, z' \in \mathbb{C}$ there exists $z_s \in \mathbb{C}$ satisfying

$$(z + K) \cap (z' + K) = z_s + K$$

So consider first the case that the boundaries of $z+K$ and $z'+K$ are disjoint. From the parallelism of their boundaries, one of these two parts is contained in the other. So, the desired complex z_s is the vertex of the smaller (for inclusion) of the two cones.

Next, consider the case where the boundaries of $z+K$ and $z'+K$ meet. By the analytical expression of $\partial(z+K)$ and $\partial(z'+K)$, it can be verified that the intersection of these two boundaries is exactly a point z_s which satisfies

$$(z + K) \cap (z' + K) = z_s + K$$

This means that z_s is the supremum of z and z' ($z_s = z \vee z'$).

It is well-known that

$$z \wedge z' = -[(-z) \vee (-z')]$$

Let $z \in \mathbb{C}$, $z^+ = z \vee 0$ and $z^- = (-z) \vee 0$. The absolute value of z related to the cone K is denoted by

$$|z|_K := z^+ \vee z^-$$

2.1. Explicit determination of the infimum and supremum

Let $z = a + ib \in \mathbb{C}$. The cone $z + K$ has two half-lines with equations $D_z: \ y = x + b - a$ and $\Delta_z: \ y = -x + a + b$ as its edges. To determine the supremum and infimum of z and another point z' in the complex plane, depending on the location of z', there are four possible cases:

(i) $z' \in z + K$, then $z \leqslant z'$ and $z \vee z' = z'$; $z \wedge z' = z$.

(ii) $z' \in z - K$, then $z' \leqslant z$ and $z \vee z' = z$; $z \wedge z' = z'$.

(iii) $z' = a' + ib'$ with $b' < b$ and $-(b-b') + a < a' < (b-b') + a$, then

$$z \vee z' = \frac{1}{2}(a + a' + b - b') + \frac{i}{2}(a - a' + b' + b)$$

and

$$z \wedge z' = \frac{1}{2}(a + a' + b' - b) + \frac{i}{2}(a' - a + b + b')$$

(iv) $z' = a' + ib'$ with $b' > b$ and $-(b'-b) + a < a' < (b'-b) + a$, then

$$z \vee z' = \frac{1}{2}(a + a' + b' - b) + \frac{i}{2}(a' - a + b' + b)$$

and

$$z \wedge z' = \frac{1}{2}(a + a' + b - b') + \frac{i}{2}(a - a' + b + b')$$

So, $z^+ = z \vee 0$, $z^- = (-z) \vee 0$ and $|z|_K := z^+ \vee z^- = z^+ + z^-$ are given by:

(i) $z \in K$ implies $z^+ = z$, $z^- = 0$ and $|z|_K = z$.

(ii) $z \in -K$ implies $z^+ = 0$, $z^- = -z$ and $|z|_K = -z$.

(iii) If $z = a + ib$ with $b < 0$ and $b < a < -b$, then $z^+ = \frac{1}{2}(a - b) + \frac{i}{2}(b - a)$, $z^- = \frac{1}{2}(-a - b) + \frac{i}{2}(-a - b)$ and $|z|_K = -b - ia$.

(iv) If $z = a + ib$ with $b > 0$ and $-b < a < b$, then $z^+ = \frac{1}{2}(a+b) + \frac{i}{2}(a+b)$, $z^- = \frac{1}{2}(-a+b) + \frac{i}{2}(a-b)$ and $|z|_K = b + ia$.

Proposition 2.1 The modulus $z \mapsto |z| = \sqrt{z\bar{z}}$ is a lattice norm; that is:

a) $|\cdot|$ is absolute, $|z| = \big||z|_K\big|$ for all z; and

b) $|\cdot|$ is monotone on the positive cone, $0 \leqslant z \leqslant z'$ implies $|z| \leqslant |z'|$.

Preuve. The assertion (a) follows immediately from the values of z according to the four cases that precede the proposition.

We must show that $|z|_K \leqslant |z'|_K$ in $\mathbb{C}$ implies $|z| \leqslant |z'|$. If $z = a + ib$ and $z' = a' + ib'$ then we have:

$$0 \leqslant z \leqslant z' \Leftrightarrow \begin{cases} 0 \leqslant a \leqslant a'; & |b| \leqslant a \\ |b' - b| \leqslant a' - a; & |b'| \leqslant a' \end{cases}$$

We must show that $|z|^2 = a^2 + b^2 \leqslant |z'|^2 = a'^2 + b'^2$. Starting from inequality $|b' - b| \leqslant a' - a$, which we will square, we get:

$$b^2 \leqslant a'^2 - 2aa' + a^2 + 2bb' - b'^2$$

So,

$$a^2 + b^2 \leqslant a'^2 + b'^2 + 2a^2 - 2aa' + 2bb' - 2b'^2$$

To achieve the desired inequality $a^2 + b^2 \leqslant a'^2 + b'^2$, it is enough to show that:

$$2a^2 - 2aa' + 2bb' - 2b'^2 \leqslant 0$$

that is,

$$a(a - a') + b'(b - b') \leqslant 0$$

For this, we distinguish four cases, according to the position of b' in $[-a', a']$:

First assume that $0 \leqslant b' \leqslant a$. Since $b'(b - b') \leqslant b'|b' - b|$, then:

$$\begin{aligned} a(a - a') + b'(b - b') &\leqslant a(a - a') + b'|b' - b| \\ &\leqslant a(a - a') + b'(a' - a) \\ &\leqslant a(a - a') + a(a' - a) \\ &\leqslant 0 \end{aligned}$$

In the case where $a \leqslant b' \leqslant a'$, we have $b \leqslant b'$ so that $b'(b - b') \leqslant 0$, and on the other hand $a(a - a') \leqslant 0$, so that:

$$a(a - a') + b'(b - b') \leqslant 0$$

In the case where $-a \leqslant b' \leqslant 0$, we obtain $b'(b-b') \leqslant |b'(b-b')|$, then:

$$\begin{aligned} a(a - a') + b'(b - b') &\leqslant a(a - a') + |b'||b' - b| \\ &\leqslant a(a - a') - b'|b' - b| \\ &\leqslant a(a - a') - b'(a' - a) \\ &\leqslant (-a - b')(a' - a) \\ &\leqslant 0 \end{aligned}$$

In the last case, $-a' \leqslant b' \leqslant -a$. Keeping in mind $-a \leqslant b$, it follows that:

$$b'(b - b') \leqslant -a(b - b')$$

Then,

$$a(a - a') + b'(b - b') \leqslant a(a - a') - a(b - b') \leqslant 0$$

as the sum of two non-positive real numbers.

Thus we have shown:

Corollary 2.1 $(\mathbb{C}; K; |\cdot|)$ is a Banach lattice.

Remark 2.1

(1) $z = a + ib \in K$ implies $z^{-1}, \bar{z} \in K$.

(2) Keeping in mind that $C(X)$ equipped with the pointwise product is a $\mathbb{C}$-algebra, the cone K exhibits a deficiency, since $K \cdot K \not\subseteq K$ (for example, $z = 2 + i \in K$ and $z' = 4 + 3i \in K$ but $zz' = 5 + 10i \notin K$).

We now establish:

Proposition 2.2 $(\mathbb{C}; K; |\cdot|)$ is Dedekind complete.

Preuve. Let $\{z_\alpha\}$ be an increasing net bounded from above in $\mathbb{C}$. We show that $\{z_\alpha\}$ has a supremum. We put $z_\alpha = a_\alpha + ib_\alpha$. By hypothesis there is $z = a + ib$ such that $a_\alpha + ib_\alpha \leqslant a + ib$ for all α. Then $a_\alpha \leqslant a$ and $|b - b_\alpha| \leqslant a - a_\alpha$ for all α. Moreover, $\alpha \leqslant \beta$ implies $a_\alpha + ib_\alpha \leqslant a_\beta + ib_\beta$ which means $a_\alpha \leqslant a_\beta$ and $|b_\beta - b_\alpha| \leqslant a_\beta - a_\alpha$. The net $\{a_\alpha\}$ is an increasing net bounded from above in $\mathbb{R}$, then $a_\alpha \to u$ in $\mathbb{R}$ and $u = \sup\{a_\alpha\}$. It follows from $|b_\beta - b_\alpha| \leqslant a_\beta - a_\alpha$ for $\alpha \leqslant \beta$ that $\{b_\alpha\}$ is a Cauchy net in $\mathbb{R}$, so $b_\alpha \to v$ in $\mathbb{R}$. It is enough to show that $u + iv$ is a supremum of $\{z_\alpha\}$. We claim that $\bigvee(a_\alpha + ib_\alpha) = u + iv$. Indeed, let $c + id$ be an upper bound of $\{z_\alpha\}$ in $\mathbb{C}$. Obviously, $0 \leqslant c - u$, and for an arbitrary $\varepsilon > 0$, we have:

$$\begin{aligned}
|d - v| &\leqslant |d - b_\alpha| + |b_\alpha - b_\beta| + |b_\beta - v| \quad \text{for } \alpha \leqslant \beta \\
&\leqslant c - a_\alpha + a_\beta - a_\alpha + |b_\beta - v| \\
&\leqslant c - a_\alpha + a_\beta - a_\alpha + \frac{\varepsilon}{2} \quad \text{for } \beta \text{ large enough} \\
&\leqslant c - a_\alpha + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for } \beta \geqslant \alpha \text{ large enough} \\
&\leqslant c - a_\alpha + \varepsilon \quad \text{for } \alpha \text{ large enough}
\end{aligned}$$

It follows that $|d - v| \leqslant c - u + \varepsilon$ and by the arbitrariness of ε, we derive that $|d - v| \leqslant c - u$, and also $u + iv \leqslant c + id$. ∎

It is well-known that every order complete Riesz space is Archimedean. Then:

Corollary 2.2 $(\mathbb{C}; K; |\cdot|)$ is Archimedean.

Proposition 2.3 The number 1 is an order unit in $(\mathbb{C}; K; |\cdot|)$.

Preuve. Let $z = a + ib \in \mathbb{C}$. Since $|z|_K = |z|$, then $\lambda = \sqrt{a^2 + b^2}$ satisfies $|z|_K \leqslant \lambda \cdot 1$. ∎

Proposition 2.4 $(\mathbb{C}; K; |\cdot|)$ is an AM-space with an order unit and its norm $\|\cdot\|_\infty$ coincides with the canonical modulus.

Preuve. It follows from Proposition 2.3 that $\mathbb{C}$ is an ideal generated by 1. For every $z = a + ib \in \mathbb{C}$, $\|a + ib\|_\infty := \inf\{\lambda > 0 : |a + ib|_K \leqslant \lambda \cdot 1\}$. Since $|z|_K = |z|$, then $\sqrt{a^2 + b^2} = \|a + ib\|_\infty$. Thus $|z|_K = |z| = \|z\|_\infty$. ∎

3. Fréchet Lattice $C(X; \mathbb{C})$

Now we will focus on the space $C(X; \mathbb{C})$ of all continuous complex functions on a topological space X.

Definition 3.1 A Hausdorff space is called a k-space if every subset intersecting each compact subset in a closed set is itself closed.

Examples of k-spaces are locally compact and first countable spaces [4, p. 231].

Note that a complex-valued function f on a k-space X is continuous iff it is continuous on each compact subset of X.

Definition 3.2 A Hausdorff space X is called hemicompact if there is a countable compact exhaustion $K_1 \subset K_2 \subset \cdots \subset K_n \subset \ldots$ of X such that for each compact subset $K \subset X$ there is $n \in \mathbb{N}$ so that $K \subset K_n$. We call such an exhaustion (K_n) admissible.

Obviously, each hemicompact space is a Lindelöf space. We will use the following theorem proven in [3, p. 69]:

Theorem 3.1 Let X be a completely regular space. Then $C(X; \mathbb{C})$ is a Fréchet space iff X is a hemicompact k-space. In this case, the topology is generated by the seminorms of uniform convergence on compacts K_n:

$$\|f\|_n = \sup\{|f(x)| : x \in K_n\}$$

In $C(X; \mathbb{C})$, the algebraic operations $(+, \times, \cdot)$ are pointwise defined. $C(X; \mathbb{C})$ becomes a unital commutative Fréchet algebra with a natural involution $f \mapsto f^*$ such that $f^*(x) = \overline{f(x)}$ for all $x \in X$.

Now, consider the positive cone L consisting of the closure of the set of sums of elements ff^* (used by Kelley and Vaught in [5]).

Lemma 3.1 For each n, $C(K_n; \mathbb{C})$ is a Banach lattice for the order induced by L.

Preuve. Obviously, the norm satisfies $\|ff^*\|_n = \|f\|_n^2$. Then $C(K_n; \mathbb{C})$ is a commutative unital C^*-algebra. According to Sherman's theorem [9], $C(K_n; \mathbb{C})$ is a Banach lattice. ∎

Thus we obtain the following:

Theorem 3.2 $C(X; \mathbb{C})$ is a Fréchet lattice for the order defined by L.

Now we come to the following automatic continuity theorem:

Theorem 3.3 The characters of $C(X; \mathbb{C})$ are automatically continuous.

Preuve. Let θ be a character of $C(X; \mathbb{C})$. It can easily be shown that $\theta(L) \subset K$. Now, it is enough to apply Theorem 9.6 [1, p.350] to obtain the desired conclusion. ∎

4. Fréchet Complex Commutative Unital Algebras

Let A be a complex commutative unital algebra and let $f : A \to \widehat{A}$ be the Gelfand map $x \mapsto \hat{x}$. It is well-known that A is semisimple iff the Gelfand map Γ is injective. Let Ω be an open set in $\mathbb{C}$. Let $H(\Omega)$ denote the set of holomorphic functions on Ω. The topology defined by the family of seminorms $p_n(f) = \sup_{z \in K_n}\{|f(z)|\}$, where (K_n) is an exhaustive sequence of compact sets in Ω, makes $H(\Omega)$ a commutative unitary Fréchet algebra.

The Gelfand transform Γ here is the natural inclusion of $H(\Omega)$ in $C(\Omega)$. Γ is injective (because if $\Gamma(f) = 0$ then $f(z) = 0$ for any $z \in \Omega$ and $f = 0$). Γ is not surjective (since not all continuous functions on Ω are holomorphic).

The image $\Gamma(A)$ is a closed subalgebra of $C(X; \mathbb{C})$ (endowed with the compact convergence topology). On A, we define the natural involution $x^* = \Gamma^{-1}(\overline{\Gamma(x)})$. Since Γ is an isometry, it follows that A is a commutative unitary C^*-algebra. According to Sherman's theorem, when ordered by the K-cone, the closure of the set of sums of elements xx^* in A makes A a Fréchet lattice.

Theorem 4.1 Let A be a complex commutative unital Fréchet algebra. Then the characters of A are automatically continuous.

Preuve. Let θ be a character of A. Using the Gelfand transform and the definition of the involution of A, we see that θ satisfies:

$$\theta(xx^*) = \theta(x)\theta(x^*) = \theta(x)\overline{\theta(x)} \in K$$

Therefore, θ will be positive from the Fréchet lattice A to the Banach lattice $\mathbb{C}$. Thanks to Theorem 9.6 [1, p.350], we deduce that θ is automatically continuous. ∎

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