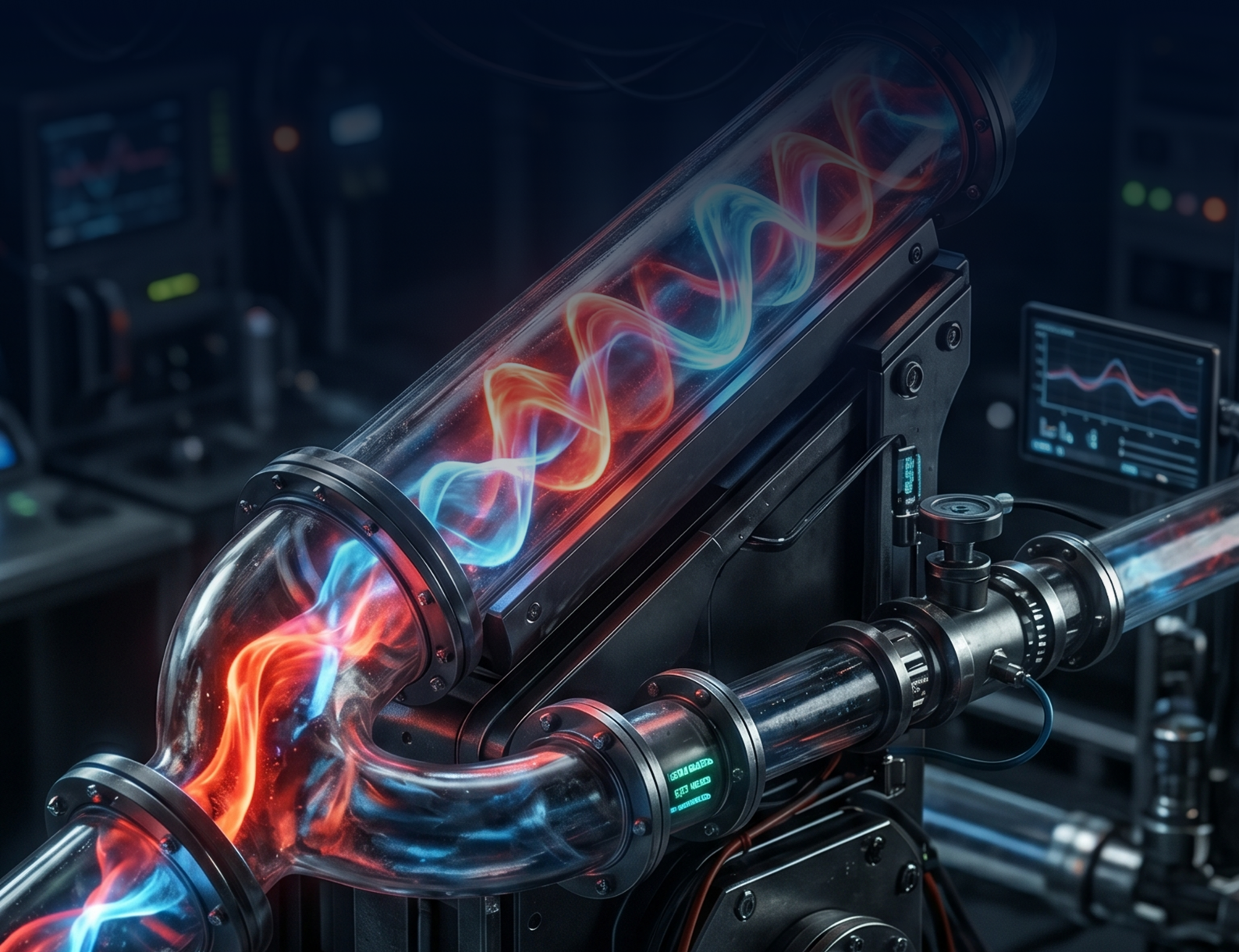


Heat Transfer with Pulsatile Flow in a Tube

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

Terry Moschandreou



Heat Transfer with Pulsatile Flow in a Tube

A Thesis Submitted in Partial Fulfillment of the Requirements for the
Degree of **Doctor of Philosophy**

in the **Department of Applied Mathematics**

at the **University of Western Ontario**

By

Terry Moschandreou

Attested by Open Association of Research Society, USA

Global Journals · February 28, 2026

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Abstract

The partial differential equations governing heat transfer with pulsatile flow in a tube, which serves as a model of a simple heat exchange device are solved in this thesis. The governing equations are the Navier Stokes equations and the convection diffusion equation. Two boundary value problems are solved. In the first case a Neumann boundary condition is specified which represents constant heat flux at the wall of the tube with fluid entering a thermal region at a constant temperature. A regular perturbation expansion is used to obtain higher order harmonics downstream for the temperature field. The main assumption is that the temperature field becomes fully developed downstream as the velocity field becomes fully developed downstream. The perturbation parameter is the ratio of pressure gradient amplitudes of unsteady flow to that of steady flow. Using a Green's function the first order solution is obtained. As a measure of heat transfer enhancement, a bulk temperature is formulated for the convective process involved and a change in unsteady Nusselt number to that of steady Nusselt number is evaluated analytically. In the second part of the thesis the more difficult problem of heat transfer in pulsatile flow with constant wall temperature is considered. Although a complete solution is not possible as in the first part, it is possible to use a combination of the Generalized Integral Transform Techniques and Laplace transforms to solve this problem upstream and downstream in the thermal region of the tube. The approximate solution indicates that a plane wave propagates down the tube and the phase of the wave defines critical values in frequency and time along the tube. Use of Dirac's distribution makes it possible to define a bulk temperature and a change in unsteady bulk temperature from that of steady bulk temperature is presented. By means of classical analysis an inequality involving the two quantities is presented. As a result a measure of heat transfer in pulsatile flow compared to that of steady flow is presented.

To My Father and Mother

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1.1 Motivation

There are several practical situations where heat is being transferred under conditions of pulsatile flow. Examples occur in industrial applications and in blood flow. Also the problem may be important in the design of control systems for heat exchanger equipment. One of the objectives in industrial applications has been to determine whether the performance of a heat-exchange device could be improved with pulsatile flow.

When pulsations are imposed on flow in a tube, it is believed that heat transfer would be changed because the pulsations would alter the thickness of the boundary layer and therefore the thermal resistance. This view was put forward by Richardson [1] who had shown that the velocity profile near the wall is steeper in pulsatile flow than in Poiseuille flow. It then follows from a consideration of Reynold's analogy that the heat transfer should increase under such conditions.

It is the purpose of this thesis to determine whether pulsatile flow can give rise to an increase in heat transfer in a tube, which is a basic element in many heat exchange devices.

1.2 Previous Work

The mechanics of pulsatile flow in a tube has received considerable attention starting with the classical work of Uchida [2], Womersley [3] and Atabek and Chung [4] which has led to an exact solution and much information about the flow problem. Studies of the corresponding heat transfer problem, however, have been far less numerous and existing results do not provide complete resolution of all the questions involved.

Siegel and Perlmutter [5] demonstrated the explicit dependence of overall heat transfer on pulsatile frequency. They found that for a constant wall temperature boundary condition, the resulting Nusselt number showed periodic axial change which could enhance heat transfer.

Barnett and Vachon [6] analyzed fully developed pipe flow in the presence of longitudinal periodic pulsations by assuming that the radial part of the temperature profile is additively independent from the axial part. Although they did not justify this assumption they were able to conclude that heat transfer effects with constant flux at the wall are amplified by

pulsations of fluids of Prandtl numbers well below unity for the limiting case of low frequency.

More recently Creff and Andre [i] solved the Navier Stokes and energy equations using a finite difference method and an asymptotic development for the dynamic and thermal quantities. Their model shows the existence of an annular effect in the entry region for the pulsatile part of velocity and temperature which is known as Richardson's effect. The work was based on the full Navier Stokes and energy equations which made it difficult to obtain an explicit expression for the Nusselt number over one cycle.

Cho and Hyun [8] solved the time-dependent laminar boundary layer equations numerically over a large range of frequencies and amplitudes of pulsation. They found that for certain frequencies the Nusselt number increased over the steady-flow value.

They also found that the Nusselt number trend is amplified as the amplitude of pulsation increases and the Prandtl number is much less than unity.

Haneke, Laschefske, Grobe-Gorgemann and Mitra [9] solved the Navier-Stokes and energy equations numerically for laminar pulsating flow in a channel. At the entrance region of the channel a sinusoidal pulse was applied. Their results indicated a positive and negative overshoot (Richardson's effect) in the axial velocity profile and flow separation near the wall. They concluded that appreciable heat transfer enhancement occurs in the channel.

Kim, Kang and Hyun [10] conducted a study of the heat transfer characteristics of fully-developed pulsatile flow in a channel and found numerically that changes in the Nusselt number were pronounced in the entrance region, and only minor far downstream. Also the effects of pulsatile frequency on heat transfer was found to be noticeable when the frequency is small or moderate but not when it is high. They concluded that oscillations may produce both heat transfer enhancement as well as reduction at different axial locations in the channel.

Base, Campbell & Hobbs [11] studied the heat transfer characteristics of pulsatile flow in a pipe experimentally and found there to be an optimal frequency at which there is an increase over the steady value in heat transfer for fluids of Prandtl number near unity.

Some other experimental work in heat transfer with pulsatile flow has been conducted by Fallen [12] showing that superposition of a pulsating flow in a pipe with zero mean velocity on steady flow increased significantly the overall Nusselt number.

Experimental work has also been done by Genin, Koval, Manchukha and Sviridov [13], Hapke [14], Andre, Creff and Crabol [15] and Peattie and Budwig [16], all of whom have reported increases in heat transfer rates with pulsatile flow.

1.3 Present Work

In this thesis the aim is to demonstrate the mechanics of heat transfer in pulsatile flow in a tube analytically. That is, we must solve the energy equation coupled with the Navier Stokes equations with a sinusoidal pressure gradient driving the flow.

In the first part of the thesis we use a regular perturbation expansion [17] to solve the convection problem subject to a constant wall heat flux. We obtain, using Laplace transforms and Green's functions, qualitative and quantitative insight into how the above phenomenon depends on the nondimensional parameters involved such as pulsation frequency and amplitude, Prandtl and Reynolds numbers.

In the second part of the thesis we use an integral transform method known as the generalized integral transform technique [23], [24], [25] to solve the problem of heat transfer with pulsatile flow in a tube as in the first part but with constant wall temperature instead of constant heat flux. An explicit formulae is obtained for temperature at any axial distance along the thermal region of the tube both upstream and downstream and the existence of a wave-like form in the solution is shown. The bulk temperature for this unsteady solution is formulated and compared to the corresponding steady one.

Governing Equations

2

2.1 Basic Assumptions

We consider fully developed laminar pulsating flow of an incompressible Newtonian fluid. It is convenient to take cylindrical coordinates with x axis coinciding with the center line of the tube. The main assumptions are that (1) the velocity field is in the x direction only and is assumed fully developed: (2) the flow is axially symmetric: (3) there is uniform pulsating flow and hence the pressure gradient becomes a function of time only; (4) viscous dissipation effects are negligible compared to the convective rate of heat transfer: (5) the fluid is entering a thermal region with a constant uniform temperature.

2.2 Governing System of Equations

The Navier Stokes equations under the above assumptions and in cylindrical coordinates are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \dots (2.1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) \dots (2.2)$$

with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \dots (2.3)$$

where x, r are axial and radial coordinates and u, v are velocity components in x and r directions respectively, t is time and ρ is density of the fluid.

If the flow direction is assumed to be parallel to the x axis, as stated in our assumptions, velocity has x component only and v is zero. The equation of continuity gives

$$u = u(r, t)$$

which indicates the velocity to be constant in the direction parallel to the center line of the tube. The pressure gradient is assumed to be a harmonic of some Fourier series in time.

The temperature T^* within the tube is governed by the simplified convection diffusion equation

$$\frac{\partial T^*}{\partial t^*} + u^*(r^*, t^*) \frac{\partial T^*}{\partial x^*} = \frac{k}{\rho C_p} \left(\frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T^*}{\partial r^*} \right) \dots (2.4)$$

where C_p, k are specific heat, and thermal conductivity of the fluid, and the starred quantities indicate dimensional form of these variables. The second derivative of T^* with respect to x that appears in the laplacian is neglected on the assumption that it is small compared with the derivatives of T^* with respect to r in the downstream region of the flow being considered.

In the first part of the thesis the solution is sought for a region of the tube downstream, where the flow field is fully developed and the heat flux along the boundary of the tube is constant and fluid is entering the region at a uniform temperature T_0^* .

Introducing the nondimensional temperature difference

$$\theta = \frac{T^* - T_0^*}{q_w a / k} \dots (2.5)$$

where q_w is the constant heat flux at the wall. a is the radius of the tube. and k is the thermal conductivity of the fluid. the problem can be put in the nondimensional form

$$\frac{\partial \theta}{\partial t} + u(r, t) \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \dots (2.6)$$

with boundary conditions for $\theta(x, r, t)$

$$\begin{aligned} \frac{\partial \theta}{\partial r}(x, 1, t) &= 1 \\ \theta(0, r, t) &= 0 \end{aligned}$$

In the second part of the thesis a constant wall temperature is prescribed on the wall of the tube with fluid entering a thermal region at a different constant temperature. The velocity field is assumed to be that of fully developed pulsatile flow as in the first part of the thesis. The governing equation is the same as in 2.6 but with

$$\theta = (T - T_w) / (T_0 - T_w) \dots (2.8)$$

and where T_w, T_0 are the wall temperature of the tube and the entry temperature of the fluid respectively.

The boundary conditions in this case are

$$\begin{aligned} \theta(0, r, t) &= 1 \\ \theta(x, 1, t) &= 0 \\ \frac{\partial \theta}{\partial r}(x, 0, t) &= 0 \\ \theta(x, r, 0) &= 1 \dots (2.9) \end{aligned}$$

2.3 Preliminary Steady State problems

In what follows we present results of steady state problems which are required for solution of the heat transfer problem with pulsatile flow.

2.3.1 Heat Transfer With Slug Flow in a Tube

In this section we consider the case of laminar flow in a circular tube with heat transfer. Let the temperature of the fluid be T_0 at the thermal entrance of the tube and the wall temperature be T_w . Also we assume in this section that at the thermal entrance to the tube the velocity is uniform over the cross-section, that is

$$u = u_m = \text{const}$$

The energy equation for a fluid element is:

$$\rho C_p \frac{DT^*}{Dt} = k \nabla^2 T^* \dots (2.10)$$

where ρ is the fluid density, C_p is the specific heat at constant pressure, k is the thermal conductivity of the fluid, $\alpha = k/\rho C_p$ is the thermal diffusivity and

$$\frac{DT^*}{Dt} = \frac{\partial T^*}{\partial t} + u \cdot \nabla \dots (2.11)$$

In terms of nondimensional temperature $\theta = (T^* - T_w) / (T_0 - T_w)$ and nondimensional quantities $x = x^* / a P_r R_e$ and $r = r^* / a$ the

equation becomes:

$$u_m \frac{\partial \theta}{\partial x} = \alpha \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) \dots (2.12)$$

with boundary conditions:

$$\begin{aligned} \theta &= 0 \text{ at } r = 1, \forall x > 0 \\ \theta &= 1 \text{ at } x = 0 \end{aligned}$$

2.3.2 Method of Solution

By separation of variables

$$\theta = X(x)R(r) \dots (2.13)$$

one obtains

$$X = e^{-\frac{n}{4m} 3^2 x} \dots (2.14)$$

$$R' + \frac{1}{r} R' + 3^2 R = 0 \dots (2.15)$$

The second equation has a solution

$$R(r) = \sum_{i=1}^{\infty} A_i J_0(\beta_i r) \dots (2.16)$$

$$J_0(\beta_i) = 0 \forall i. \dots (2.16)$$

The constants A_i are determined by the condition that

$$\theta = f(r) = 1 \text{ at } x = 0$$

Multiplying both sides of equation 2.16 by $r J_0(\beta_n r)$ and integrating from 0 to 1 gives:

$$\int_0^1 r R(r) J_0(\beta_n r) dr = \int_0^1 \sum_{i=1}^{\infty} A_i J_0(\beta_i r) r J_0(\beta_n r) dr \dots (2.17)$$

$$= \sum_{i=1}^{\infty} A_i \int_0^1 r J_0(\beta_i r) J_0(\beta_n r) dr \dots (2.17)$$

The following two results will be made use of in the solution.

If λ and μ are two different constants then.

$$\int_0^1 r J_n(\lambda r) J_n(\mu r) dr = \frac{[\lambda J_n(\mu) J_n'(\lambda) - \mu J_n(\lambda) J_n'(\mu)]}{(\mu^2 - \lambda^2)} \dots (2.18)$$

$$\int_0^1 r J_n^2(\lambda r) dr = \frac{1}{2} \left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda) \right] \dots (2.19)$$

Letting $n = 0$ and $\lambda = \beta_m$ in equation 2.19 above and given the identity

$$r J_n'(r) =$$

$n J_n(r) - r J_{n+1}(r)$ which implies that.

$$J_0'^2(r) = J_1^2(r)$$

we get

$$\int_0^1 r J_0^2(\beta_m r) dr = \frac{1}{2} J_1^2(\beta_m) \dots (2.20)$$

Thus, if $R = f(r)$ at $x = 0$ and because of equation 2.17, we get

$$\int_0^1 r f(r) J_0(\beta_n r) dr = \frac{A_n}{2} J_1^2(\beta_n) \dots (2.21)$$

where

$$A_n = 2 J_1^2(\beta_n) \int_0^1 r f(r) J_0(\beta_n r) dr \dots (2.22)$$

The uniform temperature assumption $f(r) = 1$ gives:

$$\int_0^1 r J_0(\beta_n r) dr = \frac{1}{\beta_n} J_1(\beta_n) \dots (2.23)$$

$$A_n = \frac{2}{\beta_n} \frac{1}{J_1(\beta_n)} \dots (2.24)$$

and

$$R = 2 \sum_{n=1}^{\infty} \frac{1}{\beta_n} \frac{J_0(\beta_n r)}{J_1(\beta_n)} \dots (2.25)$$

The complete solution for slug flow becomes

$$\theta = 2 \sum_{n=1}^{\infty} \frac{1}{\beta_n} \frac{J_0(\beta_n r)}{J_1(\beta_n)} \exp\left(-\frac{\alpha}{u_m} \beta_n^2 x\right) \dots (2.26)$$

2.3.3 The Graetz Problem

Heat transfer problems of forced convection in tubes have been studied for many years beginning with Graetz [18] and later with Nusselt [19] and Leveque [20].

In this section we assume poiseuille flow at the entrance of the thermal region and to the left of this region we assume the fluid is isothermal. In the thermal region a prescribed temperature or heat flux is specified at the wall of the tube. This problem is referred to as the Graetz problem.

2.3.4 Asymptotic Method to solve Graetz problem

From equation 2.6 assuming a steady state form with $u = 1 - r^2$, the Graetz problem becomes

$$\frac{\partial \theta}{\partial x} = \left(\frac{1}{1-r^2}\right) \left(\frac{1}{r}\right) \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r}\right) \dots (2.27)$$

with boundary conditions :

$$\begin{aligned} \theta &= 1 \quad x \leq 0 \\ \theta_w &= 0 \quad x > 0 \end{aligned}$$

where $\theta = (T^* - T_w) / (T_0 - T_w)$, T_w, T_0 are wall and entrance temperatures respectively.

Assume θ is in the form :

$$\theta = \sum_{n=0}^{\infty} C_n R_n(r) \exp(-\lambda_n^2 x) \dots (2.28)$$

where λ_n are eigenvalues that satisfy the following differential equation:

$$r R_n' + R_n' + \lambda_n^2 r (1 - r^2) R_n = 0 \dots (2.29)$$

satisfying the boundary equations:

$$\begin{aligned} R_n(1) &= 0 \\ R_n(0) &= 1 \end{aligned}$$

The Sturm-Liouville theory gives us orthogonality with respect to the weight function $r(1 - r^2)$, that is

$$\int_0^1 r (1 - r^2) R_m(r) R_n(r) dr = 0 \dots (2.30)$$

for $m \neq n$

It can be shown easily that

$$C_n = \frac{\int_0^1 R_n(r)r(1-r^2)dr}{\int_0^1 R_n^2(r)r(1-r^2)dr} \dots(2.31)$$

and the following identities are known to hold [18]

$$\int_0^1 R_m r (1-r^2) dr = -\frac{1}{\lambda_m^2} \left(\frac{\partial R_m}{\partial r} \right)_{r=1} \dots(2.32)$$

$$\int_0^1 r (1-r^2) R_n^2 dr = \left(\frac{1}{2\lambda_n} \right) \left(\frac{\partial R_n}{\partial r} \frac{\partial R_n}{\partial \lambda_n} \right)_{r=1} \dots(2.33)$$

where the boundary conditions above have been used.

Using these two results we get at once that

$$C_n = -\frac{2}{\lambda_n \left(\frac{\partial R_n}{\partial \lambda} \right)_{r=1}} \dots(2.34)$$

Assume a solution to the above Sturm Liouville problem equation 2.29 in the form:

$$R = e^{h(r)}$$

Substitution of R into the ordinary differential equation gives:

$$h'' + h'^2 + (1/r)h' + \lambda^2 (1-r^2) = 0 \dots(2.35)$$

An asymptotic solution is sought in the form:

$$h = \lambda h_0 + h_1 + (\lambda)^{-1} h_2 + (\lambda)^{-2} h_3 + \dots \dots(2.36)$$

where λ is the eigenvalue of equation 2.35 .

After straightforward substitutions a solution involving just h_0 and h_1 is obtained in the form

$$R = A \frac{e^{i\lambda \int_0^r \sqrt{1-\xi^2} d\xi}}{\sqrt{r(1-r^2)}^{1/4}} + B \frac{e^{-i\lambda \int_0^r \sqrt{1-\xi^2} d\xi}}{\sqrt{r(1-r^2)}^{1/4}} \dots(2.37)$$

with A, B complex numbers.

Sellars, Tribus and Klein [21] have shown that the eigenvalues are given as

$$\lambda_n = 4n + \frac{8}{3}n = 0.1.2 \dots \dots(2.38)$$

and the eigenfunctions are as follows:

Near the center of the tube. $r \approx 0$

$$R_n(r) = J_0(\lambda_n r) \dots(2.39)$$

for intermediate $r \approx 0.5$

$$R_n(r) = \sqrt{\frac{2}{\pi \lambda_n r}} \frac{\cos\left[\frac{\lambda_n}{2} r \sqrt{1-r^2} + \frac{\lambda_n}{2} \arcsin r - \pi\right]}{(1-r^2)^{1/4}} \dots (2.40)$$

and near the wall, $r \approx 1$

$$R_n(r) = \sqrt{\frac{2(1-r)}{3}} (-1)^n J_{\frac{1}{3}}\left(\frac{\lambda_n \sqrt{8(1-r)^{3/2}}}{3}\right) \dots (2.41)$$

These 3 approximations can be patched together to produce continuous results. We used a Maple routine [Appendix A] in which eigenfunctions and temperature profiles of the Graetz problem can be obtained. Refer to figure 2.1-figure 2.7 for eigenfunctions and temperature profiles.

The constants C_n are given by

$$C_n = \frac{(-1)^n \cdot 2 \cdot 6^{2/3} \Gamma(2/3) \lambda_n^{-2/3}}{\pi} \quad n = 0, 1, 2 \dots$$

table 2.1 lists several eigenvalues and important constants for the Graetz problem.

The Nusselt number for constant wall temperature is

$$Nu = \frac{qd}{\Delta \theta k} = -4 \sum_{n=0}^{\infty} \frac{C_n}{2} R'_n(1) e^{-\lambda_n^2 x} \dots (2.42)$$

table 2.1: Parameters associated with the Graetz problem.

Refer to figure 2.7 for the Nusselt number variation along the tube.

n	λ_n	λ_n^2	c_n	$-c_n R'_n/2$
0	$2\frac{2}{3}$	7.1129	+1.47989	0.7303
1	$6\frac{2}{3}$	44.489	-0.80345	0.53810
2	$10\frac{2}{3}$	113.785	0.58732	0.460074
3	$14\frac{2}{3}$	215.121	-0.474993	0.413743
4	$18\frac{2}{3}$	348.457	+0.404448	0.381785
5	$22\frac{2}{3}$	513.793	-0.355345	0.357853
6	$26\frac{2}{3}$	711.129	+0.318858	0.338988
7	$30\frac{2}{3}$	940.465	-0.290488	0.323555
8	$34\frac{2}{3}$	1201.8	+0.267691	0.310596
9	$38\frac{2}{3}$	1495.1	-0.248895	0.29950

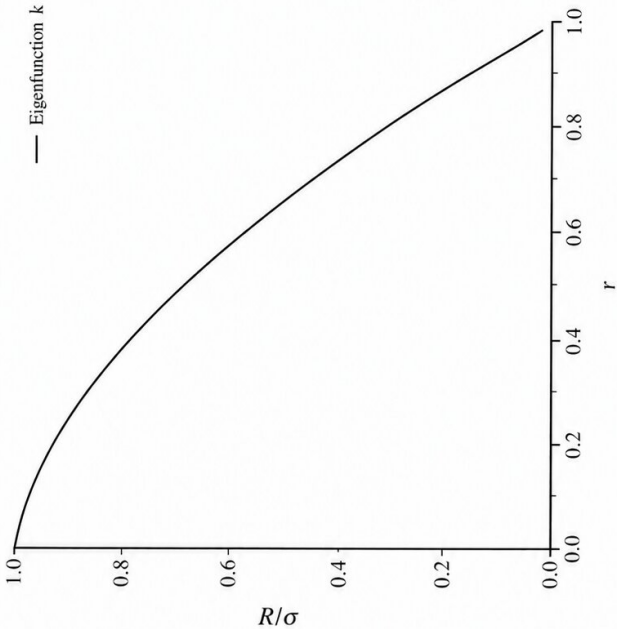


Fig 2.1

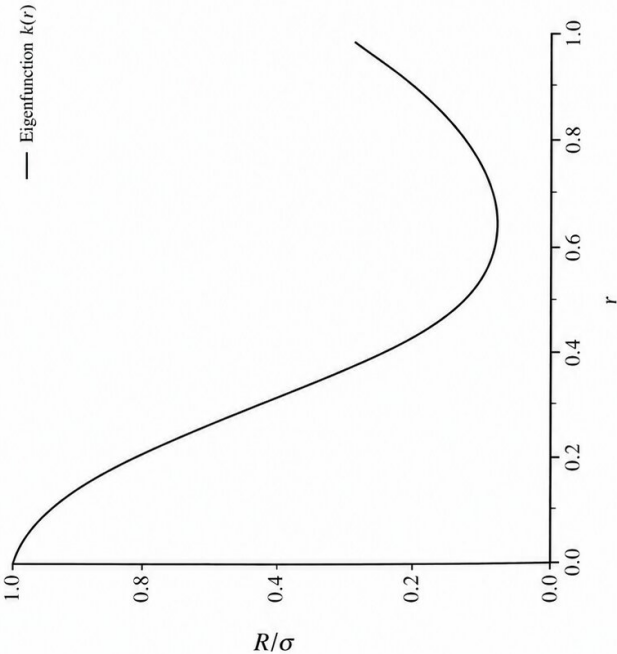
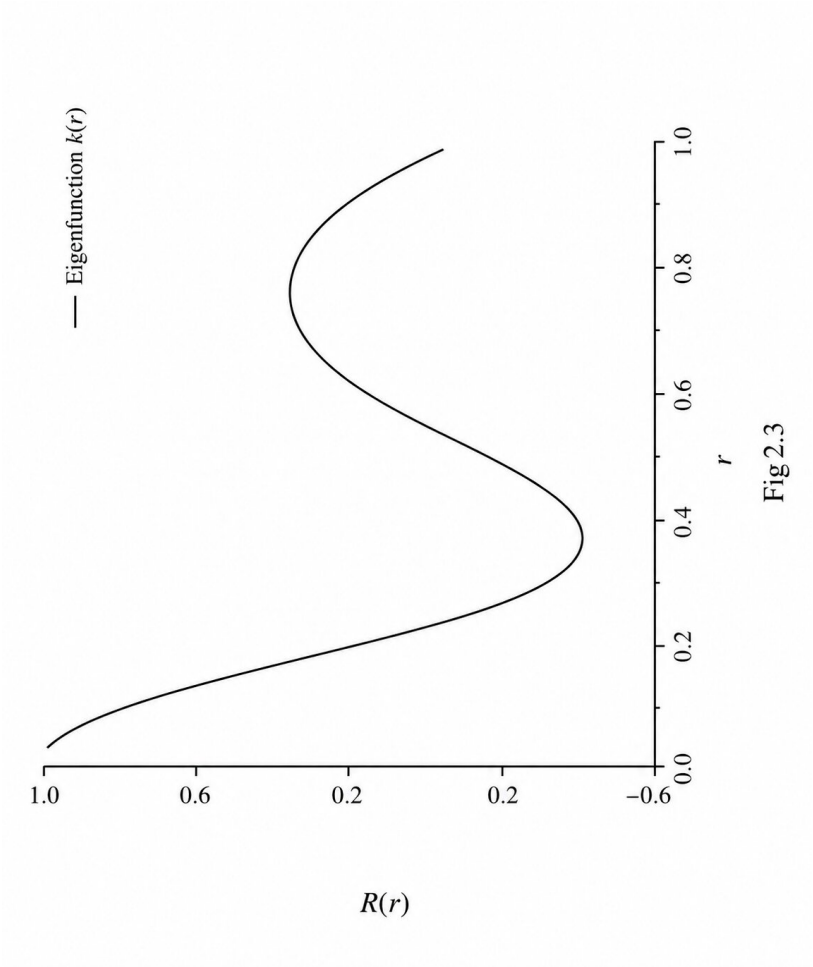
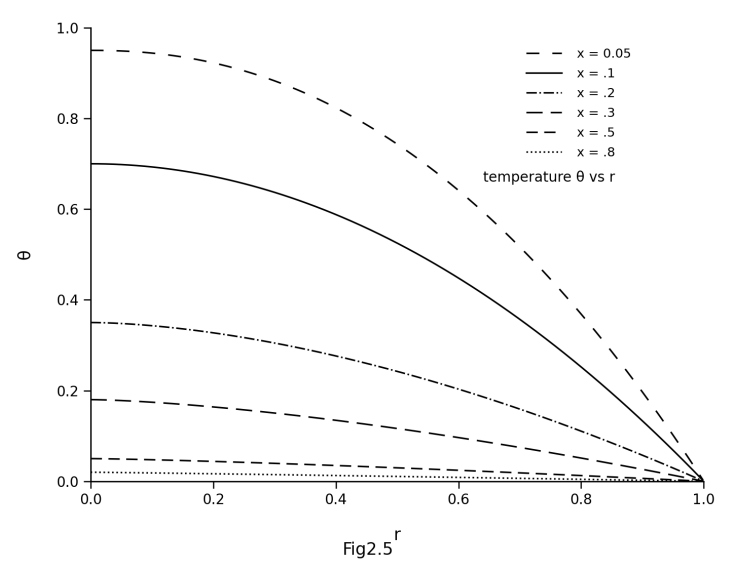
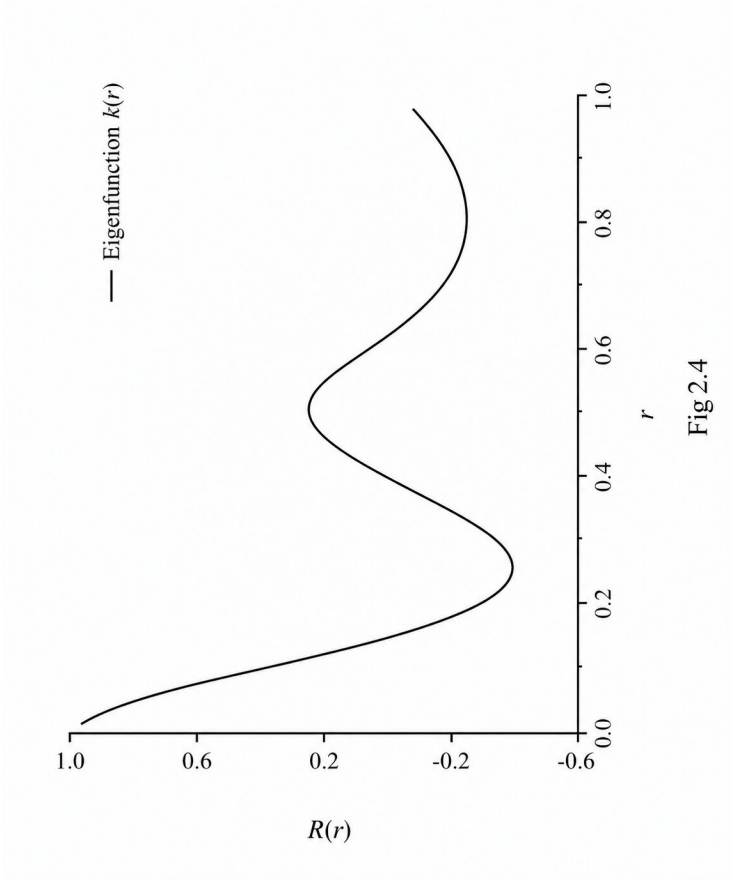
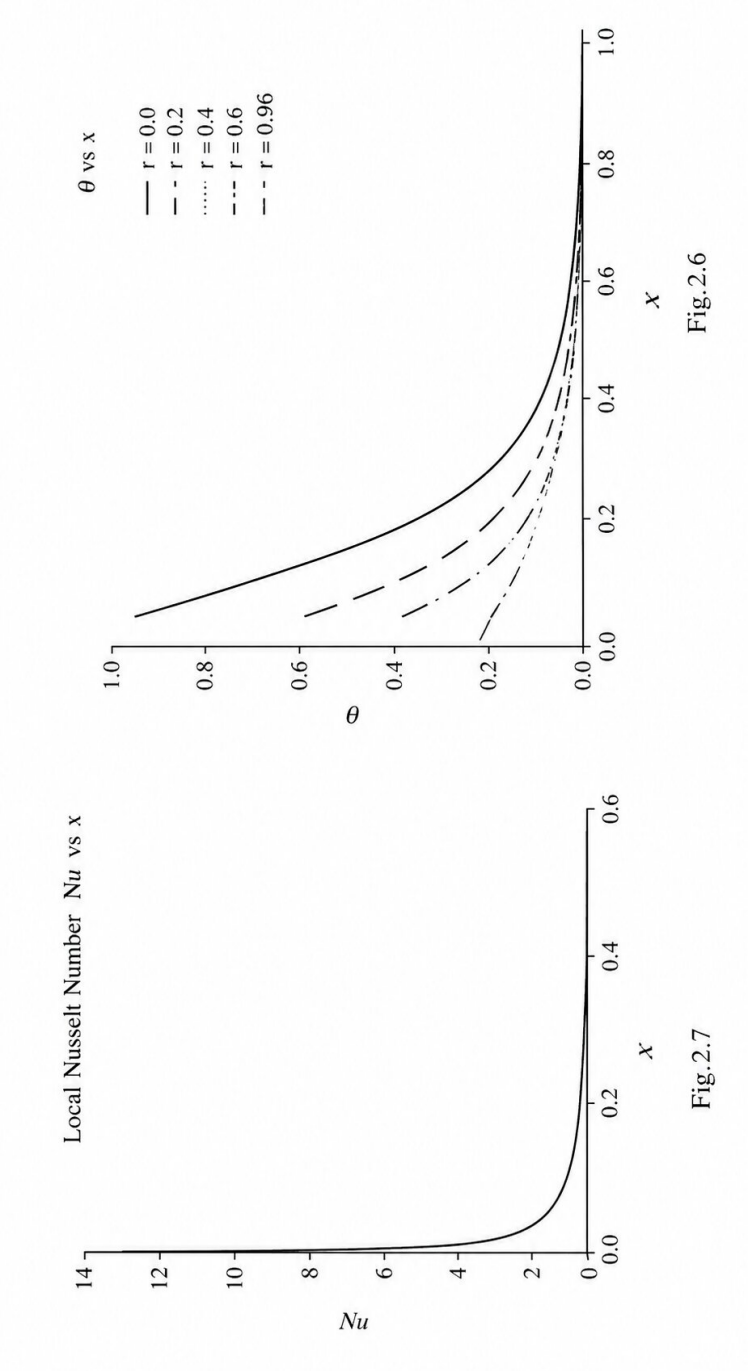


Fig.2.2







Heat Transfer with Pulsatile Flow and Constant Heat Flux

3

3.1 Overview

In this chapter an analysis of fully developed flow in a tube in the presence of longitudinal periodic pulsations with heat transfer is presented.

Using a regular perturbation method [17] an analytical solution is sought for the temperature field downstream. This expression is written as a steady term with higher order harmonic terms superimposed on it. These higher order terms are due to velocity fluctuations in pulsatile flow. It can be shown that these velocity pulsations cause harmonic oscillations in temperature to occur thereby splitting the temperature field into a steady part plus an oscillatory part.

Our aim in this thesis is to determine if the flow field for pulsatile flow enhances the overall heat transfer rate as measured by the Nusselt number.

The bulk temperature over one cycle and the change in unsteady Nusselt number to that of steady Nusselt number is shown to depend on the dimensionless frequency ω and a number which we obtain through dimensional analysis which involves a ratio of unsteady pulsation amplitude to that of steady amplitude, this number which we derive below is denoted by ξ .

3.2 Governing Equations

The Temperature T^* within the tube is governed by the following coupled partial differential equations

$$\frac{DT^*}{Dt^*} = \frac{k}{\rho C_p} \nabla_r^2 T^* \dots (3.1)$$

where the material derivative operator is defined as

$$\frac{D}{Dt^*} = \frac{\partial}{\partial t} + u^* \cdot \nabla_r \dots (3.2)$$

The function u^* satisfies the Navier Stokes equations for a sinusoidal pressure gradient driving the flow with a no slip boundary condition at the wall of the tube and a fully developed velocity profile along the tube.

The temperature of the fluid at the thermal entrance region is constant with constant heat flux at the wall of the tube. See figure 3.1 on page 23 .

The following boundary value problem results.

$$\begin{aligned} \frac{\partial T^*}{\partial t^*} + u^*(r^*, t^*) \frac{\partial T^*}{\partial x^*} &= \frac{k}{\rho C_p} \left(\frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T^*}{\partial r^*} \right) \dots (3.3) \\ T^* &= T_0 \text{ at } x^* = 0 \\ \frac{\partial T^*}{\partial r^*} &= q/k \text{ at } r^* = a \end{aligned}$$

Viscous dissipation effects as given by the dissipation function are assumed negligible compared to the convective rate of heat transfer so that no nonlinear effects are present in the energy equation.

We now introduce the following nondimensional quantities

$$\theta = \frac{T^* - T_0}{q_w a / k} \dots (3.4)$$

where q_w is the constant heat flux at the boundary of the tube, a is the radius of the tube and k is the thermal conductivity of the fluid.

$$\begin{aligned} x &= r^* k / \rho c_p a^2 = x^* / a P_r R_e \\ t &= t^* k / \rho c_p a^2 = t^* \alpha / a^2 \\ r &= r^* / a \end{aligned}$$

$$u = u^* / m_0 \quad (3.5)$$

The Prandtl number, Reynolds number and the thermal diffusivity are defined by

$$\begin{aligned} P_r &= \frac{\mu c_p}{k} \\ R_e &= \frac{\rho h_0 a}{\mu} \\ \alpha &= \frac{k}{\rho c_p} \end{aligned}$$

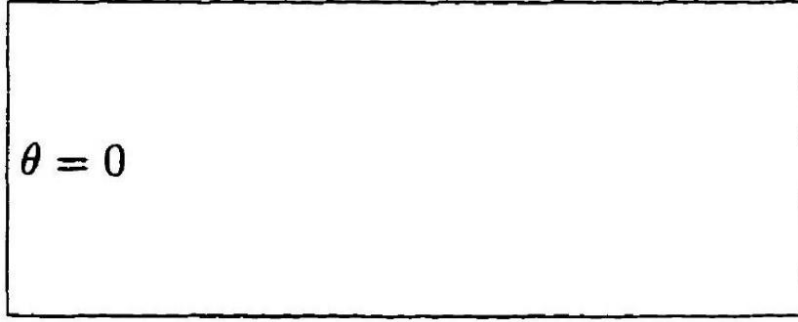
Substitution of equation 3.4, 3.5 and 3.6 into 3.3 the governing equation finally becomes, in nondimensional form

$$\frac{\partial \theta}{\partial t} + u(r, t) \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \dots (3.7)$$

with boundary conditions for $\theta(x, r, t)$

$$\begin{aligned} \frac{\partial \theta}{\partial r}(x, 1, t) &= 1 \\ \theta(0, r, t) &= 0 \\ \frac{\partial \theta}{\partial r}(x, 0, t) &= 0 \dots (3.8) \end{aligned}$$

$$\frac{\partial \theta}{\partial r} = 1$$



$$\frac{\partial \theta}{\partial r} = 0$$

figure 3.1

3.3 Regular Perturbation Method

The flow field within the thermal region of the tube is assumed to consist of a steady part represented by fully developed Poiseuille flow plus an oscillatory part represented by the classical solution for pulsatile flow [2]. Thus with one harmonic of some Fourier series expansion added to steady flow we write

$$u^*(r^*, t^*) = \frac{A_0}{4\mu} (r^{*2} - a^2) + \frac{iA_1}{\rho\omega^*} \left(1 - \frac{J_0(\sqrt{-i\rho\omega^*}a^2/\mu(r^*/a))}{J_0(\sqrt{-i\rho\omega^*}a^2/\mu)} \right) e^{i\omega^* t^*} \dots (3.9)$$

where A_0 is the pressure gradient driving the steady part of the flow and A_1 is the amplitude of the oscillatory pressure gradient driving the oscillatory part of the flow. The pressure gradient with respect to the x variable is written as a steady amplitude plus one harmonic of some Fourier series in time.

$$\frac{\partial p^*}{\partial x^*} = A_0 + A_1 e^{i\omega^* t^*} \dots (3.10)$$

Introducing a nondimensional frequency parameter,

$$\omega = \frac{\rho C_P a^2}{k} \omega^* \dots (3.11)$$

the velocity downstream can be put in the nondimensional form

$$u(r, t) = u_0(r) + \xi u_1(r, t) \dots (3.12)$$

where

$$u_0(r) = (1 - r^2) \dots (3.13)$$

$$u_1(r, t) = \frac{-i}{\omega} \left(1 - \frac{J_0(\sqrt{-i\omega/P_r}r)}{J_0(\sqrt{-i\omega/P_r})} \right) e^{i\omega t} \dots (3.14)$$

$$\xi = 4 \left(\frac{A_1}{A_0} \right) P_r \dots (3.15)$$

Variables have been nondimensionalized as before, with the normalizing velocity ψ_0 now being taken as the maximum velocity in Poiseuille flow, namely

$$u = \frac{u^*}{u_0}, b_0 = \frac{-A_0 a^2}{4\mu} \dots (3.16)$$

Due to the nonlinearity of the product of the velocity and the gradient of temperature difference in the convection equation and since the velocity for pulsatile flow depends on the ratio of A_1/A_0 , the following regular perturbation expansion is used to solve for temperature downstream.

$$\theta(x, r, t : \xi) = \theta_0(x, r) + \xi \theta_1(x, r, t) + \xi^2 \theta_2(x, r, t) + O(\xi^3) \dots (3.17)$$

where ξ is the perturbation parameter defined previously and will be assumed small (i.e. less than unity).

Substituting the velocity and temperature expressions in the governing system yields infinitely many boundary value problems for $\theta_0, \theta_1, \theta_2$, etc.

The first three are

$$\begin{aligned} u_0 \frac{\partial \theta_0}{\partial x} &= \frac{\partial^2 \theta_0}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_0}{\partial r} \\ \frac{\partial \theta_0}{\partial r}(x, 1) &= 1; \frac{\partial \theta_0}{\partial r}(x, 0) = 0 \\ \theta_0(0, r) &= 0 \\ \frac{\partial \theta_1}{\partial t} + u_1(r, t) \frac{\partial \theta_0}{\partial x} + u_0(r) \frac{\partial \theta_1}{\partial x} &= \frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_1}{\partial r} \\ \frac{\partial \theta_1}{\partial r}(1, t) &= 0; \frac{\partial \theta_1}{\partial r}(0, t) = 0 \\ \frac{\partial \theta_2}{\partial t} + u_1(r, t) \frac{\partial \theta_1}{\partial x} + u_0(r) \frac{\partial \theta_2}{\partial x} &= \frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_2}{\partial r} \\ \frac{\partial \theta_2}{\partial r}(1, t) &= 0; \frac{\partial \theta_2}{\partial r}(0, t) = 0 \end{aligned}$$

In view of the form assumed for the temperature, these boundary value problems govern the steady $\theta_0(x, r)$ and oscillatory ($\theta_1(r, t), \theta_2(r, t)$ etc.) parts of the temperature field, respectively, and their solutions are considered separately in the next two sections.

A solution to (3.19) is sought downstream. where we assume that $\frac{\partial \theta}{\partial x} = \text{const}$. The boundary condition is physically reasonable since far downstream the effects of inlet temperature are negligible where primary heating of fluid is due to the heat flux at the wall alone.

In the next section we claim that the derivative of θ_0 with respect to x downstream becomes a constant. Since the series is expanded in ξ which is variable then the partial derivatives of θ_1, θ_2 , etc with respect to x , approach 0 downstream.

3.4 Steady Temperature

The zeroth order problem, θ_0 , is solved with the use of Laplace transforms and residue theory. We have described the solution in appendix C, where we obtain the following result for the temperature

$$\theta_0(r, x) = \left[4x - \sum_m \frac{1}{\beta_m^4 \Psi'(-\beta_m^2)} + \sum_m \frac{e^{-\beta_m^2 x}}{\beta_m^4 \Psi'(-\beta_m^2)} - \sum_n C_n R_n \sum_m \frac{1 - e^{-\beta_m^2 x}}{\beta_m^2 (\lambda_n^2 - \beta_m^2)} \frac{1}{\Psi'(-\beta_m^2)} \right]$$

where some values of β_m^2 and $\Psi'(-\beta_m^2)$ are given in the following table

m	β_m^2	$-\Psi'(-\beta_m^2)$
1	25.639	8.854×10^{-3}
2	84.624	2.062×10^{-3}
3	176.40	9.435×10^{-4}

table 3.1: Zeros of the function Ψ for zeroth order perturbation.

$$\Psi(s) = -\sum_{n=0}^{\infty} C_n R_n' \frac{1}{s + \lambda_n^2} \dots (3.22)$$

We observe that for this zeroth order problem far downstream

$$\frac{\partial \theta_0}{\partial x} \approx 0 \dots (3.23)$$

We make use of this result in the solution of the first order problem.

3.5 Green's Functions

The Green's function method is one of the most important approaches for solving boundary value problems with nonhomogeneous ordinary differential equations. Sturm Liouville problems which involve self adjoint differential operators are written as

$$Lh = \frac{d}{dr} \left[\hat{p}(r) \frac{dh}{dr} \right] + [q(r)]h(r) = g(r) \dots (3.24)$$

$$\alpha_1 h(0) + \alpha_2 h'(0) = 0 \dots (3.25)$$

$$\beta_1 h(1) + \beta_2 h'(1) = 0 \dots (3.26)$$

The idea is to obtain a solution of the boundary value problem

$$Lh = g$$

in the form

$$h = Kg$$

where K turns out to be an integral operator. This kernel is called the Green's function for the boundary value problem.

Ordinarily q, g are real valued but the method can be extended to complex values for q, g . The theory of global analytic functions allows one to study the complex solution of ordinary differential equations.[22]

This extension is necessary in our problem for the solution for oscillatory temperature. Constructively, a standard technique for solving Sturm-Liouville problems is to use variation of parameters. Suppose U, V are

two linearly independent solutions of the corresponding homogeneous equation $Lh = 0$.

A solution is sought in the form $h = \phi U + \psi V$ upon which substitution gives

$$\phi(r) = \frac{1}{W} \left(\int_r^1 V(\tau)g(\tau)d\tau + A \right) \dots(3.27)$$

$$\psi(r) = \frac{1}{W} \left(\int_0^r U(\tau)g(\tau)d\tau + B \right) \dots(3.28)$$

If $W = p(UV' - VU')$ is not zero then $h = \phi U + \psi V$ is a solution for the differential equation, more explicitly,

$$h(r) = \int_0^1 G(r,\tau)g(\tau)d\tau \dots(3.29)$$

where

$$G(r,\tau) = \begin{cases} W^{-1}V(r)U(\tau) & 0 \leq \tau \leq r \leq 1 \\ W^{-1}U(r)V(\tau) & 0 \leq r < \tau \leq 1 \end{cases}$$

is the Green's function associated with the problem.

In the next section the type of Sturm-Liouville problem associated with oscillatory temperature involves a differential equation with complex valued arguments as coefficients with the nonhomogeneous part being complex valued as well. As already mentioned a solution can be obtained in the complex plane.

3.6 Oscillatory Temperature

The result of the previous section (equation 3.23) makes it possible to seek a solution far downstream for θ_1 in the form

$$\theta_1(r, t) = 4h(r)e^{i\omega t} \dots(3.30)$$

Substituting this into boundary value problem Eq 3.19 leads to the following boundary value problem

$$\begin{aligned} \frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} - i\omega h &= g(r) \dots(3.31) \\ h(1) &= 0, h'(0) = 0 \end{aligned}$$

In addition we assume that at zero frequency, the solution for h is bounded and finite.

$$g(r) = \frac{-i}{\omega} \left[1 - \frac{J_0(\sqrt{-i\omega/P_r}r)}{J_0(\sqrt{-i\omega/P_r})} \right] \dots(3.32)$$

The differential equation is of regular complex valued Sturm-Liouville type

$$\begin{aligned} Lh &= g \\ K &= h^{-1} \end{aligned}$$

$$Kg(r) = \int_0^1 G(r, \tau)g(\tau)d\tau \dots(3.33)$$

where G is the Green's function of the above Sturm-Liouville problem and K is a special integral operator

The most general solution is

$$\begin{aligned} h(r, \omega) &= c_1 J_0(\sqrt{\omega} i^{3/2} r) + c_2 Y_0(\sqrt{\omega} \cdot i^{3/2} r) + h_p(r) \\ &= c_1 U(r) + c_2 V(r) + \int_0^1 G(r, \tau)g(\tau)d\tau \\ &= c_1 U(r) + c_2 V(r) + UW^{-1} \int_r^1 V(\tau)g(\tau)d\tau + VW^{-1} \int_0^r U(\tau)g(\tau)d\tau \end{aligned}$$

The Wronskian is defined as follows

$$\begin{aligned} W &= \det \begin{bmatrix} J_0 & Y_0 \\ J_0' & Y_0' \end{bmatrix} \\ &= \frac{2}{\pi \sqrt{\omega} i^{3/2} r} \end{aligned}$$

Now the boundary conditions of symmetry and flux are

$$\begin{aligned} h'(0, \omega) &= 0 \\ h'(1, \omega) &= 0 \end{aligned}$$

The first condition gives

$$c_2 V'(0) = 0$$

and since the derivative of the Neumann function is unbounded at the centreline of the tube.

$$c_2 = 0$$

Differentiation of the function $h(r)$ with respect to r gives

$$\begin{aligned} h'(r) &= c_1 U'(r) + c_2 V'(r) + VW^{-1}U(r)g(r) + V'(r)W^{-1} \int_0^r [U(\tau)g(\tau)d\tau - UW^{-1}Vg(r)] h(r)dr. \\ &\quad - \int_1^r V(\tau)g(\tau)d\tau U'(r)W^{-1} \end{aligned}$$

The second boundary condition gives us at once that

$$c_1 U'(1) + V'(1)W^{-1} \int_0^1 U(\tau)g(\tau)d\tau = 0 \dots(3.35)$$

from which c_1 is determined as

$$c_1 = -\frac{V'(1)}{U'(1)} W^{-1} \int_0^1 U(\tau)g(\tau)d\tau \dots(3.36)$$

Hence the solution for the function h becomes

$$h = - \frac{\frac{\partial V}{\partial r}|_{r=1}}{\frac{\partial U}{\partial r}|_{r=1}} W^{-1} U(r) \int_0^1 U(\tau) g(\tau) d\tau + \int_0^1 G(r, \tau) g(\tau) d\tau \quad (3.37)$$

$$G(r, \tau) = W^{-1} V(r) U(\tau) H(r - \tau) + W^{-1} U(r) V(\tau) H(\tau - r)$$

The functions U, V are linearly independent solutions of the corresponding homogeneous differential equation

$$Lh = 0$$

The function H is Heaviside's function, defined on $[0, 1]$.

Substitution of the linearly independent solutions U, V , the function g , the Wronskian W and the Green's function G leads to the following solution:

$$h(r) = h_1(r) + h_2(r) + h_3(r) \dots \quad (3.38)$$

where.

$$h_1(r) = \frac{-\frac{d}{dr} Y_0(\sqrt{\omega}^{3/2} r)|_{r=1}}{\frac{d}{dr} J_0(\sqrt{\omega}^{3/2} r)|_{r=1}} (\pi/2) \sqrt{\omega} i^{3/2} J_0(\sqrt{\omega} i^{3/2} r) \int_0^1 J_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau \sqrt{-\frac{i\omega}{P_r}})}{J_0(\sqrt{-\frac{i\omega}{P_r}})} \right] d\tau \quad (3.39)$$

$$h_2(r) = (\pi/2) \sqrt{\omega} i^{3/2} J_0(\sqrt{\omega} i^{3/2} r) \int_r^1 Y_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau \sqrt{-\frac{i\omega}{P_r}})}{J_0(\sqrt{-\frac{i\omega}{P_r}})} \right] d\tau \quad (3.40)$$

$$h_3(r) = (\pi/2) \sqrt{\omega} i^{3/2} Y_0(\sqrt{\omega} i^{3/2} r) \int_0^r J_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau \sqrt{-\frac{i\omega}{P_r}})}{J_0(\sqrt{-\frac{i\omega}{P_r}})} \right] d\tau \quad (3.41)$$

3.7 Uniqueness of Solution

The solution obtained in the previous section involves Bessel functions Y_0 of a complex argument z , and since these in turn depend on $\log z$ each part of the solution is multivalued and thus not unique. In this section we discuss the notion of a Riemann surface upon which each part of the solution h_1, h_2, h_3 becomes single-valued and thus well-defined.

3.8 Riemann Surfaces

We introduce the idea of a branch in order to study multi valued functions properly. Two function elements (f_1, Ω_1) and (f_2, Ω_2) determine the same branch at a point in the complex plane $z_0 \in (\Omega_1 \cap \Omega_2)$ whenever $f_1 = f_2$ in a neighbourhood of the point z_0 , which holds if the functions are analytic continuations of each other. The branch at the point z_0 given

by the function element (f, Ω) is denoted as (f, z_0) The set of branches (f, z) is defined as the Riemann Surface.

A Riemann surface is simply a generalization of the complex plane where a multiple valued function is defined on a surface of many sheets where the function becomes single valued on this surface. To describe a surface of many sheets we will consider the Riemann surface for the function $\log z$ which is connected to our Green's function solution obtained previously.

3.9 A Surface for $\log z$

For z in polar form

$$\begin{aligned} z &= re^{i\theta} \\ \log z &= \text{Log} r + i\theta \end{aligned}$$

where $\text{Log} r$ is the principal value of $\log z$.

The function $\log z$ is obviously many valued at each point in the complex plane since the angle θ is increased by multiples of 2π and -2π respectively. To turn this function into a single valued function, the $\mathbf{Z}$ -plane is replaced by its Riemann surface on which new points are represented whenever the argument of the point z is either increased or decreased by 2π radians.

By considering the z - plane as a sheet R_0 , cut along the positive x axis, let the angle θ range from 0 to 2π . A second sheet R_1 is cut identically and placed in front of the sheet R_0 . The lower edge of slit in R_0 is joined to the upper edge of the slit in R_1 . On R_1 the angle θ ranges from 2π to 4π . The same is done for sheets R_2, R_3 , etc. A sheet R_{-1} placed behind R_0 on which θ varies from 0 to -2π , is cut with the lower edge of it's slit connected to the upper edge of the slit in R_0 , and this is continued for R_{-2}, R_{-3} .etc.

Now the origin is common to all sheets. In the above construction as the point completes a cycle around the origin on R_0 , the angle ranges from 0 to 2π . As it moves across $\theta = 2\pi$ the point passes to the sheet R_1 of the surface. As the point completes a cycle in R_1 , the angle θ ranges from 2π to 4π and so on. Refer to Fig3.3 for the Riemann surface for $\log z$.

All parts of the solution h that is h_1, h_2, h_3 are many valued functions in the complex plane and therefore require a Riemann surface consisting of infinitely many sheets for the logarithmic function involved. As mentioned previously the functions become single valued on the Riemann surface and hence are well defined functions.

The solution obtained however for h via the Green's function is unique. Despite the multivaluedness of each function h_1, h_2, h_3 , their sum h adds up to be single valued. A quick calculation shows that the logarithmic parts cancel out as they are summed up to obtain h .

To show this note that since $\log z = \text{Log} r + i\theta$, the multi-valued part of h_1 is

$$-i\theta \left(\frac{\pi}{2} \right) \sqrt{\omega i^{\frac{3}{2}}} J_0(\sqrt{\omega i^{\frac{3}{2}}} r) \int_0^1 J_0 \left(\sqrt{\omega i^{\frac{3}{2}}} \tau \right) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau \sqrt{\frac{-i\omega}{P_r}})}{J_0(\sqrt{-i\omega/P_r})} \right] d\tau(\tau)$$

the multi-valued part of h_2 is

$$J_0(\sqrt{\omega}i^{3/2}r)(\pi/2)\sqrt{\omega}i^{3/2} \int_r^1 i\theta \cdot J_0(\sqrt{\omega}i^{3/2}\tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{-i\omega/\text{Pr}})}{J_0(\sqrt{-i\omega/\text{Pr}})} \right] d\tau \dots (3.43)$$

and the multi-valued part of h_3 is

$$i\theta J_0(\sqrt{\omega}i^{3/2}r)(\pi/2)\sqrt{\omega}i^{3/2} \int_0^r J_0(\sqrt{\omega}i^{3/2}\tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{i\omega/\text{Pr}})}{J_0(\sqrt{-i\omega/\text{Pr}})} \right] d\tau \dots (\tau/\hbar)$$

Adding the three parts together gives us zero identically and thus the function h becomes single-valued in the complex plane. Hence we arrive at a unique solution.

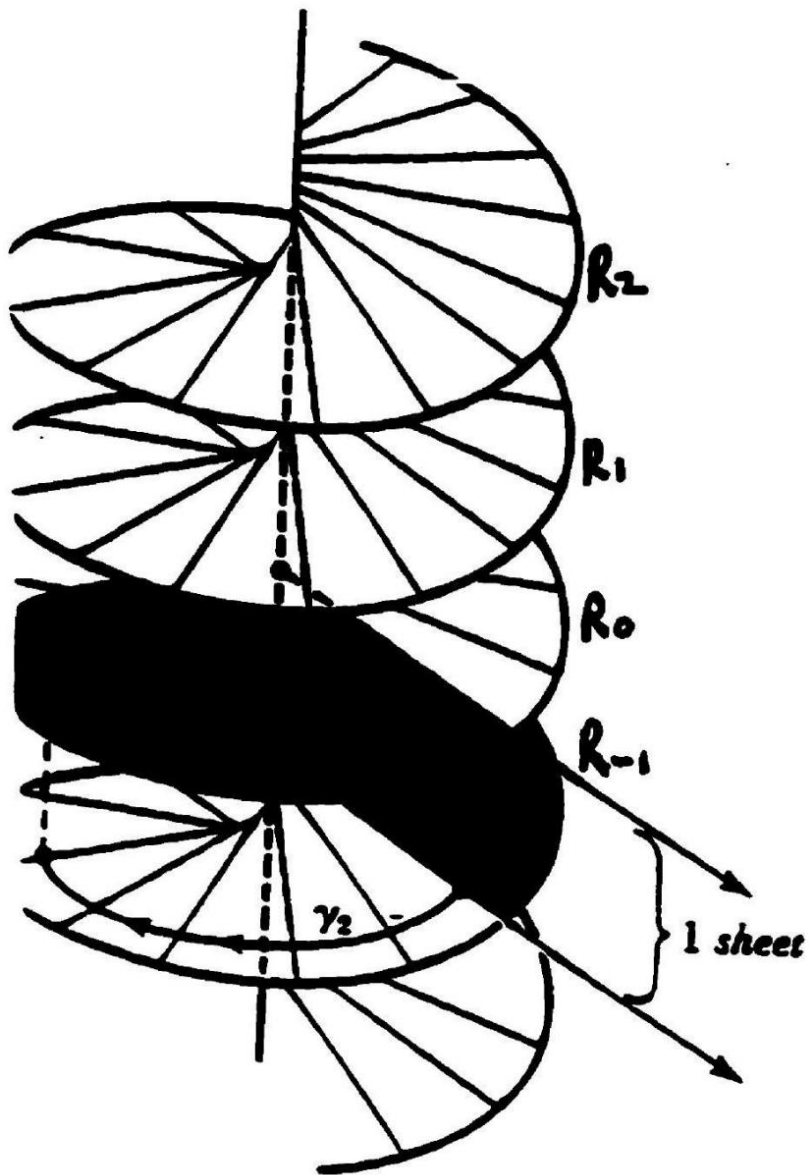


figure 3.3: Riemann surface for $\log z$.

3.10 Verification of Boundary Conditions

One of the properties of the solution for h obtained previously is that in addition to satisfying the complex differential equation, it satisfies both flux and symmetry line boundary conditions.

Differentiation and use of the first version of the fundamental theorem of calculus gives us the following

Since

$$\left. \frac{\partial J_0(\sqrt{\omega} i^{3/2} r)}{\partial r} \right|_{r=0} = 0$$

then it follows immediately that

$$\left. \frac{\partial h_I}{\partial r} \right|_{r=0} = 0$$

$$\begin{aligned} \frac{\partial h_2}{\partial r} = & -J_0(\sqrt{\omega} i^{3/2} r) (\pi/2) \sqrt{\omega} i^{3/2} Y_0(\sqrt{\omega} i^{3/2} r) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] \\ & + \int_r^1 Y_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] d\tau \frac{\partial}{\partial r} J_0(\sqrt{\omega} i^{3/2} r) (\pi/2) \sqrt{\omega} i^{3/2} \end{aligned}$$

$$\left. \frac{\partial h_2}{\partial r} \right|_{r=0} = -J_0\left(\sqrt{\omega} i^{\frac{3}{2}} r\right) \left(\frac{\pi}{2}\right) \sqrt{\omega} i^{\frac{3}{2}} Y_0\left(\sqrt{\omega} i^{\frac{3}{2}} r\right) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] \Big|_{r=0}$$

$$\begin{aligned} \frac{\partial h_3}{\partial r} = & Y_0(\sqrt{\omega} i^{3/2} r) (\pi/2) \sqrt{\omega} i^{3/2} J_0(\sqrt{\omega} i^{3/2} r) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] \\ & + \int_0^r J_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] d\tau \frac{\partial}{\partial r} Y_0(\sqrt{\omega} i^{3/2} r) (\pi/2) \sqrt{\omega} i^{3/2} \\ \left. \frac{\partial h_3}{\partial r} \right|_{r=0} = & \left[J_0(\sqrt{\omega} i^{3/2} r) (\pi/2) \sqrt{\omega} i^{3/2} Y_0(\sqrt{\omega} i^{3/2} r) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] \right] \Big|_{r=0} \end{aligned}$$

Adding up the three terms above gives

$$\left. \frac{\partial h}{\partial r} \right|_{r=0} = 0$$

Next the flux boundary condition is checked. Differentiating and using the first version of the fundamental theorem of calculus we get

$$\left. \frac{\partial h_1}{\partial r} \right|_{r=1} = - \left. \frac{\partial}{\partial r} Y_0(\sqrt{\omega} i^{3/2} r) \right|_{r=1} (\pi/2) \sqrt{\omega} i^{3/2} \int_0^1 J_0(\sqrt{\omega} i^{3/2} \tau) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] d\tau$$

$$\left. \frac{\partial h_2}{\partial r} \right|_{r=1} = -J_0\left(\sqrt{\omega} i^{\frac{3}{2}} r\right) \left(\frac{\pi}{2}\right) \sqrt{\omega} i^{\frac{3}{2}} \Big|_{r=1} Y_0\left(\sqrt{\omega} i^{\frac{3}{2}} r\right) \frac{1}{i\omega} \left[1 - \frac{J_0(\tau\sqrt{-i\omega/Pr})}{J_0(\sqrt{-i\omega/Pr})} \right] d\tau \Big|_{r=1}$$

$$\begin{aligned} \frac{\partial h_3}{\partial r} \Big|_{r=1} &= \nabla Y_0(\sqrt{\omega}^{3/2} r) \Big|_{r=1} (\pi/2) \sqrt{\omega}^{3/2} J_0(\sqrt{\omega}^{3/2} r) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/\text{Pr}})}{J_0(\sqrt{\omega}^{3/2} \text{Pr})} \right] \Big|_{r=1} \\ + \int_0^1 J_0(\sqrt{\omega}^{3/2} \tau) \frac{1}{i\omega} i^{3/2} J_0(\sqrt{\omega}^{3/2} r) \frac{1}{i\omega} \left[1 - \frac{J_0(r\sqrt{-i\omega/\text{Pr}})}{J_0(\sqrt{\omega}^{3/2} \text{Pr})} \right] d\tau \cdot \frac{\partial Y_0}{\partial r}(\sqrt{\omega}^{3/2} r) \Big|_{r=1} (\pi/2) \sqrt{\omega}^{3/2} \end{aligned}$$

Adding the above parts gives.

$$\frac{\partial h}{\partial r} \Big|_{r=1} = 0$$

Hence both boundary conditions are satisfied by the Green's function solution.

3.11 Zero Frequency

At zero frequency the term, $-V'(1)/U'(1)$, in $h_1(3.39)$ is unbounded because its imaginary part is unbounded. The real part provides a valid solution however and therefore only the real part of $-V'(1)/U'(1)$ is taken for calculation purposes.

3.12 Higer Order Perturbation terms - Convergence Criteria

The perturbation expansion in section 3.3 led to infinitely many coupled partial differential equations of which we have solved only the first two downstream.

A natural question to ask is whether or not the perturbation series converges downstream for small perturbation parameters ξ .

In order to answer this we must first solve all of the remaining higher order terms in the perturbation series starting with the second order and proceeding to the k th order one and so on.

This of course means that infinitely many boundary value problems must be solved from the second order system and so on.

Recall the second order problem

$$\begin{aligned} \frac{\partial \theta_2}{\partial t} + u_1(r,t) \frac{\partial \theta_1}{\partial x} + u_0(r) \frac{\partial \theta_2}{\partial x} &= \frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_2}{\partial r} \\ \frac{\partial \theta_2}{\partial r}(1,t) &= 0 \end{aligned}$$

$$\frac{\partial \theta_2}{\partial r}(0,t) = 0$$

Downstream this simplifies to

$$\begin{aligned} \frac{\partial \theta_2}{\partial t} &= \frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_2}{\partial r} \\ \frac{\partial \theta_2}{\partial r}(1,t) &= 0 \quad \dots(3.46) \\ \frac{\partial}{\partial r} \theta_2(0,t) &= 0 \end{aligned}$$

In fact the general k^{th} order boundary value problem downstream becomes

$$\begin{aligned}\frac{\partial \theta_k}{\partial t} &= \frac{\partial^2 \theta_k}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_k}{\partial r} \\ \frac{\partial \theta_k}{\partial r}(1, t) &= 0 \quad \dots (3.47) \\ \frac{\partial \theta_k}{\partial r}(0, t) &= 0\end{aligned}$$

The solution of this problem is readily obtained as

$$\theta_k(r, t) = \sum_{m=1}^{\infty} \frac{1}{K_m} F_m R(\beta_m, r) e^{-\beta_m^2 t}$$

where

$$\begin{aligned}F_m &= -\frac{1}{\beta_m^2} R'(\beta_m, r) \\ K_m &= \int_0^1 r [R_m(\beta_m, r)]^2 dr\end{aligned}$$

and prime denotes differentiation with respect to r .

The eigenfunctions $R(\beta_m, r)$ are the solutions to the following eigenvalue problem:

$$\frac{d}{dr} \left[r \frac{dR(\beta_m, r)}{dr} \right] + \beta_m^2 r R(\beta_m, r) = 0$$

in $0 < r < 1$

$$\begin{aligned}\frac{dR(\beta_m, r)}{dr} &= 0. \quad \text{at } r = 0 \\ \frac{dR(\beta_m, r)}{dr} &= 0. \quad \text{at } r = 1\end{aligned}$$

where the β_m are the zeroes of J_1 .

Now one sees that the k' th order solution is bounded over one cycle and across the tube radially.

Since the k 'th order problem in the perturbation expansion is bounded with $\xi < 1$ then the series converges downstream.

Hence the perturbation expansion converges downstream.

3.13 Results and Discussion

To determine the effect of the oscillations on the rate of heat transfer we define the local Nusselt number

$$\frac{q_w 2a/k}{\Re T^*(x, 1) - T_b^*(x)}$$

and the corresponding Nusselt number in steady flow

$$N_{u0}(x) = \frac{q_w 2a/k}{T_0^*(x,1) - T_{0b}^*(x)} \dots (3.49)$$

where $\Re$ denotes the real part of a complex valued function, the subscript b refers to bulk properties of the fluid, and the overbar indicates time average over one cycle, that is

$$\overline{\Re T^*}(x,1) = \frac{1}{\frac{2\pi}{\omega}} \int_0^{\frac{2\pi}{\omega}} \Re T^*(x,1,t) dt \dots (3.50)$$

$$T_b(x) = \frac{\int_0^1 \Re u(r) \Re T^*(x,r) r dr}{\int_0^1 \Re u(r) r dr} \dots (3.51)$$

In terms of the nondimensional temperature difference θ the expressions for the Nusselt numbers become

$$N_u(x) = \frac{2}{\Re \theta(x,1) - \theta_b(x)} \dots (3.52)$$

$$N_{u0}(x) = \frac{2}{\theta_0(x,1) - \theta_{0b}(x)} \dots (3.53)$$

In these, we have that

$$\int_0^1 \Re u(r) r dr = \int_0^1 u_0(r) r dr = \frac{1}{4} : (3.54)$$

and

$$\theta_b(x) = \theta_{0b} + \xi^2 \theta_{1b} + O(\xi^4) \dots (3.55)$$

where

$$\theta_{0b} = \int_0^1 u_0 \theta_0 r dr \dots (3.56)$$

$$\theta_{1b} = 4 \int_0^1 \Re u_1 \Re \theta_1 r dr \dots (3.57)$$

where only the first two terms are considered in the perturbation expansion since the expression for velocity is exact. (equation 3.12)

As a measure of the effect of oscillations on the heat transfer rate we now consider the relative difference quantity

$$\Delta N_u = \frac{N_u - N_{u0}}{N_{u0}} = \frac{N_u}{N_{u0}} - 1 \dots (3.58)$$

Substituting for the Nusselt numbers from equation 3.52 and 3.53 and for the unsteady bulk temperature from equation 3.57 this becomes finally

$$N_u = \frac{1}{\frac{1}{\xi^2 N_{u0} \theta_{1b}} - 1} \dots (3.59)$$

In figure 3.4-figure 3.11 we graphed our results for various frequencies and Prandtl numbers as well as for different values of ξ .

In particular, frequencies in the range of 0-50 are considered with Prandtl numbers P_r at 0.5 .2 .5 and 5.0. The perturbation parameter ξ is taken as 0.1 .0 .3 .0 .5 and 1.0.

3.14 Conclusions

Using a regular perturbation method, the convection equation coupled to the classical pulsatile solution was solved using Green's functions in the complex plane and Laplace transforms upon which a bulk temperature and relative Nusselt number were obtained. Since heat transfer from the tube is constant because of the boundary condition, the effect of oscillations can only occur in terms of change of bulk temperature within the tube. From the definition of Nusselt number we see that the Nusselt number and the relative Nusselt number depend on the change in bulk temperature due to the pulsations.

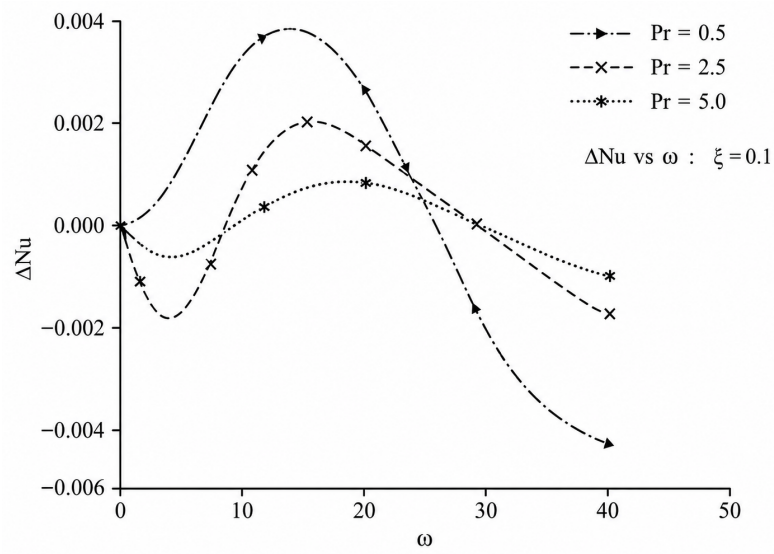
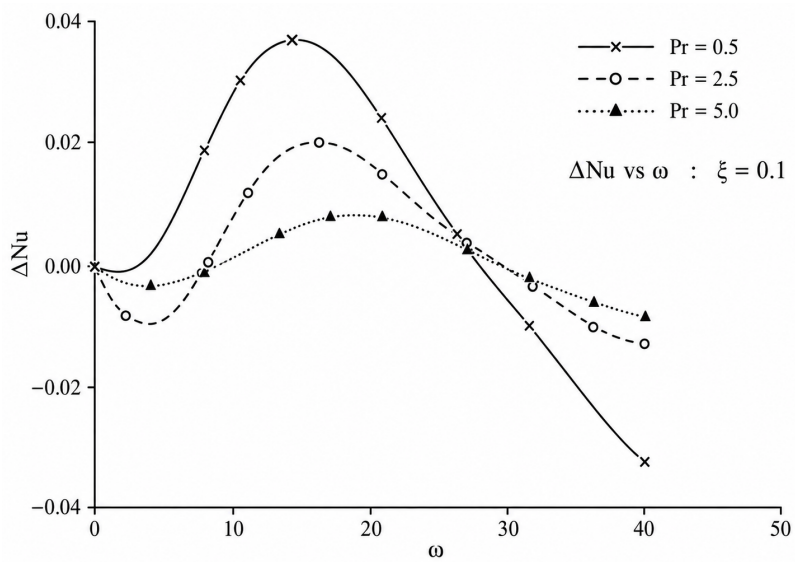
It is believed that lower viscosity and hence smaller Prandtl numbers are associated with thinner momentum and thermal boundary layers; hence, by Reynolds analogy, an increase in heat transfer should occur. For moderate Prandtl numbers near $0.5 \leq \text{Pr} \leq 1$, lowering viscosity typically increases the Reynolds number and turbulence or mixing, which tends to increase convective heat transfer. In this range, momentum and thermal diffusion occur on comparable scales, which is precisely the regime where the Reynolds analogy tends to work reasonably well. If viscosity decreases while keeping other scales roughly fixed, the Reynolds number,

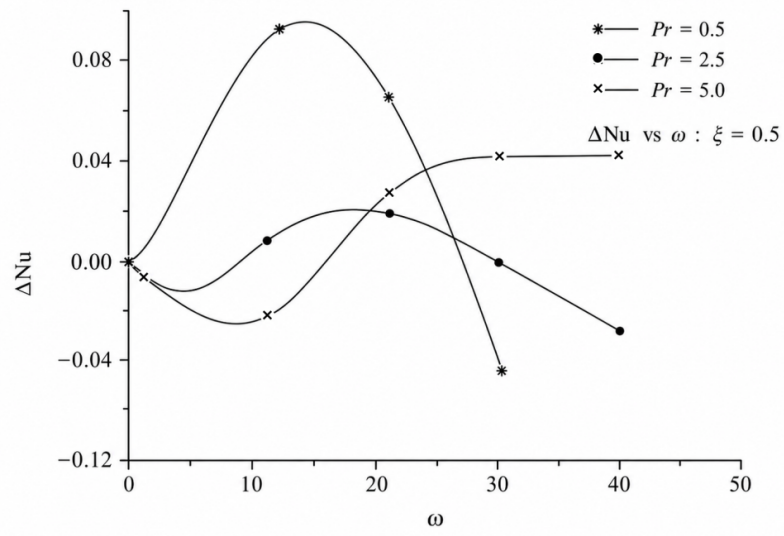
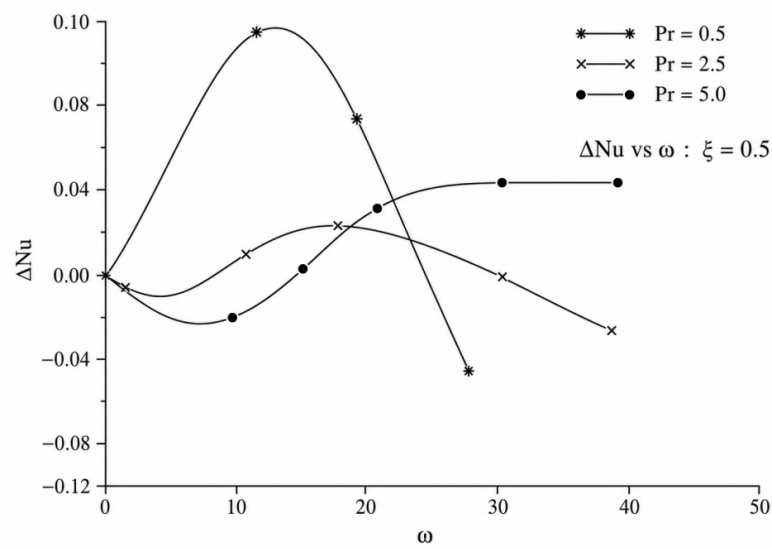
$$Re = \frac{UL}{\nu},$$

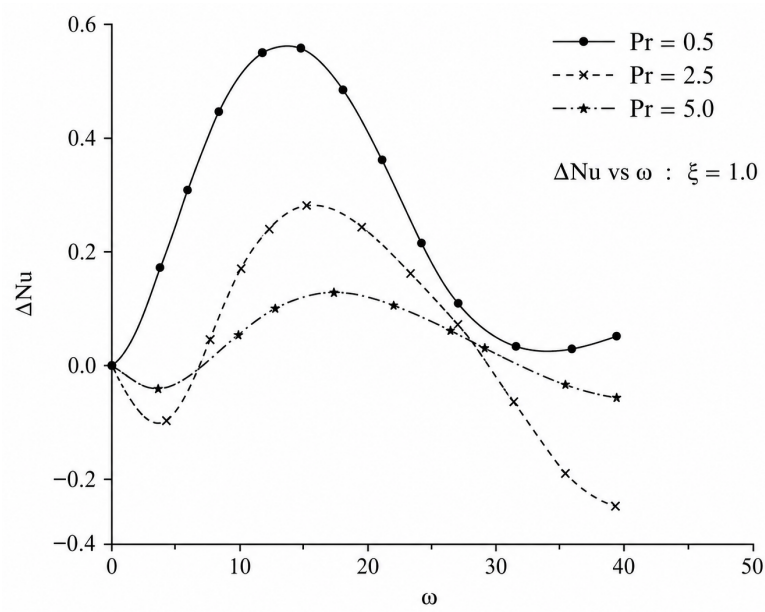
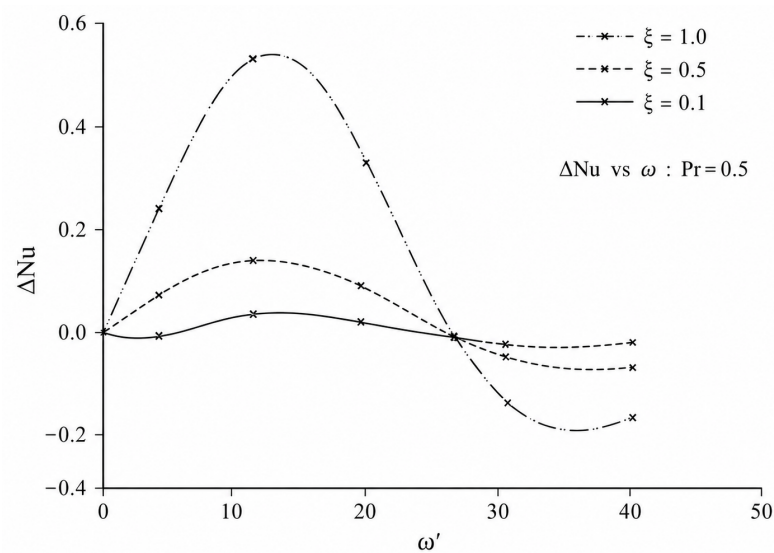
increases. A higher Reynolds number generally corresponds to a thinner momentum boundary layer, stronger inertial transport, more turbulence or stronger mixing, larger wall gradients, and therefore larger convective heat transfer coefficients.

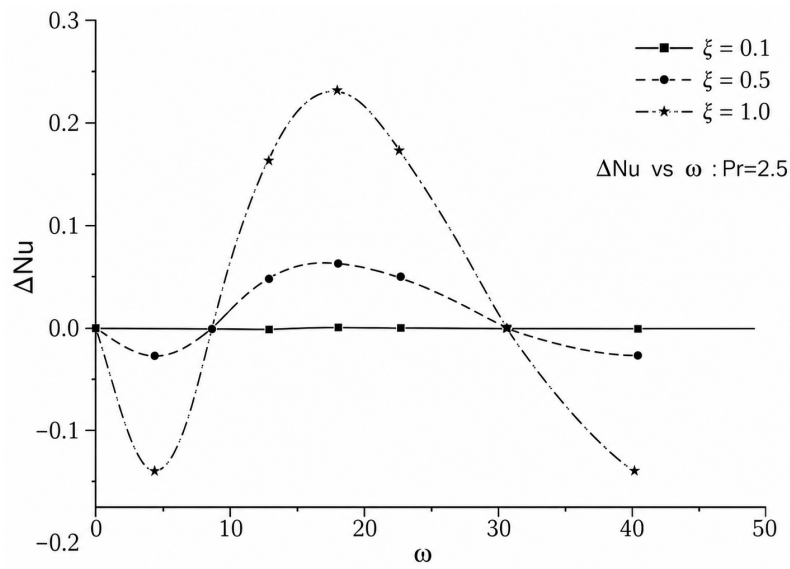
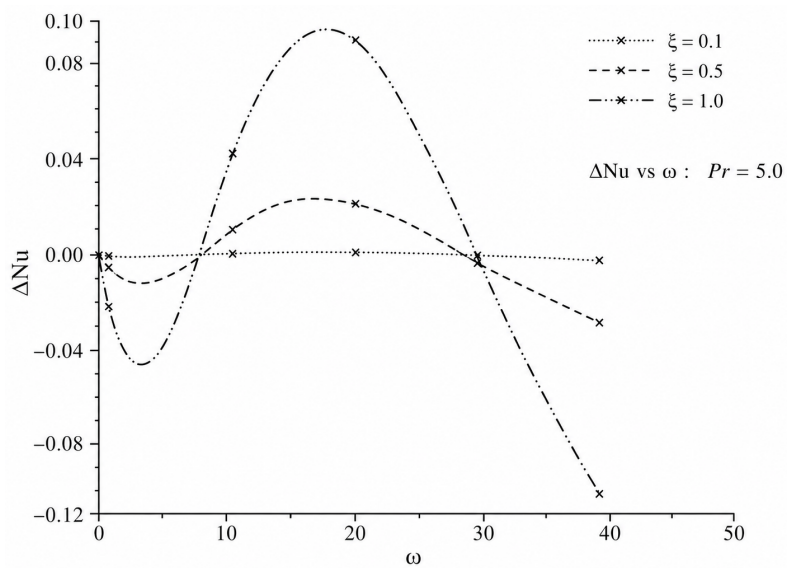
The results obtained indicate that pulsatile flow contributes to an enhancement in the overall heat transfer rate. An important observation is that there exists a critical value in frequency where a maximum occurs in the relative Nusselt number. The Prandtl number dependency is such that heat transfer increases by pulsations of fluids with Prandtl number less than unity and decreases for Prandtl numbers greater than unity.

Since the Green's function requires us to evaluate a number of tedious integrals involving Bessel functions, it was appropriate to write a Maple program to compute the bulk temperature and relative Nusselt number [AppendixB].

Fig 3.4: ΔNu vs ω .Fig 3.5: ΔNu vs ω .

Fig 3.6: ΔNu vs ω .Fig 3.7: ΔNu vs ω .

Fig 3.8: ΔNu vs ω .Fig 3.9: ΔNu vs ω .

Fig 3.10: ΔNu vs ω .Fig 3.11: ΔNu vs ω .

Heat Transfer with Pulsatile Flow and Constant Temperature

4

4.1 The Generalized Integral Transform Technique

In the previous chapter a constant flux boundary condition was specified and we were able to obtain an analytical solution. The corresponding problem with constant temperature specified at the wall is more difficult to solve and hence a complete analytical solution is not possible as in the previous section. However because of the discontinuity due to the boundary conditions at the entrance of the thermal region of the tube the use of the Laplace transform seems appropriate for this problem as well as the following integral transform method.

In the following the General Integral Transform technique [23], [24], [25] [30]-[37] is discussed which will be used to solve the convection equation with pulsatile flow. This technique is a hybrid method [34], [37], [43],[44] applied to the convection diffusion equation.

Consider the following general transient linear convection diffusion problem in the form

$$w(\vec{x}) \frac{\partial T(\vec{x}, t)}{\partial t} + \vec{u}(\vec{x}, t) \cdot \nabla T(\vec{x}, t) = \nabla \cdot K(\vec{x}) \nabla T(\vec{x}, t) - d(\vec{x}) T(\vec{x}, t) + P(\vec{x}, t) \dots (4.1)$$
$$\vec{x} \in V, t > 0 \dots (4.1)$$

where w, K, d, P have continuous partial derivatives in some closed bounded volume V , with the initial condition

$$T(\vec{x}, 0) = f(\vec{x}), \vec{x} \in V$$

and boundary conditions

$$\left[\alpha(\vec{x}) + \beta(\vec{x}) K(\vec{x}) \frac{\partial}{\partial n} \right] T(\vec{x}, t) = \Phi(\vec{x}, t), \vec{x} \in S, t > 0 \dots (4.2)$$

The technique of the generalized integral transform method is to start by choosing the following auxiliary problem.

$$\nabla \cdot K(\vec{x}) \nabla \psi(\mu_i, \vec{x}) + [\mu_i^2 w(\vec{x}) - d(\vec{x})] \psi(\mu_i, \vec{x}) = 0, \vec{x} \in V \dots (4.3)$$

with boundary conditions

$$\left[\alpha(\vec{x}) + \beta(\vec{x})K(\vec{x})\frac{\partial}{\partial n} \right] \psi(\mu_i, \vec{x}) = 0, \vec{x} \in S \dots (4.4)$$

and it is assumed that the solution of the above auxiliary problem is known.

The appropriate integral transform pair obtained from this eigenvalue problem is

Transform:

$$\overline{T_i(t)} = \int_V \frac{\psi_i(\vec{x}, \mu_i)}{N_i^{1/2}} \omega(\vec{x}) T(\vec{x}, t) dv \dots (4.5)$$

Inverse:

$$T(\vec{x}, t) = \sum_{i=1}^{\infty} \frac{1}{N_i^{1/2}} \nu(\vec{x}, \mu_i) \overline{T_i(t)} \dots (4.6)$$

Operating on the general transient convection-diffusion equation with the operator.

$$\int_V \frac{\psi_i(\vec{x})}{N_i^{1/2}} dv \dots (4.7)$$

we obtain

$$\frac{d\overline{T_i(t)}}{dt} + \int_V \frac{\psi_i(\vec{x})}{N_i^{1/2}} [\vec{u}(\vec{x}, t) \cdot \nabla T(\vec{x}, t)] dv = -\mu_i^2 \overline{T_i(t)} + \overline{g_i(t)}, t > 0, i = 1, 2, 3 \dots \dots (4.8)$$

where the function $\overline{g_i(t)}$ can be shown to be

$$\overline{g_i(t)} = \int_V P(\vec{x}, t) \frac{\psi_i(\vec{x})}{N_i^{1/2}} dv + \frac{1}{N_i^{1/2}} \int_S K(\vec{x}) \left[\psi_i(\vec{x}) \frac{\partial T}{\partial n} - T(\vec{x}, t) \frac{\partial \psi_i(\vec{x})}{\partial n} \right] ds \dots (4.9)$$

The integral appearing in the transformed convection-diffusion equation is calculated by making use of the inversion formula to give the following system of coupled ordinary differential equations

$$\frac{d\overline{T_i(t)}}{dt} + \mu_i^2 \overline{T_i(t)} + \sum_{j=1}^{\infty} A_{ij}^*(t) \overline{T_j(t)} = \overline{g_i(t)}, i = 1, 2 \dots \dots (4.10)$$

where the matrix A_{ij}^* as a function of t is

$$A_{ij}^*(t) = \frac{1}{N_i^{1/2} N_j^{1/2}} \int_V \psi_i(\vec{x}) [\vec{u}(\vec{x}, t) \cdot \nabla \psi_j(\vec{x})] dv \dots (4.11)$$

and the normalization integral, N_i , is given by

$$N_i = \int_V w(\vec{x}) \psi^2(\mu_i, \vec{x}) dv$$

The initial condition equation is also transformed using the operator

$$\int_V w(\vec{x}) \frac{\psi_i(\vec{x})}{N_i^{1/2}} dv$$

which gives us the following

$$\bar{T}_i(0) = \int_V w(\vec{x}) \frac{\psi_i(\vec{x})}{N_i^{1/2}} f(\vec{x}) dv \cdot i = 1.2 \dots \dots (4.12)$$

These equations, are an infinite system of coupled ordinary differential equations for the transformed functions. ie the $\bar{T}_i$ S.

Since the system is infinite it is not always possible to obtain an exact solution. If this system can be solved however, the inversion formula above can be used to give us the solution $T(\vec{x}, t)$ of the convection-diffusion equation. In practice the system to be solved can be truncated at the N th row and column, with N sufficiently large for the required accuracy, and then the finite system can be solved by standard numerical techniques. [38]

In matrix form. the truncated version of the system becomes

$$\begin{aligned} \vec{y}(t) + A(t)\vec{y}(t) &= \vec{g}(t), t > 0 \dots (4.13) \\ \vec{y}(0) &= \vec{f} \end{aligned}$$

$$\begin{aligned} \vec{y} &= \{\bar{T}_1(t), \bar{T}_2(t), \dots \bar{T}_N(t)\}^T \dots (4.14) \\ A &= \{A_{ij}\}, A_{ij} = \delta_{ij}\mu_i^2 + A_{ij}^*(t), i, j = 1.2 \dots \dots N \dots (4.15) \\ \vec{g} &= \{\bar{g}_1(t), \bar{g}_2(t), \dots \bar{g}_N(t)\}^T \\ \vec{f} &= \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_N\}^T \end{aligned}$$

A special case occurs when the velocity u in equation 4.11 is independent of t . In this case the matrix of coefficients, A , becomes constant, and the solution of the system of differential equations above can be explicitly written as the following

$$\vec{y}(t) = \exp(At)\vec{f} + \int_0^t \exp[A(t-t')] \vec{g} dt' \dots (4.16)$$

where the exponential matrix can be computed once the eigenvalues and eigenvectors of A have been obtained through solving the algebraic problem

$$(A - \lambda I)\zeta = 0$$

Scientific subroutines are available to accurately accomplish this task.

In applications obtaining an approximate explicit solution can be very important and of special interest in getting qualitative insight into a problem. A lowest order

solution can be obtained provided the non-diagonal elements of the coefficients matrix A are negligible as compared to those of the main diagonal which would then approximately correspond to a decoupled system.

Therefore by keeping only the diagonal elements of the matrix an approximate solution is obtained from the following initial value problem

$$y'_{i,j}(t) + A_{ij}(t)y_{i,j}(t) = \bar{g}_i(t), t > 0, i = 1, 2 \dots \dots (4.17)$$

$$y_{i,j}(0) = \bar{f}_i$$

This type of problem arises for the case of heat transfer with pulsatile flow and constant wall temperature which is considered in the next section.

4.2 Heat Transfer in Pulsatile Flow with Constant Wall Temperature

In this section the convection equation with pulsatile flow and constant temperature prescribed along the wall of the tube is solved using the integral transform technique discussed in the previous section. The governing equation derived from equation 4.1 becomes

$$\frac{\partial \theta}{\partial t} + u(r, t) \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \dots (4.18)$$

$$\frac{\partial u^*}{\partial t^*} + \frac{1}{\rho} \frac{\partial p}{\partial x^*} = \frac{\mu}{\rho} \left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial u^*}{\partial r^*} \right)$$

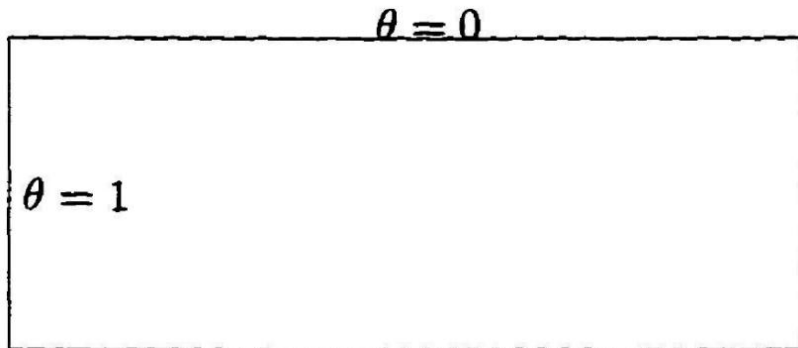
with

$$\begin{aligned} \theta(0, r, t) &= 1 \\ \theta(x, 1, t) &= 0 \\ \frac{\partial \theta}{\partial r}(x, 0, t) &= 0 \\ \theta(x, r, 0) &= 1 \end{aligned}$$

where $\theta = (T - T_w) / (T_0 - T_w)$. T_w is the constant wall temperature of the tube and T_0 is the constant temperature of the fluid at the thermal entrance region. Refer to figure 4.0 for a sketch of the model.

figure 4.0 As in chapter 2 the velocity u satisfies the Navier Stokes equations for fully

developed axial flow with the pressure gradient set equal to a harmonic of some Fourier series in time. It follows as before that



$$\frac{\partial \theta}{\partial r} = 0$$

figure 4.0

$$u = u(r, t)$$

$$= (1 - r^2) + \sum_{\omega=1}^{\infty} \frac{ik_{\omega}}{\omega} \left[1 - \frac{J_0(\sqrt{-i\omega/P_r}r)}{J_0(\sqrt{-i\omega/P_r})} \right] e^{i\omega t}$$

Taking the Laplace transform of equation 4.18 with respect to x , that is operating on this partial differential equation with $\int_0^{\infty} e^{-s^*x}(\cdot)dx$ where s is complex and $\Re s > \gamma$ for some γ positive, we get

$$\frac{\partial \theta^L}{\partial t} + u(r, t) [-1 + s\theta^L] = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \theta^L}{\partial r} \right] \dots (4.19)$$

$$\theta^L(r, 0) = \frac{1}{s}$$

$$\theta^L(1, t) = 0$$

$$\frac{\partial \theta^L}{\partial r}(0, t) = 0$$

where we have integrated by parts and used the boundary condition at $x = 0$.

Now using the Generalized Integral Transform technique [41],[42], we introduce the following auxiliary problem which is independent of the parameter s and whose solution is the zero order Bessel function as a function of β where the β 's are the zeros of J_0 .

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{d\psi(\beta, r)}{dr} \right] + \beta^2 \psi(\beta, r) = 0$$

$$\left. \frac{d\psi(\beta, r)}{dr} \right|_{r=0} = 0$$

$$\psi(\beta, 1) = 0$$

The following integral transform pair is defined as:

$$\theta^L(r, t) = \sum_{i=1}^{\infty} \frac{1}{N_i} \psi(\beta_i, r) \overline{\theta_i^L}(t) \dots (4.21)$$

$$\overline{\theta_i^L}(t) = \int_0^1 r \psi(\beta_i, r) \theta^L(r, t) dr \dots (4.22)$$

equation 4.19 is now operated on with the operator

$$L \equiv \int_0^1 r \psi(\beta_i, r) dr \dots (4.23)$$

to obtain the following:

$$\frac{\partial}{\partial t} \int_0^1 r \psi(\beta_i, r) \theta^L(r, t) dt + \int_0^1 u(r, t) r \psi(\beta_i, r) dr + s \int_0^1 r u(r, t) \psi(\beta_i, r) \theta^L dr \dots (\beta_i, r) \theta^L(r, t) = 0$$

$$= \int_0^1 r \psi(\beta_i, r) \nabla_r^2 \theta^L dr \dots (\beta_i, r) \theta^L(r, t).$$

Now, using the auxiliary problem the Laplacian on the right hand side can be written as

$$\begin{aligned}
& \nabla_r^2 \left[\sum_{i=1}^{\infty} \frac{1}{N_i} \psi(\beta_i, r) \bar{\theta}_i^L(t) \right] \\
&= \sum_{i=1}^{\infty} \frac{1}{N_i} \nabla_r^2 \psi(\beta_i, r) \bar{\theta}_i^L(t) \\
&= \sum_{i=1}^{\infty} -\frac{1}{N_i} \beta_i^2 \psi(\beta_i, r) \bar{\theta}_i^L(t) \\
&= -\beta_i^2 \sum_{i=1}^{\infty} \frac{1}{N_i} \psi(\beta_i, r) \bar{\theta}_i^L(t) \\
&= -\beta_i^2 \bar{\theta}^L(r, t)
\end{aligned}$$

From this we obtain the following initial value problem with s complex.

$$\begin{aligned}
\frac{d\bar{\theta}_i^L(t)}{dt} + \beta_i^2 \bar{\theta}_i^L(t) + s \sum_{j=1}^{\infty} \bar{\theta}_j^L(t) A_{ij} &= \int_0^1 ru(r, t) \psi \cdot dr \dots (4.25) \\
\bar{\theta}_i^L(0) &= \frac{1}{s}
\end{aligned}$$

where the matrix A_{ij} is calculated from the following

$$\begin{aligned}
A_{ij} &= \frac{1}{N_j} \int_0^1 ru(r, t) \psi(\beta_i, r) \psi(\beta_j, r) dr \\
N_i &= \int_0^1 r \psi(\beta_i, r)^2 dr
\end{aligned}$$

and the function N_i is simply the norm so that system of eigenfunctions becomes an orthonormal system.

Recall in the discussion of the generalized integral transform technique that if the matrix A is diagonally dominant then a lower order solution is possible. The main difficulty is that the matrix A is infinite and hence an infinite system of coupled differential equations in s arises for which the solution is impossible to obtain. In what follows, we assume that the off diagonal terms of the matrix A_{ij} can be neglected. This is clearly an approximation to the solution of equation 4.18. Based on this approximation, we arrive at the following system of ODE's.

$$\begin{aligned}
\frac{d\bar{\theta}_i^L(t)}{dt} + \beta_i^2 \bar{\theta}_i^L(t) + s A_{ii}(t) \bar{\theta}_i^L(t) &= \int_0^1 ru(r, t) \psi dr \bar{\psi} \dots (4.26) \\
\bar{\theta}_i^L(0) &= \frac{1}{s} \bar{\psi} \dots (4.26)
\end{aligned}$$

where u is left general for now. An existence and uniqueness theorem for this type of problem is found in Appendix D.

4.3 Distributions

The solution to equation 4.26. as will be shown later involves a generalized solution known as Dirac's distribution.

Physical quantities are usually thought of as functions. But in an experimental setting it is quite difficult if not impossible to observe instantaneous values of the function at every instant of space or time. A measuring instrument would record the effect that the function produces on it over some interval of space or time with nonzero length. An alternate description for some physical quantity is to specify it as a functional, a rule which assigns a number to each function in a set of testing functions. Continuous linear functionals defined on the space of testing functions are called distributions. [26] The distribution theory is a very powerful tool for studying linear partial differential equations with smooth variable coefficients. It turns out that using distributions allows for analyzing types

of physical phenomena more naturally than using the function concept. The Dirac delta function is an example of a distribution which arises naturally in physical settings in particular in the following section.

The delta function (δ) is defined by

$$\phi(0) = \int_{-\infty}^{\infty} \delta(t)\phi(t)dt$$

where $\phi(t)$ is a test function, an infinitely smooth function that vanishes outside a finite interval.

The delta function $\delta(t - \tau)$, selects the value of a test function at the point $t = \tau$.

Refer to the Appendix E for some important identities involving the delta function.

4.4 Method of Solution

From equation 4.26. the infinite system of complex ordinary differential equations in s with initial condition in s is

$$\frac{d\bar{\theta}_i^L(t,s)}{dt} + \beta_i^2 \bar{\theta}_i^L(t,s) + s \cdot A_{ii}(t) \bar{\theta}_i^L(t,s) = \int_0^1 ru(r,t)\psi dr = 1, 2, 3 \dots \dots (4.28)$$

$$\bar{\theta}_i^L(0) = \frac{1}{s} \dots (4.29)$$

The solution of this system is

$$\bar{\theta}_i^L(t,s) = e^{-s \int_0^t A_{ii}(\tau) d\tau} e^{-\beta_i^2 t} \left[\int_0^t e^{s \int_0^\tau A_{ii}(t) dt} e^{\beta_i^2 \tau} A_{ii}(\tau) \int_0^1 ru/ed\tau d\tau + \frac{1}{s} \right] i = 1, 2, 3 \dots \dots (4.30)$$

$$= \int_0^t e^{s \left(\int_0^\tau A_{ii}(t) dt - \int_0^t A_{ii}(\tau) d\tau \right)} e^{\beta_i^2 (\tau-t)} A_{ii}(\tau) \int_0^1 ru\psi dr d\tau + \frac{1}{s} e^{-s \int_0^\tau A_{ii}(t) dt} e^{-\beta_i^2 t} i = 1, 2, 3, \dots \dots (4.31)$$

Let

$$\Xi(t, \tau) = - \int_0^t A_{ii}(\tau) d\tau + \int_0^\tau A_{ii}(t) dt \dots (4.32)$$

$$= - \int_\tau^t A_{ii}(q) dq \dots (4.32)$$

Refer to Appendix E for the inverse Laplace transform of the exponential function in s . Hence we obtain

$$\mathcal{L}^{-1} \{ \bar{\theta}_i^L(t,s) \} = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \bar{\theta}_i^L(t,s) ds \dots (4.33)$$

$$\mathcal{L}^{-1} \{ \bar{\theta}_i^L(t,s) \} = \int_0^t \delta[x+ \equiv (t, \tau)] f(t - \tau) d\tau \dots (4.34)$$

where δ is Dirac's generalized function.

Also

$$f(t - \tau) = e^{\beta_i^2(\tau-t)} A_{ii}(\tau) \int_0^1 ru(r, t - \tau) \psi dr$$

As in the case of a Green's function the interval is extended onto the real axis by introducing the Heaviside function as a kernel.

$$H(t - \tau) = \begin{cases} 1 & t > \tau \\ 0 & t < \tau \end{cases}$$

From this we obtain

$$\mathcal{L}^{-1} \{ \bar{\theta}_i^L(t, s) \} = \int_{-\infty}^{\infty} \delta[x + \Xi(t, \tau)] H(t - \tau) f(t - \tau) d\tau \dots (4.35)$$

$$= \sum_i \frac{1}{|g'(\tau_i)|} \int_{-\infty}^{\infty} \delta(\tau - \tau_i) f(t - \tau) d\tau \dots (4.36)$$

$$= \sum_i \frac{f(t - \tau_i)}{|g'(\tau_i)|} \dots (4.37)$$

where a formula for the delta function as a function of a general argument has been used from Appendix E and the phase g is

$$g(\tau) = x + \equiv (t, \tau) \dots (4.38)$$

$$g'(\tau) = A_{ii}(\tau)$$

If we let the phase equal a constant then a set of critical points is defined in the

phase plane (x, τ) Since the velocity also depends on frequency a set of critical frequencies exist in the phase plane.

The sequence of terms $\tau_i(x, t)$ are the zeros of the function $g(\tau)$, assuming they exist for a given function $A_{ii}(\tau)$.

Tables of zeros, E and velocity are included for different axial values as well as for different values of time in a cycle. Various velocities in pulsatile flow are considered. Phase plane surface plots for frequency variation are included.

In the following section the results for pulsatile flow are discussed. Recall from the integral transform pair

$$\theta^L(r, t, s) = \sum_{i=1}^{\infty} \frac{1}{N_i} \psi(\beta_i, r) \bar{\theta}_i^L(t, s)$$

The series in the solution for θ is uniformly convergent since $\tau_j < t$ and thus the inverse Laplace transform of $\theta^L(r, t, s)$ exists.

From the generalized integral transform method, the solution for forced convection heat transfer with a general velocity term $u(t)$ becomes

$$\theta(x, r, t) = \sum_i^{\infty} \frac{1}{N_i} \psi(\beta_i, r) \sum_j \frac{f(t - \tau_j(x, t))}{|g'(\tau_j(x, t))|} \dots (4.39)$$

After terms $A_{ii}(\tau)$ cancel out we are left with the following expression for temperature difference which depends explicitly on the zeros of g .

$$\theta(x, r, t) = \sum_i^\infty \frac{1}{N_i} \psi(\beta_i, r) \sum_j e^{\beta_i^2(\tau_j - t)} \frac{\int_0^1 r \psi(\beta_i, r) u(r, t - \tau_j) dr}{|A_{ii}(\tau_j(x, t))|} \dots (4.40)$$

$$= \sum_j \sum_i \frac{1}{N_i} J_0(\beta_i r) e^{-\beta_i^2 t} e^{\beta_i^2 \tau_j} \frac{\int_0^1 r \psi(\beta_i, r) u(r, t - \tau_j) dr}{|A_{ii}(\tau_j(x, t))|} \dots (4.41)$$

where we are summing up over l the zeros of g

4.5 Results and Discussion

In this section a bulk temperature is defined as

$$\theta_{bu} = \int_0^{2\pi/\omega} \int_0^1 \int_0^1 \theta(x, r, t) dx dr dt \dots (4.42)$$

A solution for the corresponding steady problem was given in chapter 2 : the bulk temperature in that case is given by

$$\theta_{bs} = \int_0^1 \int_0^l \theta(x, r) dx dr \dots (4.43)$$

where

$$\theta(x, r) = 2 \sum_{n=1}^\infty \frac{1}{\beta_n} \frac{J_0(\beta_n r)}{J_1(\beta_n)} e^{-\beta_n^2 x} \dots (4.44)$$

As in the constant flux solution, a measure of the effect of pulsations on the overall heat transfer rate is required. The following ratio of steady bulk temperature to unsteady bulk temperature will be considered.

$$\zeta = \frac{\theta_{bs}}{\theta_{bu}} \dots (4.45)$$

From the steady state solution of chapter 2 and the generalized integral transform technique of this chapter we have

$$\frac{2}{\beta_i J_1(\beta_i)} = \frac{\int_0^1 r \psi(\beta_i, r) dr}{N_i} \dots (4.46)$$

It can be shown that there exists a zero $\tau_i = \tau$ of $g(\tau)$ for each x, t for the pulsatile flow $u(t) = 1 + \alpha \cos(\omega t)$ where g is specified in equation 4.38.

In fact we can generalize this to the case of an arbitrary Fourier series in t . It can be shown that there always exists a zero of $g(\tau)$ at any axial distance along the tube and at any time t in a complete cycle.

If there is only one zero then

$$\sum_j e^{\beta_i^2 \tau_j} = e^{\beta_i^2 \tau}$$

for some τ positive and

$$e^{\beta_i^2(\tau - t)} > e^{-\beta_i^2 t} \text{ when } \tau > 0$$

Since the r_j s are computable and positive then the following inequality results

$$\int_0^1 \int_0^1 \int_0^{2\pi/\omega} \sum_i \int_0^1 r \psi dr \frac{1}{N_i} J_0(\beta_i r) e^{-\beta_i^2 t} \left(\sum_j e^{\beta_i^2 \tau_j(x,t)} \right) dt dx dr \dots (4.47)$$

$$> \int_0^1 \int_0^1 \int_0^{2\pi/\omega} \sum_i \int_0^1 r \psi dr \frac{1}{N_i} J_0(\beta_i r) e^{-\beta_i^2 t} dt dx dr \dots (4.48)$$

$$\equiv \int_0^1 \int_0^1 \sum_i \frac{2}{\beta_i J_1(\beta_i)} J_0(\beta_i r) e^{-\beta_i^2 x} dx dr \dots (4.49)$$

This proves that ζ is strictly less than unity and thus we have heat transfer enhancement due to the pulsations in the flow field.

Numerical integration techniques can be used to calculate ζ for different forms of $u(t)$.

It is also interesting to note that if we define the bulk temperature as in equation 3.56, 3.57 of chapter 3 then we have heat transfer enhancement as well.

table 4.1: Zeros τ , Ξ of wave function and pulsatile velocity $u = 1 + \cos t$: $t = \pi/4$

$t = \pi/4$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	0.7275	.1	1.0953
0.2	0.6709	.2	1.1822
0.3	0.6153	.3	1.26198
0.4	0.5607	.4	1.3354
0.5	0.5069	.5	1.4033
0.6	0.4539	.6	1.46601
0.7	0.4016	.7	1.5241
0.8	0.3498	.3	1.5779
0.9	0.2984	.9	1.6276
1.0	0.2475	1.0	1.6736
1.1	0.1969	1.1	1.71602
1.2	0.1465	1.2	1.75503
1.3	0.0963	1.3	1.7908
1.4	0.0463	1.4	1.8234
1.5	-0.00375	1.5	1.8531
1.6	-0.05376	1.6	1.8798

table 4.2: Zeros τ, Ξ of wave function and pulsatile velocity $u = 1 + \cos t$: $t = 3\pi/4$

$t = 3\pi/2$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	4.6068	0.1	.89463
0.2	4.4871	0.2	.776635
0.3	4.3460	0.3	.641776
0.4	4.1676	0.4	.48181
0.5	3.9006	0.5	.27446
0.6	2.5791	0.6	.154097
0.7	2.2095	0.7	.40381
0.8	2.0053	0.8	.57902
0.9	1.8515	0.9	.722931
1.0	1.7241	1.0	.776635
1.1	1.6133	1.1	.95752
1.2	1.5140	1.2	1.05676
1.3	1.4233	1.3	1.14701
1.4	1.3391	1.4	1.2296
1.5	1.2602	1.5	1.3056
1.6	1.1856	1.6	1.3757

table 4.3: Zeros τ, Ξ of wave function and pulsatile velocity $u = 1 + \cos t$: $t = 3\pi/2$

$t = 2\pi$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	6.2332	0.1	1.9987
0.2	6.1831	0.2	1.9949
0.3	6.1329	0.3	1.9887
0.4	6.0825	0.4	1.9799
0.5	6.0319	0.5	1.9685
0.6	5.9809	0.6	1.9546
0.7	5.9295	0.7	1.93811
0.8	5.8777	0.8	1.91890
0.9	5.8253	0.9	1.89697
1.0	5.7722	1.0	1.87226
1.1	5.7184	1.1	1.84471
1.2	5.6638	1.2	1.814209
1.3	5.6081	1.3	1.780671
1.4	5.5514	1.4	1.743976
1.5	5.4934	1.5	1.703993
1.6	5.4339	1.6	1.660562

table 4.4: Zeros τ, Ξ of wave function and pulsatile velocity $u = 1 + \cos t$: $t = 2\pi$

$t = \pi/2$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	1.5214	0.1	1.0494
0.2	1.4731	0.2	1.0975
0.3	1.4258	0.3	1.1445
0.4	1.3794	0.4	1.1903
0.5	1.3337	0.5	1.2349
0.6	1.2887	0.6	1.2784
0.7	1.2443	0.7	1.3207
0.8	1.2443	0.8	1.3619
0.9	1.2004	0.9	1.4019
1.0	1.1142	1.0	1.4409
1.1	1.0716	1.1	1.4787
1.2	1.0292	1.2	1.5154
1.3	0.9873	1.3	1.5509
1.4	0.9454	1.4	1.5854
1.5	0.9037	1.5	1.6186
1.6	0.8622	1.6	1.6507

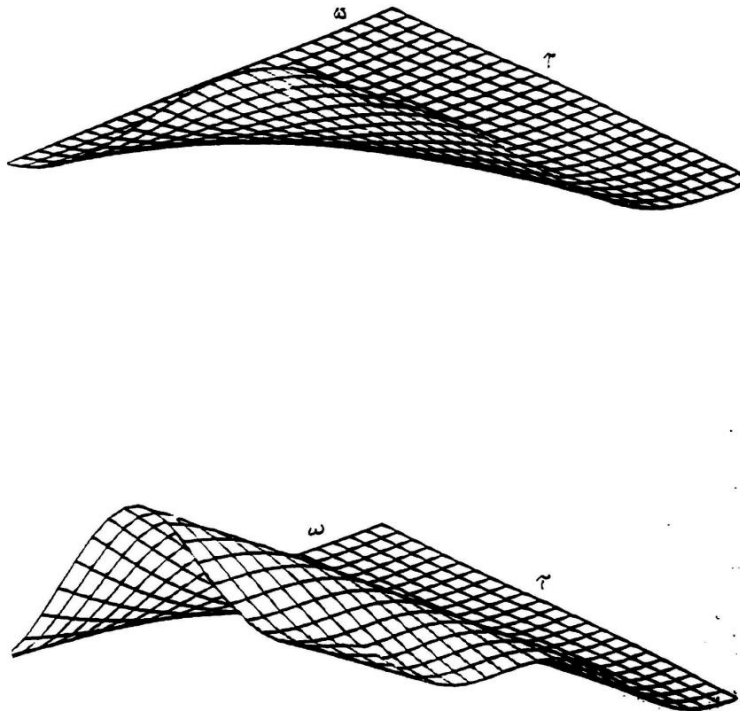
table 4.5: Zeros τ, Ξ of wave function and pulsatile velocity $u = 1 + \cos t + \sin t$:
 $t = \pi/2$

$t = \pi$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	2.7199	0.1	0.0876
0.2	2.5553	0.2	0.1669
0.3	2.4314	0.3	0.2418
0.4	2.3279	0.4	0.3131
0.5	2.2374	0.5	0.3817
0.6	2.1557	0.6	0.4479
0.7	2.0807	0.7	0.51190
0.8	2.0108	0.8	0.5739
0.9	1.9452	0.9	0.6343
1.0	1.8829	1.0	0.6929
1.1	1.8234	1.1	0.7501
1.2	1.7663	1.2	0.3057
1.3	1.7114	1.3	1.7114
1.4	1.6582	1.4	.9127
1.5	1.6065	1.5	.9643
1.6	1.5562	1.6	1.0145

table 4.6: Zeros τ, Ξ of wave function and pulsatile velocity $u = 1 + \cos t + \sin t$:
 $t = \pi$

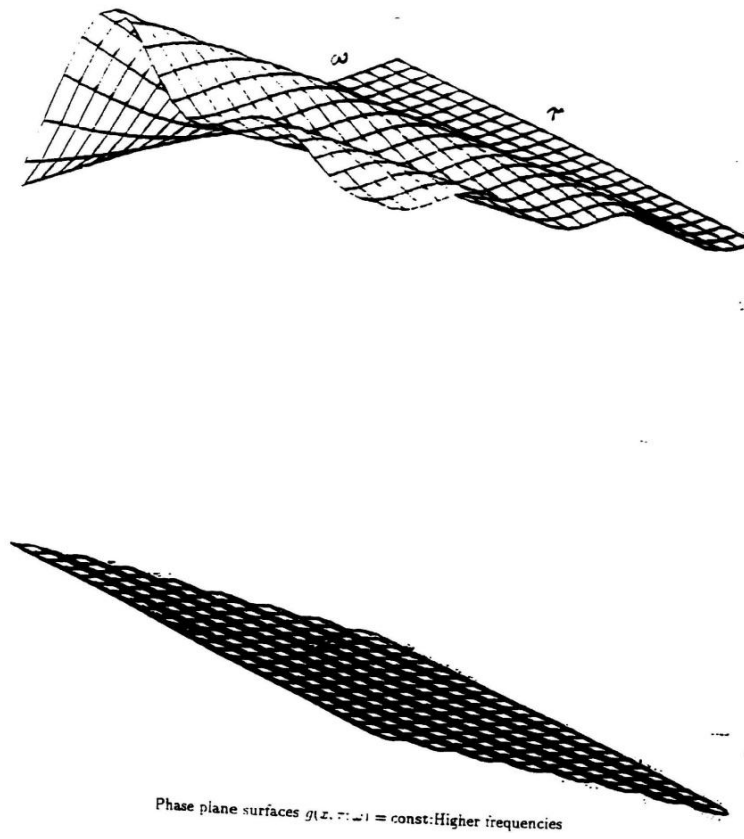
$t = 3\pi/2$			
x	Zeros τ	Ξ	Velocity $u(\tau)$
0.1	2.2127	0.1	0.4013
0.2	2.1332	0.2	0.4668
0.3	2.0598	0.3	0.5302
0.4	1.9913	0.4	0.5918
0.5	1.9266	0.5	0.6516
0.6	1.8652	0.6	0.7098
0.7	1.8065	0.7	0.7664
0.8	1.7501	0.8	0.8217
0.9	1.6956	0.9	0.8754
1.0	1.6429	1.0	0.9279
1.1	1.5917	1.1	0.9791
1.2	1.5418	1.2	1.0289
1.3	1.4931	1.3	1.0776
1.4	1.4454	1.4	1.1251
1.5	1.3986	1.5	1.1714
1.6	1.3526	1.6	1.2165

table 4.7 : zeros (τ, Ξ) of wave function and pulsatile velocity $u := 1 + \cos t + \sin t : t = 3\pi/2$



Phase plane surfaces $g(x, \tau; \omega) = \text{const}$: Moderate frequencies

figure 4.1



Phase plane surfaces $g(x, \tau; \omega) = \text{const}$: Higher frequencies

figure 4.2

Concluding Remarks

5

In the first part of the thesis, a method of analysis is presented for the problem of heat transfer in pulsatile flow with constant heat transfer in a tube. A regular perturbation expansion is used to solve for the temperature field downstream. A complex-valued Green's function is utilized to formulate a bulk temperature and a relative Nusselt number. Since heat transfer from the tube is constant the effect of pulsation only occurs by means of a change in the bulk temperature within the tube. The results indicate that a pulsating flow field enhances the overall heat transfer rate. In particular there exists a critical value in frequency where the relative Nusselt number attains a maximum value. The trend in relative Nusselt number with Prandtl number is that there is an increase with decreasing values of Prandtl number less than unity and a decrease with increasing values greater than unity. The perturbation parameter plays an important role in the mechanism of heat transfer with pulsatile flow. The results indicate that increases of pulsatile pressure gradient amplitude over that of steady pressure gradient amplitude give rise to increases in relative Nusselt number.

In the second part of the thesis, an approximate solution is presented for the problem of heat transfer in pulsatile flow with constant wall temperature in a tube. Using the generalized integral transform method and the Laplace transform a system of ordinary complex differential equations arises whose solution is presented with the aid of Dirac's distribution theory. A bulk temperature of this solution is formulated and an increase of unsteady bulk temperature over that of steady bulk temperature is presented.

Courant and Hilbert [45] have defined a progressing plane wave as a solution of the form $u = f(\beta)$ where $\beta = x - ct$ is the phase of the solution and f is any function. The form of equation 4.35 indicates that we have a plane wave propagating down the tube.

For future work any possible connections that can be made between the convection equation and the wave equation from this solution can be studied. The existence of wave-like phenomena could prove to be useful in describing physically the increase in Nusselt number of pulsatile flow over that of steady flow. Also the frequency characteristics for the convection equation coupled to the full Navier-Stokes equations with a general sinusoidal pressure gradient can be investigated. In addition solutions using the generalized integral transform technique can be studied for the Navier-Stokes equations without neglecting off diagonal terms in the method.

Appendix A: Maple Code for the Graetz Problem

6

```
Program to calculate Eigenfunctions for Graetz
Problem
and to calculate Temperature Distribution theta
R:= array(0..3.1..101);
Subroutine to calculate point of intersection of
graphs
of functions B(delta) and G(delta)
My:Newton := proc(m.delto)
deltold:= delto:
for i to m do
deltnew:= deltold - evalf(H(deltold))/evalf(K(deltold));
deltold:= deltnew:
od:
end:
Subroutine to calculate the matching point eps of the
.... solution
of the asymptotic formula and the similarity
transformation
solution of the Graetz problem.
MyNewton2:= proc(s.epslo)
epslold:= epslo:
for i to s do
epslnew:= epslold -evalf(f(epslold))/evalf(fp(epslold)
);
epslold:= evalf(epslnew);
od:
end:
Dimensionalize array for successive approximations
via Newton's .... Method
deltt:= array(1..3);
delto:= 0.11 :
Loop calculates the Eigenfunctions for the Graetz
problem
for n = 0 .. 3 using Lagrangian Interpolation.
for n from 0 to 3 do
delta:= 1 delto:
Digits:= 16 :
lamda:= 4 * n + 8/3 :
G1 := delta -> sqrt(2/( Pi* lamda* delta));
G2 := delta -> cos((lamda/2) * (delta*sqrt(1-
delta 2)
G := G1 * G2 :
E := x -> exp(-lamda * x);
```

```

F1 := delta -> diff(G(delta),delta);
B := delta -> BesselJ(O.lamda*delta);
F2 := delta -> diff(B(delta),delta);
H := unapply((G-B)(delta),delta);
K := unapply((F1-F2)(delta),delta);
:
F4 := delta -> diff(F3(delta),delta);
L := unapply((G-F3)(delta),delta);
M := unapply((F1-F4)(delta),delta);
S1 := plot (B(delta).G(delta).F3(delta),delta = 0 ..1,title =
= R (0)*).
m := 10 :
Boolean Block assigns array values deltt[n] in Newton's
Method according to the value of n.
if n = 0 then
int1:= 0.10 :
int2:=0.45:
int3 = 0.50 ;
int4:= 0.53 :
int5:= 0.65 ;
int6:= 0.935 :
elif n = 1 then
deltt[n]:= MyNewton(m.delto);
delto:= deltt[n];
int4:= 0.53 ;
int5:= 0.65 ;
int6:= 0.935 ;
else
hdel:= evalf(delto/:2):
deltt[n]:= MyNewton(m.hdel);
delto:= deltt[n];
int4:= 0.53 ;
int5 = 0.65 ;
int6:= 0.935 ;
fi:
Lagrangian Interpolating polynomials for interpolation
of
functions B(delta), G(delta), F3(delta) on [0.1].
Quadratic fit
L1:= delta ->((delta-int2)*(delta-int3))/((int1-int2)*(
int1-int3));
L2 := delta -> ((delta-int1) * (delta-int3))/((int2-int1)*(
int2-int3));
L3 := delta ->((delta-int1)*(delta-int2))/((int3-int1)*(
int3-int2));
L4:= delta ->((delta-int5)*(delta-int6))/((int4-int5)*(
int4-int6));
L5 := delta ->((delta-int4)*(mathrm delta-int6))/((int5-
int4)**(int5-int6));
L6:= delta ->((delta-int4)*(delta-int5))/((int6-int4)
**(int6-int5));
writeto('eig4.dat):
if n = 0 then
k := .01 ;
for i from 1 to 101 do

delta_ := (i - 1) * k ; 89 if delta >=0 and delta <= int1 then 90 R [n · i] :=
B (delta) ; 91 print(delta,evalf(R[n,i])): 92 elif delta > int1 and delta <= int3

```

```

then R[n,i]:= B(int1) *L1 (delta) +G (int2) *L2 (delta) + G(int3)*L3(delta);
print(delta,evalf(R[n,i])); elif delta > int3 and delta <= int4 then R[n,i]:=G(delta);
print(delta,evalf(R[n,i]));
elif delta > int4 and delta <= int6 then
R[n,i]:= G (int4) *L4 (delta) +G (int5) *L5 (delta)+
F3(int6) *L6 (delta);
print(delta,evalf(R[n,i]));
elif delta > int6 and delta <= 1 then
R[n,i]:= F3(delta);
print(delta,evalf(R[n,i]));
fi:
od:
else
k := .01 ;
for i from 1 to 101 do
delta:= (i - 1) * k :
if delta >=0 and delta <= delto then
R [n, i] := B(delta) ;
print(delta,evalf(R[n,i]));
elif delta > delto and delta <= int4 then
R [n, i] := G(delta) ;
print(delta,evalf(R[n,i]));
elif delta > int4 and delta <= int6 then
R [n . i] := G ( int4) * L4 (delta) + G (int5) * L5 (delta) + G (delta)
F3(int6)*L6(delta);
print(delta,evalf(R[n,i]));
elif delta > int6 and delta <= 1 then
R[n,i]:= F3(delta);
print(delta,evalf(R[n,i]));
fi:
od;
fi:
int4:= 0 .0 ;
int5 := 0.0 ;
int6:= 0 .0 ;
od:
Loop to print out 2 dim array R
for n from 0 to 5 do
for i from 1 to 100 do
print(evalf(R[n,i]));
od;
od;

Loop to print out temperature theta( x,r ) vaild for x
=
eps.
We can prove that eps is in the neighbourhood of 0.01
Define an array for theta( x,r )
p:= array(1..101);
q:= array(1..101);
theta:= array(1..100,1..101);
writeto('3D.dat');
k:= .01;
1:= .2;
j : = 1
for j from 1 to 50 do
axi:=  $\hat{i} * \hat{j}$ ;
for i from 1 to 101 do

```

```

i : = 101
delta: = (i - 1)*k:
init: = 0.0;
for n from 0 to 3 do
theta [ j,i]: = init +
( (2/Pi) * ( - 1)n * 6(2/3)* b GAMMA(2/3)
(4 * n + 8/3)( - 2/3 ) * R[ n ,i] * exp( - (4 * n + 8/3)( n ,i])
^ 2 * axi . );
init: = theta [ j,i];
od;
print( axi, delta,evalf(theta[j, i] ));
od;
od:
p[ i]: = delta;
q[ i] = = evalf( theta [ j,i]):
od:
od:
writeto(terminal);
writeto(' outlg.dat');
for j from 1 to 101 do
print(p[ j],q[ j]);
od:
writeto(terminal);
Routine to calculate local Nusselt modulus for
temperature distribution
given above
C := n → (-1)n * 2 * (2/3) * GAMMA (2/3) * (4 * n + 8/3) (-2/3)/3
Pi:
r:=array(1..3000);
1 := 0 . 001 ;
init: = 0 . 0 ;
for j from 1 by 10 to 3000 do
axi: = j * 1 ;
for n from 0 to 100 do
Nus = init +
(-2*C(n)*(-1)(n+1)*2(2/3)*(4*n+8/3)(1/3)/(GAMMA(4/3)
3(5/6)) ) * exp(-(4*n+8/3)2* axi);
init: = Nus;
od:
init: = 0 . 0 ;
r[j]:=(axi,evalf(Nus));
od;
writeto('Nus2.dat');
for i from 1 by 10 to 3000 do
print(r[i]);
od;
writeto(terminal);
Routine to calculate local Nusselt modulus for the
temperature
distribution theta for constant heat flux q.
y: = array(1..1000):
gm1 : = 25.639
gm2 : = 84.624
gm3 : = 176 . 40 :
H1: = 0.008854:
H2 : = 0.002062;
H3 : = 0.0009435 :

```



```

1: = 0.001 :
for j from 1 by 10 to 1000 do
axi : = j^1
Vu: = 1/( (11/48) + (1/2)^( exp( - gml *axi )/( - gml
2*H1 +
exp( - gm2^axi )/( - gm2^2 * H2 ) +
exp( - gm3^axi )/( - gm3^2 * H
3);
y[ j]: = ( axi.evalf( (Nu));
od:
print(' Local Nusselt Values');
writeto(' Nuss.dat');
for j from 1 by 10 to 1000 do
print(y[ j]);
od:
writeto('terminal');

```

Appendix B: Maple Code for Constant Heat Flux Problem

7

```
writeto('output');
hsum := proc(w,Pr,N):
lambda:= sqrt(w)* [
alpha := sqrt(w/Pr)*I I/2:
ulc:= convert(series(BesselJ(0,lambda*r), r=0 ,N),polynom) :
u2c := convert(series(BesselY(0,lambda * * b, r=0 ,N), polynom):
ulprime:= diff(ulc.r):
u2prime:= diff(u2c.r):
a := evalc(subs r=1 ,ulprime)):
b := evalc(subs(r=1,u2prime)):
ultc := convert(series(BesselJ(0,alpha*t),t=0,N),polynom):
u3tc := convert(series(BesselJ(0,lambda*t),t=0,N),polynom):
u4tc:= convert(series(BesselJ(0,alpha*r),r=0.N ),polynom):
u2tc := convert(series(BesselY(0,lambda* t),t=0,N),polynom):
cl:= evalf(sum (((-1)n * (sqrt(w/Pr)/2)4=n)/(factorial(2*n)2),n=0..25)): c2:=
evalf(sum (((-1) * * (sqrt(w/Pr)/24**n+2)/(factorial(2*n+1)2),n=0..25)): c:=
cl +I* c2:
hl1 :=Re(evalc(-b/a)):
h12 := evalc( Pi*sqrt(w)*0.5* I I/2)*ulc):
h13 := evalc(int((u3tc/(I*w))*(l-ultc/c),t=0..1)):
hl := evalc(hl1*h12*h13):
ha:= ulc* Pi*sqrt(w)*0.5* [ /2:
hb:= int((u2tc/(I*w))*(l-ultc/c),t=r..l):
h2 := evalc(ha*hb):
hc := u2c** Pi*sqrt(w)*0.5* I /2:
hd:= int((u3tc/(I*w))*(l-ultc/c),t=0..r):
h3 := evalc(hc
hl + h2 +h3 :
h := evalc(*):
hal := Re(h):
hai2:= Im(h):
g:=(l/(I*w))*(1-u4tc/c):
u := evalc(g):
ul := Re(u):
u2:= Im(u):
part0:=ul*hal:
part2:=u2*ha2: part1 := evalf(Int((part0) * (r),r=0..1.5. NCrule)):
part3 := evalf(Int((part2) * (r),r=0..1.5.NCrule)):
#print ('the bulk temperature is'):
8*(part1) + 8*(part3):
end;
for i from 1 to 10 do
P:= i*0.5:
```

```
for j from 1 by 10 to 50 do
print('The frequency w':The Prandtl No','The bulk temperature is ');
print(j,P.hsum(j,P.25)):
od:
od:
quit:
```

Appendix C: Axial Gradient of Temperature Downstream

8

In this appendix we show that the partial derivative of the zero th order solution θ_0 with respect to the x variable approaches a constant downstream. The result is used in section 3.6 and makes it possible to obtain a solution of the first order problem.

To show this let the Laplace transform of $t - t_0$ be :

$$\tau(s, r) = \int_0^\infty e^{-sx}(t - t_0)dx \dots(C.1)$$

If the wall temperature is given by $t_w(x)$, using the principle of superposition the temperature difference $t - t_0$ is expressed as the following Stieltjes Integral

$$t - t_0 = \int_{\xi=0}^x [1 - \theta(x - \xi, r)] dt_w(\xi) \dots(C.2)$$

where θ is a solution of the Graetz problem:

$$(1 - r^2) \frac{\partial \theta}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) \dots(C.3)$$

$$\theta(x, 1) = 0 \dots(C.4)$$

$$\theta(0, r) = 1 \dots(C.5)$$

Define the following transforms

$$\Theta = \int_0^\infty [1 - \theta(x, r)] e^{-sx} dx$$

$$\Psi = - \int_0^\infty \theta_r(x, 1) e^{-sx} dx$$

Applying the convolution theorem to Eq. C. 2¹ and using the boundary conditions above we get the following

$$\tau(s, r) = \int_0^\infty [1 - \theta(x, r)] e^{-sx} dx s \tau(s, 1) \dots(C.6)$$

Let

$$q(x) = \frac{k}{r_0} \int_0^x \theta_r(x - \xi, 1) dt_w(\xi) \dots(C.7)$$

Applying the convolution theorem and using the following identities

$$\begin{aligned}\mathcal{L}\{1\} &= s\tau(s,1) \dots (C.9) \\ \mathcal{L}\{\theta_r(x, 1)\} &= \Psi \dots (C.9)\end{aligned}$$

we arrive at the following

$$\mathcal{L}\left\{\int_{\xi=0}^x \theta_r(x-\xi, 1) dt_w(\xi)\right\} = \Psi s\tau(s, 1) \dots (C.10)$$

¹If $f(s) = \mathcal{L}\{F(x)\}$ and $g(s) = \mathcal{L}\{G(x)\}$ then $f(s)g(s) = \mathcal{L}\{H(x)\}$ where
 $H(x) = \int_0^x F(\xi)G(x-\xi)d\xi$

Letting

$$\Lambda(s) = \frac{r_0}{k} \int_0^\infty e^{-sx} q(x) dx = \frac{r_0}{k} \mathcal{L}\{q(x)\} \dots (C.11)$$

then:

$$\Lambda(s) = \Psi(s)s\tau(s, 1) \dots (C.12)$$

Since

$$\mathcal{L}\left\{\int_0^x (1 - \theta(x-\xi, r)) dt_w(\xi)\right\} = \Theta(s, r)s\tau(s, 1) \dots (C.13)$$

then

$$\tau(s, r) = \mathcal{L}\left\{\int_0^x (1 - \theta(x-\xi, r)) dt_w(\xi)\right\} \dots (C.14)$$

Next we let,

$$\Phi(s, r) = \frac{\Theta(s, r)}{\Psi(s)} \dots (C.15)$$

and

$$\mathcal{L}^{-1}\{\Phi(s, r)\} = \phi(x, r)$$

Also

$$\mathcal{L}\{q(x)\} = \Lambda(s) \frac{k}{r_0} \dots (C.16)$$

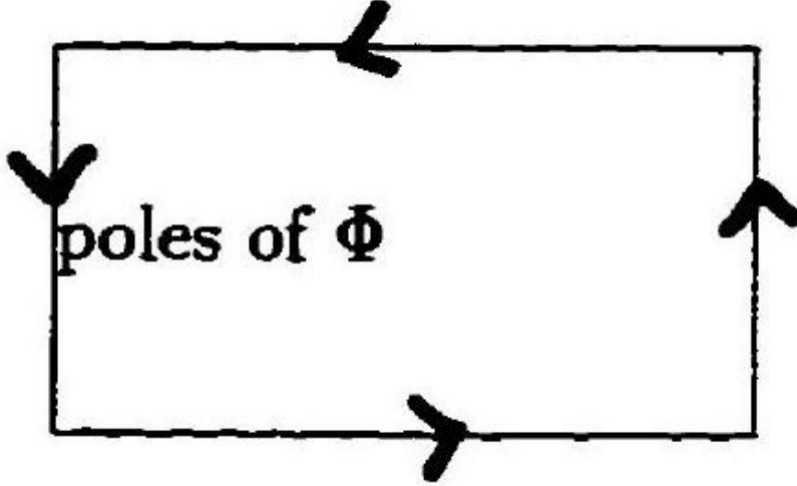
Using the convolution theorem it can be shown that

$$\mathcal{L}\left\{\int_0^x q(\xi)\phi(x-\xi, r)d\xi\right\} = \tau(s, r) \frac{k}{r_0} \dots (C.17)$$

Thus.

$$\mathcal{L}^{-1}\mathcal{L}\left\{\int_0^x q(x)\Phi(x-\xi, r)d\xi\right\} = \mathcal{L}^{-1}\left\{\tau(s, r) \frac{k}{r_0}\right\} \dots (C.18)$$

which implies directly that:



$$\frac{r_0}{k} \int_0^x \phi(x - \xi, r) q(\xi) d\xi = t - t_0 \dots (C.19)$$

Therefore, if we are given the heat flux q_r then the temperature is given by

$$t - t_0 = \frac{r_0}{k} \int_{\xi=0}^x \phi(x - \xi, r) q(\xi) d\xi \dots (C.20)$$

To find $\phi(x, r)$ we apply the following inversion formula and use residue theory in order to perform the complex integration

$$\phi(x, r) = \sum \text{Res}\{e^{sx} \Phi(s, r)\} \text{ at the poles of } (s, r)$$

$$\phi(x, r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \frac{\Theta(s, r)}{\Psi(s)} ds \dots (C.21)$$

Complex Integration is taken along the Bromwich contour:

figure 3.2 Recall that,

$$\mathcal{L}\{1 - \theta(x, r)\} = \Theta(s, r) \dots (C.22)$$

$$\theta = \sum_{n=0}^{\infty} C_n R_n(r) e^{-\lambda_n^2 x} \dots (C.23)$$

where $\lambda_n = 4n + 8/3$

Refer to section 2.3.4 and table 2.1 for the eigenfunctions R_n eigenvalues λ_n and constants C_n .

This leads to:

$$\Theta(s, r) = \frac{1}{s} - \sum_{n=0}^{\infty} \frac{C_n R_n(r)}{s + \lambda_n^2} \dots (C.24)$$

Since,

$$\mathcal{L}\{\theta_r(x, 1)\} = \Psi(s) \dots (C.25)$$

then this implies that:

$$-\sum_{n=0}^{\infty} C_n R'_n(1) \frac{1}{s+\lambda_n^2} = \Psi(s) \dots (C.26)$$

Now $\Theta(s)$ and $\Psi(s)$ have poles at $s = 0$ and $s = -\lambda_n^2$. Therefore Θ/Ψ has poles only at $s = 0$ and the zeros of $\Psi(s)$. Let β_m^2 be such that $\Psi(-\beta_m^2) = 0$.

Then the following holds:

$$e^{sx} \frac{\Theta(s,r)}{\Psi(s)} = e^{sx} \left[\frac{1}{s} - \sum_{n=0}^{\infty} \frac{C_n R_n(r)}{s+\lambda_n^2} \right] \left[-\sum_{n=0}^{\infty} C_n R'_n(1) \frac{1}{s+\lambda_n^2} \right]^{-1} \dots (C.27)$$

The zeros of $\Psi(s)$ can only be found numerically.

Now we compute the residues.

First we set.

$$G(s) = e^{sx} \left[s \sum_{n=0}^{\infty} C_n R'_n(1) \frac{1}{s+\lambda_n^2} \right]^{-1} \dots (C.28)$$

Computing the residue $\text{Res}[G, 0]$ we get:

$$\text{Res}[G, 0] = \lim_{s \rightarrow 0} -s e^{sx} \left[s \sum_{n=0}^{\infty} C_n R'_n(1) \frac{1}{s+\lambda_n^2} \right]^{-1} \dots (C.29)$$

Thus the residue of G at zero is:

$$\text{Res}[G, 0] = \frac{1}{\Psi(0)} \dots (C.30)$$

We will make use of the result, If f, g are analytic at s_0 and if $f(s_0) \neq 0$ and g has a simple zero at s_0 , then $\text{Res}\left[\frac{f}{g}, s_0\right] = f(s_0)/g'(s_0)$, to compute the residue of the entire expression associated with φ .

Since Ψ has a simple zero at $-\beta_m^2$ then the residue of G at $x = -\beta_m^2$ is

$$\text{Res}[G, -\beta_m^2] = -e^{-\beta_m^2} \left[\beta_m^2 \Psi'(-\beta_m^2) \right]^{-1}$$

Consider the other part of the term

$$e^{sx} \frac{\Theta(s,r)}{\Psi(s)}$$

in Eq. C. 27 that is

$$e^{sx} \sum_{n=0}^{\infty} \frac{C_n R_n(r)}{s+\lambda_n^2} \left[\sum_{n=0}^{\infty} C_n R'_n(1) \frac{1}{s+\lambda_n^2} \right]^{-1} \dots (C.31)$$

The residue of this part is

$$-\sum_n C_n R_n(r) \sum_m \frac{e^{-\beta_m^2 x}}{\lambda_n^2 - \beta_m^2} \frac{1}{\Psi'(-\beta_m^2)}$$

Hence we obtain the following solution for $\phi(x, r)$

$$\phi(x, r) = \frac{1}{\Psi(0)} - \sum_m^\infty \frac{e^{-\beta_m^2 x}}{\beta_m^2 \Psi'(-\beta_m^2)} - \sum_n^\infty C_n R_n(r) \sum_m^\infty \frac{e^{-\beta_m^2 x}}{\lambda_n^2 - \beta_m^2} \frac{1}{\Psi'(-\beta_m^2)} \dots (C.32)$$

where we can solve for β_m^2 numerically for all values of m from the following

$$\sum_{n=0}^\infty \frac{C_n R'_n(1)}{-\beta_m^2 + \lambda_n^2} = 0 \dots (C.33)$$

where $\lambda_n = 4n + 8/3$

It can be shown that $\Psi(0) = 1/4$ [46]

Now returning to the formula derived above for temperature $t - t_0$ in Eq. C.20.

for $q = \text{const}$.

$$t - t_0 = \frac{qr_0}{k} \int_{\xi=0}^x \phi(x - \xi, r) d\xi \dots (C.34)$$

Therefore,

$$t - t_0 = \frac{qr_0}{k} \int_{\xi=0}^x \left[4 - \sum_m^\infty \frac{e^{-\beta_m^2(x-\xi)}}{\beta_m^2 \Psi'(-\beta_m^2)} - \sum_n^\infty C_n R_n \sum_m^\infty \frac{e^{-\beta_m^2(x-\xi)}}{\lambda_n^2 - \beta_m^2} \frac{1}{\Psi'(-\beta_m^2)} \right] d\xi \dots (C.35)$$

Since.

$$\theta_0 = k [t - t_0] / qr_0$$

then

$$\theta_0(r, x) = \left[4x - \sum_m \frac{1}{\beta_m^4 \Psi'(-\beta_m^2)} + \sum_m \frac{e^{-\beta_m^2 x}}{\beta_m^4 \Psi'(-\beta_m^2)} - \sum_n C_n R_n \sum_m \frac{1 - e^{-\beta_m^2 x}}{\beta_m^2 (\lambda_n^2 - \beta_m^2)} \overline{\Psi'(-\beta_m^2)} \right] \dots (C.36)$$

Appendix D: Picard-Lindelof Theorem

9

In this section an existence theorem for first order differential equations of the form given in equation 4.26 is given.

Consider the Cauchy Problem

$$\begin{aligned}\frac{d\theta}{dt} &= f(t, \theta(t)) \\ \theta|_{t=\tau} &= \xi\end{aligned}$$

where $(\tau, \xi) \in D$ is a nonempty open subset of $\mathcal{R} \times \mathcal{C}$

Assume that f is continuous on D and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq c_R |x - y|$$

on each rectangle $R \subset D$ and c_R is a constant depending on the size of the rectangle. Then there exists a unique solution to the Cauchy problem above on $I = (\tau - \alpha, \tau + \alpha)$, where α, b are chosen appropriately small so that for the rectangle centered at (τ, ξ) ,

$$R_{ab} \subset D$$

and

$$\alpha = \min\{a, b/M\}$$

where

$$M = \max |f(t, x)|$$

and the maximum is taken over $(t, x) \in R_{a,b}$

9.1 Proof

We show that it is possible to define

$$\phi_0(t) = \xi \dots (D.1)$$

$$\Phi_{k+1}(t) = \xi + \int_{\tau}^t f(s, \phi_k(s)) ds \dots (D.2)$$

$k = 0, 1, 2, \dots$) for $t \in \bar{I} = [\tau - \alpha, \tau + \alpha]$ and derive the estimates

$$|\phi_{k+p}(t) - \phi_k(t)| \leq M \alpha \frac{(c\alpha)^k}{(k+1)!} \dots (D.3)$$

for some $c > 0$. This would prove that $\{\phi_k\}$ is a Cauchy sequence in the normed linear space of continuous complex valued functions $C(\bar{I})$, with the uniform metric norm. Now since this space is known to be a complete metric space (ie. Banach Space) then

$$\phi_k \rightarrow \phi, \text{ for some } \phi \in C(\bar{I})$$

Taking limits of the functions ϕ_k it follows that

$$\phi(t) = \xi + \int_{\tau}^t f(s, \phi(s)) ds$$

It is a simple matter to see that ϕ provides the required solution.

We prove the above rigorously.

Define ϕ_0 on $\bar{I}$ using the initial condition already given.

The sequence of functions ϕ_k defined above continue to successively define $\phi_1, \phi_2, \phi_3, \dots$ on $\bar{I}$. Trivially, we have that

$$|\phi_0(t) - \xi| \leq M|t - \tau|$$

for all $t \in \bar{I}$. Hence, because of the definition of α , the curve $(t, \phi_0(t)), t \in \bar{I}$ remains in $R_{a,b}$ and, in particular, the integral to be used in defining Φ_1 exists. Induction is used for all k . Thus assume that

$$|\phi_k(t) - \xi| \leq M|t - \tau|, t \in \bar{I}.$$

Now ϕ_{k+1} must be shown to satisfy the same inequality. But from the first step of induction method, Φ_{k+1} is defined on $\bar{I}$, and using the definition of the maximum M , leads to

$$|\phi_{k+1}(t) - \xi| \leq \int_{\tau}^t |f(s, \phi_k(s))| ds \leq M|t - \tau|$$

The next step is to establish the estimate

$$|\phi_k(t) - \phi_{k-1}(t)| \leq \frac{c^{k-1}M}{k!} |t - \tau|^k$$

($k = 1, 2, \dots$), $t \in \bar{I}$, where c is the Lipschitz constant associated with $R = R_{a,b}$.

Now using the Lipschitz inequality, it follows that

$$\begin{aligned}
|\phi_{k+1}(t) - \phi_k(t)| &\leq \left| \int_{\tau}^t [f(s, \phi_k(s)) - f(s, \phi_{k-1}(s))] ds \right| \\
&\leq \int_{\tau}^t |\phi_k(s) - \phi_{k-1}(s)| ds \\
&\leq \frac{c^k M}{k!} \left| \int_{\tau}^t (s - \tau)^k ds \right| \\
&\leq \frac{c^k M}{(k+1)!} |t - \tau|^{k+1} \quad (D)
\end{aligned}$$

Hence the above inequality is proven by induction. Reapplication of this inequality gives:

$$\begin{aligned}
|\phi_{k+p}(t) - \phi_k(t)| &\leq |\phi_{k+p}(t) - \phi_{k+p-1}(t)| + |\phi_{k+p-1}(t) - \phi_{k+p-2}(t)| \\
&\quad + \dots + |\phi_{k+1}(t) - \phi_k(t)| \\
&\leq \frac{c^{k+p-1}}{(k+p)!} M |t - \tau|^{k+p} + \frac{c^{k+p-2}}{(k+p-1)!} M |t - \tau|^{k+p-1} \\
&\quad + \dots + \frac{c^k}{(k+1)!} M |t - \tau|^{k+1} \\
&\leq \frac{M \alpha (c\alpha)^k}{(k+1)!} \left[1 + \frac{(c\alpha)}{2!} + \dots + \frac{(c\alpha)^{p-1}}{p!} \right] \\
&\leq \frac{M \alpha (c\alpha)^k}{(k+1)!} e^{c\alpha}
\end{aligned}$$

Hence the sequence of functions ϕ_k is a Cauchy sequence and the sequence must converge to a solution ϕ .

The uniqueness of the solution follows readily using the Lipschitz condition for f .

Appendix E: Properties of the Dirac Delta Function

10

$$\begin{aligned}\int_a^b f(t)\delta(x-x_0)dx &= f(x_0) \quad \text{if } a < x_0 < b \\ &= 0 \quad \text{if } x_0 < a \text{ or } x_0 > b\end{aligned}\tag{E.1}$$

$$\delta(t) = \delta(-t)\tag{E.2}$$

$$t\delta(t) = 0\tag{E.3}$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \quad a \neq 0\tag{E.4}$$

$$f(t)\delta(t-a) = f(a)\delta(t-a)\tag{E.5}$$

$$\delta[g(t)] = \sum_j \frac{1}{|g'(t_j)|} \delta(t-t_j)\tag{E.6}$$

$$\mathcal{L}\delta^{(k)}(t-\tau) = s^k e^{-s\tau}, \quad -\infty < \Re s < \infty\tag{E.7}$$

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$^1 K^-$ is a compact complex Hermitian operator acting on $L^2(0,1)$, $K^2 : L^2(0,1) \rightarrow L^2(0,1)$

3 If f, g are analytic at s_0 and if $f(s_0) \neq 0$ and g has a simple zero at s_0 , then $\text{Res}[f/g, s_0] = f(s_0)/g'(s_0)$



Save our Planet

Heat Transfer with Pulsatile Flow in a Tube

Terry Moschandreou

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