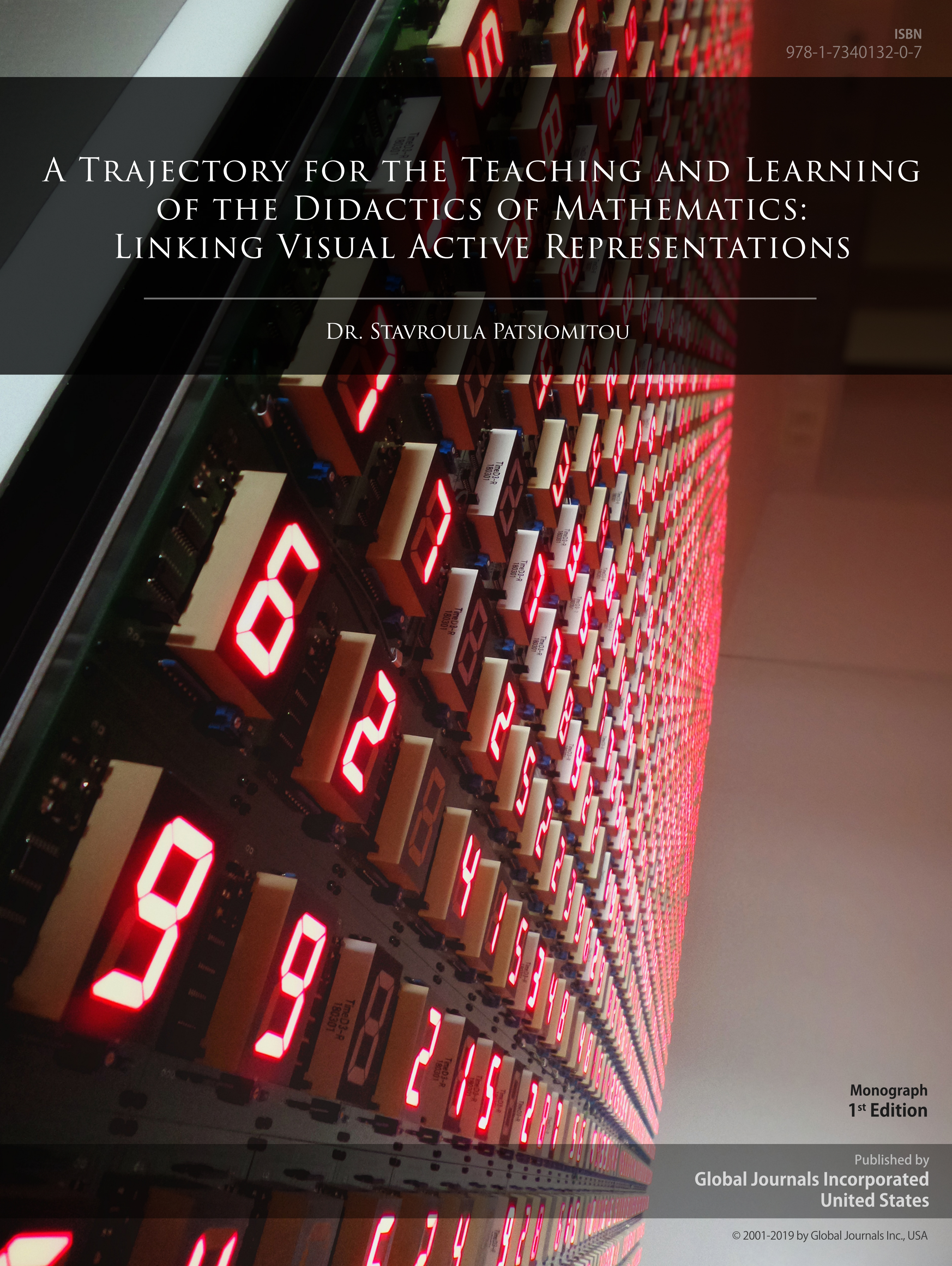

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A Trajectory for the Teaching and Learning of the Didactics of Mathematics: Linking Visual Active Representations

Dr. Stavroula Patsiomitou

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A Trajectory for the Teaching and Learning of the Didactics of Mathematics [using ICT]

Linking Visual Active Representations

STAVROULA PATSIOMITOU

A Trajectory for the Teaching and Learning of the Didactics of Mathematics [using ICT]

Linking Visual Active Representations

Stavroula Patsiomitou
(Ph.D, ME.d)

MONOGRAPH

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Preface

A monograph is the goal of every author who seeks to share with others crucial ideas on a specialized subject s/he considers to be important. For me, the development of students thinking is a goal that I have served all my life as a teacher of mathematics; since my high school years, when my schoolmates asked me to help them with mathematics. I have understood that every student has his/her own personal way of understanding mathematics: their own personal *learning style*.

And that it is in our mind that the information we receive is translated into the codes we understand. I have examples from my personal life that helped me grasp the importance of learning theories: My mother, for example, was trying to help me learn to ride my bicycle by running along beside me (helping me to balance). But I could not learn to use my bicycle alone; not until the morning I woke up and tried to *instrumentally decode* my own actions. *Preexisting knowledge* played an important role, but the *non-conceptual behavioral repeats* of actions on my bicycle did not help me. When teaching me to tell the time, my father used his own watch, moving the hands back and forth, to teach me *dynamically*, refusing to believe my grandmother, who said I was too *biologically immature* to understand time yet. As you can understand, I learned to tell the time more easily using my father's watch as *a dynamic manipulative*. *Cognitive conflicts, instrumental decoding and dynamic actions* were synthesizing my own learning style.

As part of the leaning process we have to understand what the mathematical objects are, how to use them and how to represent them in static or dynamic means. Language also plays a crucial role in the teaching and learning process. Do we learn alone as individuals, or with others in a social context? Do we learn using traditional means, or through e-learning and computer software? Both are important for students. As teachers, we have to choose the road, the learning path our students will follow, by using a *thought experiment* to construct a *hypothetical learning path* that predicts their progress and their thought development.

The key concept in this monograph is the idea of *Linking Visual Active Representations* (LVAR) (Patsiomitou, 2008a, b, 2012a), which I conceived when I was writing my Master's thesis (Patsiomitou, 2005a), and which I would subsequently develop and expand during the research I conducted for my Ph.D. thesis. A second important concept is the *dynamic hypothetical learning path* (DHLP) (Patsiomitou, 2012a, b). When I started my PhD, I did not know what I had conceived was actually a DHLP. I called it a didactic scenario, a didactic sequence…. But after reading the related bibliography I understood that I had been constructing DHLPs for my students all my teaching life to scaffold their knowledge construction and to help them develop their thinking.

The research underlying the concrete theoretical framework was conducted in accordance with the methodology I apply every time I write a paper (e.g., Patsiomitou, 2015b). Specifically, the research with related "key" words was conducted in: (1) Databases listing international literature (ERIC, Scopus, etc.), (2) Libraries of International Universities, (3) open e-journals with pedagogical-educational subject-matter, (4) Conferences proceedings, (5) Self-publishing texts (6) Reports from surveys of international organizations (e.g UNESCO, OECD), or Reports on international programs or programs funded by international organizations[qualitative data A type]. The gathering of the material focused on studies in English language and applied a strict criterion of publication within the last two decades, however important articles from > 25 years ago were also considered [qualitative data B type]. The initial screening of the texts (i.e. the evaluation performed as part of the concrete methodical survey) was followed by a second screening with a strict limitation in terms of similarities or differences in certain characteristics [qualitative data C type]. From these texts, a number of extracts were selected which refer to the concepts dealt with in this work, raising issues and creating incentives for further research and study (qualitative type D data). Summaries of ancient texts (e.g., by Euclid, Plato and Aristotle) were added to the set of qualitative data in the light of their contribution to the definition of terms used in the monograph. The present work is a meta-analysis of the qualitative type D data, on the basis of which conclusions are drawn utilizing material contained in already published works [with reference to the source].

The concrete monograph includes a substantial amount of references to the bibliography and incorporates excerpts from papers by many important scholars. I therefore hope the current work will become the starting point of a "hypothetical" learning path in Didactics which I designed, a detailed reading of which will allow my University students to "discover" many important theories for themselves. Also, the value of this book is to motivate teachers of secondary education, as well as pre-service teachers of mathematics and provide them with the theoretical constructs they need to embark on their own investigations.

This is the key idea of the work in question: searching for more…..discovering the undiscovered and ….seeing the unseen. As you read the book and understand the theories, the connections between them become increasingly obvious. Mathematical knowledge and understanding as well as representational systems are all tightly bound up with the teaching and learning of mathematics.

However, an investigation of traditional curricula reveals an overwhelming emphasis on working with symbolic representations. In the context of my search for "windows" or "keys" which can facilitate the students' cognitive development, I shall be discussing dynamic geometry environments and microworlds in general with a view to understanding their potential and how they can be combined to make the learning of mathematics more interesting and relieving students of their fear of the subject.

Every chapter is written with the goal of addressing overarching research issues, providing guidance for future research that involves technology. The aim is to inculcate in students of mathematics a greater awareness of the theory and research into the Didactics of Mathematics, taking into account the impact representational technological environments have had on mathematics learning and teaching.

However, the current work reflects my understanding of teaching and learning as communicating vessels, allowing teachers to communicate their ideas to their students and to "learn" from their responses. My work have influenced from the discussions I have had with my students both in and out of class, and from the results of my own research. Furthermore, I would like to acknowledge all my teachers for the knowledge they shared with me so generously, particularly since they motivated me to continue and search for more.

I wish to thank my family for supporting me all the years of my life spent studying and writing. The monograph is dedicated to my parents for believing in me, to my children Alexandros, Loukia-Ioli and Theano-Magdalene for their love and patience, and who encouraged me to realize my dreams. I thank you from the heart!

Athens, July 2019

Stavroula Patsiomitou

About the Author

I was born in Larisa, Greece and have lived the last 35 years in Athens, Greece. I am a teacher of mathematics in secondary education and, since 2005 a researcher. I was awarded my Ph.D. in the Didactics of Mathematics at the University of Ioannina, Department of Primary Education, in 2012. The topic of my thesis was "*The development of students' geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment: Linking Visual Active Representations.*" I have also introduced the meanings *Reflective Visual Reaction, theoretical and experimental dragging, instrumental decoding* etc. though many international papers that have been published at PME conferences or refereed journals. My Ph.D. thesis is held by the Library of the Hellenic Parliament, the National Library of Greece and other institutions. It has also been freely available to anyone from the scientific website of the National Documentation Center since 2013 (webpage [1]). Prior to this, in 2005, I was awarded my Master's degree in Education, specifically, in the Didactics and Methodology of Mathematics, in the inter-university program of the University of Athens (NKUA) and the University of Cyprus. My Master thesis topic was *"Fractals as a context of comprehension of the meanings of the sequence and the limit in a Dynamic Software environment"* available on the scientific website (webpage [2]).

I have taught mathematics many years: (a) in private high schools where I worked as teacher of Mathematics from 1983-1997 and (b) in state secondary education, appointed in 1998. I got a positive evaluation ("Excellent") in the context of the evaluation process to be appointed in the Model Schools. I was seconded in the Ministry of Education acting as Researcher, for the "Authority for Quality Assurance in Primary and Secondary Education" for three years (2014-17). For this time, I was evaluated as "Excellent" in the context of civil servants assessment.

I taught for the academic year 2016-17 and 2017-18 the course MEM321: *Didactics of Mathematics* at the University of Crete, Department of Mathematics and Applied Mathematics. I have been also chosen for the academic year 2017-18 at the University of Crete to teach the course "MEM322: *Using ICT in Mathematics teaching and learning* (spring semester: 2017-18). Furthermore, I was chosen as visiting Lecturer (according to the PD 407/1980), for the course "*Practical training of prospective teachers of Mathematics*" of the Department of Mathematics and Applied Mathematics of the University of Crete (spring semester: 2016-17). (https://orcid.org/0000-0002-7102-4582)

At the end of the course, my students presented their work in workshops organized by the University and myself. For many years I was also collaborating with academics at the National and Kapodistrian University of Athens, Dept. of Mathematics, for the Practical training of prospective teachers of Mathematics and in supervising their training at the secondary-level.

I have written an algebra textbook in Greek for 16- to 18-year-olds, titled "A collection *of Algebra problems for students and math-teachers*" (250 pages). I have also written a dynamic geometry textbook (two volumes-600 pages) in Greek (titled "*Learning mathematics with the Geometer's Sketchpad v4*") which was approved by the Greek Pedagogical Institute and has been sent to experimental Model schools in Greece. Moreover, a didactic portfolio, entitled: "*Didactic approaches to teaching Mathematics to students with different learning styles*" (in Greek), which includes many indicative didactic approaches I use, demonstrating also to the results of my teaching on my students, is free at the following link (web page [3]).

I have authored and presented more than 50 papers at conferences in Greece and abroad, published more than 35 articles in refereed journals in both the Greek and English languages. I have organized many interdisciplinary exemplary teachings (or teaching demonstrations) and exhibitions for teaching Mathematics in secondary-level education. Furthermore, I strongly encourage the use of e-learning in class or out of class. For this, I created e-learning material (15 electronic lessons) for the secondary-level (web page [4]) and the tertiary-level of mathematics. I also created an electronic journal, wrote articles and managed e-learning material created by secondary students (web page [5]).

I was involved in the Greek translation of the Geometer's Sketchpad v.4 dynamic geometry software, advising the translation team on the suggestion of its Chief Technology Officer, Nicholas Jackiw. My name is included on the splash screen "Special Thanks to ..." of the Greek version of the Geometer's Sketchpad v.4 dynamic geometry software.

I was a NCTM (National Council of Teachers of Mathematics) Member for the years 2007-2012. I am also a PME (*International Group for the Psychology of Mathematics Education*) Member [2008-2019] and act as Reviewer of the *International Conference for the Psychology of Mathematics Education* (for the years 2009-2019).

I am a mother of three children who have studied at University: a son who has awarded a postgraduate degree from the National Technical University of Athens in Mechanical Engineering, a daughter, who has awarded a graduate degree in the Chemistry Department of the National and Kapodistrian University of Athens, and a daughter, who has awarded a graduate degree in the History and Archaeology Department of the National and Kapodistrian University of Athens. My scientific writing work and my career in Education took place in parallel with raising my three children, without any help from others.

My aim was (/is) to help students develop their understanding, their cognitive thinking and their intelligence as they interact with the software environments. My research interests centre on computer assisted mathematics learning and teaching in general and dynamic geometry software in particular, gender's equality and gender and leadership.

Outline of the Monograph

In my investigation of learning theories I tried to find the "*key words*" and common characteristics they shared. In the following paragraph I shall try-briefly, and in a very simplistic and superficial way- to treat complex learning theory issues as if they were much simpler than they are. However, the order in which they are discussed can provide a path, or illustrates a trajectory in didactics. For example, in Piaget, the notion of the *schema* and what Piaget considers to be *assimilation and accommodation* are important notions in his theory. What Piaget's theory has in common with other approaches is its investigation in the theoretical construct "*development of a pupil's thinking*". Where they differ is in respect of Piaget's view that a child's development corresponds with their *biological mature*. The age of 7, the age at which a child can distinguish materials from more than two characteristics in accordance to Piaget's *developmental stages*, is also important for Vygotsky as the age at which a child can develop a close interaction between *language and thought*. Vygotsky also introduced developmental stages. Vygotsky assigns an important role to the social construction of knowledge and the role of language and how a student can express his/her thinking. The *development of language* as the student progresses through levels is also a feature of the van Hiele model. The van Hiele model considers the *perturbations and cognitive conflicts* which appear as a student develops their thinking up through the different levels. How can a student develop his/her thinking according to van Hiele? Using *manipulatives* at the first stages in their development, following an *instructional sequence* that seeks to scaffold students' language. *Scaffolding* is a notion introduced and developed by Vygotsky, who considers the *use of tools* to provide an important scaffold for students' thinking. Manipulatives are *external representations,* and the role of *visualization* has been discussed by many scholars. As a student develops their thinking, they develop the way in which they use language and formulate: at first, they use mostly *inductive reasoning*, but as time passes and they follow a concrete course of *instruction*, they start to formulate with *abductive and deductive reasoning*. The role of *microworlds* as cooperators or "antagonist" environments has been discussed extensively by many scholars, as has the ways in which the incorporation of tools in the digital environment scaffolds students' thinking during *instrumental genesis*. What is the role of the student in the instructional and teaching process? Is s/he a *passive audience or an active one*? Can s/he participate actively in the learning process? *Hypothetical learning trajectories* or paths are theoretical constructs that give the student the advantage of being able to construct the didactic sequence and adapt it according to his/her needs. Can a student develop his/her thinking and how can s/he achieve it? As I discussed in previous works, using *Linking Visual Active Representations* a student can develop *mental linking representations* that connect the new knowledge with existing in their mental structures.

In the current work, I shall describe the theories that led me to the theoretical constructs of my Ph.D. It is very important to point out that this study is not a translation of my Ph.D thesis. In my thesis I have tried to restrict the theoretical background to the absolute minimum required to analyze the results of the experimental process and arrive at new theoretical constructs. My post-doctoral empirical research has led me to "discover" several additional notions which are not included in my thesis; for example, the notion of *hybrid- dynamic objects*. I will also incorporate part of my previous research in the form of a meta-analysis, in the sense of an analysis that includes those meanings. Most of these results will be included in my next study, which will be published in the near future. Moreover it was very difficult for me to include in this text all the notions and theories that has been posited and developed since researchers, scholars and psychologists started to write about and investigate the fundamental notions of knowledge, understanding, development, learning, teaching and everything else is considered part of the Didactics of Mathematics.

Also, as I aforementioned, the value of this book is to motivate students to start investigations, giving them the necessary theoretical constructs for their beginning. In the figure below I have tried to indicatively connect the theoretical constructs included in the current work.

Introduction: I start my introduction to the book with the notion of hypothetical learning trajectories, providing ways in which the geometry curriculum can invoke a dynamic reinvention process through teaching for the construction of geometrical objects. The core idea in the main part of the book is that students will learn in the most profound way possible when something happens that makes them love the particular knowledge being studied and are responsible for constructing their own knowledge, as Papert argues. After the short introduction, the theoretical underpinnings will be presented over five chapters in which I try to incorporate the most important and essential meanings, notions and concepts—the presuppositions for describing and analyzing research studies in the didactics of mathematics

Chapter 1. This chapter provides essential answers to the questions: "What are mathematical objects? What are diagrams, figures and diagrammatic representation? What is diagrammatic reasoning?" I also present the pairings of knowledge types: conceptual-procedural, relational-instrumental, operational-structural along with the concept of reflective abstraction.

Figure I. Indicative connections between the theoretical constructs included in this book (Patsiomitou, 2012a, 2018b, p.49, modified)

CHAPTER 2. Representations, representational systems and visualizations of mathematical objects are discussed extensively in this chapter. Both internal and external representations and multiple external representations are presented with multiple examples and excerpts from the work of important scholars. I also introduce a connection between multiple external representations and mental images for the development of understanding, by taking into account technological-digital representations; I offer by this a different perspective, updating the Lesh's model of multiple representations. I also discuss indicative representational environments used for the teaching and learning of mathematics and their role or capabilities in knowledge construction.

CHAPTER 3. An extensive analysis of the literature regarding dynamic geometry systems is presented in this chapter. I explain what I mean with the notions "*instrumental decoding*", "*instrumental obstacles*". Moreover what is an artifact and what is meant as instrument during instrumental genesis. Dynamic transformations and their role in the construction of mathematical meanings are presented as a crucial particularity of DGS environments. I also discuss the notion of *hybrid-dynamic objects* as well as the notion of *procept-in-action*.

CHAPTER 4. Assisting/ Encouraging students' cognitive growth is a major aim of mathematics education. The Piagetian notions (*cycle of equilibration*, *assimilation and accommodation*) are discussed in the chapter, along with my opinions on this. I also introduce a spiral cycle of equilibration regarding students' number construction at different ages, as well as a spiral curriculum for the learning of numbers, taking into account the notions of Piaget and Bruner. The theory developed by van Hiele is analyzed extensively. Argumentation, proof and proving process are also discussed, with examples of both Toulmin's model of argumentation, and the pseudo-Toulmin model which I introduced to incorporate the impact the use of the tools has on the construction of arguments. I also present an example of my research in which the students used a custom tool to develop their thinking. At the end the chapter I discuss my version of the "house of quadrilaterals", in which I incorporate the non-convex quadrilaterals.

CHAPTER 5. The word "problem" is derived from the Greek word "provlema". What is an open problem, what are the four problem-solving phases developed by Polya, and what are the factors involved in a successful problem-solving process are issues which I discuss in the chapter in question. I present a theoretical construct-namely an *empirical classification model for sequential instructional problems* in geometry- a cognitive trajectory, which relates to the importance of students building a representation of a problem and the role which modeling a real-world problem plays in students' gradual investigation of a problem.
Horizontal and vertical mathematization and the modeling process as it has been developed in the international literature are also addressed in this chapter. The notions of hypothetical learning trajectories, paths and progressions are discussed as well as my adaptation on *Mathematics Teaching Cycle,* based on the work of Simon. I briefly present the DHLP for the research study of my PhD. The chapter ends with an extensive analysis on the notion of Linking Visual Active Representations (LVARs) and their importance on students' thinking.

IN PLACE OF AN EPILOGUE: I have presented a short history of my Ph.D. study. My advice to you is this: tenacity, knowledge, ambition and passion to succeed are the keys to reach your goals. And never give up!
Are LVARs a new theory for teaching and learning?
I leave the answer to this question to you, as well the option of commenting on my work. Please do not hesitate to communicate via e-mail and/or to send your comments to the following e-mail address (spatsiom@gmail.com). Thank you in advance. I wish you pleasant reading!

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'To Comenius, first of all, we owe it that pedagogy was regarded as a science and teaching as an art'

Ossian Lang (1892)

Introduction

I. Curricula and dynamic active learning trajectories: The geometry curriculum as a *dynamic reinvention process* for the construction of geometrical objects

Simon (1995) defined hypothetical learning trajectories as "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). A hypothetical learning trajectory is hypothetical "because […it] "is not knowable in advance" (Simon, 1995, p. 135). He used the metaphor of a sailor to explain the difference between a trajectory and a hypothetical learning trajectory:

> "You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your 'hypothetical trajectory'." (pp. 136-137)

In this thoughtful paragraph, I recognized my own experiences with my every year students in class. The way that my students interacted with the pre-prepared material (digital and otherwise) which I had planned for them, changed the whole path we followed, as I added paths to explain something that was not understood or helped students overcome their misconceptions by using a different path. This was the same feeling I had when I read how Clements & Sarama (2004) defined *learning trajectories* as

> "descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (p. 83).

Moreover, in their article "*Learning Trajectories: Foundations for Effective, Research–Based Education*" in section "What, if anything, is "new" in the learning trajectories construct?", Clements & Sarama (2014) discuss what is new in learning trajectories, reporting the common characteristics the learning trajectories have with psychological and educational theories "for example, Bloom's taxonomy of educational objectives and Robert Gagne's conditions of learning and principles of instructional design, information-processing theories, information- processing models, developmental and cognitive science theories" (p.8-9).

Remillard (1999) supports that as teachers interact with their students, they feel the need to understand their thinking and find methods of guiding their students towards understanding. Remillard (1999) also emphasizes "the substantial role that teachers play in shaping the curriculum experienced by students" (p.316).

Officially, curriculum includes instructions, informing the teacher how to manage the teaching process in class. Teachers follow these instructions, but often not in a detailed way as a curriculum is only 'an [official] plan for teaching" and instruction. (e.g., van den Akker, 1998, cited in Zulkardi, 2002). As Zulkardi (2002) supports

> "The plan can be found at different levels of various educational settings. At the micro level (classroom), the curriculum refers to a plan for concrete instructional activities. At the meso level (school or institutional) it refers to a course or an educational program and at the macro level it is used to indicate a more general curricular framework for a district, province or nation" (p. 23-24).

According to Zulkardi (2002) "there are several types of curriculum proposed by Goodlad, et al. (1979) and adapted by van den Akker (1998) (Zulkardi. 2002, p. 24):

- *"ideal curriculum*, the original assumptions and intentions of the designer;
- *formal curriculum*, the concrete curriculum documents, such as student materials and teacher guides;
- *perceived curriculum*, the curriculum as interpreted by teachers;
- *operational curriculum*, the actual instructional process as realized in the classroom (also referred to as curriculum-in-action or the enacted curriculum);
- *experiential curriculum*, the curriculum as it is experienced by the pupils;

- *attained curriculum*, the learning outcomes of the pupils. In some studies the term intended curriculum is used, which refers to a combination of the ideal and formal curriculum while implemented curriculum refers to a combination of the perceived and the operational curriculum".

The development of the curriculum in class by means of a constructivist process focuses on an *active learning process* (Piaget, 1937/1971), fuelled by the interaction between their experience, the mental processing of their knowledge (Vygotsky, 1978) and the students' sequential construction of this knowledge (Terwel, 1999). This kind of knowledge construction is facilitated by the teachers and instructors, who scaffold students' mathematical thinking, facilitate mathematical discussions in class, use mathematical representations, and reinforce alternative learning methods (Hiebert & Carpenter, 1992 cited in Fuson, Carrol & Drueck, 2000, p.277). Remillard (1999) considers that curriculum materials were in the 1950s "the primary vehicles used […] to stimulate *curricular change* [and] to change the nature of students' mathematics learning opportunities" (p.315). Teachers can develop the curriculum in class, as it is they who have to identify and deal with their students' difficulties and needs. According to Remillard (1999)

> "Regardless of how teachers draw on and use curriculum materials, their work in relation to planning and teaching mathematics can be viewed as curriculum development—the processes by which teachers develop curricular plans and ideals and translate them into classroom events. Through the curriculum development process, teachers plan and shape students' experiences in the classroom. The term "*curriculum development*" is often used to describe the writing of curriculum materials. In referring to teachers as curriculum developers, I suggest that the curriculum development process does not stop when textbooks are printed, but continues in the classroom" (p.319).

This is in accordance with Freudenthal's proposed *educational development* of mathematics, his own alternative to curriculum development which centres on the development of curriculum materials, and seeks to foster actual change in classroom teaching (Gravemeijer and Terwel, 2000, p.779).

Gravemeijer & Terwel (2000) highlight what Freudenthal claims:

> "As viewed by Freudenthal, curriculum theory is not a fixed, pre-stated set of theories, aims and means, contents, and methods. Rather, it is always related to processes. Understood positively, the word 'curriculum' is more often than not used in combination with change or development, for example, as in curriculum development or developmental research. For Freudenthal, *curriculum theory* was a practical endeavour from which new theoretical ideas might arise as a kind of scientific by –product" (p. 779).

Many researchers (e.g., Cobb & Bauersfeld, 1995; Fuson, Carrol and Drueck, 2000, p.277) agree that problem solving is a fundamental process that can help teachers introduce meanings as they encourage their students to investigate the problem. Moreover, students can *reinvent* what is mentioned in textbooks through the problem-solving process, as Fuson et al. argue:

> "in contrast to traditional textbook instruction focused primarily on rote learning and practice of skills, instruction is envisioned through which students construct meaning for the mathematical concepts and procedures they are investigating and engage in *meaningful problem-solving* activities" (Fuson, Carrol and Drueck, 2000, p.277) (ibid).

Dubinsky (1991a) in his study "*Reflective Abstraction in advanced mathematical thinking*" reports the process of memorization without understanding on the part of students as they are obliged or accept to follow the traditional learning process following the instructions of the teacher who translates the curriculum into classroom instruction. Dubinsky supports that:

> "Our conjecture is that this is due to the overall approach in the traditional classroom, where the goal, as presented and defended by the teacher, is for the student to develop skills in computational procedures, to display on examinations, and to "get a good grade". […] the student cannot learn these procedures through understanding, whereas he or she is presented by the teacher with a conflict-free way out - imitate and memorize. Unsurprisingly, most students accept the offer and take this route. *But imitation and memorization do not lead to cognitive constructions and the result is that the students' desire to learn through growth is suppressed*. He or she is "turned off mathematics" (p. 117) .

On the other hand, Corcoran, Mosher & Rogat (2009) state that a learning path/trajectory differs from a curriculum in that, the latter is not based on an analysis of research results regarding how students learn a concrete idea. Additionally, Corcoran et al. state that curricula are not validated by empirical research results.

Moreover, a learning path can help teachers by providing them with a conceptual structure that allows them to adapt their instruction to their students' needs (Corcoran et al., 2009, p.23).

When students construct a learning path, it is a meaningful way for them to construct meanings, since it avoids the traditional pedagogy of memorization without understanding and proving theorems in geometry that are already known, exactly as Ausubel (1962) reports below, distinguishing between the two types of learning, the *rote and the meaningful*:

> "The *rote learning* of lists of nonsense syllables and arbitrarily paired adjectives is representative of few defensible learning tasks in modern classrooms. […] *Meaningful learning* of verbally presented materials constitutes the principal means of augmenting the learner's store of knowledge, both within and outside the classroom. Hence, no research program purporting to advance this objective can avoid coming to grips with the fundamental variables involved in meaningful learning" (Ausubel, 1962, p. 215)

Can we view the learning trajectories or a set of learning trajectories as an evolution of the meaning of a curriculum? Certainly, the learning trajectory process allows the same students to determine what the next sequential instructional activity will be, whether it is to overcome an obstacle or to form the next cognitive step in their understanding of a concrete concept. Brousseau (1986) argues that

> "Students start their learning process in an environment that is unbalanced and full of difficulties and obstacles just like human society. The new knowledge comes from the skill to adapt to the new circumstances and stimuli and a new reaction to the environment is the proof that a learning process has taken place." (cited in Manno, 2005, p.23)

Can a teacher working in cooperation with his/her students become the designer of a hypothetical learning trajectory (HLT) in mathematics or a sequence of HLTs? While this is not the aim of the concrete study, I nonetheless agree with Gravemeijer, Bowers & Stefan (2003)

> "To start developing a sequence of instructional activities, the designer first engages in a *thought experiment* to imagine a route the class might invent (Gravemeijer, 1999). Here, knowledge of the history of mathematics as well as prior research concerning students' invented mathematical strategies can be used to develop what Simon (1995) has called a Hypothetical Learning Trajectory, or a possible taken-as-shared learning route for the classroom community." (p. 52).

Corcoran, Mosher & Rogat (2009) also stress that there are common characteristics shared by curricula based on research and the learning trajectories. The most important is that both maintain a close connection between the tasks and the students' mathematical thinking. Corcoran, Mosher & Rogat (2009) defined a *learning progression* in science based on an NRC (2007) report: "[…] empirically grounded and testable hypotheses about how students' understanding of, and ability to use, core scientific concepts and explanations and related scientific practices grow and become more sophisticated over time, with appropriate instruction." (p. 8)

According to Freudenthal (1973) mathematics education should be a process of *guided reinvention*. *Guided reinvention* for Freudenthal means a faithful reproduction of a scientific activity by the student, and is thus an elaboration on the Socratic Method.

> "Freudenthal saw the reinvention approach as an elaboration of the Socratic method and to illustrate the Socratic method, he spoke of 'thought experiments', i.e.the thought experiment of teachers or textbook authors who imagine they are teaching students while interacting with the man dealing with their probable reactions. One part of the thought-experiment, therefore, lies in anticipating student reactions. The other part consists in the design of a course of action that fits anticipated student reactions. More precisely, the idea is that teaching matter is re-invented by students in such interaction" (Gravemeijer & Terwel, 2000, p. 786).

The method of guided reinvention is linked epistemologically with the Socratic Method ("maieftiki" in Greek) by which teachers ask questions designed to elicit the correct answer and reasoning processes. The questioning process thus helps students determine and extend their underlying knowledge. Guided reinvention differs qualitatively form the Socratic method because the aim of the method is the students to completely participate, undertaking active role by self-acting for the construction of meanings. "Though the student's own activity is a fiction in the Socratic Method, the student should be left with the feeling that it [i.e. understanding and insight] arose during the teaching process; that it was born during the lesson, with the teacher only acting as midwife". (Freudenthal, 1991, p. 100-101, cited in Gravemeijer & Terwel, 2000, p. 787)

Gravemeijer (2004) also supports that

> "The teachers can influence their students' inventing activity only in a more indirect manner. To do so, teachers [adding here: or the designers of a learning trajectory] will have to put themselves in the shoes of the students. This asks for a shift ***from an observer's point of view to an actor's point of view*** (Cobb, Yackel and Wood, 1992), where the actor is the student, and the observer the teacher. The challenge for the

teacher—and also for us—**is to try to see the world through the eyes of the student**. How much these worlds may differ may be illustrated by other pictures of Watson's strip about Calvin and Hobbes." (p.8) (Figure I).

Many researchers argue that working in a dynamic geometry environment allows students to reinvent their personal knowledge by interacting with the other members of the group or with the teacher (or the participating researcher). For example, Furringhetti & Paola (2003) support that "in this case, the reinvention is guided, […] by the use of the [dynamic geometry] environment".

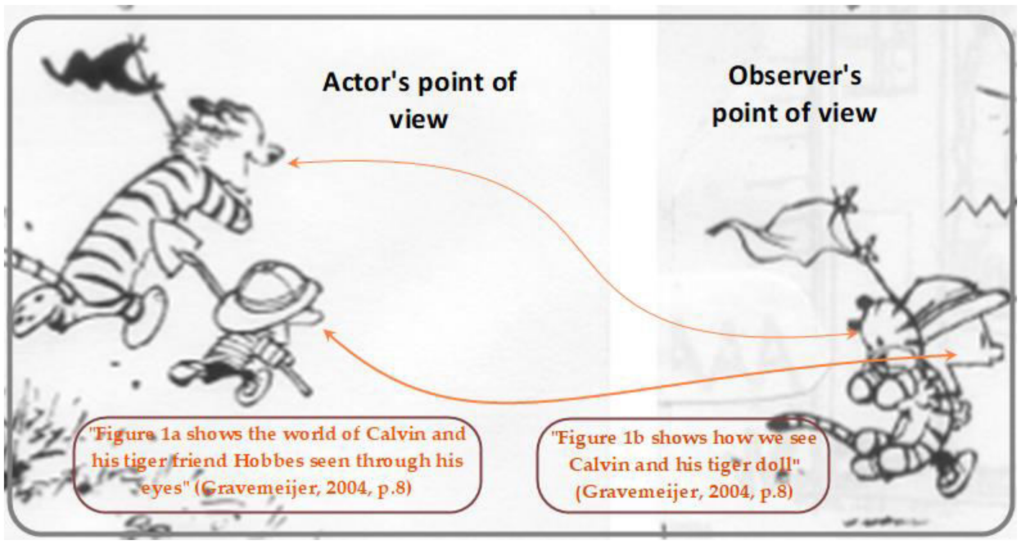


Figure I. "Actor's point of view" vs "observer's point of view"
(Copied from Bill Watson (1996), It's a magical world, Kansas City: Andrew and McMeal, page 58 and 82, cited in Gravemeijer, 2004, p.8) (adapted)

Looked at from this point of view, learning geometry is a human activity and learning becomes a process of *dynamic reinvention* (Patsiomitou, 2012a, b, 2014), following on from the *guided reinvention* posited by Freudenthal (1973).

Papert (1984) in his paper "*Microworlds: Transforming Education*" describes the experience of a little girl who discovered number "zero" as she played with a microworld. This was a crucial point for her understanding, as she understood that the command "S0" made the microworld stop moving. As Papert argues (1984, p. 81)

> "I think she was excited because she had discovered zero. They tell us in school that the Greek mathematicians, Pythagoras and Euclid and others, these incredibly inventive people, didn't know about zero. […]The fact that not every child discovers zero this way reflects an essential property of the learning process. No two people follow the same path of learnings, discoveries, and revelations. *You learn in the deepest way when something happens that makes you fall in love with a particular piece of knowledge*."

These words of Papert made me think of my own process with my students over the years teaching in class. They loved particular pieces of knowledge, presented in static or dynamic geometry software (DGS environment), with its *active-"alive" representations* (e.g., Patsiomitou, 2005a, 2012a, 2018b, 2019a, b) that made different students discover concepts in several different ways, at different times over the years. I also fell in love with the particular incidents, which have played an important role in my thinking process since then. The role the *active-"alive"* representations play in the learning trajectory which, though it may take several different routes to reach it, has the same learning goal, made me think of a way to define what a dynamic active learning trajectory is, based on the previous definitions of Simon (1995) and Clement & Sarama (2004, 2014): *Dynamic Active Leaning trajectories* (Patsiomitou, 2018a, p. 244) *are sequential instructional tasks and activities engaged in [with] a learning goal and designed with dynamic active linking representations to engender mental linking representations which help students develop their thinking in the specific math domain.*

II. A trajectory for the teaching and learning of the "Didactics of Mathematics" [using ICT]

Biehler, Scholz, Sträßer, and Winkelmann (1994), in the Preface of "*Didactics of Mathematics as a Scientific Discipline",* argue that *"Didactics of mathematics* is an applied area of activity: As in engineering, (applied) psychology, and medicine, the boundary between scientific work and (constructive) practice is – to say the least – "fuzzy". Didactics of mathematics shares a certain type of (social) problem with the above-mentioned disciplines, namely mathematics education; and it uses a multiplicity of methods" (p. 3).

Novák (2003) states that "Didactics of math is usually considered a *special didactics* (a subject, possibly branch didactics), in a sense of educational theory in math. It is a science with its own structure, logic and the way of thinking. We can distinguish four dimensions in it: content, pedagogical, psychological, and constructive."(cited in Blažková, 2013, p.5)

Chevallard (2005) in his study "*Steps toward a new epistemology in mathematics education*" determines what *didactics* is. As he argues, "It derives from the Greek *didaktikos*, which means (or meant) "skilful at teaching". […] The idea behind didactics is that someone attempts to do something so that someone – generally, someone else – learns something. The adjective "didactic" refers to a cultural posture existing from time immemorial" (p.1).

Chevallard's (2005) definition on didactics is included in the following paragraph:

> "Didactics should, in my view, be defined as *the science of the diffusion of knowledge* in any social group, such as a class of pupils, society at large, etc. This "definition" requires some comments. In the first place, let me emphasise that its referring to a *science* is no writing automatism. It points to the fact that research – in mathematics education, for example – is not enough. Science is both *a process* of gaining knowledge, and the *organised body of knowledge* gained by this process. (It happens that, in didactics, the knowledge gained and organised is about… the diffusion of knowledge!) Doing didactics is therefore not only just "doing research", and, consequently, producing pieces of knowledge; it is also, inseparably, organising these pieces into a body of knowledge – *didactics* –, with an experimental (or clinical) basis *and* a theoretical superstructure endowed with a paradoxical capacity, that of strengthening its empirical foundation" (p.2) (italics by the author).

Tchoshanov (2013) defines Didactics "*as a science, engineering, and art of teaching and learning*" (p.18). Tchoshanov (2013) agrees with Chevallard and other scholars that *Didactics* is not only the science, but also the art of teaching and learning. Tchoshanov additionally considers didactics to be "an engineering of teaching and learning", namely "the analysis, design and construction of teaching products for learning" (p.17-18) (Figure II). Tchoshanov (2013) also adopts D'Angelo's (2007) view of didactics which defines it as "*e-Didactics*", an ICT-integrated didactics (p. 21).

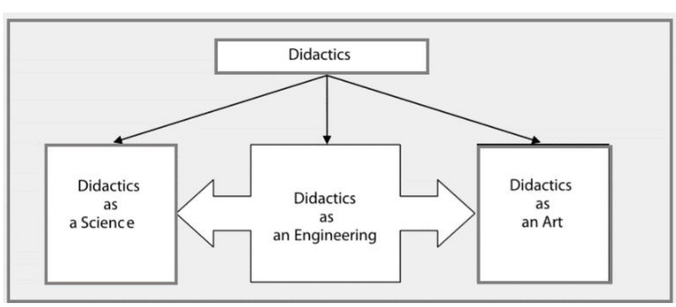


Figure II. *Didactics* as a science, engineering, and art of teaching and learning (Tchoshanov, 2013, p.18) (adapted)

For the current work, I have adopted a blending, an amalgam of the aforementioned definitions.

For me '*Didactics of Mathematics'* [using ICT] is the science and art of teaching and learning mathematics, designing and implementing teaching and instructional products for the learning of mathematics in static or computing environments, incorporating the content of the subject of mathematics, mathematics pedagogy, the history of mathematics, and psychological theories of learning, teaching and human-computer interactions.

In the current monograph, my aim is to organize these instructional products into a body of knowledge, a trajectory for the teaching and learning of the 'Didactics of Mathematics" [using ICT]. Trying to synthesize everything I have read or heard I found myself "entangled" in knowledge items that can interconnect, or contradict one another. A small part of this body of knowledge will be presented over the next five chapters. I have tried to present the content incorporating many illustrating figures-- the *radiance of thoughts and wisdom of the cited scholars*-- which is itself “piece of the art for Didactics of Mathematics”. It is thanks to their efforts and ideas that the Didactics of Mathematics [using ICT] is a scientific discipline as important as Mathematics, Pedagogy, Engineering, Medicine or Psychology.

Chapter I.

1.1. What are Mathematical Objects?

Dörfler (2002) in his study "Formation of Mathematical objects as decision making" asserts that the question "what is a mathematical object" can be answered from different viewpoints:

(a) Mathematical objects are "exemplified above an apriori existence outside of time and space and independent of human thinking", an answer offered by Plato (360 B.C.);
(b) Mathematical objects "are or arise from structures, patterns and regularities in the physical world (Kitcher, 1984)";
(c) Mathematical objects "are, or reflect, structures, patterns and regularities in and of human actions and mental operations (like counting, measuring, comparing, moving), according to genetic epistemology of Piaget" (p.340).

According to Dörfler "*All these philosophical or epistemological positions have in common that they in one way or the other take a referential view on the mathematical objects as they occur in mathematical texts and discourse in general" (p.340).*

When a student endeavors to interpret the word "mathematical object", s/he could consider it through different lenses: as something material we can perceive through our sensory system, as something that we can act on, or/and as something we can think about. Mathematical objects are a particular kind of object (e.g., functions, operations on functions, spaces of all kinds-for example Banach spaces, geometrical figures).

Numerous researchers have investigated the nature of mathematical objects and tried to define them (e.g., Davis, 1983, 1984; Piaget, 1985; Gray & Tall, 1991, 1994; Dubinsky, 1991a, b; Dubinsky & McDonald, 2001; Sfard, 1987, 1989, 1991, 1992; Tall et al., 2000). As we know, since Plato, a mathematical object has been considered as something abstract. Portnoy et al. (2006) report Plato's (360 B.C.) perspective on the figural constructions of geometers as a connection between the figural objects (perceived objects) and the corresponding conceptual objects (conceived objects):

> "they are not thinking about these figures but of those things which the figures represent; thus it is the square in itself and the diameter in itself which are the matter of their arguments, not that which they draw; similarly, when they model or draw objects, which may themselves have images in shadows or in water, they use them in turn as images, endeavoring to see those absolute objects which cannot be seen otherwise than by thought. (Plato's Republic, 360 B.C., p. 391, reported in Portnoy et al., 2006, p. 199).

A large amount of researchers pointed out that a mathematical object can be represented using different models and representations (e.g., Chevallard, 1989; Janvier, 1987a, b, c) or semiotic systems (e.g., Duval, 1993, 1995a, b, 1999, 2000). As Duval (1993) argues "[…] on the one hand, the learning of mathematical objects cannot be other than a conceptual learning and, on the other hand, it is only by means of semiotic representations that an activity on mathematical objects becomes possible" (p. 38). Moreover, according to Duval (1999) "the only way of gaining access to mathematical objects is using signs, words or symbols, expressions or drawings"(p.60).

On the other hand, what is a *mathematical concept*? In the words of Peirce (1894): "We think only in signs. These mental signs are of mixed nature; the symbol-parts of them are called concepts […]" (Peirce, 1894, reported in Stewart, 2008, p. 12). In order to develop an understanding of a concept, the students have to create a transitional bridge between the 'external' and the 'internal or mental' representation of this concept (e.g, Kaput, 1999; Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001; Tchoshanov, 2013). Tchoshanov (2013) also argues that "the development of students' representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images" (p. 74).

Moreover, students' visualization of an object may differ from their perception of it, while the important thing is to understand which mathematical concept or relationship is being represented. A computer microworld can encourage students to interact with visually represented mathematical concepts and ideas, promotes dynamic imagery and can help them to translate between mathematical representations or interpret information received from a real world environment (e.g., Battista, and Borrow, 1997). Kaput (1991) reporting Vergnaud (1987) explains and depicts the relation between mental representations (i.e. the *signified*) and material representations or physically instantiated symbols (i.e. the *signifier*), for example pictorial, diagrammatic notations, mathematical symbols, diagrams, graphic representations (**Figure 1.1**). According to Kaput (1991)

"When using such material notations, we build and/or elaborate our mental structures in cyclical processes that go in opposite directions". (p. 57)[…] The directionality of the reference depends on the cognitive operations involved, which in turn depend on the context, and hence is not fixed".(p.59).

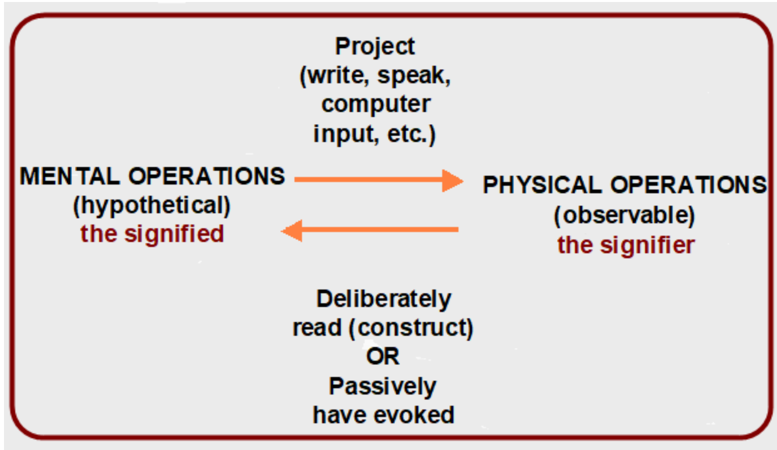


Figure 1.1. Kaput's (1991) relation between mental representations and physically instantiated representations (p. 57) (an adaptation for the current study)

"A science that studies the life of signs within society is conceivable. It would be part of social psychology and consequently of general psychology. I shall call it *semiology* (from Greek *semeion* "sign"). Semiology would show what constitutes signs, what laws govern them." (Ferdinand de Saussure (1857-1913), cited in Danesi, 2004).

Peirce (1933) conceptualized a semiotic triad consisting of three components: *sign, object, and interpretant.* Kaput (1991) clarifies Peirce's (1933) semiotic behavior as involving an interaction among "*sign, object and interpretant*", giving an example: "a numeral A-the *sign*, that refers to the numerosity of a set of objects B-*the object* and the mind in which the integration takes place-*the interpretant* […]" (p.59). Similarly, Duval (2000) supports that "interpretant is emphasized in such a way [in the triadic conceptualisation of Peirce] that representations are mainly mental phenomena and individual beliefs" (p.58).

In other words the *sign/'representamen'* represents somebody or something in a given way or capacity, the 'representamen' conveys an equivalent sign in the mind of someone else. This equivalent sign we call the *'interpretant'* of the initial sign and the 'interpretant' represents the *'object'* or 'idea' of the first 'sign', which we call a *'referent'*. A representamen is the 'vehicle' for the sign, the interpretant is the 'sense' and the referent is the 'object'. A representamen thus corresponds to Saussure's 'signifier'—it is a perceptible object which functions as a sign. A 'referent' is an object the representamen stands for. The image the referent creates in the mind of another is the interpretant. According to Adda (1984)

"First of all, being abstract, the objects of mathematics that are treated, the properties and the relations that are studied can never be seen (in contrast, for example, with the objects studied by the physical and natural sciences) and so the distance between the *signified* and the *signifiers* plays here a role that is more crucial than for any other type of discourse. […] By studying the «misunderstandings» brought about by this confusion between signifier and signified we have observed the responsibility they bear not only in a very great number of errors but also in the impossibility of acquiring the concepts themselves" (p.58).

Saenz-Ludlow & Kadunz (2016) elaborated on Peirce's semiotics. In their study "*Constructing Knowledge seen as a semiotic activity*" they discuss issues of signs, sign use, and communication. As Saenz-Ludlow & Kadunz argue

- "[…] semiotics elucidates the way knowledge and experience of mathematics students can co-construct each other;
- […] shows how students' construction of mathematical knowledge is linked to successful communication mediated by visible signs with their rule-like transformations" (p. 1).

Saenz-Ludlow & Kadunz (2016) used the vertices of two joined triangles to position the three components *sign, object, and interpretant* (**Figure 1.2**). According to Saenz-Ludlow & Kadunz (2016)

"In the counter-clockwise direction (represented by the interior triangle), the sign vehicle materializes certain aspects of the real Object. […] The sign-vehicle evokes an interpretant in the mind of the Person who perceives it and who is willing to make some kind of sense. This interpretant gives rise to an object, in the mind of that Person,[…] Peirce calls this object a dynamic object. This dynamic object is continually modified in the mind of the interpreting Person […] Put it differently, the sequence of dynamic objects is the result of the Person's ongoing process of conceptualization" (p. 9).

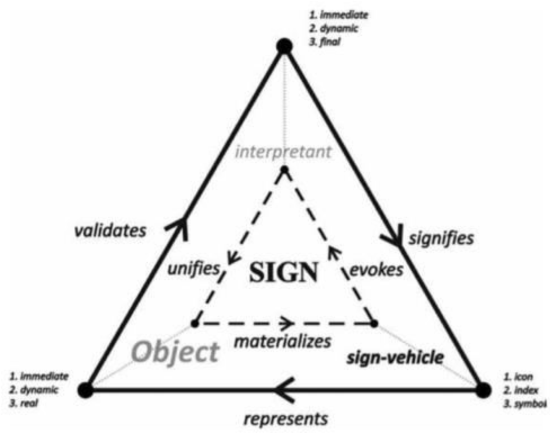


Figure 1.2. The sign-vehicle mediates between the object and the interpretant (Sáenz-Ludlow & Kadunz, 2016, p. 9)

Signs can be classified into three categories: *icon, index and symbol* (Yeh & Nason, 2004, p. 4):

- "A *"Sign"* can only represent certain aspects of the object and in addition, it has aspects that are not relevant to the object (Yeh & Nason, 2004, p.4; Cunningham, 1992).
- An *"Icon"* stands for an object by resembling or imitating it.

  The key characteristic of an icon is similarity to its object. Its main function is to represent relations. Icons represent things by imitation, […] (Peirce, EP II, 17; NEM III, 887, cited in Bakker & Hoffmann, 2005, p.338).
- An *"Index"* refers to the sign which is the effect produced by the object.

  The main function of indices is to direct someone's attention to something, exactly as in everyday language when we use the indices 'here', 'there', 'now', 'tomorrow', 'next', or the letters we use in geometry or the variables in algebra […](Peirce, 1.369; NEM III, p. 887, cited in Bakker & Hoffmann, 2005, p.339).
- *A symbol* refers to objects by virtue of a law, rule or convention. In this case, language could be a prototype of symbols. (Yeh & Nason, 2004, p.5).

  A Symbol is a sign which refers to the Object that it denotes by virtue of a law, usually an association of general ideas, which operates to cause the symbol to be interpreted as referring to that Object. (Peirce, EP II, 292, cited in Bakker & Hoffmann, 2005, p.339)

Kadunz and Straesser (2004) define sign "*as an entity, which stands for something else, which points to something else*" (p. 242). They add that "*it is not the sign, which points to something, but the person looking onto the sign who links it to the object*".

Johnson-Laird (2004) in his study "*The history of mental models*" presents an alternative view of signs: "Peirce distinguished three properties of signs […] First they can be iconic and represent entities in virtue of structural similarity to them. Visual images, for example are iconic. Second, they can be indexical and represent entities in virtue of a direct physical connection. The act of pointing to an object, for example, is indexical. Third the can be symbolic and represent entities in virtue of a conventional rule or habit. A verbal description, for example, is symbolic. The properties can co-occur: a photograph with verbal labels for its parts is iconic, indexical, and symbolic" (p. 181)

I shall try to explain the meanings of *symbol and sign* with simple examples. If we ask the question "*What is a quadrilateral?*", while pointing at a figure of a quadrilateral on the board, the object quadrilateral becomes the signifying form for the word "quadrilateral". This is to say that the word acquires a meaning when we point to a correspondent object. In other cases the word can be used to represent the object, in order to communicate with other persons. If we have for example written an article on quadrilaterals in which we try to explain the mathematical meaning, the article is a sign which represents the object of our knowledge, which is something we want to share with other people. A quadrilateral constructed on a computer screen or on the blackboard can be characterized as an *image*, a *diagram*, a *metaphor* or a *figure*. According to Peirce*, 'images', 'diagrams', and 'metaphors'* are three subcategories of Icons. Diagrams in mathematics are "*Icons of a set of rationally related objects*" in the words of Peirce.

> "[…] a Diagram is an Icon of a set of rationally related objects. By rationally related, I mean that there is *between them*, not merely one of those relations which we know by experience, but know not how to comprehend, but one of those relations which anybody who reasons at all must have an inward acquaintance with. This is not a sufficient definition, but just now I will go no further, except that I will say that the Diagram not only represents the related correlates, but also and much more definitely represents the relations between them, as so many objects of the Icon." (Peirce, 1906, 'PAP [Prolegomena for an Apology to Pragmatism]', NEM 4:316, c. 1906, cited in Kadunz and Straesser, 2004, p. 245).

Building on the aforementioned researchers' viewpoint, one might wonder: Are the students able to grasp logical operations on abstract mathematical objects? What does it mean to obtain access to an abstract mathematical object or a mathematical entity? What about their conceptions of geometrical objects?

1.2. Geometrical Objects: Drawings, Figures, Constructions

A number of researchers (for example Dina van Hiele, in Fuys et al, 1984; Parzysz, 1988; Fischbein, 1993; Bartolini Bussi, & Mariotti1998; Mariotti, 1995, 1997; Pratt & Ainley, 1997; Jones, 1998; Mesquita, 1998; Hollebrands, 2007; Battista, 2007; Patsiomitou, 2009a, b, 2011, 2012a, b) report distinguish among *figures, constructions, drawings and diagrams* when they report geometrical representations.

Dina van Hiele made clear in her writings the distinction between the 'drawing' and the 'construction' of a shape. She distinguished the notion of *construction* from the notion of *drawing* in order to express the difference between the images that a student constructs (in a paper/pencil environment) when s/he tries to externalize his/her mental representation, using geometry rules (or not in correspondence). According to Dina van Hiele "the teacher [in order] to reach his goal [has] to refine [to his/her students] that there is a clear distinction between *the drawing of figures* and the *constructing of figures*" (Fuys et al., 1984, p. 36).

Laborde (1993 quoted in Hollebrands, 2007) describes the *drawing* as referring to the material entity, and the *figure* as the set of discursive representations and diagrams which links the drawing to the *abstract mathematical meaning* (Hollebrands, 2007, p.167). Pratt & Ainley (1997, p.296) also argue that "a *drawing* incorporates many relations which are to be disregarded when considering the corresponding figure […]. Furthermore, a drawing is fixed as a single case, whereas the *figure* is often intended to represent an infinite set of cases." Pratt & Ainley use the term "construction […] as a way of incorporating both the drawing and the figure" (p.297).

Mesquita (1998) considers that the representation of a figure or a situation in geometry can suggest two different possibilities:

- The possibility to conceive "its '*finiteness*', in the sense of finite and diversified forms (Gestalten) in its spatio-temporality";
- The possibility to conceive "its '*ideal objectiveness*' detached from the material constraints linked to external representation" (p. 185-186).

This consideration is very close to the notion of *figural concepts* formulated by Fischbein (1993) in his study "*The theory of figural concepts*". Fischbein argues that:

> "The objects of investigation and manipulation in geometrical reasoning are then mental entities, called by us *figural concepts*, which reflect *spatial properties* (shape, position, magnitude), and at the same time, possess *conceptual qualities* -like ideality, abstractness, generality, perfection" (p. 143).

Building on Fischbein's *figural concepts*, Dvora and Dreyfus (2004) declare that

> "the conceptual nature of the geometrical figures includes characteristics such as *completeness, abstraction* and *generalization* while the figural nature includes characteristics such as *colour, size and shape*. The conceptual and figural characteristics used when proving depend both, on the conceptual system that includes abstract ideas and concepts and on the figural system that includes mental representations and images."(Dvora &Dreyfus, 2004, p. 311).

Parzysz (1991) in his study "Representation of space and students' conceptions at high school level" mentions the main purposes which can be fulfilled by drawings:

- "*they illustrate definitions or theorems* […]. This is due to the nature of geometry […], whose objects are obviously linked with material realizations (drawings, or models which can be drawn).
- *they sum up a complex set of information*: the "figure", drawn in order to solve a geometrical problem, allows a simultaneous glance at most of the data present in the wording.
- *they help in conjecture*: the "figure" also makes it possible to suggest potential relations between its elements, which will have to be demonstrated afterwards […]" (p. 576).

A crucial issue concerning geometrical meanings relates to the nature of the geometric reasoning students employ to solve construction problems. During the problem-solving process, students develop different kinds of reasoning including *inductive, abductive, plausible and transformational reasoning* (e.g, Harel & Sowder, 1998; Peirce, 1992; Simon, 1996). Mariotti (1997) as far as geometrical reasoning is concerned, distinguishes between *geometrical figures* as mental objects and *visual images*. *Geometrical reasoning* deals with a

> "mixture of two independent, defined entities that is abstract ideas (concepts), on the one hand, and sensory representations reflecting some concrete operations, on the other" (Fischbein, 1993, p. 140).

The perception of a visual image of a geometric object does not coincide with the mental object a student has in mind. For example, the orientation of the geometrical object could play an important role to students understanding of the geometrical figure. I use an example every year with my students, in the light of the following episode that occurred one year in class. I was very surprised when, as I was using a material classroom triangle tool (a right and isosceles triangle-tool) to investigate their understanding of "triangles' classification", a student answered as follows (**Figure 1.3**):

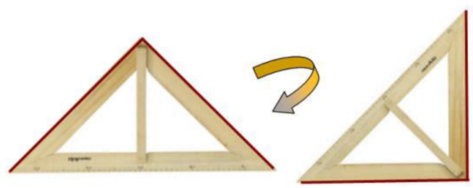


Figure 1.3. A right and isosceles material-triangle

> Researcher: *What kind of triangle is this?"*
> Student: *It is an isosceles triangle.*
> Researcher: (Turning the triangle through 90 degrees) *Now, what kind of triangle is this?*
> Student: *It is a right-angle triangle*
> Researcher: *So, what kind of triangle is it?*
> Student: ....*It depends on the way you hold it!*

It was the same object, but the orientation of the right angle played an important role in my student's answer. The way the student answered was also affected by his mental image of the right triangle, which Mesquita calls "*prototypical figure*" (1998, p. 189) which is to say an internal representation recalling a *prototype image* (e.g., Hershkovitz, 1990) that s/he has shaped from a textbook or other authority.

Mesquita (1998) states that the term "figure"' can be considered "as a synonym for external and iconical representation of a concept or a situation in geometry. A concept in the words of Fischbein (1993) "*expresses an idea, a general, ideal representation of a class of objects, based on their common feature*. *In contrast, an image (we refer here to mental images) is a sensorial representation of an object or phenomenon*" (p. 139)

Parzysz (1988) in a similar way defined *a drawing* as a material representation of a geometrical object and *a figure* as the "text defining it [the geometrical object]" (p. 80).

Fischbein (1993, p. 139) explains how one can prove a known geometrical proposition *"using descriptions of apparently practical operations*": "consider the isosceles triangle ABC with AB = AC. We want to prove that <B = <C" (**Figure 1.4**).

> "In this proof one has used a certain amount of knowledge expressed conceptually: the two sides AB and AC have been declared to be equal. One has used the concepts of point, side, angle and triangle. One has mentioned verbally the process of reversion. But, at the same time, one has used figural information and figurally represented operations - mainly the idea of detaching the triangle ABC from itself, reversing it and superposing it upon the original one" (Fischbein, 1993, p. 140) [...] What we assume is that, in the special case of geometrical reasoning, one has to do with *a third type of mental objects* which simultaneously possess both conceptual and figural properties. (Fischbein, 1993, p.144).

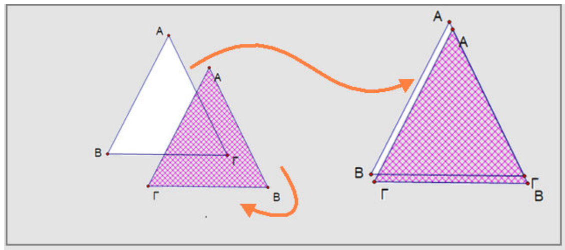


Figure 1.4. Reversing and superposing the triangle ABC

Figure 1.4 illustrates the reversing and superposing of the triangle in Fischbein's example mentioned above (which I created to make the proof obvious).
In other words it is crucial for the students' cognitive development to improve their ability to transform the visual image or drawing they perceive, into a construction with concrete properties. The investigation of problems in the dynamic geometry environment provides the feedback for the students to acquire a theoretical background, necessary for the conceptual development in Euclidean geometry.

1.3. Diagrams, Diagrammatic Representations and Diagrammatic Reasoning

Diagrams are an important medium (or 'vehicle', to use Peirce's terminology) in mathematics.
They are visual representations that can transfer information from the problem into a static or dynamic environment. Mesquita (1998) states that the term "diagram" sometimes is used in the [same] sense" we use the term figure (p 183). Bakker & Hogffmann (2005) argue that geometrical figures are diagrams as they represent relations among the lines and the points-vertices, indicated by letters. According to Bakker & Hogffmann (2005):

> "Peirce defines a 'diagram' as a sign "which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used." (Peirce, 4..418, 1903). Thus, a diagram is a complex sign which includes *icons, indices, and symbols* (as indicated by the hint at conventions). Most important, however, is its iconic character, which results from the fact that a diagram, first of all, is supposed to represent relations. Thus, geometrical figures such as triangles are diagrams because they represent particular relations of lines and vertices that are indicated by letters. Logical propositions are diagrams, because they represent certain relations of other propositions, symbols and indices (e.g. the modus ponens)" (Bakker & Hoffmann, 2005, p. 339).

Furthermore, scholars use the terms "image" or "metaphor" to refer to the material diagram we need to denote relations among objects (or to turn a verbal or symbolic expression into a different representation (mental or iconic). Kadunz and Straesser (2004) in their study "Image-Metaphor-Diagram: Visualization in Learning Mathematics" define

- "*images* as potential representations (i.e.: a not necessarily material means to speak about something), which can - by means of analogy - present a multitude of relations. […] images - as analogous representations - offer the heuristical part of learning […] an image relates to something, it 'denotes' something.
- *metaphor* as a pattern, which transports the meaning of a word into a meaning, which is valid only by means of a mental comparison (Du Marsais, 1730, cited in Kadunz and Straesser, 2004, p. 243)

Diezmann (2005, p.281) considers that diagrams have three key cognitive advantages in problem solving:

- "They facilitate the conceptualisation of the problem structure, which is a critical step towards a successful solution (van Essen & Hamaker, 1990)".
- "They are an inference-making knowledge representation system (Lindsay, 1995) that has the capacity for knowledge generation (Karmiloff-Smith, 1990)".
- "They support visual reasoning, which is complementary to, but differs from, linguistic reasoning (Barwise & Etchemendy, 1991)" (p. 281).

Reasoning through a diagram is called *diagrammatic reasoning,* namely diagrammatic reasoning is reasoning through a diagram. Students often fail to generate accurate diagrams in mathematics as they do not have experience or competence in what Peirce (1903), Bakker & Hoffman (2005) and others call "*diagrammatic reasoning*". For Peirce, *diagrammatic reasoning* involves three steps (Bakker & Hoffman, 2005):

- "*The first step is to construct a diagram* […] Such a construction of diagrams is motivated by the need to represent the relations that students consider significant in a problem. This first step may be called 'diagrammatization'".
- "*The second step is to experiment with the diagram* (or diagrams). Any experimenting with a diagram is being executed within a representational system and is a rule or habit-driven activity. […]"
- "*The third step is* to observe the results of experimenting and *reflect on them* […]" (pp. 340-341).

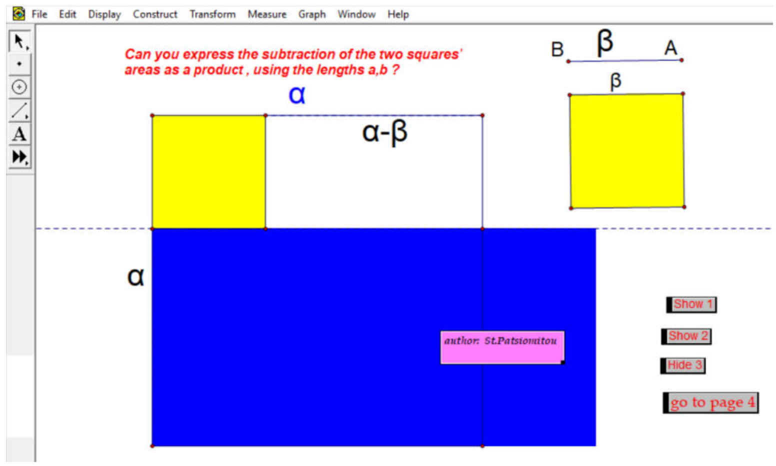


Figure 1.5. A metaphor for the Proposition 5 (the algebraic identity a^2-b^2), from Euclid's "Elements", BOOK II, created by the author in a DGS (Patsiomitou, 2008d, p. 199)

And although students have knowledge, they cannot use it effectively to represent a diagram that stimulates their ability to make sense of mathematics. This is to say that diagrams as both representations encourage students to reflect both on the structure of the problem they have been presented with, and on their own pre-existing mathematical knowledge—meaning that the diagrams the students produce can serve as a window through which to view their mathematical strengths and weaknesses. However, while diagrams can help students to conceptualise a problem, they cannot make up for a lack of fundamental mathematical knowledge. Dvora & Dreyfus (2004) similarly support that diagrams in geometry can *become obstacles* that can be divided in three types:

- *Particularity of Diagrams*: [...] This obstacle causes students to be trapped by the one case concreteness of an image or diagram which may contain irrelevant details or may even introduce false data
- *Prototypical Diagrams as Models*: [...] a prototypical image may induce inflexible thinking thus preventing the recognition of a concept in a non-standard diagram.
- *Inability to "See" a Diagram in Different Ways*: [...] It is only at level 2 [van Hiele level –analysis] that the student can focus on parts of a diagram and analyze properties of figures. (p.311-312)

Dvora & Dreyfus, suggest that:

> "In order to prevent the development of misconceptions regarding this phenomenon, teachers should be equipped with appropriate tools for working with their students [...]" (p.318)

Figures 1.5 and 1.6 are snapshots of a diagram that I created in Geometer's Sketchpad (Jackiw, 1991) to represent in an interactive way the Proposition 5, in Euclid's "Elements". The conceptualization of its construction is reported in details in the study "*Do geometrical constructions affect students algebraic thinking"* (Patsiomitou, 2008c) and in extended version in the study "*The impact of Structural Algebraic Units on students' algebraic thinking in a DGS environment*" (Patsiomitou, 2009a).

> "Netz's (1999) study of the practices of lettering diagrams in Greek geometry allows the observation that Greek geometers would produce their diagrams at the same time that they would conceive their proofs. In other words, the diagram would not be drawn at the end to merely illustrate the written proof; nor would the diagram be drawn in its entirety before the production of the argument. Rather, the Greeks would use the argument to complexify a diagram by adding new constructions, or at least complexify the reading of a diagram by adding new signs to focus attention on previously ignored features of a diagram" (Herbst, 2004, p. 134).

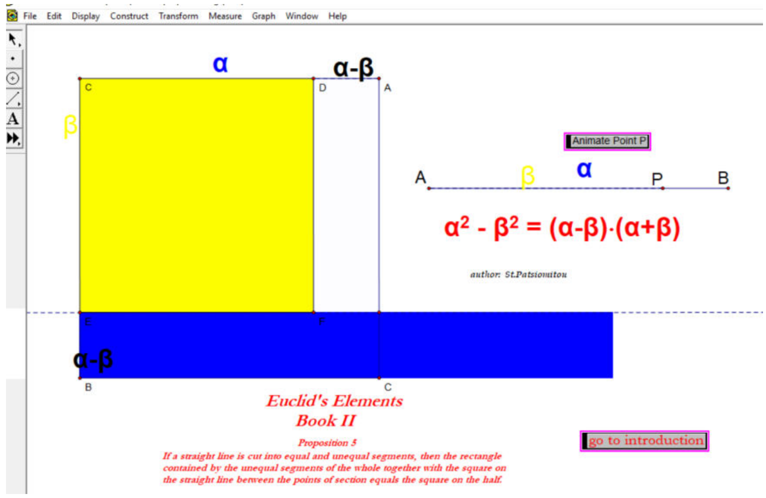


Figure 1.6. Interaction with the dynamic diagram (Patsiomitou, 2008c, p. 199)

Scholars have debated the effectiveness of diagrams in reasoning. Barwise and Etchemendy (1998) conclude that diagrams play an important role in reasoning as a diagram can assist students visualize the steps of a proof.

> [...] Diagrams, like sentences, carry information: they carve up the same space of possibilities, though perhaps in very different ways.[...] Maps. charts. diagrams, and other nonsentential forms of representation can be and often are, of equal importance to sentences. (p. 109, cited in Sinclair, 2001, p. 27).

Herbst (2004) has proposed four modes of interaction with diagrams: *empirical, representational, descriptive, and generative;* these are also reported and clarified in the study of Gonzalez & Herbst (2009, p. 157).

> "Within the *empirical mode* of interaction, a student uses proximal, physical experiences with diagrams as resources for making statements about geometric objects of discourse. These statements are the symbols that point to the properties of diagrams as referents. Conversely, within the *representational mode,* the agent uses distal physical experiences (oral declarations and questions) to describe how diagrams as symbols represent abstract geometric objects of discourse. These two modes of interaction, the empirical and the representational, portray two opposite views about how students may work with diagrams when solving problems in geometry.[…]".

Michal Yerushalmy (2005) in her study "*Functions of Interactive Visual Representations in Interactive Mathematical Textbooks*" argues that

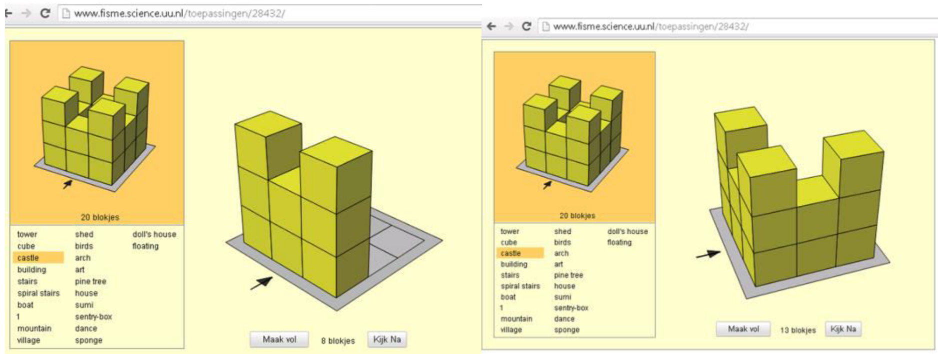
> "While any diagram presents information and point of view (thus implicitly engaging the viewer in meaningful interpretations), the *interactive diagram* [like the *interactive math applets* accessed across the Web], explicitly requires from the viewer to take action, to change and inscribe the diagram within given limitations" (p.228).

Gadanidis (2000) also argues that "well designed interactive applets enable students to engage in investigations of mathematical relationships without having to spend a lot of time learning how t use the tool that creates the various representations of these relationships"(p.1).

"Building with blocks" (**Figures 1.7a, b**) is a math applet provided by the Freudenthal Institute for Science and Mathematics Education (FI). It is available from the Institute's website (Webpage [6]). Students of any age can use this applet to play and develop their spatial reasoning. Parts of the diagram are hidden, but the student can change the orientation of the diagram to better view another option. Students can also add or remove blocks to "build" a construction (e.g., a castle).

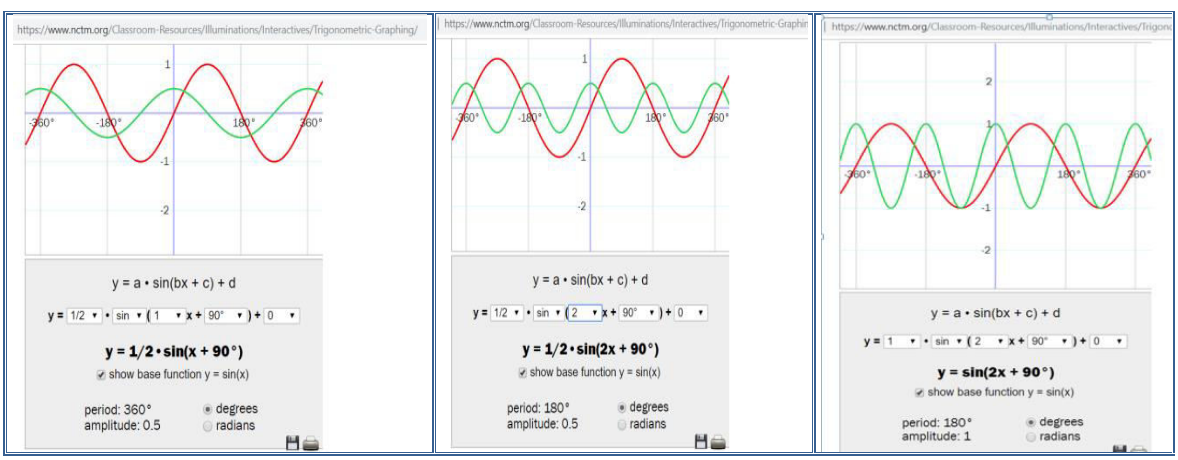
Boon (2006) in his article "*Designing didactical tools and micro-worlds for mathematics education*" has drawn a distinction between three different kinds of applets:

- *"Applets that offers a 'virtual reality'.* These applets are used for representing and simulating real-world objects and processes that form the basis of mathematical reasoning.
- *Applets that facilitate the use of 'models'.* These applets offer interactive models that can be helpful in building and understanding the more abstract mathematical objects and concepts.
- *Applets that offer a mathematical microworld.* In these applets mathematical objects like formulas, equations and graphs can be constructed and transformed" (p.1).



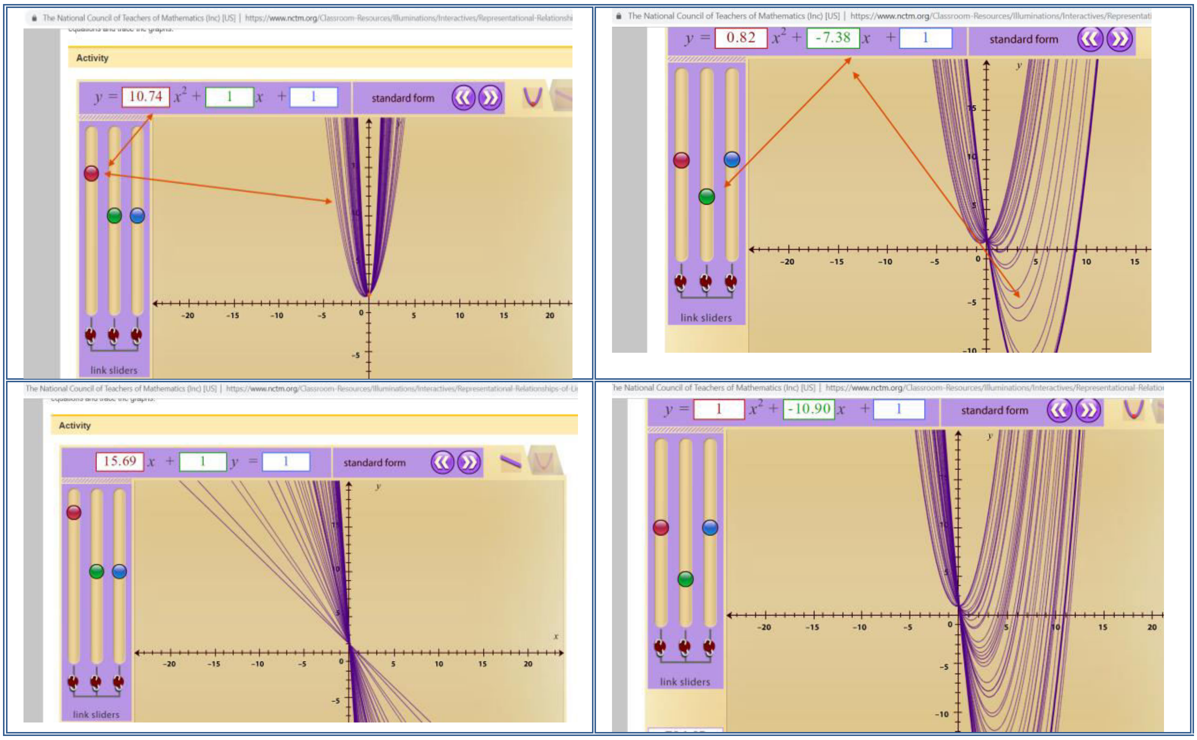
Figures 1.7a, b. "Building with blocks" math applet (Freudenthal Institute for Science and Mathematics Education) (Webpage [6])

According to Boon (ibid.) "*the block building environment* [**Figures 1.7a, b**] *gives the user freedom in making his own constructions, but the environment also enforces a cubic structure that draws the attention more easily to orthogonal co-ordinates as a means to model space*" (p.2).
Students can visualize the effect of modifying the coefficients of the trigonometric functions in the NCTM interactive diagrams (**Figures 1.8 a, b, c, d**). This action on interactive diagrams helps students to acquire a *direct perception* of transformations of the mathematical objects (Patsiomitou, 2006g, in Greek); they also prompt the students to examine the role which the coefficients play in the graphic representation of the trigonometric function.



Figures 1.8 a, b, c. Trigonometric functions and their graphic representations (Webpage [7])

Teachers can use the interactive applets to create an interactive assignment. Students can use them to scaffold their understanding: the applets let them focus their attention on the modified objects and the reasons for the modifications; most importantly, the students can save time as they can experiment at home--the diagrams are web-based and easy to understand.
The NCTM interactive math applet (**Figures 1.9.a, b, c, d**) allows students to modify the graphic representations and trace the changes to the families of the quadratic function which result from the modification of its coefficients. The coefficients are the same colour as the sliders, which help the students to focus and *directly perceive the role played by the coefficients* of the functions in relation to the graphic representation. They can articulate this, thanks to the direct manipulation of the sliders and the effect they have on the interactive diagram.



Figures 1.9 a, b, c. Investigating the families of functions (Webpage [8])

The same is true in the graphs below (**Figures 1.10 a, b, c, d**); the student can construct a graphic representation from the three points that are the roots of the polynomial function. Then, s/he can view the graphic representation of its derivative, as well as the calculation of the area, representing a definite integral.

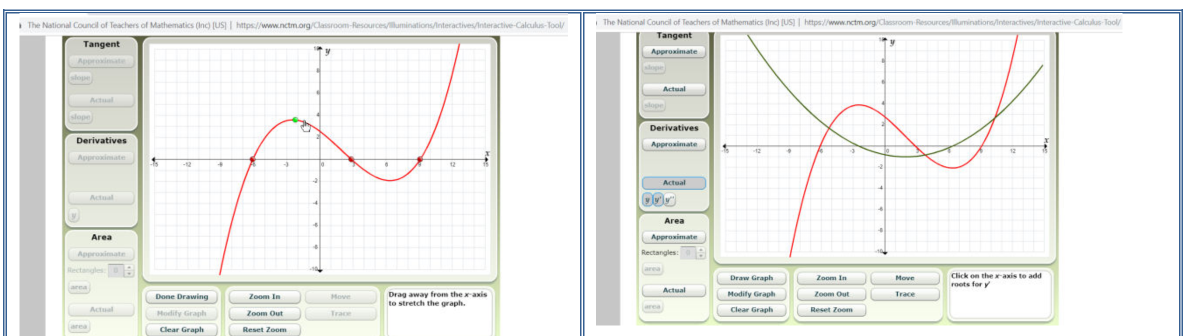
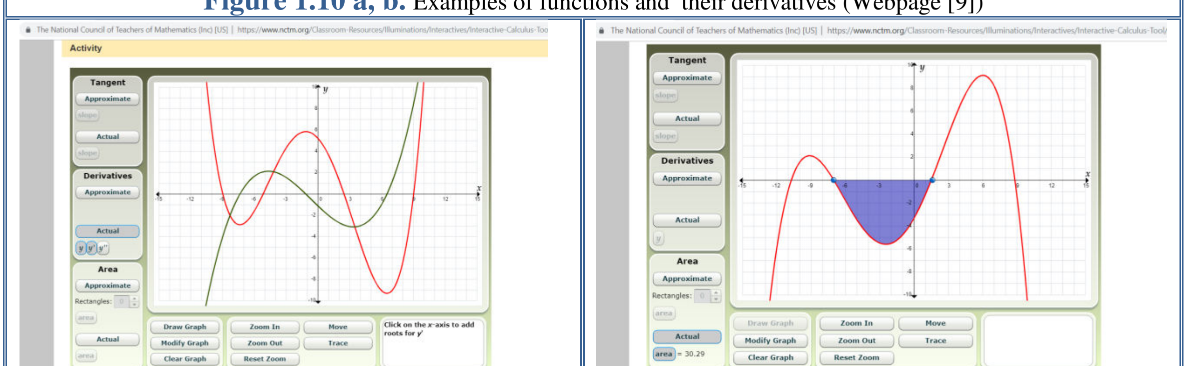


Figure 1.10 a, b. Examples of functions and their derivatives (Webpage [9])



Figures 1.10 c, d. Examples of functions and their derivatives –Investigating definite integrals (Webpage [9])

Sinclair (2001) in her Thesis "*Supporting Student Efforts to Learn with Understanding: An Investigation of the Use of JavaSketchpad Sketches in the Secondary Geometry Classroom*" argues that "if we expect students to develop [reasoning] based on a given diagram, we must ensure that they are able to interpret what is shown (p.27)[…] The challenge will be to use or create [diagrams] that help students concentrate on important details (p.28).

Sinclair (2001) concludes that "[her] study results show that JavaSketchpad [pre-constructed applets] motivates and engages students. It helps students strengthen their geometric thinking skills-especially at the visualisation and analysis levels, by supporting student exploration, visual reasoning, and communication activities" (p. 136). On the other hand, Sinclair (2001) states that "colour and motion [of pre-constructed diagrams in JavaSketchpad] attracted the students' interest, but this was not always enough to help them interpret visual details. Students needed to be prompted to notice particular features and relationships" (p. 134). Generally speaking, students face it difficult to notice relations among objects in a diagram, whether it is constructed in a static of in a dynamic environment. This is because the students are working in the spatio-graphical field of geometry, while their teachers are teaching them --and expecting them to answer-- in the "axiomatic" or theoretical field of geometry.

This is in accordance with what Parzysz (2002) and Jore & Parzysz (2005) assert. The way of teaching geometry at the beginning of junior high school can be distinguished between the:

- *'spatio-graphical' geometry* (Parzysz, 2002) [...which] is a formalisation of the physical space; in this geometry, the objects (e.g., models, drawings on a sheet of paper, or a blackboard, or a computer screen have a physical nature); the actions are actually carried out on the objects [...];
- *'proto-axiomatic' geometry* (Parzysz, 2002) can be considered as a geometry partially theorized, the implicit reference of which is a Euclidian axiomatic theory[…]. Its objects (configurations) have a theoretical nature; the actions refer to these theoretical objects and the validations are of a 'hypothetic-deductive' type (mathematical proofs) (Jore & Parzysz, 2005, p.113)

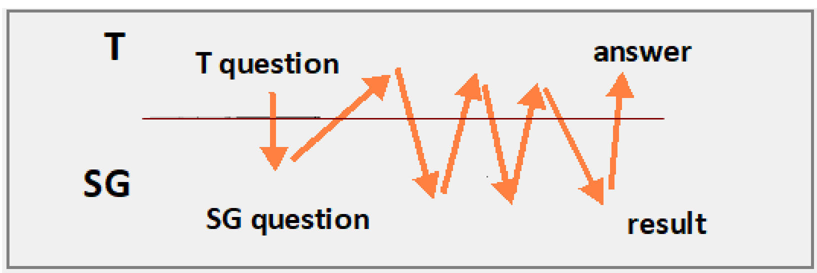


Figure 1.11. Illustration of the activity of the problem solver (Laborde, 2005, p. 162) (adapted)

Laborde (2005) in her study "*The hidden role of diagrams in students' construction of meaning in geometry*" distinguishes between *robust and soft* diagrams created in a *DGS environment*, placing emphasis on difficulties of students to connect their construction with the theory of geometry. As Laborde claims "*diagrams in two-dimensional geometry play an ambiguous role: on the one hand, they refer to theoretical geometrical properties, while on the other, they offer spatio-graphical properties that can give rise to a student's perceptual activity*" (Laborde, 2005, p. 159)

In a DGS, students can construct either *a robust or a soft* diagram. In a DGS milieu "robust constructions are constructions for which the drag mode preserves their properties" (Laborde, 2005, p.22). Laborde (2005) made a distinction between the domain of geometrical objects and relations (which she denoted by **T**, referring to **T**heoretical) and that of spatio-graphical entities (which she denoted by SG, referring to **S**patio-**G**raphical), instantiated by diagrams on a static or a dynamic environment.

Laborde (2005, p.162) illustrates the activity of the problem solver according to this view in the case of a problem that starts and ends in the T domain (**Figure 1.11**). Laborde (2005) constructed the diagram to explain that the way in which figures /or diagrams are used in school problems requires "the use of both domains and several moves between them" (p.162).

According to Laborde, a continuous interplay between the T domain (e.g. a theoretical question posed by the teacher) and the SG domain (e.g. an experimental process in a DG environment relating to the issue) scaffolds students' answer in the theoretical field.

> "[…] *our thinking is performed upon signs of some kind or other, either imagined or actually perceived. The best thinking, especially on mathematical subjects, is done by experimenting in the imagination upon a* [dynamic] *diagram or other scheme, and it facilitates the thought to have it before one's eyes.* (Peirce, NEM I, p.122, cited in Bakker & Hoffmann, 2005, p.335).

Paraphrasing Peirce's argument, I think that dynamic diagrams facilitate thought "*to have it before our eyes*". Dynamic diagrams make it easier than static diagrams to experiment, since students are provided with feedback (or receive feedback) from the on-screen results.
The reason for this is the continuous interplay between the spatiographical and theoretical aspects of the environment, which helps students to overcome the expected difficulties. These difficulties also have to do to the students' competence at relating *procedural knowledge* with *conceptual understanding*.

1.4. Kinds of Knowledge Pairs

High-school students' ability to rightly and accurately construct a figure using static or dynamic means relates to two factors: whether they *know how* to construct it, and whether they *know why* the concrete method of construction results in a figure with concrete properties and not a drawing—which is to say a shape that *looks like* a figure. The answer to the question "How do we construct it?" relates to what is called *procedural knowledge*. The answer to the question "Why to construct it in this way?" relates to what is called *conceptual knowledge*. Which is to say there is a *duality or polarization* in mathematical knowledge between "Knowing how" and "Knowing why" (Scheffler, 1965; Hiebert & Lefevre, 1986). Even and Tirosh (2008) state that the notions "knowledge" and "understanding" are the focal interest and subject under analysis of many researchers. According to them:

> "Different forms of knowledge and various kinds of understanding are described in the mathematics education literature (e.g., instrumental, relational, conceptual, procedural, implicit, explicit, elementary, advanced, algorithmic, formal, intuitive, visual, situated, knowing that, knowing how, knowing why, knowing to)" (p. 206).

Many theorists and researchers in the field of developmental psychology, educational psychology, cognitive science etc. have for various reasons investigated why students cannot apply their previous conceptual knowledge (in other words, knowledge of the concepts and the relations among them) to solve unfamiliar problems, or use concrete concepts to accomplish procedures (e.g., Byrnes & Wasik, 1991; Kitcher, 1984; Hiebert, 1986; Rittle-Johnson, & Alibali. 1999; Carpenter, 1986; Carpenter et al.,1999; Kadijevich, & Haapasalo, 2001; Schneider & Stern, 2010; Rittle-Johnson, & Schneider, 2014). According to Hiebert & Lefevre (1986):

- "*Conceptual knowledge* is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to otherpieces of information. The development of conceptual knowledge is achieved by the construction of relationships between pieces of information." (Hiebert & Lefevre, 1986, pp. 3-4).
- "*Procedural knowledge* of mathematics encompasses two kinds of information. One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols" (Hiebert & Lefevre, 1986, pp. 7-8).

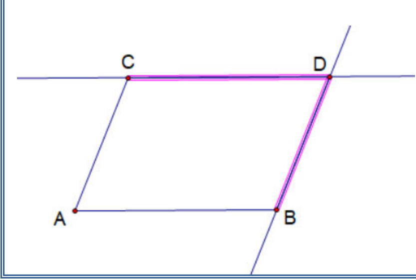
Haapasalo and Kadijevich (2000) suggest the following "*dynamic*" characterizations for conceptual and procedural knowledge (cited in Haapasalo, 2008, p.55):

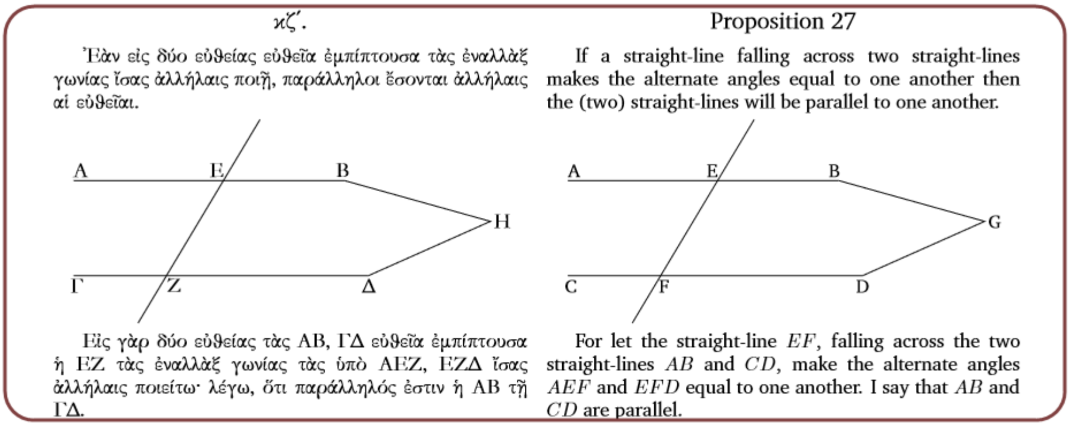
- "*Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation forms. This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms".

Baroody, Feil & Johnson (2007) define procedural knowledge as the "mental actions or manipulations, including rules, strategies, and algorithms, needed to complete a task." (p. 123).

In the words of Schneider & Stern (2010, p. 179) "procedural knowledge can be automatized to different degrees, depending on the extent of practice. Automatized procedural knowledge can be used with minimal conscious attention and few cognitive resources (Johnson, 2003)".
The point of investigation is: how conceptual and procedural knowledge influence each other? What kind of knowledge must be developed first during the teaching and learning of mathematics if students are to understand mathematics? Do students have to learn the concepts before they apply them during procedures or vice versa?
In my opinion, procedural knowledge can support the conceptual knowledge and vice versa. How does this occur?
The students use their conceptual knowledge to construct a figure in different ways. For example, they can use a definition or a theorem as the basis for the construction of an equilateral triangle: thus, according to the definition, an equilateral triangle is a triangle all of whose sides are congruent. This means that students can use their rulers to construct a triangle with three equal sides. Alternatively, if the student knows the theorem "an equilateral triangle has three angles equal to 60 degrees", they can also use the information incorporated in it and use a protractor to construct a triangle whose angles are equal to 60 degrees. In the DGS software, students have to cooperate with the environment in order to accomplish their constructions. They cannot touch the tools, but they can create constructions using the mouse in accordance with their mental representation.
However, if a student has not grasped the concept of "equilateral triangle" but knows how to construct an equilateral triangle, then s/he can perceive the properties of the figure and can be guided, through proper questioning, to discover and formulate them (e.g., Patsiomitou, 2008a).
One of the cognitive aims in my teaching is my students to actively construct the properties of a figure and the connections between them -in other words I want them to be able to link conceptual and procedural knowledge.
In the table below, I present an example of the conceptual and procedural knowledge needed to construct a parallelogram in a DGS environment, The Geometer's Sketchpad (Patsiomitou, 2012a, p.125, in Greek).

Table 1.1: Construction of a parallelogram

Procedural knowledge	Conceptual knowledge
1. Constructing a segment AB 2. Constructing a point C above segment AB 3. Constructing a parallel line from point C to AB. 4. Joining points A and C with a segment. 5. Constructing a parallel line from point B to AC. 6. Constructing the intersection point D 7. Hiding the parallel lines 8. Joining points C and D, and D and B, with segments CD and DB. (**Figure 1.13a, b, c, d, e**) 	The following Propositions in Euclid's "Elements", BOOK I, support students' conceptual understanding. *Proposition 27: If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.* (reported in Fitzpatrick, 2007, p. 30) (**Figure 1.12**) *Proposition 31:* To draw a straight-line parallel to a given straight-line, through a given point. (reported in Fitzpatrick, 2007, p. 33) *Proposition 33:* Straight-lines joining equal and parallel (straight lines) on the same sides are themselves also equal and parallel. (reported in Fitzpatrick, 2007, p.35) Definition 23 (κγ΄) in Euclid Elements, BOOK I : Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions). (reported in Fitzpatrick, 2007, p.7)



κζ΄.

Ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

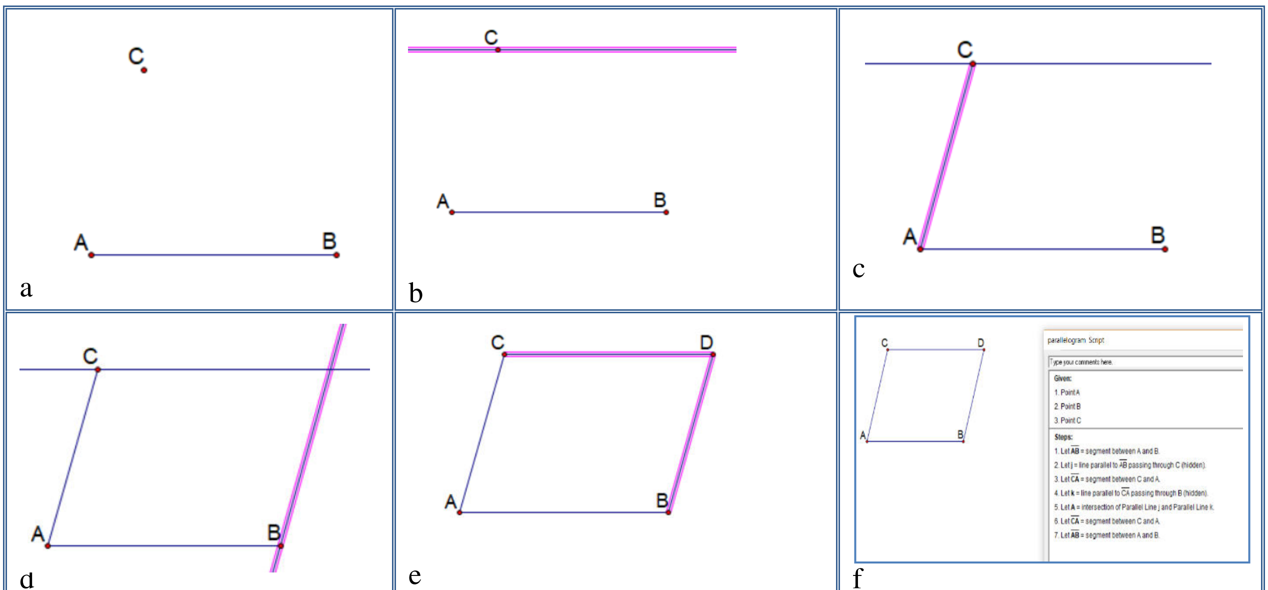
Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΕΖ, ΕΖΔ ἴσας ἀλλήλαις ποιείτω· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Proposition 27

If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.

For let the straight-line EF, falling across the two straight-lines AB and CD, make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

Figure 1.12. The concept of parallelism [Proposition 27 (κζ΄) in Euclid Elements, BOOK I] (Fitzpatrick, 2007, p.30)



Figures 1.13a, b, c, d, e, f: A step-by-step construction of a parallelogram –[and] a script describing the process of constructing a parallelogram

The steps of the construction are also described in a script (a custom tool created using Sketchpad), as it is illustrated in the **Figure 1.13f.** Lopez-Real and Leung (2004) argue that DGS environments promote links between procedural and conceptual knowledge. In order to construct a parallel line using the software, one has to select two objects: a straight object (for example a line) and the point from which the line parallel to the initial line will be drawn. I intentionally familiarize the students with the software, "'step by step', in parallel with the corresponding theory" (Mariotti, 2000, p. 41): all too often, students make purely mechanical use of the software, which makes it impossible for them to understand the logic underlying the command options. Furthermore they would not be able to construct the connections between the *spatiographical field and the theoretical field of the software* (Laborde, 2005). Through the procedure of constructing a perpendicular or parallel line, the student is led to understand the necessity of two given objects: the point and the straight object (line). Therefore, is the construction that leads the student to "shape" the respective notion (for example the meaning of perpendicularity or parallelism) as well as their connection to the Euclidean proposition. Furthermore, the construction of the parallel line using the software's tools (point and straight line) is related to the notion of the figure as theoretical object. In this case, students use the definition of the parallelogram to construct the figure. The construction is a drawing (or a perceptual object), since the starting point is random and the lines drawn do not necessarily form a parallelogram, or dragging may mess the construction up as it does not maintain its properties.

The notion of “knowledge” is closely related to the notion of “understanding”. Skemp (1978) was a pioneer who investigated "*What does it mean to understand mathematics*?" (Byers & Herscovics, 1977, p. 24). Skemp (1978) presented his view on the distinction between two kinds of understanding in mathematics: *relational and instrumental*.

- *Relational understanding* is described as knowing both what to do and why. This kind of understanding denotes the ability of the student to infer particular rules or procedures by considering some general relationships.
- *Instrumental understanding* entails “rules without reasons” (Skemp, 1978, p. 9). This kind of understanding denotes the ability of the student to apply /utilize rules without knowing why they work (see also Even and Tirosh, 2008, p.206; Haapasalo, 2013, p.2).

Skemp (1978) proposes three advantages for the “*instrumental understanding*”:

- “Instrumental mathematics is usually easier to understand […]
- The rewards are more immediate and more apparent […]
- One can often get the right answer more quickly and reliably by instrumental thinking […]” (p.12)

Skemp (1978) also proposes four advantages for “*relational understanding*”:

- "It is more adaptable to new tasks […]
- It is easier to remember […]
- Relational knowledge can be effective as a goal in itself […]
- Relational schemas are organic in quality"[…] “very much like a tree extending its roots” (p.12-13)

Moreover, ‘*logical understanding’* (Skemp, 1986) “is the ability of the student to reason deductively or the ability to connect mathematical symbolism with relevant mathematical ideas and to combine these ideas into chains of logical reasoning” (p.166). Given that these kinds of knowledge differ so much, he argues, should we perhaps distinguish between instrumental mathematics and relational mathematics, in the same way we do between instrumental and relational understanding? Looked at thus, learning instrumental mathematics entails learning a number of maps showing us how to get from A to B, while learning relational mathematics means constructing a conceptual structure that will allow us to generate an infinite number of ways of getting from any A to any B within a structure. White and Mitchelmore (2002) in their study “*Teaching and learning mathematics by abstraction*” discuss Skemp’s ideas (1986) and “how concepts are formed through an abstraction process” (p.236). According to Skemp (1986) abstracting is "an activity by which we become aware of similarities [...] among our experiences" and a concept as "some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class" (p. 21 cited in White and Mitchelmore, 2002, p. 236).
Skemp also distinguished between “primary and higher-order concepts, explaining that higher-order concepts are abstractions of earlier abstractions and so progressively removed from experience of the outside world" (1986, p. 24, cited in White and Mitchelmore, 2002, p. 237).
In one way or the other, researchers have developed theories that seek to explain how students develop abstract processes which encompass an experience broader than the primary concept developed previously. For example, “green” or “red” is a *primary concept* developed from sensory experience while “colour” is a *secondary concept*, developed through a generalization, a synthesis of the primary concepts which ultimately becomes an abstract concept which incorporates all the primary concepts (Skemp, 1986, p. 24). “*Generalizing, synthesizing and abstracting*” is a sequence also for Dreyfus (1991) that a student has to follow as “abstraction may be seen as a many-to-one function where generalisations about the base contexts are synthesized to form a new abstraction” (White and Mitchelmore, 2002, p. 236).

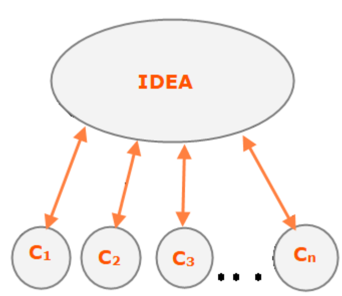


Figure 1.14. Links between an idea and Ci concrete objects (White, & Mitchelmore, 2002, p. 239) (adapted)

White and Mitchelmore (2002) illustrate with figures how they conceive the links between an "idea" and Ci concrete objects, based on Skemp's (1986, p.20) notion of concept. As White and Mitchelmore make clear "The word "idea" […] could refer to any mathematical object such as a concept, an operation or a relation[…] These links enable the learner both to recognize the idea in each Ci and to call up a variety of contexts in which the abstract idea is found" (p.239) (See also **Figure 1.14**).
White and Mitchelmore (2010) developed a teaching model called "*Teaching for Abstraction*" which consists of four phases:

- *"Familiarity.* Students explore a variety of contexts where a concept arises, in order to form generalizations about individual contexts and thus become familiar with the underlying structure of each context.
- *Similarity.* Teaching then focuses on helping students recognise the similarities and differences between the underlying structures of these contexts.
- *Reification.* The general principles underpinning the identified similarities are drawn out, and students are supported to abstract the desired concept into a mental object that can be operated on in its own right.
- *Application.* Students are then directed to new situations where they can use the concept." (cited in White, Wilson & Mitchelmore, 2012, p.761)

Pirie and Kieren (1989) characterize understanding as follows: "Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication […] each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without". (p. 8).

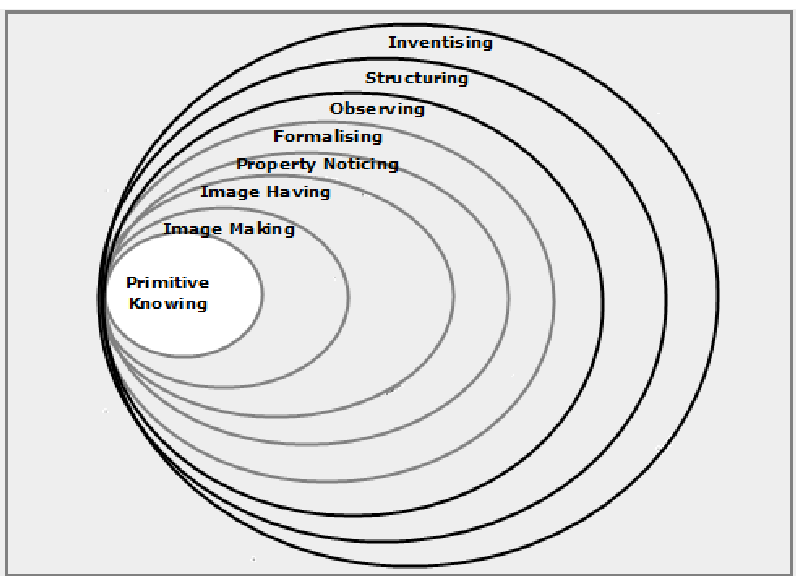


Figure 1.15. The Pirie & Kieren (1994, p.186) model for the growth of mathematical understanding (adapted)

Pirie & Kieren (1994) consider "[…] understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories" (Pirie & Kieren, 1994, p. 165). They developed a model for the growth of mathematical understanding. They identified eight levels of understanding and depicted their model as *nested rings or embedded layers* (see also, Sinclair, 2001 p.12; Slaten, 2006, p.32).
Pirie & Kieren model support that students can move back and forth between the rings: they come to the learning task with "primitive knowledge", their understanding is informal when they are operating in any of the three next modes, but can ultimately become more abstract. According to Pirie & Kieren (1994) the model explains how a student understands is an interactive process of organizing and reorganizing his/her conceptual structures (**Figure 1.15**).
Primitive Knowing refers to the starting knowledge, at the beginning of instruction. For me it is the preexisting knowledge that a student has in his/her mind at the beginning of the teaching and learning process.
Image making refers to the mode of understanding that is developed through actions and reflections on those actions.

Image having refers to the mode of understanding that is developed without having to act on the objects. Now the student can use his/her mental representations of the involved concept.
Property noticing refers to the mode of understanding when a student can construct properties, combining aspects of images relevant to the objects.
Formalising in the next level of understanding when the student constructs formal or abstract methods from the previous images, from which s/he has constructed properties.
Observing refers to the mode of understanding where a student can reflect on his/her formal understanding.
Structuring is the next level where a student can use deductive reasoning and logical argumentation.
Inventising is the level where a student can create a new concept from his/her structural understanding.
Sfard (1991, 1994) identified the dual meanings of "*operational understanding*", which is to say knowledge of the operations that can be performed on mathematical objects, and "*structural understanding*", meaning knowledge of the structure of a mathematical object.

Sfard (1991, p.5) in her study "*On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin*" illustrates through examples the duality of structural and operational understanding (**Figure 1.16**), presenting in this way the "*dual nature of mathematical conceptions*". It seems that processes and objects are what are conceptualized as "*different sides of the same coin*" (p.1).

Sfard (1989, 1991, 1992) argues that a mathematical object, or an abstract object generally, can be conceived or interpreted both operationally, when it is considered as a performed process or a process to be carried out, and structurally when it is interpreted as a permanent object with concrete properties. She identifies the meaning of *reification* as the next step in the mind of learner as "it converts the already condensed process into an object-like entity" (Sfard, 1992, pp. 64-65, in Davis, Tall & Thomas, 1997, p.133). In Sfard's opinion mathematical objects can be seen as discursive objects within a mathematical discourse occurred or taking place in a classroom.

	Structural	Operational
Function	Set of ordered pairs (Bourbaki, 1934)	Computational process or Well defined method of getting from one system to another (Skemp, 1971)
Symmetry	Property of a geometrical shape	Transformation of a geometrical shape
Natural number	Property of a set or The class of all sets of the same finite cardinality	0 or any number obtained from another natural number by adding one ([the result of] counting)
Rational number	Pair of integers (a member of a specially defined set of pairs)	[the result of] division of integers
Circle	The locus of all points equidistant from a given point	[a curve obtained by] rotating a compass around a fixed point

Figure 1.16. Structural and operational descriptions of mathematical notions (Sfard, 1991, p. 5) (adapted)

Concretely, according to Sfard (1991) "seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing […] it also means being able to recognize the idea "at a glance" and to manipulate it as a whole[…]" (p. 4).
Sfard describes a historical example regarding "the development of the notion of number as a cyclic process […] whenever a new kind of number was being born" (p.13). Sfard summarized schematically in a figure the whole process of the development of the concept of number in the **Figure 1.17** below.
As Sfard (1991) concludes and summarizes: "the history of number is a long chain of transitions from operational to structural conceptions […] processes performed on already abstract objects have been converted into compact wholes, *or reified*[…] "(p. 14).
Sfard (1991) distinguishes three stages in concept development: *interiorization, condensation and reification.*

- *Interiorization* is the stage through which “a learner gets acquainted with the processes which will give rise to a new concept” (Sfard, 1991, p. 18) […] A process has been interiorized if it can be carried out through mental representations “(Piaget, 1970, p.14).
- *Condensation* is the stage through which a learner “becomes more and more capable of thinking about a given process as a whole […] (Sfard, 1991, p. 19).
- *Reification* is defined “as an ontological shift –a sudden ability to see something familiar in a totally new light […] “ (Sfard, 1991, p. 19).

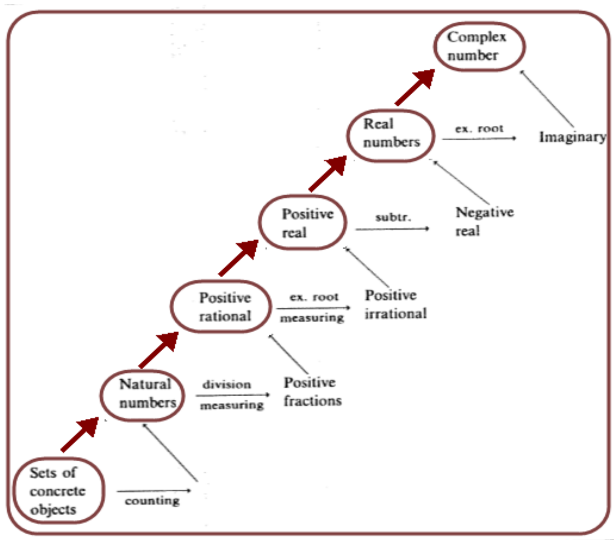


Figure 1.17. The Development of the concept of number (Sfard, 1991, p. 13) (adapted)

Sfard (1991) argues that “the terms “operational” and “structural” refer to inseparable, though dramatically different, facets of the same thing” (p.9). Thus, unlike “conceptual and procedural” or algorithmic and abstract” she argues that “we are dealing here with *duality*, rather than *dichotomy”* (p.9).
Even and Tirosh (2008) in their article: “*Teacher knowledge and understanding of students' mathematical learning and thinking*” have investigated among others the meanings of *instrumental understanding* and *relational understanding*, also trying to clarify if they consist a “dichotomy or a continuum” (p.206). According to Even & Tirosh,

> “Skemp argued that although instrumental mathematics is easier to understand within its own context, its rewards are more immediate and apparent, and one can often obtain the right answer more quickly and reliably, relational mathematics has the advantages of being more adaptable to new tasks, being easier to remember and capable of serving as a goal in itself” (p.206).

Eventually, Even & Tirosh (2008), conclude that “While Skemp assumes a dichotomy between instrumental and relational knowledge, and Nesher (1986) and Resnick and Ford (1981) question its usefulness, Hiebert and Carpenter (1992) and other researchers suggest that absolute classifications are impossible” (p. 207). Concretely:

- Nesher(1986) does not consider there to be a dichotomy between performing procedures with algorithms and learning through understanding concepts. In his view students must acquire the competence to use both algorithms and concepts.
- Resnick and Ford (1981) consider competence with algorithms to help students extend their working memory.
- Hiebert and Carpenter (1992) argue that both conceptual and procedural knowledge are important for the acquisition of competence in mathematics.

Moreover, Mason and Spence (1999), determined a special form of knowing: “Knowing-to act in the moment”, which is “the type of knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behavior” (cited in Even and Tirosh, 2008, pp. 207-208).
Schneider & Stern (2010, p. 190) report also other theorists who have proposed numerous other pairs of knowledge kinds, for example,

- “competence and performance (Chomsky, 1965),

- structures and procedures of the mind (lnhelder & Piaget, 1980),
- declarative and procedural knowledge (Anderson, 1983),
- explicit and implicit knowledge (Schacter, 1987)".

Schneider & Stern (2010) argue that "researchers are far from understanding how these kinds of knowledge relate to each other and how they shape development. Valid empirical measures are an indispensable precondition for scientifically investigating these questions rather than merely speculating about them." (p. 190)

Stein and Smith (1998) state that "tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for students thinking" (Tchoshanov, 2013, p. 67).

Research into mathematical education has long concerned itself with the transition from a process to a concept. Many researchers (e.g. Dienes, 1960; Piaget, 1972 a, b; Davis, 1983, 1984) also, "focused on the idea of a process becoming a mental object […] as a fundamental method of cognitive development in mathematical thinking" (Davis, Tall & Thomas, 1997, p.132). On the other hand, in the words of Sfard (1989)

> "Although ostensibly incompatible (how can anything be a process and an object at the same time?), they are in fact complementary. The term "complementary" is used here in much the same sense as in physics, where entities at subatomic level must be regarded both as particles and as waves to enable full description and explanation of the observed phenomena […]"(Sfard, 1991, pp. 4–5)

The process-object duality is important for the learning of mathematics. If a student has developed his/her conceptual understanding then s/he has also developed the ability to see both the process-facet and the object-facet of a concept. This development is called *encapsulation* (Dubinsky, 1991a) or *reification* (Sfard, 1991) as a redefinition of the notion of "*conceptual entity*" introduced by Piaget (1977). Beth & Piaget (1966) consider the notion of *encapsulation* to be a "dynamic" process which transitions into a mental object when "[...] a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower" (Beth & Piaget 1966, p. 247). Gray & Tall (1991) defined the meaning of 'procept' as a combination of the words "pro-[cess] + [con]-cept", "to be the amalgam of process and concept in which process and product is represented by the same symbolism" (Gray &Tall, 1991, p. 73). A procept, "is consisted of a collection of elementary procepts which have the same object" (Gray & Tall, 1994 reported in Davis, Tall & Thomas, 1997, p.134). The meaning of an *elementary procept* is according to them "an amalgam of […]: a process which produces a mathematical object and a symbol which is used to represent either process or object […]" (Gray & Tall, 1994 reported in Davis, Tall & Thomas, 1997, p.134). Gray and Tall (1994) "hypothesise that successful mathematical thinkers can think proceptually, that is, they can comfortably deal with symbols as either process or object. An operational orientation would thus interpret $2(a + b)$ and $2a + 2b$ quite differently, whereas proceptually the two expressions would be seen as identical" (White and Mitchelmore, 2002, p. 236).

Kadijevich & Haapasalo (2001) argue that, using computers, students can spend less time on procedural skills and more on developing their *conceptual understanding* (Fey, 1989).

Moreover, Kadijevich (2018) in his study "*Relating procedural and conceptual knowledge*" reports the ways that promote relations between *procedural and conceptual knowledge*. As he argues:

- "Links from procedural to conceptual knowledge may be established through the *elaboration and coordination of several microworlds* […].The links in question can be promoted through replicating solutions with technology on the basis of technology-generated partial solutions[…]
- *By applying some general problem solving productions* (i.e. if-then rules), links from conceptual to procedural knowledge may be established […]Problem solving through the development of expert system knowledge bases comprising if-then rules (the so-called knowledge engineering)[…]
- *By using the notion of procept* (i.e. "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both")[…]. Relatingdifferentproblemrepresentationswouldestablishlinksbetweenprocedural and conceptual knowledge […]
- Procedural and conceptual knowledge may be unconnected, […] and it is *big ideas* (e.g. equal partitioning) that, applied as overarching concepts, connect concepts and procedures […]Using comparisons (e.g. comparing methods whereby problems are solved; comparing problems solved with the same procedures) may be a way to promote links between procedural and conceptual knowledge […]" (pp.19-20).

Given the core role in mathematics education of developing procedural and conceptual knowledge and forging links between the two, a key question is how different technologies affect the relationship between the two.

1.5. The Concept of Reflective Abstraction

Piaget introduced the concepts of *empirical abstraction*, *pseudo–empirical abstraction* and *reflective abstraction* "to describe the construction of logico–mathematical structures by an individual during the course of cognitive development" (Dubinsky, 1991a, p.95).

- *"Empirical abstraction:* a subject (e.g., a student) proceeds to this kind of abstraction after the observable experience with a few objects through which the subject understands that these objects have a common property or in the words of Dubinsky "the subject observes a number of objects and abstracts a common property" (p.98) *Empirical abstraction derives knowledge from the properties of objects* (Beth & Piaget, 1966, pp.188–189).
- *Pseudo–empirical abstraction:* a subject (e.g., a student) proceeds to this kind of abstraction after the experience with actions performed on the objects (p.98). *Pseudo–empirical abstraction* "*is intermediate between empirical and reflective abstraction and teases out properties that the actions of the subject have introduced into objects*" (Piaget, 1985, pp.18–19).
- *Reflective abstraction*, "is completely internal" (p.97). *Reflective abstraction is drawn from what Piaget (1980, pp. 89–97) called the general coordinations of actions and, as such, its source is the subject and it is completely internal*.

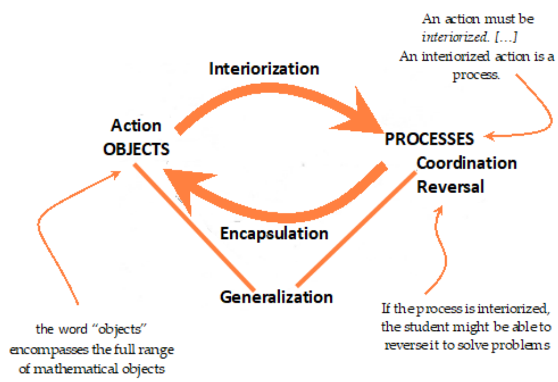


Figure 1.18. Schemas and their construction (Dubinsky, 1991a, p.105) (adapted)

According to Piaget, "The development of cognitive structures is due *to reflective abstraction*" (Piaget, 1985, p. 143)"*Reflective abstraction* is the construction of mental objects and of mental actions on these objects. Piaget found that the development of children's logical thinking could be described in terms of five sub-operations or forms of construction in *reflective abstraction*: *interiorization, coordination, encapsulation, generalization, and reversal* (Dubinsky, 1991a, p. 103).

- *Interiorization* "is the translation of a succession of material actions into a system of interiorized operations" (Beth & Piaget, 1966, p. 206, cited in Dubinsky, 1991, p.100).
- *Coordination* "of successive displacements can form a continuous whole" (Piaget, 1980, p. 90, cited in Dubinsky, 1991a, p.100)
- *Encapsulation* "of actions or operations become thematized objects of thought or assimilation" (Piaget, 1985, p. 49, cited in Dubinsky, 1991, p.100).
- *Generalization "is the passage from "some"to"all, from the specific to the general* (Piaget & Garcia, 1983, p. 299, cited in Dubinsky, 1991, p.97)".

Paschos & Farmaki (2006) analyzed the mental operations of university students, employing the Piagetian theory of *reflective abstraction*. As they conclude

> "[…] the mental mechanism and operations of the students are gradually revealed. Understanding this mechanism will allow us to decide and distinguish whether the students come to a true understanding of the definition of the definite integral concept, as opposed to having just an empirical perception of integration, by which they can act effectively only in a limited and particular framework. The methodology

developed here may have a wider applicability in guiding our actions to help students develop advanced mathematical thinking" (p. 343-344).

A schema is a reasonable, consistent and coherent collection of actions on objects and processes (**Figure 1.18**). I attempted to briefly give the following description of the organization and construction of a *schema* based on the above figure, which can, in my opinion, be used in several mathematical areas, and not only at the advanced level of mathematics:

"[…the word] "objects" encompasses the full range of mathematical objects […] each of which must be constructed by an individual at some point in her or his mathematical development. […] At any point in time there are a number *of actions that a subject can use* (italics used by Dubinsky) for calculating with these objects […] an action must be interiorized. *[…]* An interiorized action is a process. *Interiorization* permits one to be conscious of an action, to reflect on it and to combine it with other actions.[…] If the process is interiorized, the student might be able to reverse it to solve problems[…]" (Dubinsky, 1991a, p.105-106)

Lehtinen & Repo (1996) elaborated on the Piagetian theory of reflective abstractions. They conducted a study, aiming to investigate the construction of advanced mathematical concepts in a computer-based environment. According to them, a student can solve typical problems with the help of "horizontal generalization" (empirical) but "is not able to construct an adequate conceptual understanding because their medal models are limited to the level of concrete mathematical knowledge" (p.108).

According to Lehtinen & Repo (1996) "*reflective abstraction refers to a process in which the student tries to construct abstract structure and operations by reflecting on his /her own activities and the arguments used in social interaction*" (p.106)

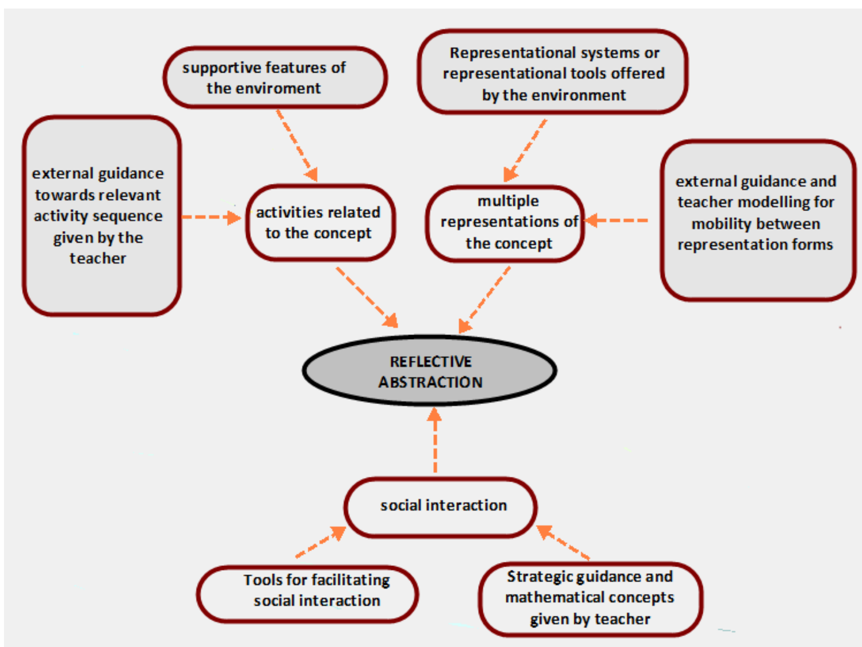


Figure 1.19. Presuppositions of adequate reflective abstraction (Lehtinen & Repo, 1996, p. 113): (an adaptation for the current study)

Lehtinen & Repo in their study systematically analyze the prerequisites for effective abstraction with a focus to "(a) critical activities, (b) multiple representations, and (c) challenging and facilitating social interaction" (p.108). In Lehtinen & Repo's opinion, the basic activities should be of optimal difficulty and allow time for the construction process. The activities should also relate to the concept to be learned in a way that activates the student's relevant prior knowledge and provides opportunities for all the sub-operations of reflective abstraction Continuous guidance is also needed from the teacher in the form of direct or indirect intervention, as is the utilization of multiple representations and the continuous shifting between different representational systems with expert modeling of the use of digital tools. Lehtinen & Repo "*elaborated a model that summarizes the previous described presuppositions of adequate reflective abstraction in the following figure" (p. 112)* (**Figure 1.19**)

Finally, Lehtinen & Repo concluded that

- “good school achievement in mathematics is not always a valid indicator of a high –level understanding of mathematical concepts and operations” (p.124)
- “the average level of conceptual understanding can be improved noticeably by involving students in a sequence of critical activities and by changing the quality of their social interaction” (p. 125).

Chapter II.

2.1. Visualization and Dynamic Visualization

Trying to understand more deeply the activities of teaching and learning, a number of educators, researchers and psychologists have turned their attention to representations and systems of representations of mathematical and scientific objects and ideas (e.g., Goldin, 1988; Greeno, 1991; Kaput, 1987; Janvier, 1987a, b, c).
Representations, representational systems and visualization of mathematical objects are reported as being fundamental in the international literature. Most researchers, educators and teachers agree that representations of mathematical and scientific objects positively impact on students' understanding and on the way they communicate and share mathematical meanings; they also help students develop their mathematical reasoning during the problem-solving process (e.g., Palmer, 1977a, b; Vinner, 1983; Presmeg, 1986a, b; Janvier, 1987a, b, c; McCormick, DeFanti,& Brown, 1987; Vergnaud, 1987; Glasensferd, 1991; Zimmermann & Cunningham, 1991; Goldin, 1998 a, b; Boulton-Lewis, 1998; Kaput, 1987, 1989, 1991, 1992, 1998, 2001; Lakoff, 1993; Duval, 1993, 1995a, b; Arcavi, 2003; Ainsworth, 1999a, b, 2006; Clements & Sarama, 2007, 2009; Lavy, 2006; Duval, 1998, 1999, 2006; Goldin, 2003, 2008; Hitt, 2002; Zazkis, & Liljedahl, 2002; Patsiomitou, 2008a, b, 2012a, b, 2013a, b). The increasing research interest regarding representations and representational systems, is a result of the need to face practical and theoretical issues concerning the difficulties students encounter when they try to translate from one form of representation to another (e.g., to transform a verbal expression in a geometrical problem into a figure using static or dynamic means, or to transform an algebraic type of a function into a graphical representation). Specifically, a problem representation is "a cognitive structure which is constructed by a solver when interpreting a problem" (Yackel, 1984, p. 7, cited in Cifarelli, 1998). Word problems are a kind of representation. In the words of Susan Gail Gerofsky (1999) "The word problems represent a final test of students' competence in recognizing problem types […] and translating those problems into tractable diagrams and equations which can be solved using taught algorithmic methods. School word problems are not social events not part of an oral culture. They are ideally meant to be solved silently, individually, using pencil and paper".
From a cognitive psychological point of view a major problem in constructing a representation of a problem is that we need to know which lines go together to form objects (Anderson, 1983/2015, p.34). In other words how to organize the components of the figure in geometry. Anderson (2015) states that "we organize objects into units according to a set of principles called the *gestalt principles* of organization, after the Gestalt psychologists who first proposed them (e.g., Wertheimer, 1912/1932)" (p.34).

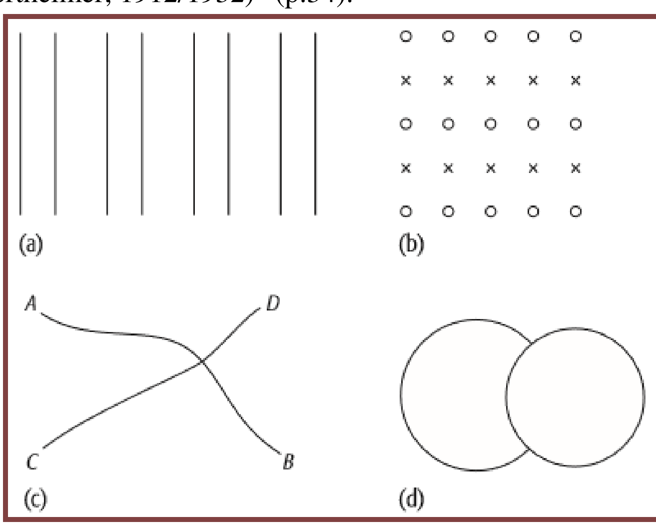


Figure 2.1. Illustration of the gestalt principles of organization (Anderson, 1983/2015, p. 34).

Anderson (2015) defines gestalt principles of organization as "the principles that determine how a scene is organized into components. The principles include proximity, similarity, good continuation, closure, and good form" (p. 368). **Figure 2.1** illustrates the gestalt principles (Anderson, 2015, p. 34):

- “Figure 2.1a *illustrates the principle of proximity*: Elements close together tend to organize into units. Thus, we perceive four pairs of lines rather than eight separate lines.
- Figure 2.1b *illustrates the principle of similarity*: Objects that look alike tend to be grouped together. In this case, we tend to see this array as rows of o’s alternating with rows of x’s.
- Figure 2.1c *illustrates the principle of continuation*. We perceive two lines, one from A to B and the other from C to D, although there is no reason why this sketch could not represent another pair of lines, one from A to D and the other from C to B. However, the lines from A to B and from C to D display better continuation than the lines from A to D and from C to B, which have a sharp turn.
- Figure 2.1d *illustrates the principles of closure and good form*. We see the drawing as one circle occluded by another, although the occluded object could have many other possible shapes. The principle of closure means that we see the large arc as part of a complete shape, not just as the curved line. The principle of good form means that we perceive the occluded part as a circle, not as having a wiggly, jagged, or broken border” (Anderson, 2015, p. 35).

Another source of difficulty for many students during the problem-solving process in geometry is that they compare the image with a prototype which they have in their mind - (an *archetype*, a prototype which differs for each individual student). “*A prototype* is a mental representation which is a good example of a category” (Lakoff, 1987, p. 43, cited in Presmeg, 1992, p. 597). Mesquita (1998) in her study “*On Conceptual Obstacles Linked with External Representation in Geometry*” defined also the notion of “*Prototypical Figures”* as

> “those ones corresponding to a regular organization of contour, orientation and form; prototypical figures tend to respect enclosure laws (closed borders are preferentially perceived), privileging some directions (such as horizontal and vertical ones) and forms (which tend to be regular, simple and symmetric); the components of the figure (sides, angles, for instance) have approached dimensions. Stability and aesthetic preoccupations may reinforce the perception of these prototypical figures. In opposition to them, we can consider the limit-cases figures” (p. 189)

In view of the fact that most students face *cognitive obstacles* when a part of their knowledge, generally effective for their problem-solving processes is inadequate and cannot be adapted to the process at hand (Brousseau, 1992, 1997), the utilization of proper representations helps students overcome obstacles (Goldin & Shteingold, 2001). According to Brousseau (1997)

> “Students start their learning process in an environment that is unbalanced and full of difficulties and obstacles just like human society. The new knowledge comes from the skill to adapt to the new circumstances and stimuli and a new reaction to the environment is the proof that a learning process has taken place.[…] “ the problem s/he has to face has been chosen in order to make him learning and gaining a new knowledge, this knowledge is justified by the inner logic of the situation”(cited in Manno, 2006, p.23)

In addition, Mesquita (1998) distinguishes two roles of external representations in geometrical problems: *a descriptive* one and *a heuristical* role.

- “an external representation is *descriptive* when its sole function is to give a synoptical apprehension of the properties mentioned in the problem statement” (p. 191)
- “an external representation has a heuristical role if it acts as a support for intuition, suggesting transformations that lead to solution” (p.191).

Difficulties in mathematics generally are associated with visual processing and may be overcome. The role of visualization and visual reasoning in geometry and generally in mathematics understanding have been the focus of interest for many researchers, educators and psychologists (e.g., Tall & Vinner, 1981; Vinner, 1983).

There is a substantial bibliography on visualization, spatial visualization, spatial ability, visual thinking, mental imagery and their relation with students’ mathematical performance (e.g., Presmeg, 1986a, b, 1992, 1997; Zimmerman & Cunningham 1991; Goldenberg, 1992). A few researchers use the terms *visualization* and *mental imagery* alternatively (e.g., Drake, 1996). Guttierez (1996) argues that

> "There is no general agreement about the terminology to be used in this field: It may happen that an author uses, for instance, the term "visualization" and another uses "spatial thinking", but we find that they are sharing the same meaning for different terms. On the other hand, a single term, like "visual image", may have different meanings if we take it from different authors. Such an apparent mess is merely a reflection of the diversity of areas where visualization is considered relevant and the variety of specialists who are interested in it " (p. 4)

Norma Presmeg (1986b) in her study “Visualization in High School mathematics” defines the notion of visual image, as “*a mental scheme depicting visual or spatial information*” (p.42). Presmeg (1986b) classified the kinds of imagery used by students/visualisers in her study as follows (p.43):

(i) *Concrete pictorial imagery*: pictures created in the mind by the learner […] Concrete imagery is effective in alternation with abstract non visual modes such as analysis, logic, or a facile non visual use of formulae (p. 45).
(ii) *Pattern imagery*: pure relationships depicted in a visual-spatial scheme […]
(iii) *Memory images of formulae*: visualisers “see” a formula in their minds […]
(iv) *Kinaesthetic imagery*: imagery involving muscular activity […]
(v) *Dynamic (moving) imagery*: use of moving images […]Dynamic imagery is potentially effective (p.45)

Abraham Arcavi (2003) in his study “*The role of visual representations in the learning of mathematics*” introduces the notion of visualization as a way we can “see” what is unseen. He makes a metaphor and reports examples of the way we can “see” what is unseen through the use of technology (for example, the zoom function in computer environments is a way to “see” the unseen). According to Arcavi (2003)

> “In a more figurative and deeper sense, seeing the unseen refers to a more “abstract” world, which no optical or electronic technology can “visualize” for us. Probably, we are in need of a *“cognitive technology*” (in the sense of Pea, 1987, p. 91) as “any medium that helps transcend the limitations of the mind … in thinking, learning, and problem solving activities.” Such “technologies” might develop visual means to better “see” mathematical concepts and ideas” (p.26).

Arcavi defines visualization “blending and paraphrasing the definitions of Zimmermann & Cunningham (1991, p.3) as well as Hershkowitz et al. (1989, p.75)” as follows:

> “*Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings*.” (p.26)

In the **Table 2.1.** below I have brought together indicative definitions of visualization reported in the international literature, as well as definitions of related notions.

Table 2.1. Visualization

Author	Definition of visualization
Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore, & Vinner, S. (1989)	“Visualization, generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information” (p. 75).
Cunningham (1991)	defines visualization as “the ability to focus on specific components and details of very complex problems, to show the dynamics of systems and processes, and to increase the intuition and understanding of mathematical problems and processes”. (p. 70, cited in Elliot, Hudson & O'Reilly, 2000, p. 152).
Presmeg (1986b)	argues that “a visual image is a mental scheme depicting visual or spatial information” (p. 42).
Presmeg (1997)	defines visualization as “the process involved in constructing and transforming visual mental images…” (p. 304).
Zimmerman and Cunningham (1991)	define visualization as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand drawn or computer-generated" (p. 1). […] Consider visualization to be “the process to form a mental image” (with paper and pencil, or with the aid of technology)” (p. 3).
Goldenberg (1999)	“Visualizing [is] picturing (and drawing) what is inherently visible as well as that which is not (either because it is an abstract object or relationship, or because it is a concrete object that has not

	yet been built)" (p.197).
Duval (1999)	argues that "Vision refers to visual perception and, by extension, to visual imagery. The *epistemological function* of vision consists in giving direct access to any physical object […] The *synoptic function* of vision consists of apprehending simultaneously several objects or a whole field" (p. 12)[…] visualization is based on the production of a semiotic representation" […which] does not show things as they are. *A semiotic representation shows relations or, better, organization of relations between representational units* (p. 13).

Spatial ability and spatial visualization are defined as the ability to perceive and mentally manipulate visual images, as the following researchers support:

Kelly (1928)	defines *spatial ability* as the combination of two ingredients: (1) the ability to percept and reserve visual images; and (2) the ability to mentally manipulate these images (cited in Lawrence Joseph Pleet, 1990, p. 17).
Lohman (1979)	defines *spatial ability* as the ability to generate, retain, and manipulate abstract visual images"(p. 188).
Chien (1986)	defines *spatial visualization* ability as: "the individual's ability to mentally manipulate, act upon, and transform visual stimuli. The ability to anticipate mentally a series of object movements is also involved in this process". (p. 11, cited in Lawrence Joseph Pleet, 1990, p. 17)

In the international bibliography we read also about 'dynamic imagery' (Presmeg, 1986a, b), 'dynamic reasoning, dynamic visualization, or dynamic imagery', (Goldenberg, 1992). There is also a substantial bibliography investigating the inter-relationships between visualization, mental imagery, and mathematical performance.
A visual image in the words of Presmeg is "a mental construct depicting visual or spatial information" (Presmeg, 1992, p. 596). Moreover, visual reasoning is legitimated as a way of reasoning through visualization, which is recognised as fundamental to mathematical reasoning. Barwise and Etchemendy (1991, p.16) consider that visual reasoning can be considered as valid reasoning: 1. visual information is part of the given information from which we reason; 2. visual information can be integral to the reasoning itself; 3. visual representations can play a role in the conclusion of a piece of reasoning (cited in Elliott, Hudson, O' Reilly, 2000, p.152)
Goldenberg (1992) suggested visual representations as a mean for the students to discover the properties of geometrical figures. Goldenberg considers

> "that by ignoring visualization and qualitative reasoning, curricula not only fail to engage a powerful part of students' minds in their mathematical thinking, but also fail to develop students' skills at visual exploration and reasoning" (cited in Rahim & Olson, 1998, p. 374).

Goldenberg (1999) incorporates visualization among other "habits of mind" as a close interaction with skills (p.197). Cuoco, Goldenberg & Mark (1996) in their study "Habits of mind: an organizing principle for mathematics curriculum" support that there are many kinds of visualization in mathematics (pp.381-382):

- One involves *visualizing things* that are inherently visual […]
- A second involves *constructing visual analogues to ideas or processes* that are first encountered in non-visual realms […].
- Finally, there are, for some people, *visual accompaniments* (not analogues, exactly) to totally non-visual processes […].

Then they subdivided these three categories to more categories (for example, visualizing data, relationships, processes, change, calculations)
Cuoco, Goldenberg & Mark (1996) consider the following repertoire of habits of mind that students should have (pp. 3-8): Students should be *pattern sniffers, experimenters, describers, tinkerers, inventors, visualizers,*

conjecturers. Cuoco, Goldenberg & Mark argue that "high school curricula should strive to develop these habits" (p.3).

Zazkis, Dubinsky and Dautermann (1996a) expand the notion of visualization and define it as a dynamic process, meaning the action of alternating transformations between external media (/stimuli) and a student's mind. Zazkis, Dubinsky, and Dautermann (1996a) define visualization as "an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses" (p. 441).

My review of the related literature, makes me view visualization as a *dynamic process* whose dynamism is further expanded /or enriched in a computer environment, in which a student-user can make transformations on screen that have an impact on his/her mental transformations (Patsiomitou, 2012a, b; 2019a). Visualization functions as a microscope through which to view an abstract idea, or to dynamically transform ideas or processes using visual or non-visual means. In other words, visualization is a person's competence to "move" images in mind, even if s/he is working in a paper-pencil environment, operating by thus dynamically. A student's competence at transforming mental images is rooted in *dynamic visualization*. Dynamic visualization can become a mediator in the problem-solving process, as it can be a very powerful *instrument* for the students to gain a greater understanding of the mathematical concepts embodied in the problem. In other words, the peculiar property of dynamic visualization is that individuals who possess this ability can reason.

2.2. Representations and Representational Systems

> "*Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics, the syntax and semantic of which are rich, varied, and universal, but also for two strong epistemological reasons: (1) Mathematics plays an essential part in conceptualizing the real world; (2) Mathematics makes a wide use of homomorphisms in which the reduction of structures to one another is essential*". (Vergnaud, 1987, p. 227, cited in Goldin, 2008)

Goldin (1998b) in his study "*Representational Systems, Learning, and Problem Solving in Mathematics*" through a brief but comprehensive and in depth discussion regarding the evolution of theories for the learning of mathematics argues

> "In my study of mathematical problem solving, learning, and development over the past 25 years, I have become persuaded that the notion of representational systems and their construction can provide the foundation for a model incorporating and synthesizing all the above ideas[…]" (p. 140).

Researchers in the sphere of the Didactics of Mathematics take different approaches to conceptual determination, the theoretical interpretation of the notion of representation, and the ways that representations are used. Indicatively, I shall report issues 1 and 2 of the *Representations and the Psychology of Mathematics Education* journal and Vol. 17, No. 1 and 2, of the *Journal of Mathematical Behavior*, in which the researchers approach the matter in different ways. For example:

Goldin (1998b) denotes the notion of "Representational systems" or "representational modes," as that systems "which include systems of spoken symbols, written symbols, static figural models or pictures, manipulative models, and real world situations, discussed by Lesh (1981)[…]" (p.143). He terms them as "external systems of representation". Goldin (1998b) also mention that the term *representational system* bears some resemblance to what Kaput (1987, p.162) calls "*symbol scheme*".

> "What Kaput (1985, 1987), following Palmer (1977), called a "representation" or "representation system" corresponds most closely in my terminology to a relationship of symbolization between two representational systems." (Goldin, 1998b, p. 143).

Kaput (1998) in his study "*Representations, Inscriptions, Descriptions and Learning: A Kaleidoscope of Windows*" defines internal representations as hypothesized mental constructs and the "external representations" as material notations of one kind or another. Kaput also defines the term "notation system" in an interchangeable way with the meaning of "representation system" and even "symbol system" (p. 270). Kaput adds that "We now turn to an illustration of how the computational medium offers notational opportunity for instructional design within a curricular context" (p. 272).

Vergnaud (1998) in his study "*A Comprehensive Theory of Representation for Mathematics Education*" argues that "representation is not a static thing but a dynamic process that borrows a lot from the way action is organized. This leads to strong objections to the metaphor of the triangle (**Figure 2.2**), on which many authors

have commented, in one way or another, since Aristotle. It is too static, and does not offer any insight for the representation of relationships, while most scientific concepts are relational." (p. 167)
Similarly, Vergnaud (2009, p.93) argues:

> "Representation is a dynamic activity, not an epiphenomenon that would accompany activity without feeding it or driving it. […] it organizes and regulates action and perception; at the same time, it is also the product of action and perception. Therefore, the operational form of knowledge must be considered as a component of representation. Schemes are essential: they organize gestures and action in the physical world, as well as interaction with others, conversation, and reasoning.[…]".

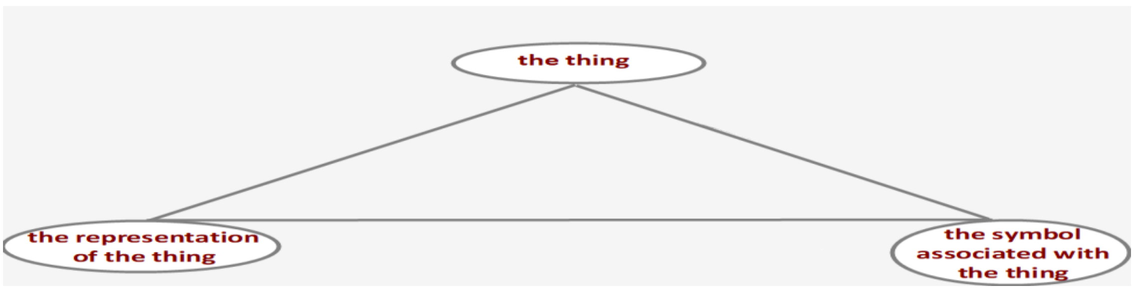
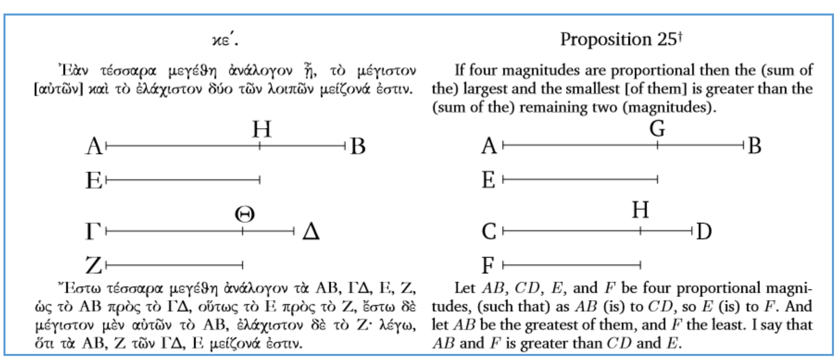


Figure 2.2. The metaphor of the triangle (Vergnaud, 1998, p. 168) (adapted)

Duval (1988a, b) "coined the term "register" in order to refer the different semiotic systems used to present information or to objectify a representation. […] Basically, in geometry, three registers are used: the register of natural language, the register of symbolic language, and the figurative register. This register is linked to the perceptual visual system, which has its own organization laws" (Mesquita, 1998, p.183).
Duval (2000) in his study "*Basic Issues for Research in Mathematics Education*" supports that when we talk of "representations" the four following aspects must be taken into account:

- "*the system by which representation is produced* […]
- *the relation between representation and the represented object* […]
- *the possibility of an access to the represented object apart from semiotic representation* […]
- *the reason why representation using is necessary* […]"(p.58)

I shall cite a few examples to explain the notion of representational systems with which students can express, communicate and/or share ideas in mathematics. For example the Proposition 25, in BOOK 5, of Euclid's "Elements" ["*If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes)*" (reported in Fitzpatrick, 2007, p. 154)], expresses an abstract idea for which diagrams of the reported objects can provide considerable support (**Figure 2.3**).


κε΄.

Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν.

Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ ΑΒ, ΓΔ, Ε, Ζ, ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ Ε πρὸς τὸ Ζ, ἔστω δὲ μέγιστον μὲν αὐτῶν τὸ ΑΒ, ἐλάχιστον δὲ τὸ Ζ· λέγω, ὅτι τὰ ΑΒ, Ζ τῶν ΓΔ, Ε μείζονά ἐστιν.

Proposition 25†

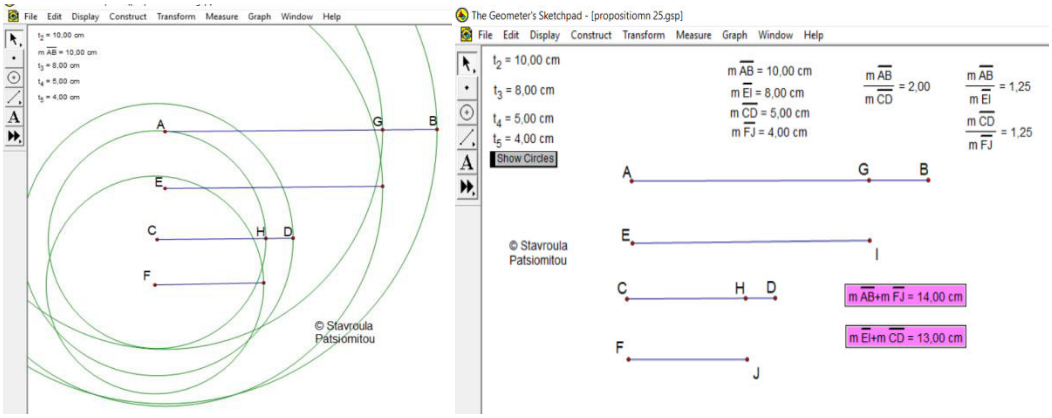
If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).

Let *AB*, *CD*, *E*, and *F* be four proportional magnitudes, (such that) as *AB* (is) to *CD*, so *E* (is) to *F*. And let *AB* be the greatest of them, and *F* the least. I say that *AB* and *F* is greater than *CD* and *E*.

Figure 2.3. Proposition 25, in BOOK V of Euclid's "Elements" (Fitzpatrick, 2007, p. 154)

The formulation of the proposition belongs to a "verbal" representation system (or is a written symbol) while the figures belong to another representation system: "the pictorial".
If we transfer the proposition into a DGS environment using parameters supported by the environment and following the mode of construction I describe in my study "*Hybrid-dynamic objects: DGS environments and*

conceptual transformations" (Patsiomitou, 2019b), then we have a representation system which supports iconic representations in an interactive way (**Figures 2.4 a, b**).



Figures 2.4.a, b. Proposition 25, using DGS tools

I know from my classroom experience that students find it difficult to translate a formal Euclidean proposition into a figure on screen, which is to say they encounter difficulties translating between different systems of representation.
Sakonidis (1994) also argues that *"representations which are too abstract for the child lead to rote manipulation of symbols and rules, and to excessive concern with learning the representations at the expense of the concept represented*" (p. 42).

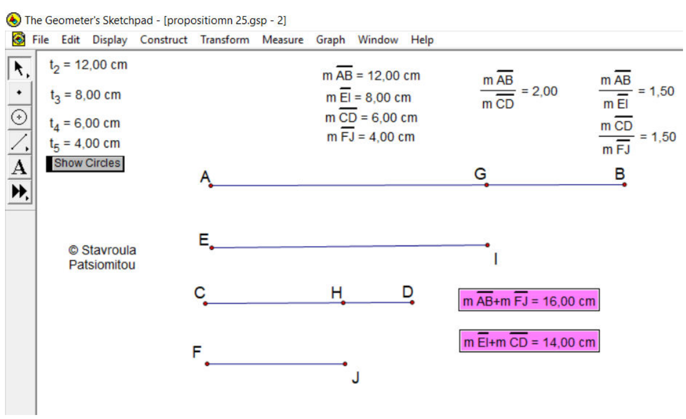


Figure 2.5. Investigating and validating Proposition 25 in a DGS environment

We can validate the truth of the Proposition by changing the values of the parameters in the figures constructed in the DGS environment (**Figures 2.4b, 2.5**), --something that can also be done in a paper and pencil environment using a compass and a ruler for construction.
Different semiotic systems will produce different representations for any mathematical object. Each new representational system (or semiotic system in the words of Duval) provides new means of representation, new ways to process mathematical representations and consequently new ways to mathematical thinking. Suppose we try to explain the Proposition 25 mentioned above in a paper pencil environment. We will produce different representations for the same mathematical object. We have, therefore, to adapt Peirce's triadic conceptualization thus:
[Object, "representamen" (sign), "interpretant] to [Object, one of the various semiotic systems, composition of signs] (Duval, 2000, p. 59).
Duval (2000) constructed a diagram (**Figure 2.6**) to visualize what he supports: "In that perspective, deeper causes of misunderstanding appear. Whenever a semiotic system is changed, the content of representation changes, while the denoted object remains the same. But as mathematical objects cannot be identified with any of

their representations, many students cannot discriminate the content of representation and the represented object: objects change when representation is changed!" (p. 59)

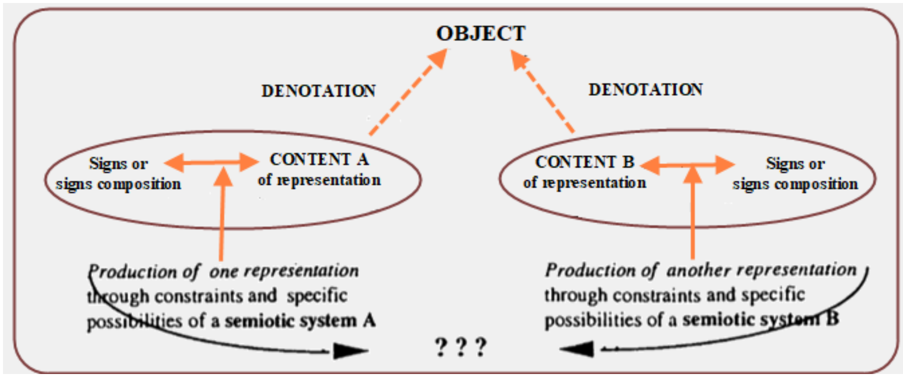


Figure 2.6. Representation and understanding for mathematical knowledge (Duval, 2000, p. 59) (adapted).

Goldin (1998a) in his study "*The PME Working Group on Representations*" noted several different meanings that have been given to the notion of representation "in connection with the learning, teaching, and development of mathematics (Goldin, 1998, p.285):

- "A. *External physical embodiments* (including computer environments)-any physical situation or set of situations external to the individual, which can be described mathematically or seen as embodying a mathematical concept; e.g., (1) a number line, drawn and labeled, illustrating order relationships among numbers; (2) a configuration of pegs on a peg-board providing an array model for multiplication; or, more broadly, the peg-board apparatus itself, (3) a calculator- or computer-based environment, within which mathematical constructs such as functions and graphs can be displayed and manipulated.
- B. *External linguistic embodiments*-we also took "representation" to include verbal, syntactic, and related semantic aspects of the commonly shared language in which mathematical problems are posed and mathematics is discussed.
- C. *Formal mathematical constructs*-still with emphasis on a problem environment external to the individual, a different meaning of "representation" is that of a formal structural or mathematical analysis of a situation or set of situations; e.g., (1) state-space representations of problems or games such as the Tower of Hanoi, Nim, etc.; (2) representations of mathematical entities, such as groups, rings, functions, etc., by means of other mathematical entities, such as linear operators on vector spaces representing elements of groups, graphs representing elements of function-spaces, etc. Though there is a sense in which all mathematics can be regarded as "internal" to individuals, the emphasis here was on "representation" as an analytical tool for formalizing or making precise mathematical ideas or mathematical behavior.
- D. *Internal cognitive representations*- we considered a very important meaning of the term "representation" to refer to internal, cognitive configurations of learners and problem-solvers. Thus we could talk about a student's internal, individual representation(s) of or for mathematical ideas such as "area," "functions," etc. We also considered systems of cognitive representation in a broader sense, as constructs to assist in describing the processes of human learning and problem solving in mathematics".

2.3. External and Internal Representations

Goldin (2008) in his study "*Perspectives on representation in mathematical learning and problem solving*" argues that "to discuss representation, we must be able to consider at a minimum configurations of symbols or objects *external* to the individual learner or problem solver, configurations *internal* to the individual, *relations* between them, and *structures* within and across them. These basic notions are essential to characterizing the nature of the patterns that mathematics is about" (p. 178).

Goldin (2008) defines the term "representation" using the notion of configuration. He defines representations by means of a number of synonymous verbs, also used by different researchers and scholars when they deal with the term "*representing configuration*".

> "*a representation is a configuration that can represent something else in some manner. [...] the representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest, or symbolize the represented one*". (Goldin, 2008, p.179)

A mathematical object is a creation in a person's mind that is formed as we have defined /or determined it through our experience or has been formed previously. The mathematical concept as it has been mentioned previously embody a web of relations between objects; they cannot be touchable through our daily experience neither through our sensory system just like the real of natural objects of the environment around us. In other words the mathematical concepts as mathematical objects are touchable only through their signs and the semiotic representations. Sakonidis (1994) considers that students / learners acquire the ability to use a representation in a gradual process which involves the following steps:

- "Identification of the elements of the representing world
- Establishment of relationships between the elements of the representing world
- Transformation of the above relationships to the ideas for which these elements stand for, that is, to relationships between elements of the represented world

Moreover, includes, the ability to move between representation systems". (p.42)

Verhoef & Broekman (2005) in their study "*A process of abstraction by representations of concepts*" consider that experience with objects in the real world is important for the development of students' knowledge and can be divided into *direct experience* with objects and *mediated experience* through media. They support that during the representation process "the representing medium (the representation) is related to the represented object (the reality) through a set of mapping principles that maps elements of the reality to elements in the representation".

- The term *pictures* have been chosen by Verhoef & Broekman to characterize the kind of representations "if they are (almost) similar to the represented object, such as photographs or statues. In these cases, [...] there is a one-to-one mapping or isomorphism between the two" (p. 274).
- The term *icons* have been chosen by Verhoef & Broekman to characterize the kind of representations that "[...] represent the represented object to some extend of similarity. [...] An example of this is the figure of a man or a woman on a toilet door. The relationship between an icon and the represented object depends on their 'mode of correspondence'" (p. 274).
- The term "*symbol*" has been chosen by Verhoef & Broekman to characterize the kind of representations when "[they] have no similarity at all with their represented object. These are chosen arbitrarily by convention. Examples of these are the letters of the alphabet, or numerals" (p. 274).

De Vries, Demetriadis and Ainsworth (2009) identify "a pervasive underlying distinction into dyadic and triadic views of representation:

- From a cognitive perspective a representation can be characterized as dyadic, referring to Palmer's definition: a representation is something that stands for something else.
- From a triadic perspective, "a representation involves three entities: [...] the referent or object existing in the world, the signifier or representamen (i.e., a mark, an idea, a word, an image, a sound, a smell), and the signified or interpretant (the idea evoked in someone's head), referring to Peirce's definitions of a sign" (de Vries, Demetriadis and Ainsworth, 2009, p. 139).

The most researchers and scholars agree that a potentially useful distinction can be drawn between *external representations* and *internal representations*. Others, think there is no such distinction. According to Sakonidis (1994)

> "Mason (1987) and von Glasersfeld (1987) criticize the internal/external distinction, on the grounds that for the child inner representations are not a representation of the real world but of a child's inner world. Von Glasersfeld suggests that is more appropriate to talk about inner experiences, and their expression in terms of pictures, diagrams, words or symbols as a presentation of an inner world"(Sakonidis, 1994, p.41)

Tschoshanov (2013) also states that "scholars claim that representation could refer to both internal and external manifestations of concepts (Pape & Tchoshanov, 2001)" (p.73).

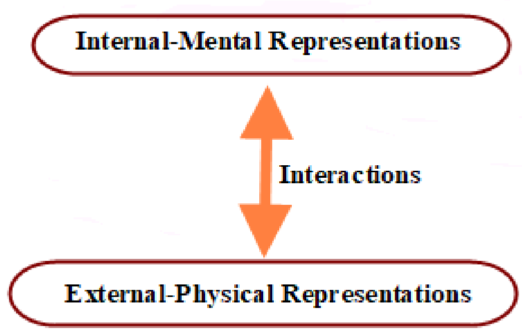


Figure 2.7. Internal versus external representations (Goldin & Kaput, 1996, p. 399) (adapted)

Goldin & Kaput (1996) in their study "*A joint perspective on the idea of representation in learning and doing mathematics*" provide a "sound basis for further development" with regard to the concept of representation in the psychology of mathematical learning and problem solving. They distinguish internal from external representation:

- With the term *internal representation* Goldin & Kaput (1996) "refer to possible mental configurations of individuals, such as learners or problem solvers. Of course, being internal, such configurations are not directly observable" (p. 399). Also, they do not "refer to the direct object of introspective activity […] although the experience of introspection is subjective, the descriptions that result from introspection are observable as, for example, verbal and gestural behavior" (p.400).
- With the term *external representation* Goldin & Kaput (1996) "refer to physically embodied, observable configurations such as words, graphs, pictures, equations, or computer microworlds. These are in principle accessible to observation by anyone with suitable knowledge" (p. 400).

Goldin & Kaput (1996) depict *an interaction between mental representations* ("as those […] that are encoded in the human brain and nervous system and are to be inferred from observation") (p. 402) *and external representations* (as those accessible to direct observation, for example, written words, speech, formulas, concrete manipulatives, computer microworlds as they appear on a screen […]). Figure 2.7 presents a correspondence between what is accessible by and what is in the human brain.

For example, if a teacher writes on a computer screen the formula sin (2x-1) (an external representation) the resulting function should plot a sinusoidal curve (Figure 2.8). The students may mentally relate the formula with the (internal) visual image of the graph as a sinusoidal curve representing the graphic representation of the function written with the symbolic expression.

According to Goldin & Kaput (1996) "[…] intrinsically, an interaction or at of interpretation is involved in the relation between that which is representing and that which is represented (von Glasersfeld, 1987) (p. 399).

Such correspondence "involve complex prior constructions achieved through representational acts" (Goldin & Kaput, 1996, p. 401). If the student has developed an interaction between the external and the internal representation of the concept then s/he has developed the level of understanding of this concept.

The above example reminds me of an example provided by Karadag (2009) who explains the "*dynamically nested RBC model of abstraction*" introduced by Hershkowitz, Schwarz, and Dreyfus (2001) [1] (see also, Schwarz, Hershkowitz, & Dreyfus, 2002). Karadag (2009) clarifies that

> " […] in order to calculate sin 2x by knowing the value of sin x or cos x, students need *to recognize* (or is guided to recognize) that they can use sin (A+B) as a reference point. By taking summation identity of the trigonometry, they can build sin 2x identity with sin (A+B) = sin (x+x). After obtaining sin2x=2sinxcosx, they can construct this knowledge to produce the formulas for sin 3x, cos 2x, etc. *In order to produce new knowledge structures, the process starts from the beginning*" (p. 24).

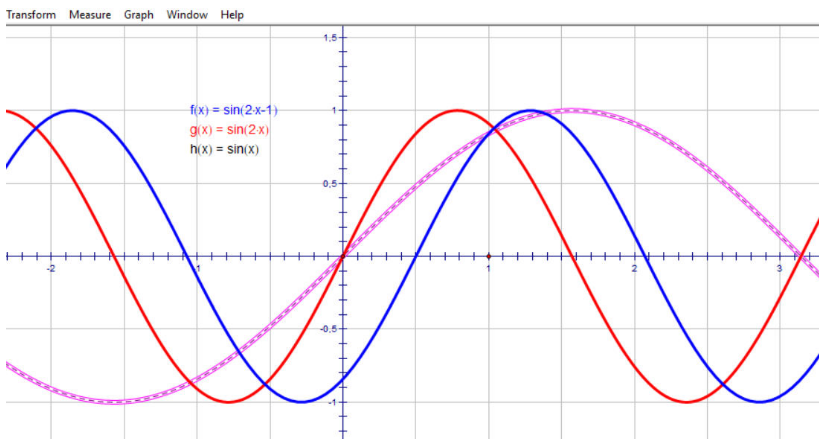


Figure 2.8. Sinusoidal curves graphically representing the symbolic form of the trigonometric functions (constructed in Sketchpad dynamic geometry software)

Similarly, Piaget (1937/1971) claims that we understand new constructs by assimilating or accommodating them into our pre-existing cognitive structures. Piaget and Inhelder (1956) also pointed out that a student has developed his/her *perceptual thinking* when s/he can internally manipulate his *mental images*. They emphasize that in this case the students has been fully developed their *representational thinking*. According to Pape & Tschoshanov(2001)

> "[…] *representational thinking* [is] the learner's ability to interpret, construct, and operate (communicate) effectively with both forms of representations, external and internal, individually and within social situations" (p. 120).

Tschoshanov *(2013)* poses a key question concerning the relationship between external/ internal representations in learning and the meanings of *assimilation/accommodation*: *"how students' internal schemata assimilates external representations, and how new external representations help students to accommodate their emerging internal representations". (p. 74).* This assumption is depicted in the **Figure 2.9** (Tchoshanov, 2013, p. 74), namely an interplay between students' external and internal representations in developing understanding of a concept (the concept five). According to Tchoshanov (2013) "the development of student's representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images" (p. 74)

[1] "Hershkowitz, Schwarz, and Dreyfus (2001) presented a theoretical and practical model for the cognitive analysis of abstracting in mathematics learning.[…] Processes of knowledge construction are expressed in the model through three observable and identifiable epistemic actions: *Recognising*, *building-with, and constructing (*RBC). *Recognising* takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building-with* is an action comprising the combination of recognised constructs in order to achieve a localised goal, such as the actualisation of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* consists of assembling and integrating previous constructs by vertical mathematisation to produce a new construct. […] *Vertical mathematisation* represents the process of constructing new mathematical knowledge within the mathematics itself and by mathematical means" (Hershkowitz et al., 2007, p. 44) .

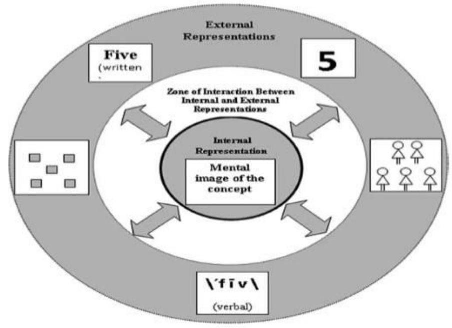


Figure 2.9. Tchoshanov's (2013, p.74) illustration for the relationship between external and internal representations in developing understanding of the concept five (Tchoshanov, 2013, p. 74, modified from Pape and Tchoshanov, 2001)

In the words of Mesquita (1998) the terms "external" and "iconical" representation in geometry, are used in the following senses: "External, embodied materially on paper or other support; Iconical, or figurative: centered on visual image (in opposition to other possible semiotic systems). We also use the term "figure"' as a synonym for external and iconical representation of a concept or a situation in geometry" (p. 183). Mesquita (1998) argues that the "external representation of a geometrical problem, per se, does not enable one to solve the problem, but it may contribute to the definition of the structure of the problem in order to facilitate treatments" (p.184).
Cifarelli (1998) examined the development of mental representations during the problem solving situations, involving a constructivist point of view. The mental representations occur as a mathematical conceptualization during problem solving. Cifarelli (1998) claims that students develop three increasing abstract levels of solution activity (**Figure 2.10**).

TABLE 3. Levels of Conceptual Structure

	Level of Activity	Attributes	Examples
High	**Structural Abstraction** (4)[a]	Solver can "run through" potential solution activity in thought and operate on its results	Solver can draw inferences from results of potential activity without the need to carry out solution activity
	Re–Presentation (2)	Solver can "run through" prior activity in thought	Solver can anticipate potential difficulties
	Recognition (2)	Solver encounters new situation and identifies activity from previous tasks as relevant for solving current task	Solver recognizes diagrammatic analysis activity as appropriate for solving Tasks 2–9
Low			

a. Indicates number of solvers achieving given level.

Figure 2.10. Levels of Conceptual Structure (Cifarelli, 1998, p. 246) (adapted)

According to Cifarelli (1998) "The construct of problem representation has played a central role in describing the knowledge that learners bring to mathematical problem solving situations […]." (p. 239). Cifarelli complementary states that "we need to reconsider traditional views of representation, and adopt a perspective which:

- acknowledges both the *constructive function* of representation in the development of conceptual knowledge and the *resulting mental objects* which solvers can then reflect on and transform as they interpret problem situations. […]
- the process of representation appears much more *dynamic* than previously articulated by traditional theories of mathematics learning.[…]
- the finding that the solvers demonstrated *increasingly abstract levels* of solution activity while solving the problems suggests the need to address qualitative aspects of mathematical performance seldom considered as important in the study of representations in mathematical problem solving.[…]" (p.262)

The term ''representation'' in the words of Scaife and Rogers (1996)

"has a variety of different meanings, depending on the context. A common distinction is between representation as process, and representation as product, as the outcome of this process. Process concerns the transformations and preservations that occur in deriving the representation from what is being represented" (p. 190)

In the Table 2.2. below, I have brought together indicative definitions of the notion of representation and how researchers conceive its role.

Table 2.2. **The Notion of Representation**	
Author /s	Definition of the notion of representation
Stephen Palmer (1977b)	A representation is "first and foremost something that stands for something else" There are five elements/aspects involved in a representation: (i) [what] the represented world is (ii) [what] the representing world is (iii) [what] aspects of the represented world are being represented /[modeled] (iv) [what] aspects of the representing world are doing the representing /[modeling] (v) [what] are the correspondence between the two worlds (cited in Sherin, 2000, p.404)
Johnson-Laird (1983)	• *Propositions* are strings or symbols that correspond to natural language • *Mental models* are structural analogues of the world • *Images* are perceptual correlates of models from a particular point of view (cited in Sakonidis, 1994).
Lesh, Post & Behr (1987)	"The term representation is interpreted in a naive and restricted sense as external (and therefore observable) embodiments of students' internal conceptualizations-although this external/internal dichotomy is artificial" (p.33).(Webpage [10])
Kaput (1991)	has distinguished between • *mental structures* as means by which an individual organizes and manages the flow of experience, and • *notation systems* as materially realizable cultural or linguistic artifacts shared by a cultural or language community.(p.55)
Seeger, Voight & Werchescio (1998)	"[…] a mental reproduction of a former mental state" "a structurally equivalent 'presentation' through pictures, symbols or signs," and "something 'in place of' something else" (Seeger, 1998, p. 311 cited in Pape & Tchoshanov, 2001, p. 120).
Pape & Tchoshanov (2001)	"use the term representation(s) to refer to both the internal and external manifestations of mathematical concepts. [They] write representation(s) with the parenthetical "s" to emphasize that, [they] are speaking of both the act of representing (the verb, to represent) and the external form of the representation (the noun form)" (p. 118).
Tschoshanov (2013)	"as external stimuli (numerals, equations, graphs, tables, diagrams, etc.) of concepts or internal cognitive schemata — abstractions of ideas that are developed by a learner through experience. Representation could also refer to the act of externalizing an internal, mental abstraction" (p.73). defines *representational thinking* "as the ability of the student to construct, interpret, and communicate effectively with both

	forms of representations, external and internal, individually and in social context” (p. 75)

In other words, researchers consider representations to be actions developed within the mind of the learner, but also the object created / brought into being during the action, using static or dynamic means. Computer microworlds can be viewed as specific forms of *external representations* or *external representational systems*. Having taken on board the aforementioned definitions in the literature, I think that a representation is both (a) *an external entity* (such as a verbal expression, a graph, a figure, a map, a picture), which is to say an external correspondence of objects or processes with the objects that are represented by the entities brought into being as representing objects by the modelling process, and (b) *an internal mental entity,* meaning a structurally equivalent modification of physical/mental objects/processes which are constructed in the mind as a result of the processing/elaboration of information and the manipulation of objects and concepts due to the cognitive schemes which have developed in the subject’s mind.

2.4. Multiple External Representations

Students’ mathematical thinking can be “represented” using different modes. Bruner (1966) proposed three *sequential modes of representation* that can be used to build a hypothetical learning path for the learning of a concept (for example, the number 7):

- *Enactive representation*
  This mode of representation is based on actions. For example, the pupil uses his/her pencils to understand the number 7. S/he has to touch them, put them in order, count them, act on them and experiences the correspondence between the seven pencils and number “7”. S/he acts on them through *direct action*. The same happens if the pupil counts his/her fingers.
- *Iconic representation*
  This mode of representation is based on images. For example, the student uses an image of 7 pencils. S/he has to look at them and count them. S/he recalls the material objects-pencils in his mind. Before the age of 6 the pupil cannot classify the materials from more than one characteristic (e.g., combining color and size, or color and shape).
- *Symbolic representation*
  This mode of representation is based on symbols. For example, the student uses the symbol 7 that has replaced the image with the seven pencils, which has already been replaced by the seven material objects or manipulatives used to understand the concept. Now, s/he has constructed a mental representation of the numbers and begins to understand the concept of the number.

Bruner (1966) argues that a child’s development has to follow the sequential learning representational path for the mastering of the concepts: from concrete real-world objects representation through iconic representation to symbolic representation. The concrete materials help students develop connections between conceptual and procedural knowledge and makes new learning easier and more meaningful. These kinds of representation take under consideration an increasing degree of abstraction.

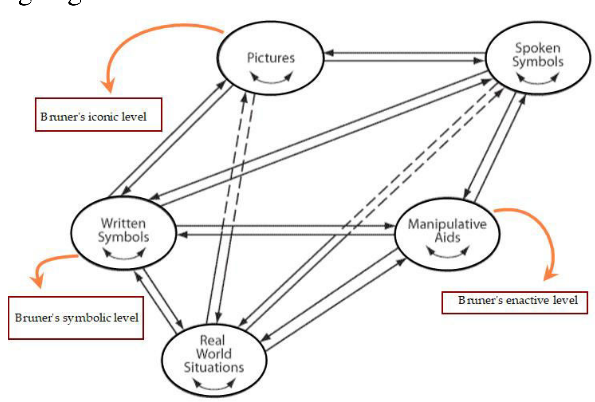


Figure 2.11a. Lesh’s model (1979) for translation between modes of representation (cited in Post, 1988, p.11) (an adaptation for the current study)

Post (1988) in his study “*Some notes on the nature of mathematics learning*” examines the implications that behavioral and cognitive theories have for the teacher in the mathematics classroom, as “*two broad theoretical umbrellas under which the vast majority of learning theories can be classified*” (p.1). According to Post (1988)

> “when learning a new concept, it is important that students “see” the concept from a variety of perspectives or interpretations. […] These modes, [shown **in** **Figure 2.11a**] represent an extension of Bruner’s early work in representational modes (Bruner, 1966). The term "*manipulative aids*" in this figure relates to Bruner's enactive level, "pictures" relates to Bruner's iconic level, and "written symbols" relates to Bruner's symbolic level. Lesh (1979) added verbalization ("spoken symbols") and "real-world situations" to Bruner's model and stressed the interdependence of these modes. Expanding (to five) the number of modes of representation and stressing the various translations within and among these modes are the two most important contributions of this model” (p. 10) […] Mathematical problem solving requires a move from the real-world situation to mathematical symbolism. Manipulative aids are in a sense halfway between the concrete real world of problem situations and the world of abstract ideas and mathematical symbols (written or oral). They are symbols in that they are made of physical materials, which in turn represent real-world situations” (p. 13).

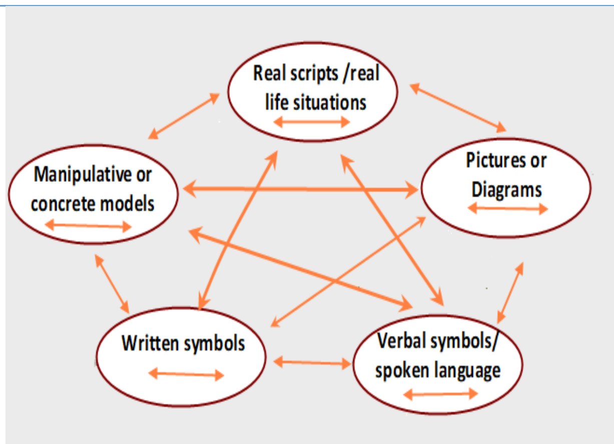


Figure 2. 11b. The Lesh (1979) multiple representation Translation model, adapted from Lesh, Post, and Behr (1987, p.34) (an adaptation for the current study).

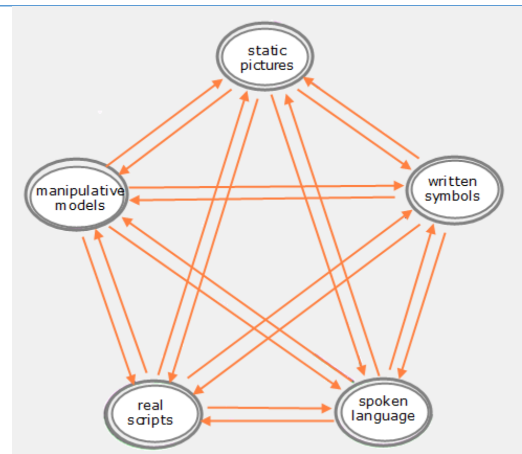


Figure 2.11c. Interplay among distinct types of representation systems (Lesh, Post, and Behr, 1987) (Webpage [10]) (adapted)

Behr, Lesh, Post, & Silver (1983) have identified five distinct types of representation systems that occur in mathematics learning and problem solving:

- *“experience-based "scripts*"-in which knowledge is organized around "real world" events that serve as general contexts for interpreting and solving other kinds of problem situations;
- *manipulatable models*-like […] arithmetic blocks, fraction bars, number lines, etc., in which the "elements" in the system have little meaning per se, but the "built in" relationships and operations fit many everyday situations;
- *pictures or diagrams*-static figural models that, […] can be internalized as "images";
- *spoken languages*-including specialized sub languages related to domains like logic, etc.;
- *written symbols*-which, like spoken languages, can involve specialized sentences and phrases (X+3=7, AUB) as well as normal English sentences and phrases’ (reported in Lesh, Post & Behr, 1987) (Webpage [10])

Lesh, Post and Behr (1987) proposed a multiple representation model in which they suggest a student understands a concept if s/he has the competence to translate between different modes of representation of the concept. Many similar figures have been constructed. For example Lesh & Doerr (2003) replaced the “Real scripts/or Real life situations” mode of representation with the “Experienced-based Metaphors”, adding by this new information in the multiple representation figure (**Figures 2.11b, c**).

Lesh (1979) considers that translation among representations in problem solving process occurs in three steps: “translating from the given situation to a mathematical model; transforming the model so that the desired results are apparent; translating the model based result back to the original problem situation to see if it is helpful and makes sense" (cited in Shavelson et al., 1987, p.4). Lesh Post and Behr (1987) identify five steps in the translation process, concerning to modeling a problem in mathematics: "simplifying the problem by ignoring irrelevant information; mapping between the givens and the "model"; transforming the properties of the model to arrive at a result; translating the result back to the givens; evaluating the fit of the result to the givens" (cited in Shavelson et al., 1987, p.4).

Lesh, Landau & Hamilton (1983), Behr, Lesh, Post, & Silver (1983), Lesh, Post & Behr (1987) emphasize the "*translation among these distinct types of representational systems and transformations within them*" (Figure 2.11d). Lesh, Post & Behr (1987) argue that a student *understands a concept or an idea* (for example what does it mean 1/3) if s/he: "(a) can recognize the concept in different representational systems ; (b) can flexibly manipulate the idea within given representational systems and (c) can accurately translate the idea from one system to another" .

For example if a student reads a mathematical word problem, s/he understands it if s/he can reformulate it in his/her own words. This is *a transformation* within the same representational system. S/he can also use a symbolic expression to express it [the problem] using mathematical symbols. This is *a translation* among different representational systems.

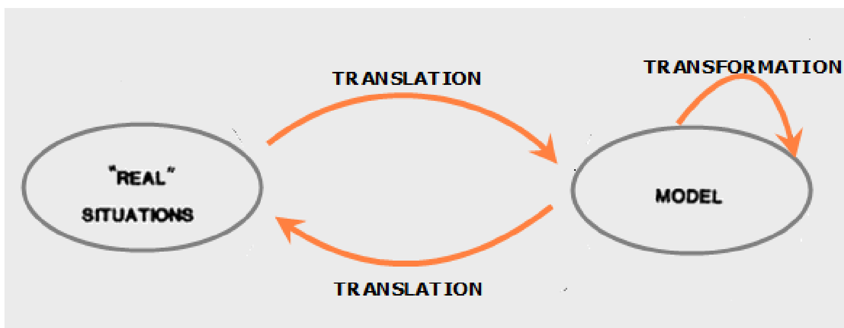


Figure 2.11d. Translations and transformations during problem solving (Lesh, Landau & Hamilton, 1983) (Webpage [11]) (adapted)

As mentioned above, we can represent a concept with multiple representations, such as pictorial representations, verbal representations, real-world representations, manipulatives or concrete representations, and symbolic representations.

• ***Pictorial representations or iconic representations***: Pictorial representations are any two–dimensional pictures generated in a paper-pencil or computer environment which represent concrete objects. (e.g., Ainsworth, 1999a, b; Ainsworth et al., 2002; Tabachneck-Schijf & Simon, 1998; Gagatsis & Elia, 2004; Gagatsis, Spyrou, Kapetanidou, Patsiomitou & Evangelidou, 2004, in Greek). These pictures (e.g., a photograph, a picture, a graph, a map) can be generated by the teachers, the students or they form part of a problem in textbooks. Students/teachers can also construct their own pictures in a static or dynamic environment in order to experience several aspects of mathematical ideas and meanings kinesthetically.

Carney and Levin (2002) study has proved that the function of pictures/images in mathematics can be very influential. They identify five different functions for pictorial representations in mathematics problems and tasks (e.g., *decorative, representational, organizational, interpretational and transformational*)(reported in Finesilver, 2014, p. 72):

- *"Decorative pictures* simply decorate the page, bearing little or no relationship to the text content,
- *Representational pictures* mirror part or all of the text content.
- *Organizational pictures* provide a structural framework for the text content.
- *Interpretational pictures* help to clarify difficult text, and
- *Transformational pictures* include systematic mnemonic components designed to improve recall of text information".

Many studies implemented this model (e.g., Gagatsis & Elia, 2004) to investigate how students perceive the pictorial representations in mathematics. Finesilver (2014) concludes that there is a "relationship between the development of representational strategies and multiplicative thinking." (p.2).

Lesh, Landau & Hamilton (1983) put pictorial representations in the centre in the following figure (**Figure 2.11e**) during problem solving process because a picture can help a student to understand fractions as "s/he can express fraction ideas presented with circular regions using rectangular regions, or using written symbols" (Webpage [11]).

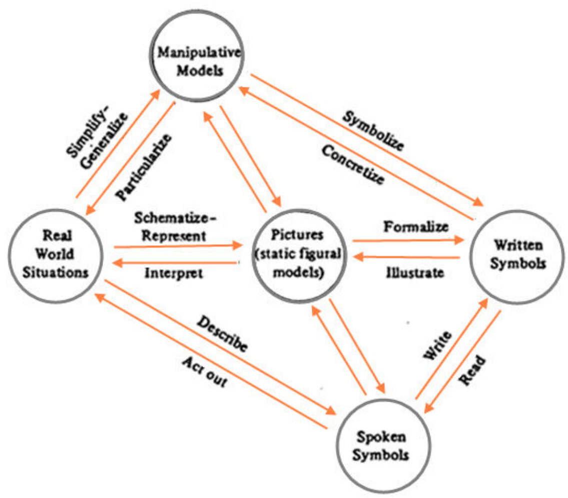


Figure 2.11e. Translations among modes of representation during problem solving (Lesh, Landau & Hamilton, 1983) (Webpage [11]) (adapted)

Johnson-Laird (2010) in his study '*Mental models and human reasoning*" states that iconic representations help persons to visualize a verbal expression of a problem and how the different objects mentioned in the problem relate to each other. As he writes:

> "A visual image is iconic, but icons can also represent states of affairs that cannot be visualized, for example, the 3D spatial representations of congenitally blind individuals, or the abstract relations between sets that we all represent. *One great advantage of an iconic representation* is that it yields relations that were not asserted in the premises (24, 28, 29). Suppose, for example, you learn the spatial relations among five objects, such as that A is to the left of B, B is to the left of C, D is in front of A, and E is in front of C, and you are asked, "What is the relation between D and E?"[…] "You could use formal rules to infer this relation, given an axiom capturing the transitivity of "is to the left of." (p. 2)

An iconic representation can also be used in a problem presented in a DGS environment. If we copy-paste a picture into a DGS environment, we can process it using the tools provided by the software. In this case, the picture becomes an illustration that can help students to understand and organize the objects in the picture (e.g., Patsiomitou, 2014).

• ***Verbal representations***: These are representations that are generated through the language and verbal expressions we use while discoursing in a mathematics class. Examples of verbal representations include the definitions, theorems or geometrical properties, that a student formulates in support of his/her logical reasoning as s/he tries to solve an equation or a geometrical or mathematical problem. But the students fail to support their thinking when they do not know the exact terminology in mathematics, or when they confuse the meanings. Moreover, when students do not understand a concept, they cannot "speak" about it. Vergnaud (2009) in his study "*The Theory of Conceptual Fields*" considers that the linguistic and symbolic expressions are a "part" of a concept which can be developed during *didactic situations*:

> "Because language and symbols play an important role in the conceptualizing process, many researchers identify conceptualization and symbolization, as if the wording and symbolizing activity were sufficient roots of knowledge, particularly mathematical knowledge. This is not the case. The analysis of situations and schemes shows that the conceptualizing process already takes place in the simplest forms of activity (even without language): the reason is that no action can be efficient without the identification of some objects and their properties. Even more complex concepts, to gain sense and operationality, need to be contextualized and exemplified in situations. Therefore, from a developmental point of view, a concept is altogether: a set of situations, a set of operational invariants (contained in schemes), and a set of linguistic and symbolic representations" (p. 94).

Skinner (1957/1992) in his monograph "*Verbal Behavior*" poses a very important issue concerning the correspondence between verbal expressions and the things or situations these verbal expressions represent:

> "It has been tempting to try to establish the separate existence of words and meanings because a fairly elegant solution of certain problems then becomes available. Theories of meaning usually deal with corresponding arrays of words and things. How do the linguistic entities on one side correspond with the things or events which are their meanings on the other side, and what is the nature of the relation between them called "reference"? Dictionaries seem, at first blush, to support the notion of such arrays. But dictionaries do not give meanings; at best they give words having the same meanings. The semantic scheme, as usually conceived, has interesting properties. Mathematicians, logicians, and information theorists have explored possible modes of correspondence at length. For example, to what extent can the dimensions of the thing communicated be represented in the dimensions of the communicating medium? But it remains to be shown that such constructions bear any close resemblance to the products of genuine linguistic activities" (p.41).

This is a very important issue and one that every teacher may find themselves facing when s/he tries to teach a concept in class using only verbal expressions (e.g., a lecture). A few students will be unable to understand the teacher, because they cannot translate the information in their mind according to their pre-existing structures, or because they simply cannot imagine it. Certainly, when we teach geometry or mathematics, a figure (or a graphic representation) generally contributes to a better understanding of the concepts. This issue is supported theoretically by the theory developed by Paivio (1986), as well as by Baddeley's (1986) model of the *architecture of memory*.

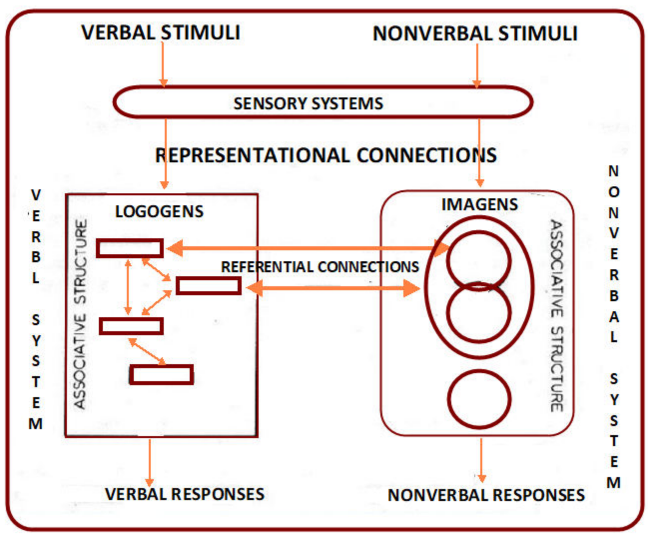


Figure 2. 12: The Dual Coding Theory (Paivio, 1986, cited in Gilbert, 2010, p.4) (adapted)

Paivio (1986) in his "*Dual Coding Theory*" proposes that when a person /a student is studying a subject, s/he encounters the meaning of the concrete subject (e.g. quadrilaterals) in a network of words and ideas (Paivio attaches a common label 'logogens' to the verbal information) and separately in the images or non-verbal information (e.g., information received through touch, sight, sound, taste) relating to the concrete subject (Paivio attaches a common label "imagens" to the non-verbal information). They can be linked together to provide an understanding of the subject (Figure 2.12). Gilbert (2010) states that

> "Paivio proposes that verbal stimuli – those which come in verbal form– and non-verbal stimuli – are processed in different ways by sensory systems that are in common to them both. […]These can be linked together to provide an enriched understanding of that system. Most importantly, the two types of associative structures are capable of 'cross-linking' to form 'referential connections'. […]When called upon to do so, an individual will either produce a verbal or a non-verbal output based on the relevant associative structures, or will produce one or both of them based on the referential structures that have been developed. As the presentation of a comprehensive account of verbal stimuli, non-verbal stimuli, their associations and referential connections would be very lengthy, this introductory paper is only concerned with those non-verbal stimuli presented in visual form" (pp. 3-4).

Moreover, according to Baddeley's (1986) model, if a verbal expression is accompanied by a visual picture of the object, their relation will be strengthened. De Vries, Demetriadis and Ainsworth (2009) distinguish internal representations to: *propositional representation, mental images and mental models.*

> "Following Paivio (1971, 1990), cognitive psychology has typically distinguished two types of internal representations depending on the type of correspondence relations: *propositional representation*, which is a verbal or text-like mode, and *mental images* which correspond to a visual-pictorial mode of representation. In addition a third kind is often postulated which are *mental models* as structural or logical analogues of the word (Johnson-Laird, 1983)" (de Vries, Demetriadis and Ainsworth, 2009, p. 139)

Mayer and Anderson (1992) claim that a student has to construct three types of connections during a meaningful learning process, connecting iconic with verbal representations:

- "*representational connections between verbal information* that is presented and the learner's verbal representation of that information;
- *representational connections between pictorial information* that is presented and the learner's visual representation of that information; and
- *referential connections between corresponding elements* in the learner's verbal and visual representations." (Reported in Sullivan, 2004, p. 9).

Mayer and Moreno (1998) propose also that meaningful learning occurs when "five active cognitive processes are involved in learning from multimedia presentations: *selecting words, selecting images, organizing words, organizing images, and integrating words and images*. This has become known as the SOI (Select, Organize, and Integrate) model of meaningful learning. Selecting words and images equates to building mental representations in verbal and visual working memory (respectively). Organizing words and images consists of building internal connections among either the propositions or the images (in that order). Integrating implies building external connections between a proposition and its corresponding image"(Sullivan, 2004, p. 7).

• ***Symbolic representations***: These are representations which include/incorporate symbols such as letters, numbers, other symbols, formulas, operations on numbers and formulas, arithmetic, algebraic or geometric symbols (e.g., Vergnaud, 1988; Ainsworth, 1999a, b; Johnson, 2017).

For example, the solution of an equation represents the structure of a symbolic representation in which a student performs calculations between numbers of different variables. The symbols can have different meanings, depending on the framework in which we implement them. For example, the symbol "<" has different meanings depending on whether it is implemented in an algebraic or geometric utterance (e.g., $3x + 2 < 5$, $< xOy = 90^{o}$). Symbolic representations may be produced in a static or a computer environment.

Kalavasis (2018) in his study "*Mathematics and the real world in a systemic perspective of the school"* presents examples of the history and epistemology of mathematics, (e.g. the figurate numbers) and their symbolic representations as they have been conceived by Pythagoreans (**Figure 2.13**).

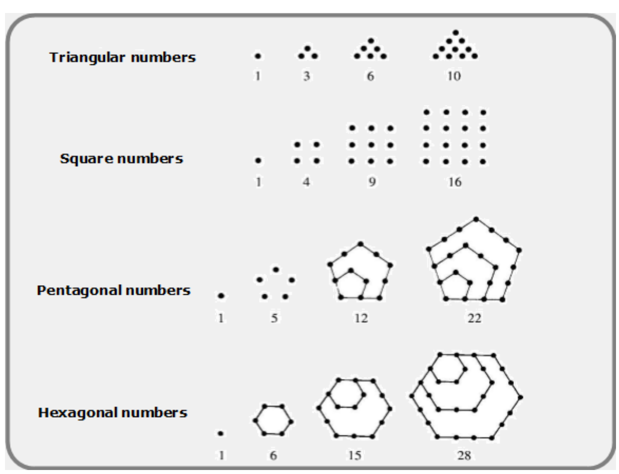


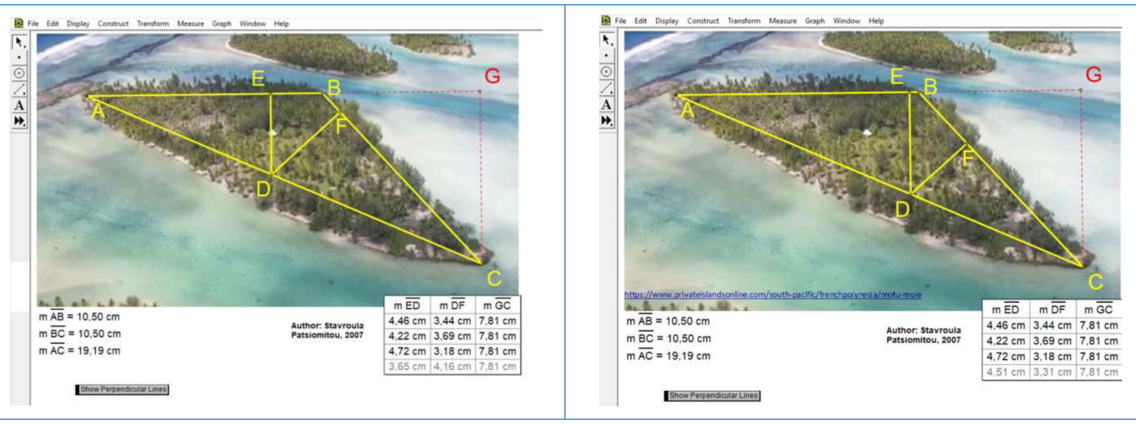
Figure 2. 13: Symbolic representations of the figurate numbers (conceived by Pythagoreans, cited in Kalavasis, 2018, p.17) (adapted)

Kalavasis (2017) states that the figurate numbers "evolved their representational constructions using the practical and noetic instrument of the *gnomon" (p.16).* As Kalavasis argues:

“The role of representations and symbolic languages, playing a crucial role in mathematics, becomes an obstacle in the interdisciplinary learning path of the students in the everyday school timetable across their differentiated uses in the different disciplines. Thus, the widely studied didactical transposition is effectively enriched with the praxeological transposition” (p. 9).

• ***Real-world representations****:* These representations are correlated with situations, events and objects that take place in the real world. The students who use these representations are supported to make mathematical connections among the objects in the real world and the abstract mathematical meanings (e.g., Lesh, Post & Behr, 1987). Real-world representations may be produced in a static or a computer environment.

For example, in the **Figures 2.14a, b,** I have pasted a picture of an island into a DGS environment; it is an isosceles triangular shape and can support the solution to Viviani’s theorem [Vincenzo Viviani (1622-1703)]. Point D lies on the base of the isosceles triangle. As we know, the sum S of the length of the perpendiculars from the point D to the sides is equal to the altidute h (CG in the **Figures 2.14 a, b**). The students can experimentally prove that the sum S will not be modified if we change the position of point D.



Figures 2.14a, b. A real world representation for Viviani’s theorem in a dynamic environment (Patsiomitou, in press)

Viviani’s problem could be reformulated as following: “A man lives in a triangular island. His house is located on a side of the equilateral triangle. Every morning he starts out from his house to buy several things and walks along two paths that are perpendicular to the other sides of the triangle. He counts his steps and finds that even if he does not always starts from his house, but from the side of the triangle on which his house stands, and walks along perpendiculars to other sides, he always walks the same distance in total. Could you explain this?”

This representation is complex as it combines pictorial, symbolic and real-world options, which I implemented in a dynamic environment.

Tünde Kántor (2013) investigates many occasions of Viviani’s Theorem. Tünde Kántor (2013) gives among others the following benefits of using historical problems (p.81):

- “We can show the continuity of mathematical concepts and processes over past centuries […]
- We motivate learning process in the classroom, because our pupils deal with problems which were objects of investigation centuries ago. […]
- Pupils connect mathematics to various cultures and other intellectual developments in science […]”

A real-world representation can be an interpretation of a real-world problem. Such problems are incorporated in the “*Nine Chapters on the Mathematical Art” (Jiuzhang suanshu).* According to O’Connor and Robertson (2003) *“Jiuzhang suanshu* is a practical handbook of mathematics consisting of 246 problems intended to provide methods to be used to solve everyday problems of engineering, surveying, trade, and taxation.” (Webpage [33]).

I chose to set the following problem for my university students last year, as I think it is very interesting:

“*There is a square town of unknown dimensions. There is a gate in the middle of each side. Twenty paces outside the North Gate is a tree. If one leaves the town by the South Gate, walks 14 paces due south, then walks due west for 1775 paces, the tree will just come into view. What are the dimensions of the town*”.

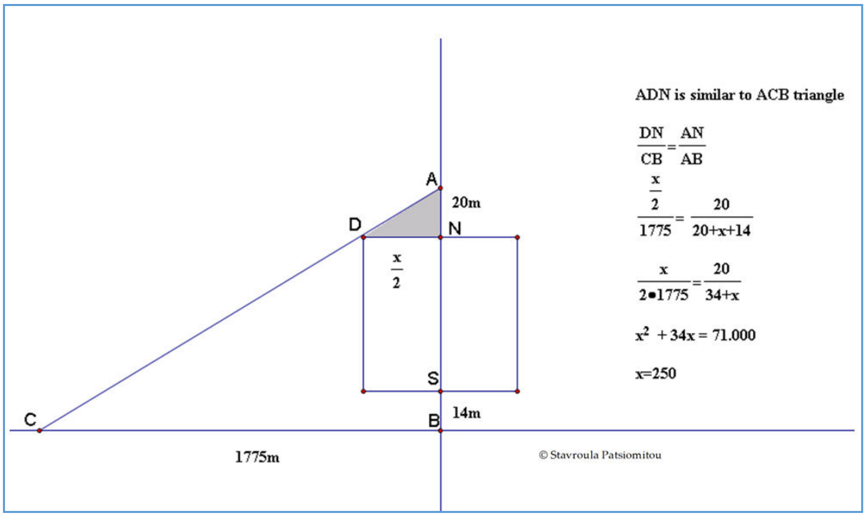


Figure 2.15. Solution to the real-world problem in a DGS environment

My questions on the problem concern the way that proactive teachers of mathematics would use the concrete problem in class. What kind of representations would they use, how would they model the problem etc.? **Figure 2.15** is an image of the solution to the problem resulting from the interpretation of the problem into the DGS environment.

- ***Manipulatives, or concrete representations***: These are objects (e.g., Dienes cubes, geoboards, pattern blocks, fraction pieces) which are designed to mediate between a particular mathematical concept and the way students learn the concept. Students can manipulate them by touching or moving, and thus are concrete means (Dienes, 1960; Baroody, 1989; Van de Walle, 2005; Johnson, 2017). Ross (2004) defines manipulatives as follows: "[…] materials that represent explicitly and concretely mathematical ideas that are abstract. They have visual and tactile appeal and can be manipulated by students through hands-on experiences" (p. 5).

Clements & Mcmillen (1996) in their study "*Rethinking "concrete" manipulatives*" argue that "attidutes towards mathematics are improved when students are instructed with concrete materials by teachers knowledgeable about their use[…]" (p.270).

Clements & Mcmillen (1996) in an extended and substantial study present the advantages/ key benefits of using computer manipulatives, and rethink the meaning of "concrete" manipulatives. They argue that" " (1) *Computers offer a manageable and clean manipulative, (2) Computers afford flexibility, (3) Computer manipulatives allow for changing the arrangement or representation, (4) Computers store and later retrieve configurations, (5) Computers record and replay students' actions, (6) Computer manipulatives link the concrete and the symbolic by means of feedback, (7) Computer manipulatives dynamically link multiple representations,* and *(8) Computers change the very nature of the manipulatives" (p.272-274).* Clements & Mcmillen highlight also the advantages of computer manipulatives for teaching and learning "*Computer manipulatives link the specific to the general, encourage problem posing and conjecturing, build scaffolding for problem solving, focus attention and increase motivation and encourage and facilitate complete, precise explanations "(p. 275-276).* They finally support that

> "Now when teachers close their eyes and picture children doing mathematics, manipulatives should still be in the picture, but the mental image should include a new perspective on how to use them" (p. 278).

Janvier (1987b) considers a representation to be a combination of both ingredients: external objects, as "written symbols and real objects" and "mental images". He created an illustration to present "a visual resemblance between a representation and a star" (**Figure 2.16**).

A strong argument that a student cannot understand a concept from one type of representation of the concept alone is that this type of representation cannot describe a mathematical concept thoroughly-- each representation has its own distinct advantages. The core of mathematical understanding can thus be reached /achieved through the use of multiple representations.

Janvier's (1987b) Model of multiple representations incorporates "Tables, Graphs, Formulations, Verbal Descriptions and Object". Janvier (1987b, c) considers that the translation (meaning the psychological process mediating between different forms of representations) occurs as the star turns around to appear another foot.

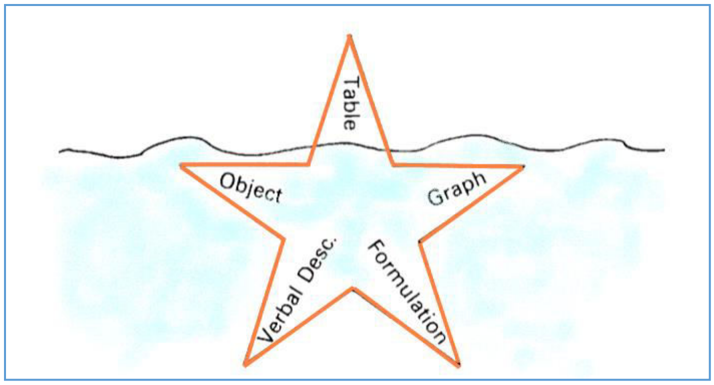


Figure 2.16. A visual resemblance between a representation and a star (Janvier, 1987b, p.69, cited in Coskun, 2011, p.33) (adapted)

Arcavi (2003) states that:

> "Another cognitive difficulty arises from the need to attain flexible and *competent translation back and forth between visual and analytic representations of the same situation*, which is at the core of understanding much of mathematics. Learning to understand and be competent in the handling of multiple representations can be a long-winded, context dependent, non-linear and even tortuous process for students (e.g. Schoenfeld, Smith and Arcavi, 1993). The sociological difficulties, include what Eisenberg and Dreyfus (1991) consider as issues of teaching. Their analysis suggests that teaching implies a "*didactical transposition*" (Chevallard, 1985) which, briefly stated, means the transformation knowledge undergoes when it is adapted from its scientific, academic character to the knowledge as it is to be taught" (p.38).

Johnson (2017) in her study "*A New Look at the Representations for Mathematical Concepts: Expanding on Lesh's Model of Representations of Mathematical Concepts*" expanded Lesh's model including the "technological type of representations". Johnson (2017) created an exagon to incorporate this model (p. 6). As Johnson argues "future research on representations should directly include technology as a distinct representation" (p.7).

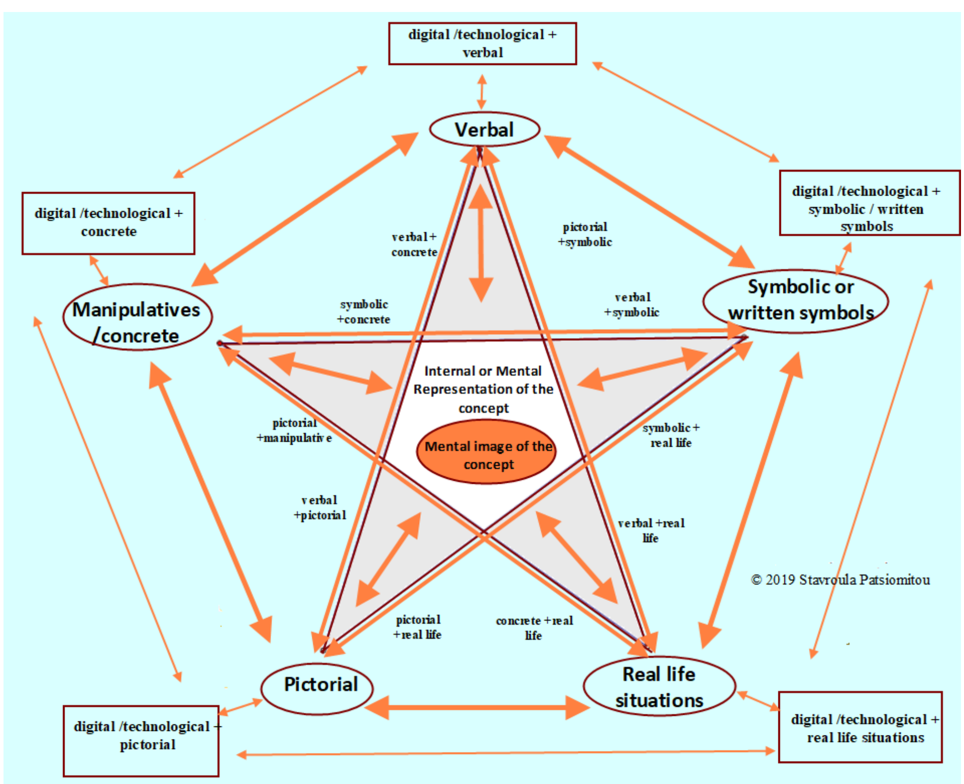


Figure 2.17. My proposal for the connections between multiple external digital representations and mental images of the concept for the development of understanding of the concept

Figure 2.17, which I created to illustrate connections between external and internal representations of a concept, incorporates Janvier's, Lesh's and Tchoshanov's translational model of multiple representations. I think that technological and digital representations that can be developed on several computer have the potential to change the way the students perceive the manipulation of objects, the written or oral language, as well as the symbolic and graphic representations provided to them. We can still provide animated real-life situations that enrich the problem-solving with an external representation which does not stand as an obstacle exactly as Mesquita (1998) reports.

In Figure 2.17 arrows connect the different modes as well as the different technological modes, as I think that every mode can be expanded to encompass its technological/digital version.

According to Kaput, Noss & Hoyles (2002) in their article "*Developing New Notations for a Learnable Mathematics in the Computational Era*" the aim to introduce and incorporate digital infrastructures in the teaching and learning of functions "is to put phenomena at the center of the representation experience, so children can see the results, in observable phenomena, of their actions on representations of the phenomenon, and vice versa. These are

- The definition and direct manipulation of graphically defined and editable functions, especially piecewise-defined functions […]
- Direct, hot-linked connections between functions and their derivatives or integrals. […]
- Direct connections between these new representations and simulations to allow immediate construction and execution of variation phenomena. […]
- Importing physical motion-data […] and reenacting it in simulations […] to drive physical phenomena (including cars on tracks)". (p. 19)

As Kaput, Noss & Hoyles (2002) conclude

> "Thus we wish to challenge our community to focus attention on the design and use of representational infrastructures that intimately link to students' personal experience. This is a necessary step if we are to move away from a 19th century school mathematics concentrating on isolated skills based on static representational systems in a tightly-defined curriculum (with only a minority able to engage in independent problem solving). Our contention is that knowledge produced in static, inert media can become learnable in new ways, and new representational infrastructures and systems of knowledge become possible, serving both the learnability of previously constructed knowledge and the construction of new knowledge" (p.39).

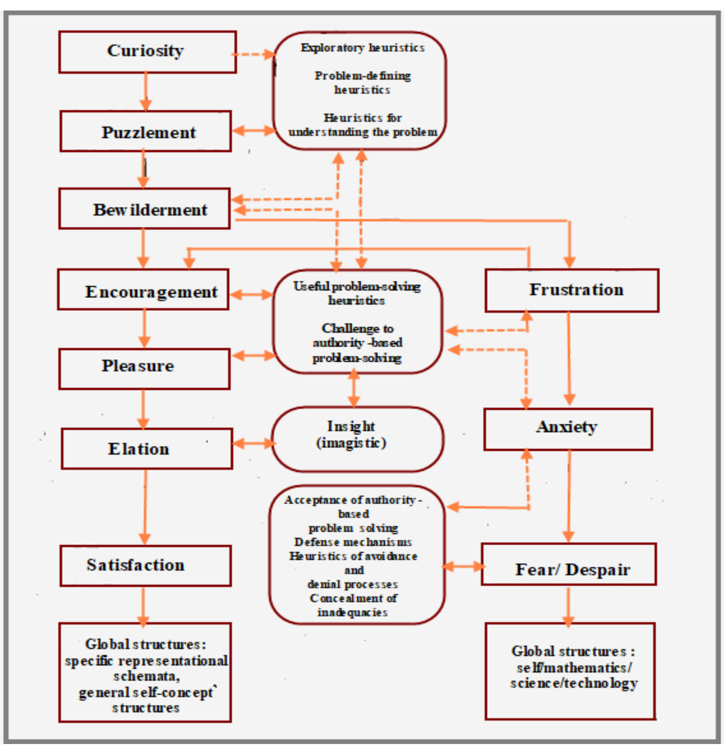


Figure 2.18. Affective states interacting with heuristic configurations (Goldin, 2000, p. 213) (an adaptation for the current study)

Goldin (2008) in his study considers to be five types of mature systems of internal representation (Goldin, 1987, 1992, 1998), psychologically fundamental, extending earlier “dual code” and “triple code” models (Paivio, 1983; Zajonnc, 1980). These are (Goldin, 2008, p. 184)

- *“Verbal /syntactic systems*, that include natural language capabilities-lexicographic competencies, verbal association, as well as grammar and syntax;
- *Imagistic systems,* including visual/spatial, tactile/kinesthetic, and auditory /rhythmic encoding;
- *Formal notational systems,* including the internal configurations corresponding to learned, conventional symbol-systems of mathematics (numeration, algebraic notation, etc.) and how to manipulate them;
- *A system of planning, monitoring and executive* control that guides problem solving, including strategic thinking, heuristics, and much of what are often referred to as *metacognitive* capabilities; and
- *An affective system* that includes not only the “global” affect associated with relatively stable beliefs and attitudes, but also the changing states of feeling as these occur during mathematical learning and problem solving. The characterization of *affective structures* is emerging as an important way to help understand students’ mathematical engagement and motivation” (p. 184)

Goldin has elaborated on the role that affective states play in the problem–solving process in numerous articles. Goldin (2000) in his study “*Affective Pathways and Representation in Mathematical Problem Solving*” constructs a realistic model from problem-solving competence. He outlines in the above figure (**Figure 2.18**) and discusses in the article, “two major affective pathways, one favorable and one unfavorable, together with conjectured relationships between affective states and useful or counterproductive heuristic configurations” (p. 209)
According to Goldin (2000)

> “The affective states described are not global attitudes or traits, but local changing states of feeling that the solver experiences and can utilize during problem solving-to store and provide useful information, facilitate monitoring, and evoke heuristic processes. Thus affect, like language, is seen as fundamentally representational as well as communicative)(p. 209)[…] affect is not incidental but fundamental, and it cannot be handled simply by a commitment to make mathematics fun or enjoyable.

Learning style is how a learner process information and prefer to learn. There are four main learning styles: (a) *Visual* (a person learns more effectively through seeing) (b) *Auditory* (a person learns more effectively through hearing) (c). *Kinesthetic* (a person learns more effectively through feeling) (d). *Tactile* (a person learns more effectively through touching). The terms learning style and cognitive style differ among scholars. Kordaki (2005) in her study “*The role of multiple representation systems in the enhancement of the learner model in open learning computer environments”* states that “learners seem to arrive at schools with different learning styles, such as: *intuitive, visual, holistic, field dependent, reflective, rational, analytic and field independent*” (p. 253). Hartley (2008) defines “*learning styles*” as the ways that the subjects/students/learners conduct their learning tasks. He also defines “*cognitive styles”* as the ways that the subjects conduct their cognitive tasks.
Ainsworth (1999) in her study “Designing effective multi-representational learning environments” supports that “Multi-representational learning environments are used by a wide range of learners in a number of domains and many advantages are claimed for their use. By using multiple external representations (MERs), it is hoped that learners can benefit from the properties of each of the representations and that ultimately this will lead to a deeper understanding of the subject being taught. However, research that has evaluated how effectively multi representational environments support learning has produced mixed results. A number of studies have shown that learners find working with MERs to be very difficult (e.g., Tabachneck, Leonardo & Simon, 1994; Yerushalmy, 1991).” (p. 1).
A few difficulties that can occur relate to the format of the representations as well as to the operators that act on them (Ainsworth, 1999b, p. 34). These kinds of difficulties are presented in the following table:

Table 2.3. Difficulties with MERS (Ainsworth, 1999b, p. 34)

Difficulties with MERS have to do with: (Ainsworth, 1999, p. 34)	**Referring to:**
1. “the modality of the representations – (propositional v graphical)” 2. “the levels of abstraction (e.g. concrete to symbolic representations) […]” 3. “the type of representation (e.g.	“differences in the format of representations (and hence their operators)”

histogram, equation, table, line-graph)[…]" 4. "the specificity of representations[…]" 5. "whether representations are static or dynamic […]" 6. "differences in labeling and symbols on the representations […]" 7. "alternative uses of representations […]"	
8. "the interface to the representations […]" 9. "self-constructed & selected representations versus pre-determined representations […]" 10. "whether the representations encourage different strategies […]"	"differences in operators as the format of these representations need not necessarily differ".

Consequently, designers of multi-representational learning environments are faced with the question of how to develop a system where the learners can benefit from the advantages of MERs. Ainsworth (1999a, 2006) introduced taxonomy of the functions of MERs and created a diagram to visualize theses functions (**Figure 2.19**). According to Ainsworth (1999a) in her study "The functions of multiple representations" supports that

> "A conceptual analysis of existing multi-representational learning environments suggests there are three main functions that MERs serve in learning situations – to complement, constrain and construct. *The first function* is to use representations that contain complementary information or support complementary cognitive processes. *In the second,* one representation is used to constrain possible (mis)interpretations in the use of another. *Finally, MERs* can be used to encourage learners to construct a deeper understanding of a situation" (p.3).

Complementary functions: MERs differ either in the processes each supports or in the information each contains (Ainsworth. 2006, p.188):

- *"Individual differences*: if learners are presented with a choice of representations, they can choose to work with the representation that best suits to their learning style
- *Task*: […] learners given MERs can benefit from choosing the best representation for the current task […].
- *Strategy:* Different forms of representation can encourage learners to use more or less effective strategies" […] "as each strategy has inherent weaknesses, switching between strategies made problem solving more successful by compensating for this"(p.188)

Constraining functions: "A second advantage of using MERS is that certain combinations of representations can help learning when one representation constrains interpretation of a second representation […]" (Ainsworth. 2006, p.188)

Constructing [deeper understanding] *functions*: MERs support deeper understanding "when learners integrate information from MERs to achieve insight […]" (Ainsworth. 2006, p. 189)

- *"Abstraction* is the process by which learners create mental entities that serve as the basis for new procedures and concept at a higher level of organization […]";
- *"Extension* can be considered as a way of extending knowledge that a learner has form a known to an unknown to representation, but without fundamentally reorganizing the nature of that knowledge […]"and
- *"Relational understanding* is the process by which two representations are associated again without reorganization of knowledge […]" (p. 189).

Ainsworth (2006) argues that "multiple external representations can provide unique benefits when people are learning complex ideas […] the effectiveness of multiple representations can best be understood by considering three fundamental aspects of learning: the design parameters […], the functions that multiple representations serve in supporting learning and the cognitive tasks that must be undertaken by a learner interacting with multiple representations" (p. 183)

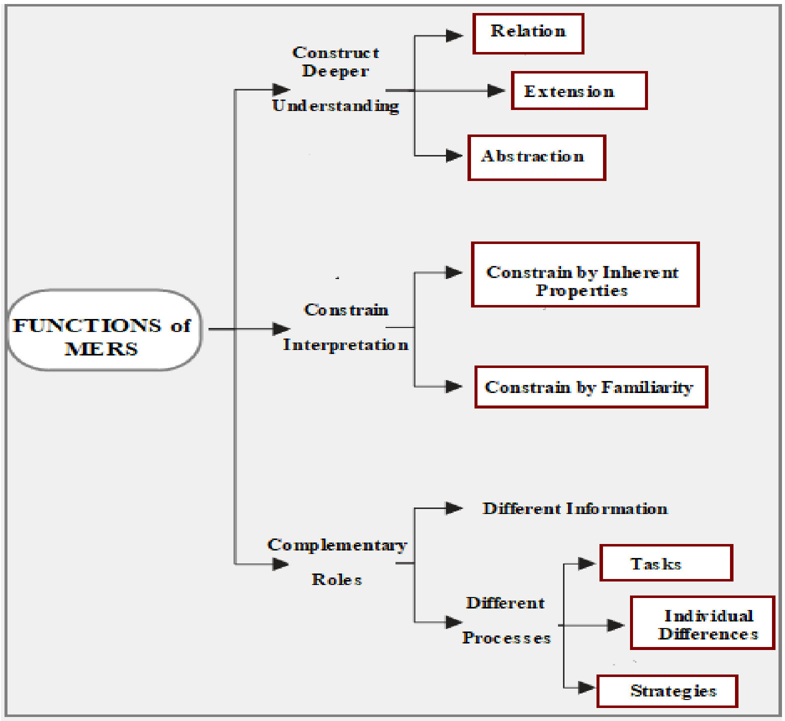


Figure 2.19. A taxonomy of functions of MERs (Ainsworth, 1999a; 2006, p. 187) (an adaptation for the current study)

A very powerful way to facilitate and enhance students' understanding can be achieved with the use of multiple representations, particularly in computer-based learning environments (e.g., Moreno 2002; Mayer& Moreno, 2003). A few examples of multiple representations in a computer-based learning environment include "interactive diagrams with embedded transcripts, […] video presentations, interactive graphs and forms, audio explanations of concepts, and still images" (Sankey, Birch and Gardiner, 2011, p. 20). Sankey, Birch and Gardiner (2011) argue that "students reported very favorably on their use of the multimodal learning elements and perceived that these had assisted comprehension and retention of the material" (p. 18). Wong, Yin, Yan and Cheng (2011) also in their study "*Using Computer-Assisted Multiple Representations in Learning Geometry Proofs*" propose and use a multimedia learning environment to let students interact with multiple representations relevant to a geometry proof. Concretely, they propose a "computer-assisted learning environment called MR Geo to help students in learning to do theorem proving, with the help of multiple representations including problem description, static figure, dynamic geometry figure, formal proof and proof tree" (p. 43). (**Figure 2. 20**)

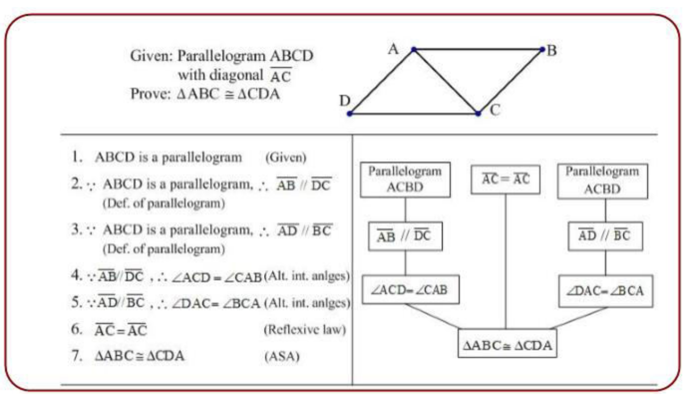


Figure 2.20. A formal proof and its proof tree provided by the MR Geo computer-assisted learning environment (Wong, Yin, Yang, & Cheng, 2011, p. 47)

According to Wong, Yin, Yang, & Cheng (2011, p. 52) "The connection between formal proof and proof tree raised students' comprehension of geometry proof. Some LG students indicated that after understanding the geometry proving process, they no longer hated geometry classes. The above results indicated that MR Geo might offer an attractive, alternative approach to geometry education with multiple representations in a computer-assisted learning environment, comparing to traditional classroom teaching".

2.5. Duval's Cognitive Model of Geometrical Reasoning

Duval (1995b, p.145-147) provides an analytic framework for analyzing the semiotics of geometric objects as theoretical and abstract objects. Duval identifies or distinguishes four types of cognitive apprehension, namely how we perceive (with our sensory system) and conceive (in our mind) a figure. These types of cognitive apprehension are the following (reported also in Jones, 1998; Deliyianni, Elia, Gagatsis, Monoyiou & Panaoura, 2009; Patsiomitou, 2011, 2012a, b, 2018b, 2019a, b; Forsythe, 2014):

- *perceptual apprehension:* this is what is recognised at first glance; how one perceives a figure, what are the sub-figures in the figure; in other words what one can view in the figure or perceive in regard of the objects that belong to the figure.
- *sequential apprehension*: how one understands the order of the construction steps; what are the geometric properties and definitions used for the construction of the figure. Using a DGS or computing environment generally a student can enrich his understanding of the different paths that can be used for the same construction of a figure (see also Gomes and Vergnaud, 2004, cited in Forsythe, 2014, p.40)
- *discursive apprehension*: how one verbalizes the construction steps and explicate/interpret the construction steps using reasoning; "the definition of a geometrical object and a description of its construction are part of discursive apprehension" (Forsythe, 2014, p.40)
- *operative apprehension*, how one operates the figure "which involves manipulating the figure mentally or physically to provide an insight into a problem" (Jones, 1998, p. 31). "Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way" (Deliyianni et al. 2009, p. 697).

Duval (1999) in his study "*Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning*", describes three kinds of operations delimited by how a given figure is transformed:

- "*The mereologic way:* you can divide the whole given figure into parts of various shapes [...] and you can combine these parts in another whole figure or you can make appear new subfigures.[...] We call «reconfiguration» the most typical operation.
- *The optic way:* you can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently [...].
- *The place way:* you can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and horizontal lines" (Duval, 1988, pp. 61-63; 1995, p.147).

The mereologic, the optic way and the place way constitute what Duval defined as "the *operative apprehension"* of the figure, which according to him differs from the *perceptual apprehension* "because perception fixes at the first glance the vision of some shapes and this evidence makes them steady" (p.19) [...] Operative apprehension is [also] independent of discursive apprehension"(p.21)

Duval (1995b) supports that "a mathematical way of looking at figures only results from co-ordination between separate processes of apprehension over a long time, something that is supported with work with computers, if the software has been defined having this in mind" (reported in Jones, 1998, p.31). Duval (1998, p.38) proposes "that geometrical reasoning involves three kinds of cognitive processes which fulfil specific epistemological functions, namely (**Figure 2.21**):

- *visualisation processes*, with regard to *space representation* (italics by the author) for the illustration of a statement, for the heuristic exploration of a complex geometrical situation, for a synoptic glance over it, or for a subjective verification" (p.38).
- *construction processes*, by tools (e.g., ruler, compass, protractor) or dynamic tools (e.g., a DG software's primitives): "construction of configurations can work like a *model* in that the actions on the representative and the observed results are related to the mathematical objects which are represented" (p.38);
- *reasoning processes* "in relation to discursive processes for extension of knowledge, for proof, for explanation" (p.38).

Duval argues, "[...] these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry" (ibid. p38)

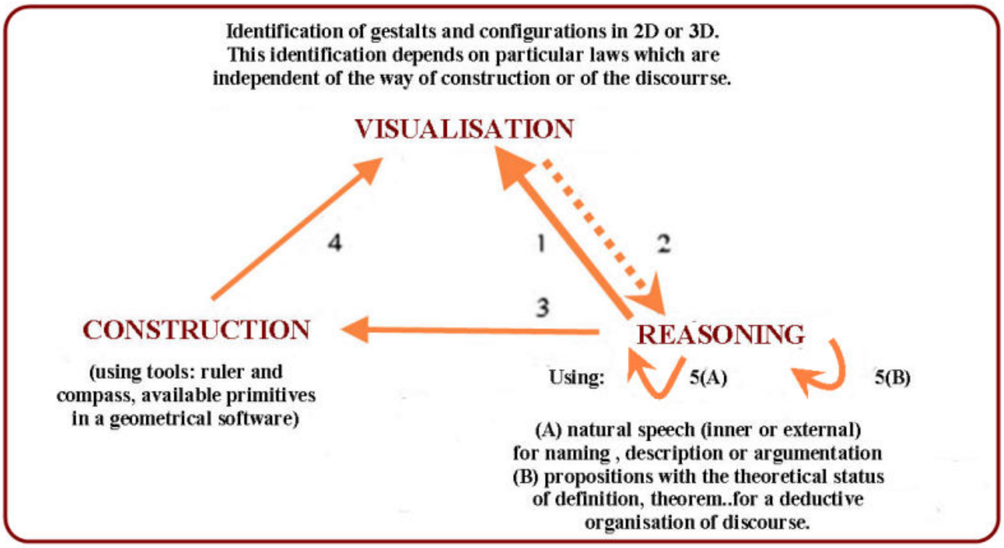


Figure 2.21. The cognitive interactions involved in geometrical activity (Duval ,1998, p.38)(Webpage [12]) (an adaptation for the current study)

In the **Figure 2.21**, Duval illustrates the different cognitive processes and the arrows that represent the way that one of these can support another in any geometrical activity. For example, an arrow starts from the 'construction' cognitive process towards 'visualization' but this arrow is not reversed. Namely, Duval points out that (a) these different processes can be performed separately and (b) a cognitive process (e.g., visualization) does not necessarily depend on another cognitive process (e.g., the construction process). The arrow 2 is dotted as Duval considers that visualisation does not always help students to reason or formulate an argumentation. Arrows 5A and 5B show how that reasoning can emerge along a path separate from the processes of construction or visualisation. Of course, construction can leads to visualisation, but even then the actual processes of that construction stem from links between pertinent mathematical properties and from the limitations/or constraints imposed by the tools used. In the same way, even if visualisation can help students formulate their thinking by, guiding them in the direction of a proof, it can still be misleading at times (Jones, 1998, p. 32).
To facilitate visualization Duval suggests the student has to develop the operative apprehension of the figure, namely the mereologic, the optic way and the place way of the figure and its subfigures. This will happen physically by manipulating the figures in a static or a dynamic environment or mentally when a student has developed the competence to achieve it. It is very crucial for the teachers to find ways to trigger and elicit it through proper activities.

2.6. Linking Visual Active Representations

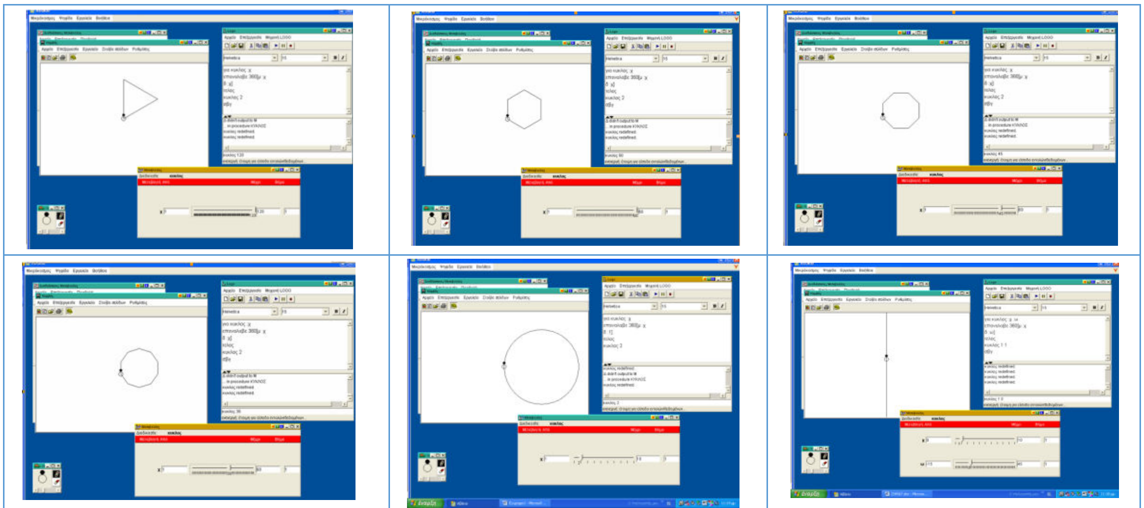
The topic of LVAR is discussed extendedly in Chapter 5.

2.7. Indicative Representational Environments used for the Teaching and Learning of Mathematics

Edwards (1998) argues that "we can speak of a microworld as "embodying" a sub domain of mathematics or science: not because of some reifying link between the representation and the mathematical or scientific entity, but because of the opportunity that such environments provide for learners to kinesthetically and intellectually interact with the designers' construction of these entities, as mediated through the symbol system of a computer program" (p. 74).
Kynigos (2007) introduces the term "half-baked" microworlds

> "to describe digital media designed to facilitate communication between researchers, technicians, teachers and students as they become engaged in changing them. Microworlds have been the main Logo-based vehicles through which the key ideas of generation of meanings through communicational and

constructionist activity have been mediated within the field of instructional design (Goldenberg, 1999)" (p.335).



Figures 2.22 a, b, c, d, e, f. Multiple Linked Representations created in the E-slate microworld (Patsiomitou, 2012c, p.144)

E-slate (Kynigos, 1997; Kynigos et al., 1997) (http: //e-slate.cti.gr) is a logo-based microworld, used in Greek schools in the teaching and learning process of mathematics at several levels, but also for investigating different aspects of educational practice. E-slate consists of three distinct but interlinked work areas, the *components* of E-slate. According to Kynigos (2004)

> "In the E-slate environment, components are black boxes in that the user cannot alter their main functionality and in that they are developed primarily to be technically efficient. However, each component is designed so as to be as generic as possible in the sense that it can be used for a family of activities and not just a few activities" (p.31).

The linked representations I have constructed using the E-slate microworld (**Figures 2.22 a, b, c, d, e, f**) reveal an approach to the concept of the circle which uses an increasing number of circumscribed regular polygons. E-slate "emphasizes connectivity by adopting a variety of ways to connect components" (Kynigos, 2004, p. 33). Sliders also have been designed "in order to allow the user to manipulate some value by changing it continually through the slider" (Kynigos, 2004, p. 35).

The mathematical component which is called "the variation tool" "extends traditional Logo to the role of a scripting language and on a database component" (Kynigos, 1997; Kynigos et. al., 1997). According to Kynigos (2002) (paper available at Webpage [35])

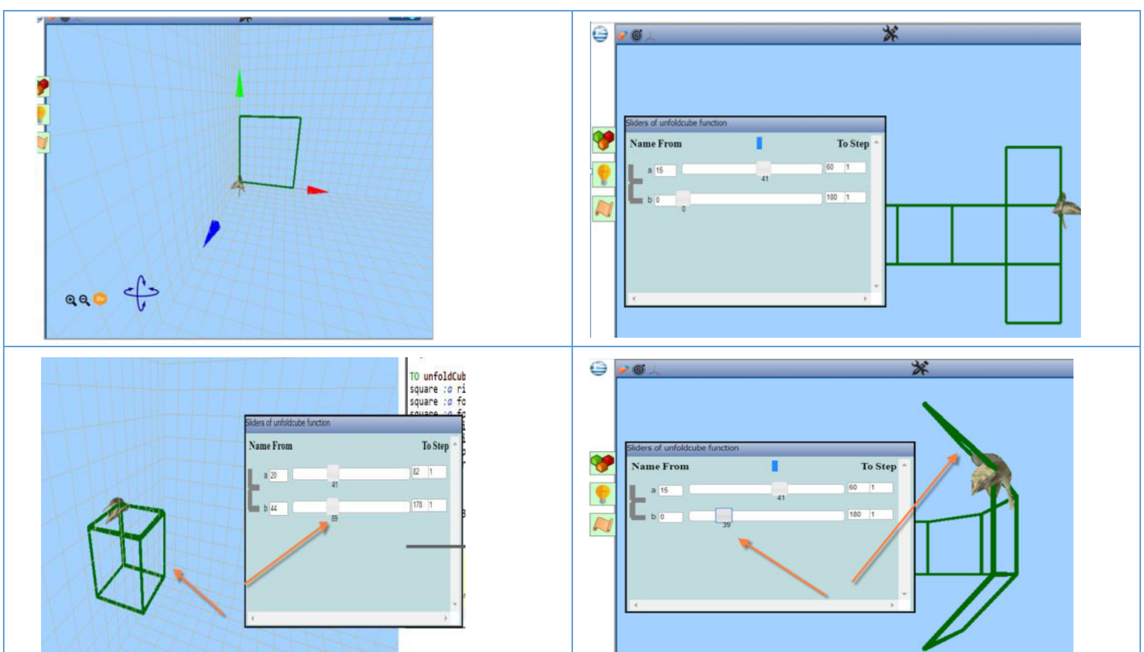
> "The variation tool is designed so that it provides a kinesthetic means for continually changing the independent variable of the respective world to which it is connected and observing what remains constant and what changes. In this case, when the language, turtle, canvas and variation components are connected to each other, execution of a variable procedure with any value for the variable(s) and clicking on the turtle's trace "energizes" the variation tool which recognizes which command resulted in that particular trace (fig. 1). A slider appears for each variable with editable range and step. Dragging the slider results in a continual reshaping of the figure according to the corresponding variable value. The effect is that of the same figure dynamically changing form (in a way similar to that of Geometry Sketchpad). More important, it gives a feeling of the way things change and the rate of change" (p. 15).

MaLT is also a constructionist microworld environment (Kynigos & Latsi, 2007) widely used in Greek schools (especially in Model Schools). Other packages also used in the teaching and learning process are Cabri II (Laborde et al., 1988), Function Probe (Confrey & Smith, 1992), Geogebra (Hohenwarter, 2001), Geometer's Sketchpad (Jackiw, 1991), Web Sketchpad (McGraw Hill, 2019) etc.

Many activities have been constructed in the MaLT environment and are available online through Digital School Platform (Webpages [13, 14]). Teachers and students can even access them (in class during the lesson or out of the class) using their mobile phones (e.g., Geogebra, Web Sketchpad) or tablets for the teaching and learning process of mathematics. Furthermore, MaLT is a 3D programming environment that enables dynamic manipulation; it is a very useful Web tool for the construction of meanings. According to MaLt Manual (retrieved in May 2018):

> "MaLT+ (MachineLab Turtleworlds) is an online tool of symbolic expression in mathematical activity by means of programming for the creation and tinkering of 3D dynamic graphical models […] On the left side of MaLT+ appears the component of the '3D scene', which also includes the avatar. The avatar is a 3D object that you can move it in the 3D space by executing some Logo commands" (p.4). (See also, Webpage [15])

For example, constructing a rectangular shape or a cube in the MaLT environment along with its 2D or 3D transformations makes MaLT a very important tool for the intuitive perception, construction and deep understanding both of meanings in 2D and 3D figures and of the figures' properties. In the **Figures 2.23 a, b, c, d** the screenshots of four linked representations could constitute part of a sequence of actions and processes for the construction of the meaning of rectangle and cube. The rectangle is projected along the axis system of the grid. A slider appears for each variable (i.e., for every side of the rectangular shape). Students can also measure the sides of the quadrilateral, using the grid provided by the MaLT environment, and then use formulas to continue their calculations, combining symbolic, graphic and visual representations. The MaLT environment also provides dragging facilities through the manipulation of the sliders or the object on screen. According to Kynigos & Latsi (2007) "Studying in a dynamic way 3D geometrical objects students have to analyze a 3D figure, break it into smaller parts and determine angle measures and lengths of line segments. Projecting themselves into the place of the turtle and moving from the visual to the descriptive level of thought students have to search for ways to reconceptualize 3d objects in terms that can be explained to the 3d turtle through logo commands. Moreover through the use of sliders students are provided with a direct manipulation metaphor for sequentially changing variables' values and simultaneously observing the variation both of 3d object and of their place in 3d space" (p.360).



Figures 2.23 a, b, c, d: Screenshots of figures in the 2D or 3D MaLT constructionist microworld (Webpage [16])

When a representational environment is combined with another environment (e.g., a DGS environment), the two can complement one another, offering to learners advantage through the properties of different representations. This will ultimately lead to a deeper understanding of the meaning under investigation. Research has evaluated how effectively representations constructed in different environments can support learning by operating complementarily.

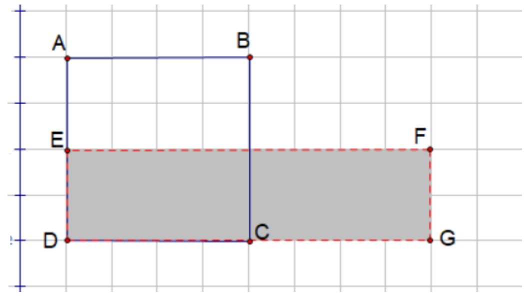


Figure 2.24. Construction of two rectangular figures in a DGS environment.

For example, constructing two rectangular shapes--a square with 4 cm sides and a rectangle with sides a, b equal to 8cm and 2cm respectively--in a DGS environment (e.g., Sketchpad, Geogebra or other DGS software) and experimenting with them helps students by allowing dragging and *direct manipulation of the object,* to move from visualizing a square to describing and analyzing it as that a square is the rectangular shape with the minimum perimeter among rectangular shapes that all have the same area. (Figure 2.24, Area of square A_S = $16cm^2$, area of rectangle A_R= $16cm^2$, perimeter of square P_S= 16cm , perimeter of rectangle P_R= 20cm).
At an advanced level the students can solve the problem "*Suppose a rectangle has a fixed area of c square meters. Find the dimensions that minimize the perimeter*" (Kreider and Lahr, 2002, p.1).
If we suppose that a and b are the lengths of the sides, then the area of the rectangle is A = a b, and the perimeter P of the rectangle is P = 2a + 2b.

$$A = a\cdot b \Rightarrow b = \frac{A}{a}$$

If we substitute it b into the equation of rectangle's perimeter then

$$P = 2a + 2\cdot\frac{A}{a} \quad (a>0)$$

Taking the derivative of P, we get

$$P'(a) = 2 - 2\frac{A}{a^2} = 0 \text{ which implies } a = \sqrt{A}$$

Thus, the perimeter will be a minimum when the rectangle is a square ant its side is $\sqrt{A}$ meters.

The use of Cabri3D (Laborde, 2004) also enhance the visualization of a figure's properties as well as experimentations with "real" object on screen, --importantly, this is true not only for students in the first classes of Secondary or in Primary education, as every student needs to directly manipulate a geometrical object to understand it (Figures 2.25a, b). Cabri 3D is a three-dimensional interactive software package for exploring geometry. It was launched in 2004. According to El-Demerdash (2010, p. 22-23) the key features of Cabri 3D can be summed up in the following points:

- "Create solid geometric construction with just a few clicks of the mouse.
- Integrate numeric data using measurements and calculation tools.
- Manipulate and animate constructions and reshape objects using only the mouse.
- Print out patterns from virtual constructions and transform them into real objects".

Cabri 3D also allows the construction, transformation and on-screen unfolding of three-dimensional objects (such as cubes, cones and pyramids). New objects may also be formed when planes intersect with each other: the intersection of a cone with a plane that does not go through the vertex of the cone, for instance, generates *conic sections* (see also Kösaa, & Karakus, 2010, p.1386)
I shall focus on this favorite to me example, which combines History of mathematics and the use of technology: the conics sections. According to Bogomolny (2004) "*Menaechmus* (c. 375-325 BC), a pupil of *Eudoxus*, tutor to Alexander the Great, and a friend of *Plato* (*Smith*, p. 92), is credited with the discovery of the *conics*. A more revealing term is *conic sections* on account of their being found as the intersections of circular cones by planes. If

the planes pass through the vertex of the cone, the conics are said to be *degenerate*, otherwise they are not. There are three non-degenerate conics: the *ellipse*, the *parabola*, and the *hyperbola"* (Webpage [36]*)*.

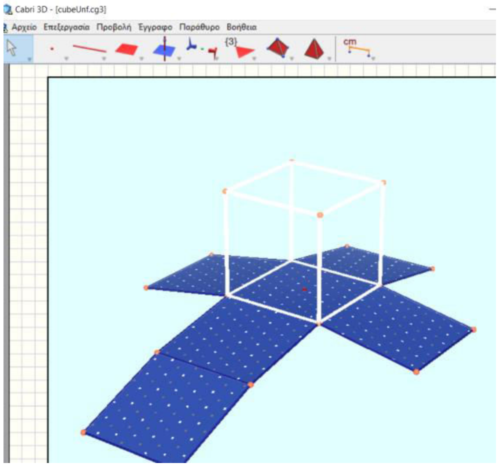
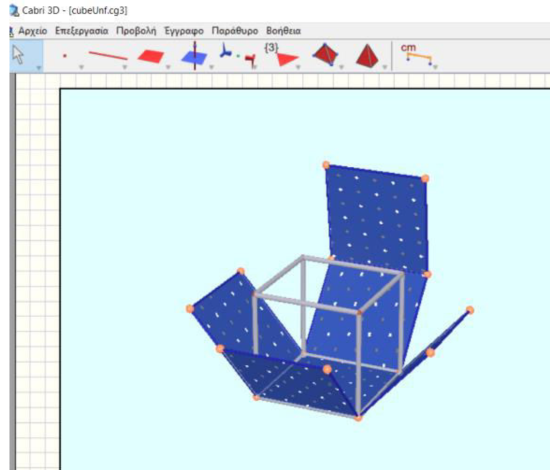
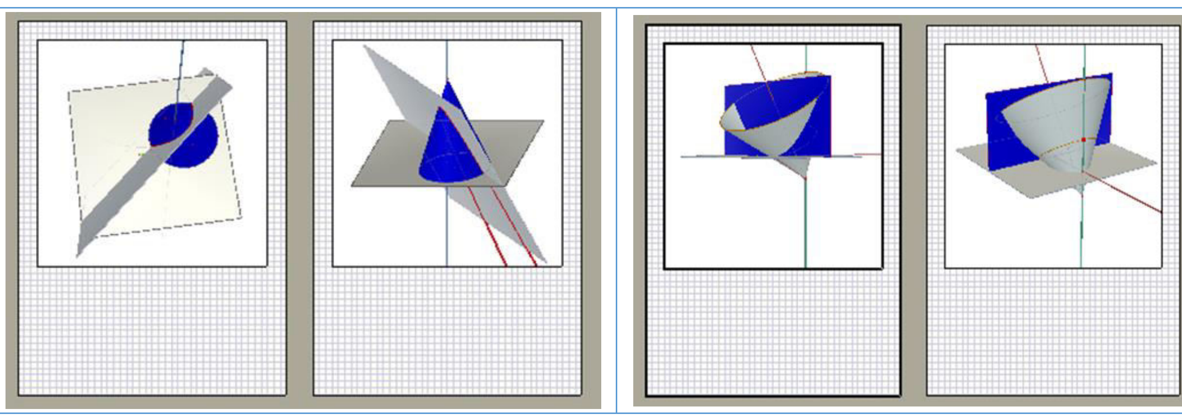
	

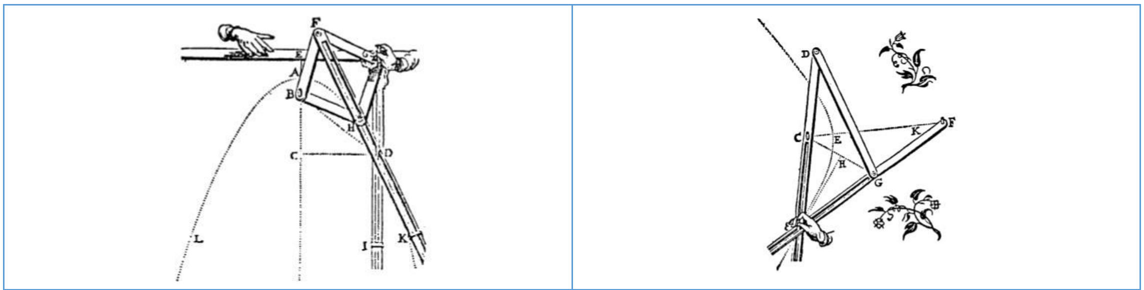
Figure 2.25a. Unfolding a cube in the Cabri 3D dynamic geometry environment (Webpage [17])	**Figure 2.25b.** Implementing direct manipulation using the mouse (Webpage [17])

Conic sections have remained at the epicentre of interest since antiquity. If we try to answer the question "what is the definition of conic sections" we will receive answers depending of the frame within we are investigating the construction. Thus, a conic section is a curve in the plane, a locus of points or, in a 3D plane, the intersection of a cone (Patsiomitou, 2007d).
Bartolini Bussi & Mariotti (1999) argue "Since the age of Apollonius, a deep understanding of the properties of conic sections has been achieved. However, most of the properties were expressed through relationships, which are neither immediately related to the shape of the cone to be cut nor to the shape of the section[…] in addition to the historical point of view, the relationship between the arguments used in theoretical and in practical geometry seems interesting to investigate from a cognitive perspective." (p.28)



Figures 2.26 a, b, c, d. Creating conics sections using Cabri3D (Patsiomitou, 2007d, p. in Greek)

In the **Figures 2.26 a, b, c, d,** I have constructed a few illustrations of conic sections using Cabri3D (Patsiomitou, 2007d, p. 40, in Greek). It is crucial for students directly manipulate the representation and subsequently the abstract object, as Laborde & Laborde (2011) in their study "*Interactivity in dynamic mathematics environments: what does that mean?"* argue: "*Direct manipulation* has proven to be a key feature to facilitate creative user interaction with computer and has slowly generalized to most of computer platforms." (Laborde and Laborde, 2011, p.1).



Figures 2.27 a, b: Creating conic sections using linkages (cited in Masami Isoda, 1998, p. 87)

Isoda (1998) has written the article "*Developing the Curriculum for Curves Using History and Technology*" in which, history of mathematics is combined with technology, meaning how linkages can be combined with new technological systems for the teaching and learning of mathematics. According to Isoda (1998) "in the age of Descartes, curves were only figures defined by geometry and drawn using devices such as ruler and compass, linkages, and mechanics, etc (p.86)." For example in the **Figures 2.27 a, b** which are van Schooten's linkages (1657, cited in Masami Isoda, 1998, p.87) a tangent of a parabola is constructed by the mechanical linkage. Isoda (1998) argues that "[…] students should know the reason why we can draw a parabola using the linkage […as] the visual and manipulative feature of these devices helps student to reflect on their own experiences" (p.87).

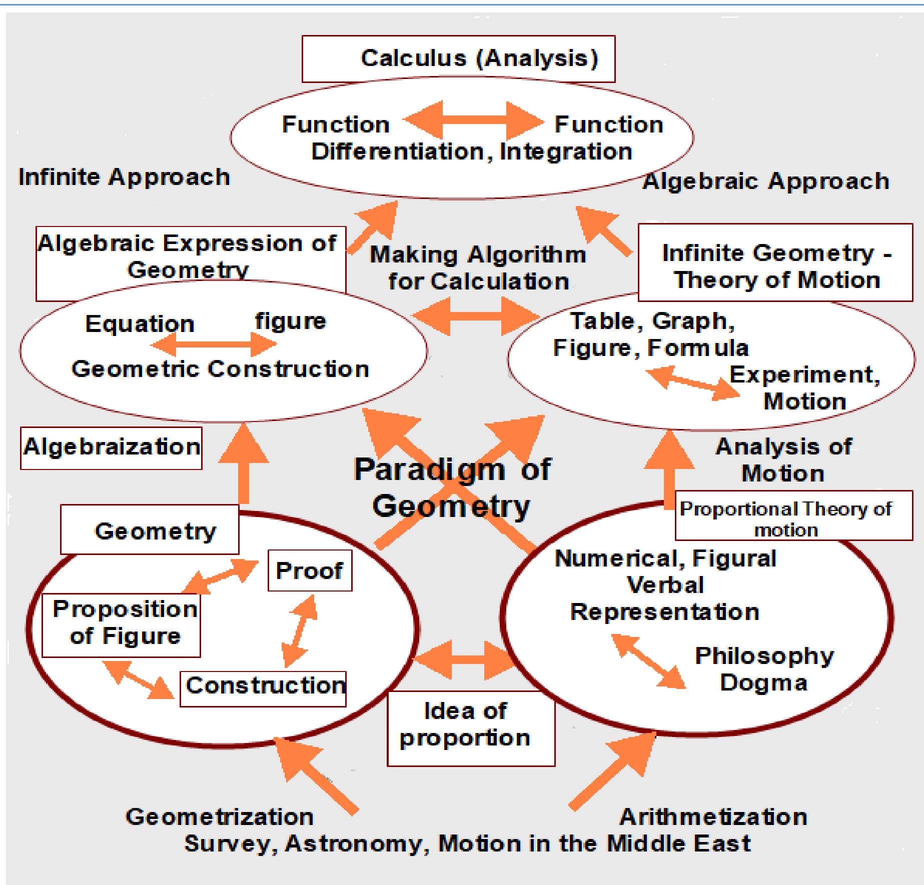


Figure 2.28. A Historical Root of Calculus from Ancient Greek Mathematics to the 17th Century Focused on Mediterranean and European Area (Isoda, 1996, cited in Isoda, 1998, p.84) (an adaptation for the current study)

Figure 2.28 depicts "*a historical root of calculus from Ancient Greek Mathematics to 17th century focused on Mediterranean and Europe Area*", created by Isoda (1996) and reported in Isoda (1998, p. 84). The figure depicts a brief but meaningful history of the evolution of calculus since antiquity. As the evolution of calculus does not fall within the ambit of the current work, I shall only mention what Isoda (1998) highlights "*Dynamic Geometry*

Software enhances and realizes Descartes dream" (p.82) as DG is an evolution of ancient drawing tools, such as the linkages are.

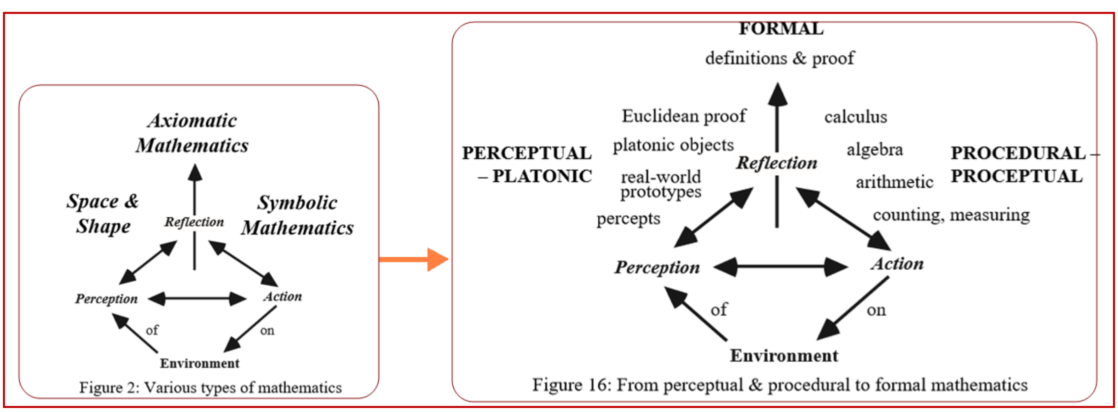
On the other hand, Isoda & Matsuzaki (1999) discussed the roles of old technology and new technology in the teaching of mathematical modelling. As they argue:

> "But does new technology alternate old technology? For example if we use DGS in geometry, can we discard a ruler and a compass? Of course we cannot, but we have to consider how the roles of a ruler and a compass should be changed" (p.268).

Bartolini Bussi (2005) also claims that it is very difficult to build a concept only through a one-sided process, for example through the algebraic definition. What must be also mentioned is the instructional sequence that will a teacher follow in class and the activities that will be used in the learning trajectory exactly as Laborde & Laborde (2011) support: "learning [can] emerge from the interactions between the students and appropriate tasks to be done with the machine". Laborde & Laborde highlight also the important role of the teacher in the teaching and learning process for the development of abstract ideas on the part of the students.

> ***Direct manipulation*** **has proven to be a key feature to facilitate creative user interaction with a computer and has slowly generalized to most of computer platforms. For educational software nevertheless *Direct manipulation* cannot be designed by chance and has to follow some additional principles, one of them is called *epistemic fidelity*: the representation of mathematical objects have to avoid any contradiction with the abstract object they are supposed to represent; and this has to be true to the graphical level as at the level of their behavior under direct manipulation.** (Laborde & Laborde, 2011, p.1)

Tall, Gray, Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas, Yusof (2001) also argue that the development of abstract concepts "begins from the ability to *perceive* things, to *act* on them and to *reflect* upon these actions to build theories" (p.81) (Italics by the authors). They constructed sequential figures in their study "*Symbols and the Bifurcation between Procedural and Conceptual Thinking*" to illustrate that. The **Figure 2.29 b** below right (Figure 16, page 98 in Tall et al (2001)) is an evolution of the **Figure 2.29a** on the left (Figure 2, page 82 in Tall et al (2001)).



Figures 2.29a, b: From perceptual to formal mathematics and advanced mathematical thinking (Tall et al, 2001)

According to Tall et al (2001)

> "The transition to advanced mathematical thinking makes a complete shift in focus from the existence of perceived objects and symbols representing actions on the objects to new theories based on specified properties of formally defined *mathematical structures*. Geometric experiences can be used to focus on certain properties (points, lines, intersections, curves, continuity, etc) to formulate new axiomatic systems such as non-euclidean geometry, topology and analysis. Properties of arithmetic and algebraic symbols are formulated and generalised to give axioms for groups, rings, fields, vector spaces, and so on. […]

However, the essential quality that makes advanced mathematical thinking different from elementary mathematics is the introduction of *formal definitions and proof*" (p.98)

The concept of a function is a mathematical object that cannot be smoothly understood by high school students, especially by students who find maths difficult. Function Probe is a multi-representational software package which can be used to teach functions to students (Confrey & Smith, 1992). Function Probe is a Java –based, cross-platform software which opens with three separate but linked windows: a Table window, a Graph window and a Calculator window. According to Confrey & Maloney (2008, p.183) "Function Probe was designed to support student thinking about, and exploration and understanding of families of functions, including linear, quadratic, exponential, polynomial, rational and trigonometric. The software was built to permit students to explore the contrasting and complementary appearance and behavior of these functions using different representations" (p. 183).

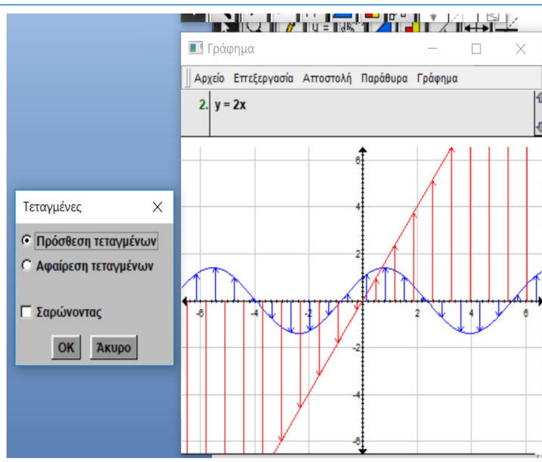


Figure 2.30a: Turning Graphs from equations and a preparation for the prediction of added ordinates

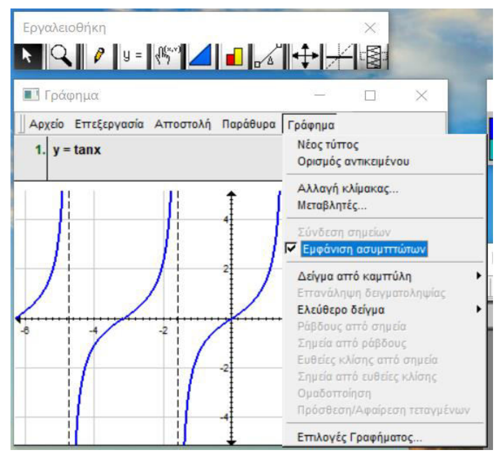


Figure 2.30b: Showing asymptotes of the function y=tanx

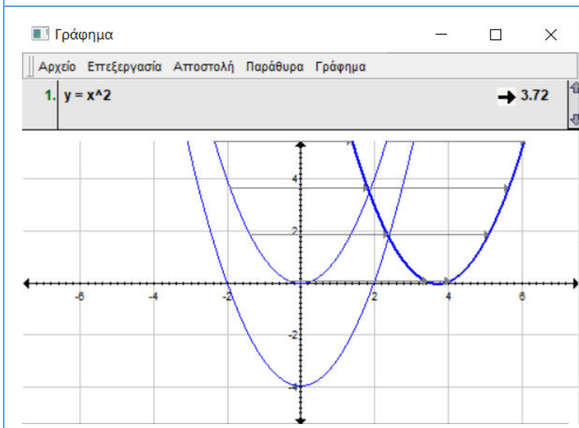


Figure 2.30c: Visualizing the transformation (translation) of the function $y=x^2$ to generate $y=x^2 + 3.72$; the original remains on screen, while the translation appears on screen after being moved horizontally 3.72 units to the right (having previously been moved vertically 2 units).

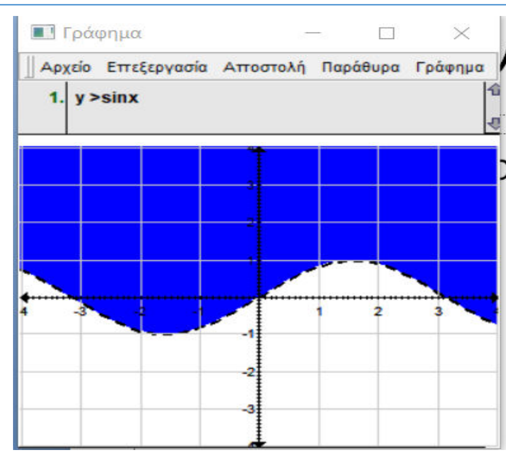


Figure 2.30d: Visualizing the graph of the inequality y> sinx

Using the software's features, anyone can easily construct graphs of functions from equations, show asymptotes, visualize the transformations of functions or visualize the graph from inequalities --all of which were very extremely hard to visualize using traditional means. Transformations of functions were --and still are-- a very important issue in the teaching and learning process, given both the difficulty the students have in moving between different families of functions, and the way in which the translation from the symbolic/ tabular to graphic representations occurs.

In the **Figures 2.30 a, b, c, d** I created a few graphs using Function Probe's features, absolutely agreeing with what Borba & Confrey (1996) claim:

> "[…] new forms of representation change the mathematics to be taught (Confrey, 1993a, 1993b). Mathematics does not exist independently of its representational forms; it exists through those forms" (p. 335).

A translation between representations --that many learning environments have been designed to embody in their features-- helps the students to what Ainsworth (1999a) calls "dyna-linking" or "automatic translation", through which

> "[if] a learner acts at one representation the effects of their actions are shown on another[…] the cognitive load placed on learners should be decreased and so free them to learn the relation between representations (e.g., Kaput, 1992; Scaife & Rogers, 1996)" (p.133).

Artigue (1997) in their paper "Teaching and Learning elementary analysis: what can we learn from didactical research and curriculum evolution" mention some main categories of difficulties that students face when they learn functions (Artigue, 1997, p. 208-209): (a) "*Difficulties in identifying what really a function is and in considering sequences as functions. (b) Difficulties in going beyond a process conception of functions and being able to link flexibly the process and the object dimension of this concept, and develop with respect to it a perceptual view (Tall & Thomas, 1991). (c) Difficulties in linking the different semiotic registers (Duval, 1995) which allow us to represent and work out functions and (d) Difficulties in going beyond numerical and algebraic modes of thinking"* (Artigue, 1997, p. 208-209).

Even (1998) also in her study "*Factors Involved in Linking Representations of Functions*" illustrated "how knowledge about different representations [of a function] is not independent, but is interconnected with knowledge about different approaches […] knowledge about the context of the presentation, and knowledge of underlying notions." (p. 120). Moreover, "the ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allow one to see rich relationships, develop a better conceptual understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems" (p. 105).

Even (1998) reports three factors involved in linking representations of functions that can be extended in other areas. These factors are:

(a) *"Different ways of approaching functions*: An important aspect of knowledge about a mathematical concept is the different ways of approaching or conceiving the concept. A common distinction today is between an *operational approach to a concept* as a process, and *a structural approach* as an object (e.g., Dubinsky, 1991; Sfard, 1991). […] Flexibility in moving from one representation to another is intertwined with flexibility in using different approaches to functions. […] (p.108-109)

(b) *Context of the presentations*: Another critical aspect that intertwines with the ways representations come into play in the understanding of a concept is the context of the problem presentation. […] (p.115).

(c) *Underlying notions*: The quality of the knowledge of underlying notions of the functions being dealt with, is also intertwined with the ability to translate from one representation to another […]" (p.117).

Students face many difficulties when they have to deal with the concept of function. I shall mention a path concerning the concept of function, based on my experience as a teacher of mathematics, which can scaffold secondary-level students learning process and allow them to gradually grasp abstract mathematical objects (Patsiomitou, 2019b, p. 33):

Elementary level arithmetic and algebraic approach: "1 kg of apples costs 2 Euros, 2 kg cost 4 Euros […] x kg cost y Euros. What is the relationship between x and y?" The appearance of the variables x and y reveals a limited understanding on the part of students because x and y are symbols used as signifiers referring to objects; in the words of Piaget (1952/1977), they are "intentionally chosen to designate a class of actions or objects." (p.191). The question is how the relationship between different kinds of objects can be shown? Which procedure/or procedures can we apply so that the concept of function is easily understandable for students? Do these procedures or processes lead to an understanding of the concept of function?

1st level. The variable's approach: I continue: 1 kg of apples costs 2 Euros, 2 kg costs 2*2 Euros etc.; […] the number 6 is represented /signified by the product 2*3 and the symbol y is represented /signified by the product 2*x. (i.e., x kg cost 2x Euros). The expression 2*3 is the same notation to represent both a process and the product of that process. In other words it "could be used both *operationally,* as denoting an operation, and *structurally,* as signifying an object (the result of an operation). The fact, however, that the

same signifier had to be employed in two seemingly incompatible roles, operational and structural, certainly aggravated the difficulty of *reification"* (Sfard, 2000, p.50).
2nd level. A diagrammatic approach: The next step is the construction of Venn diagrams in which arrows connect the A set of numbers representing kilograms with the B set of numbers representing Euros. When representing objects in Venn diagrams, we use dots for objects. Constructing Venn diagrams allows students to think about the classification of objects, while the arrows help them to describe relations between objects and understand meanings such as "one to one" and "onto".

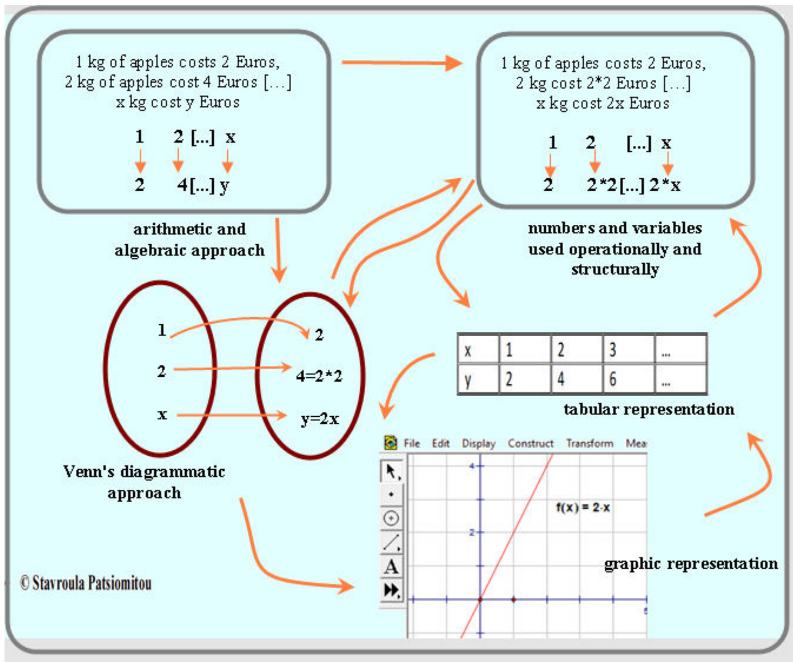


Figure 2.31: Linking the different kinds of representation of a function (Patsiomitou, 2019b, p. 33) (modified)

3rd level. A graphic and tabular approach: A function is used to describe the expressed relationships between variables. Replacing the numbers 1, 2, 3… that represent the kilograms with the variable "x" and constructing a function (y=2x) in which we determine a rule for a sequence of objects, ultimately provides us with a definition of the concept of function and its graph. Thus, in response to the symbol of the function y=2x ('*representamen*' in the words of Peirce, 1955) one can draw a line which would be the *interpretant* of the symbol y=2x (**Figure 2.31**).

The prerequisite here for students is the structural knowledge of numbers which allows them to use numbers to build a more complex concept. In **Figure 2.31**, we can view both *treatments and conversions* (Duval, 2002, p.3) between the aforementioned semiotic representations:

- *"Treatments* are transformations of representations which happen within the same register […] (Duval, 2002, p.3)
- *Conversions* are transformations of representation which consist of changing a register without changing the objects being denoted […]" (Duval, 2002, p.4).

Duval (2002) in the **Figure 2.32** clarifies what he means with the notions *treatment* and *conversion* between different semiotic representations.

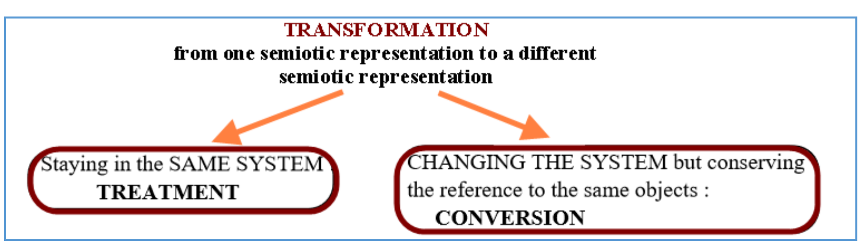


Figure 2.32. Types of transformations of semiotic representations (Duval, 2002, p.3) (an adaptation for the current study)

Treatments and conversions express connections or links between different modes of representation. In a short literature review I shall summarize in the next table how different researchers examine, and report the role of linking representations in learning and understanding of mathematical concepts.

Table 2.4. Linking Representations

Kaput (1989)	"The cognitive linking of representations creates a whole that is more than the sum of its parts."(p.179)
Even (1998)	"Connectedness between different representations develops insights into understandings of the essence as well as the many facets of a concept" (p. 105).
Ainsworth (1999b)	reports the way that linking representations affect students' –users' thinking. As she mentions: "One question facing designers of learning environments is whether to provide automatic (dynamic) linking between representations. Here, one acts in one representation and sees the results of these actions in another. Thus, it is hoped that the relation between the representations is made more explicit and hence understandable to learners than has traditionally been possible with static media" (p. 39).
Ainsworth (2006)	"Dynamic linking or representations is assumed to reduce the cognitive load upon the student –as the computer performs translation activities, students are freed to concentrate upon their actions on representations and their consequences in other representations". (p. 194)
Hähkiöniemi (2006)	investigated how students development of understanding of the concept of derivative. He found that "students had two kinds of connections: they changed from one representation to the other or they explained one representation with the other" (p.18).
Hegedus & Kaput (2004)	report the importance of linking representations in Simcalc –a long time project—and they stated "We are confident, however, that by combining the two key ingredients, dynamic representations and connectivity technology, students can better understand fundamental, core algebra ideas by forming new, personal identity relationships with the mathematical objects that they construct individually and collaboratively with their peers". (p. 136)

The growth of digital resources that allow interaction with mathematical content has enriched the ways in which teachers and students engage by employing new kinds of representations: the "dynamic representations" or "dynamic diagrams". Ainsworth (1999) mentions the kind of "dynamic representations as follows (p. 35):

> "The introduction of information technology into the classroom has brought a new type of representation to learning situations - *dynamic representations*. These include animations which have been defined as a series of rapidly changing static displays giving the illusion of temporal and spatial movement (Scaife & Rogers, 1996).

Ainsworth, & Van Labeke (2004) define *dynamic representations* as those which "display processes that change with respect to time" (p.241).
GeoGebra (Hohenwarter, 2001, 2002) is an open source mathematics education software tool which is used by millions of users worldwide. It allows for experimentation even in a web browser in full HTML5 mode. (Botana, & Kovács, 2016) http://www.geogebratube.org/student/b128631)
Geogebra dynamic geometry software (http://www.geogebra.org), is also a multi-representational dynamic tool, especially when learners use software's CAS (Computer algebra systems) features to perform procedures in algebra and calculus.
According to Hohenwarter, Hohenwarter, Kreis, and Lavicza (2008, p.1):

> "The multi-platform, open-source dynamic mathematics software GeoGebra (Hohenwarter & Preiner 2007) tries to combine the ease-of-use of dynamic geometry software with the versatile possibilities of computer algebra systems. The basic idea of the software is to join geometry, algebra, and calculus, which other packages treat separately, into a single easy-to-use package for learning and teaching mathematics

from elementary through university level. GeoGebra is available free of charge on the Internet, has been translated to 36 languages by volunteers, and gathers a rapidly growing worldwide user community".

In the Figure 2.33a below, on the left we can see the symbolic expression of a polynomial, how we can turn it into a function; on the right, we can see the graphic representation of the function and its roots. "Through the Geogebra environment and also by using different kinds of instructional materials (such as worksheets on paper, interactive applets etc.) students can be guided towards discovering the concepts of derivative and /or integral and to explore, visualize and understand basic calculus concepts" (Hohenwarter et al, 2008, p.8).

According to Caligaris, Schivo, Romiti (2015):

> "the incorporation of the GeoGebra Applets, and the teaching situations arising there from, is a much more effective teaching methodology than traditional one to facilitate the learning of the fundamental concepts of Calculus […]The graphics in books, as well as on the blackboard, are static and require students' imagination adequately trained. When thinking about teaching strategies to discuss the fundamental concepts of Calculus, both its dynamic characteristics and the study of change and movement, have to be kept in mind. Nowadays, the existence of free programs with versatile capabilities and interactive representation helps to improve the presentation of content taught in this area of knowledge, allowing dynamic visualization" (p. 1188).

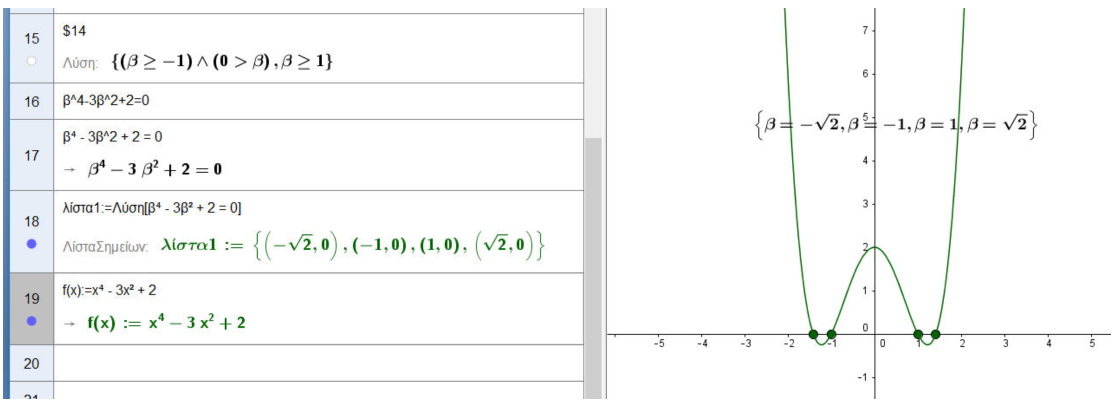


Figure 2.33a: Symbolic and graphic representations of a function using Geogebra

It is also possible to find derivatives and integrals of functions. Teachers can use Geogebra to help their students understand meanings: generating the graph of any function on screen from its symbolic representation makes a strong metacognitive visual impact on their students' thinking.

For the teaching of calculus, a teacher must have experience to achieve "The transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt, […]" what Chevallard (1988, p.6) has termed the *didactic transposition of knowledge* (see also, Chevallard, 1999, 2005).

Botana, & Kovács (2016) also argue that "classroom demonstrations and deeper investigations of dynamic analytical geometry are ready to use on tablets or smartphones as well. […] The covered school topics include definition of a parabola and other conics in different situations like synthetic definitions or points and curves associated with a triangle. Despite the fact that in most secondary schools, no other than quadratic curves are discussed, simple generalization of some exercises, and also everyday problems, will smoothly introduce higher order algebraic curves" (p. 1).

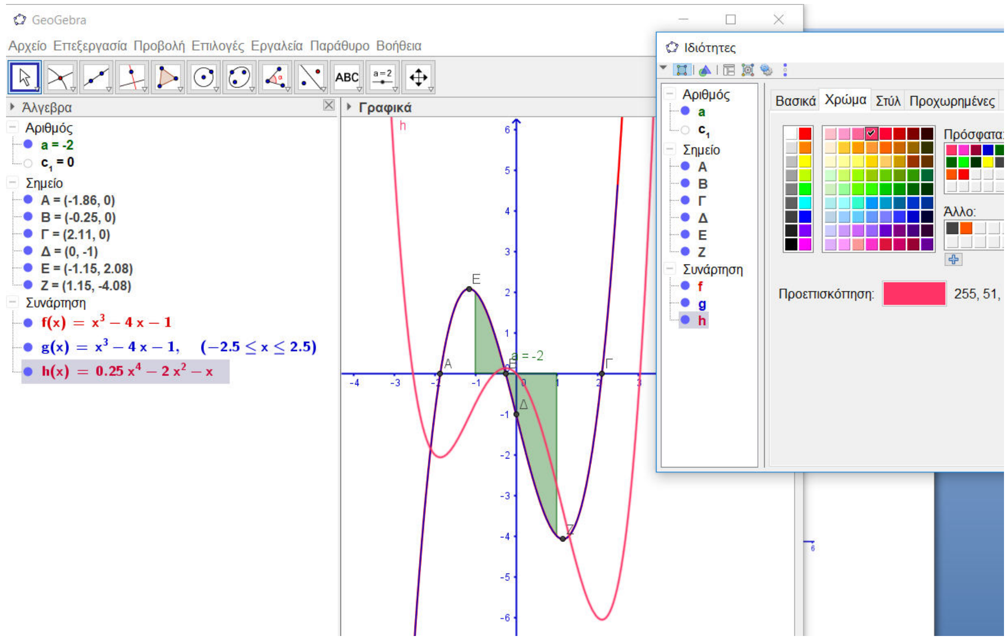
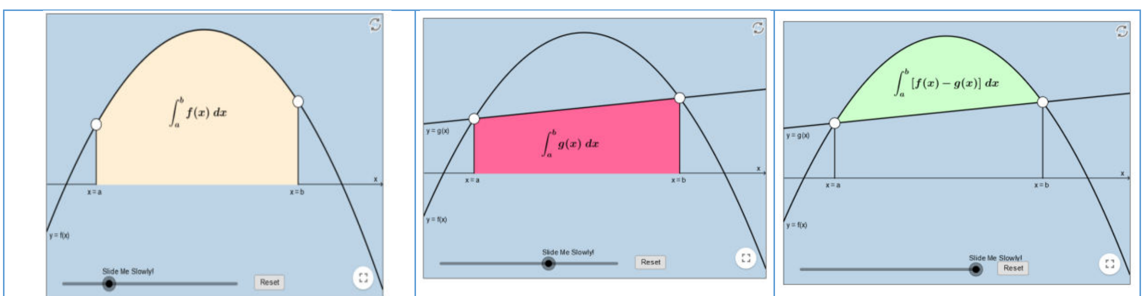
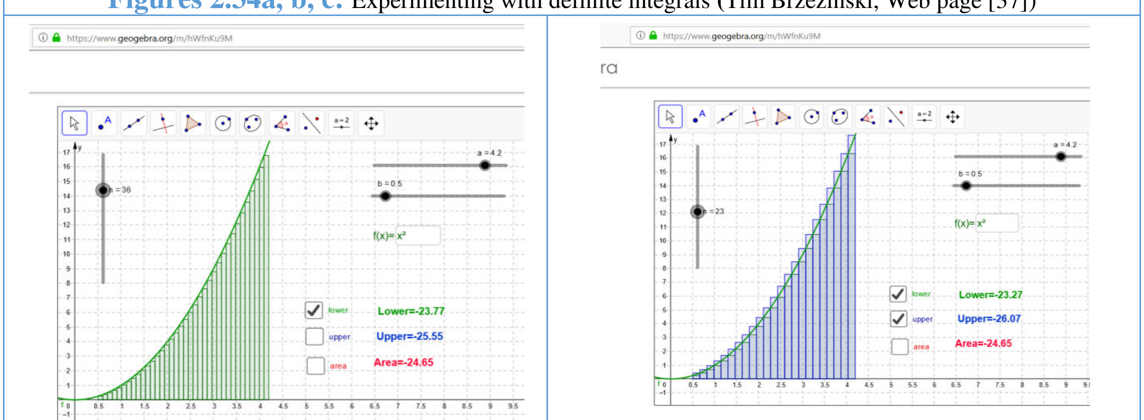


Figure 2.33b. Creating a function on Geogebra Calculus, its definite integral and connecting their symbolic representations with graphic representations

Definite integrals represent the exact area under a given curve, linking a graphic representation of a function with its symbolic representation. Riemann sums are also used to approximate those areas. GeoGebra applets also are embedded in HTML5 mode and sliders can help the visualization of important meanings (**Figures 2.33 a, b, c, 2.34 a, b**).



Figures 2.34a, b, c: Experimenting with definite integrals (Tim Brzezinski, Web page [37])



Figures 2,35a, b: Experimenting with definite integrals (Mark Willis, Web page [38])

Pre-calculus activities at a young age can help young learners to intuitively conceive concepts they will learn in calculus courses later. The students' improved understanding in pre-calculus topics will enhance the gradual development of an understanding of concepts in calculus at any age.

Sinclair (2018) in her study "*Time, Immersion and Articulation: Digital Technology for Early Childhood Mathematics*" states that she has been involved in childhood research projects and

> "three novel and significant themes have emerged in this work: *the temporalizing* of early childhood mathematics (time); the exposure of young children to advanced mathematics (*immersion*); and, the relations between digital technologies and paper-and-pencil technology (*articulation*)" (p. 205) .

The interactive Web Sketchpad (McGraw Hill, 2019) environment encourages students to experiment with open-ended tasks, during the teaching and learning process, in class or out of class. Daniel Scher, Scott Steketee and others have built web sketches that allow anyone to experiment (Webpage [18]). According to Daniel Scher (personal e-mail communication on July, 23, 2019):

> "*Web Sketchpad is dynamic mathematics technology from the creators of The Geometer's Sketchpad software. It began as part of the NSF-funded DRK-12 funded Dynamic Number project and brings over 25 years of Sketchpad development and innovation to the web and electronic textbooks, requiring only HTML5 and JavaScript. Unlike its desktop counterpart, Web Sketchpad has no default set of mathematical tools; instead the teacher or activity developer chooses tools to support each activity, providing the tools needed for the activity at hand. Thus, a geometry activity might feature the familiar Point, Straightedge, and Compass tools found in desktop Sketchpad's toolbar while a calculus activity might put tools for exploring Reimann sums front and center. This style of tool presentation enables less-prescriptive and more open-ended student tasks, encouraging the student to be more self-reliant: instead of following step-by-step worksheet directions she concentrates on how to use a small set of manageable tools to accomplish a mathematical task. When a student taps a tool icon, the entire result of the tool appears onscreen*.[…]".

Using Web Sketchpad anyone can create Web-sketches that can be linked procedurally and conceptually. Crucially, this permits the development of sequences of activities that can be saved and then shared with students using Google Drive, email etc.

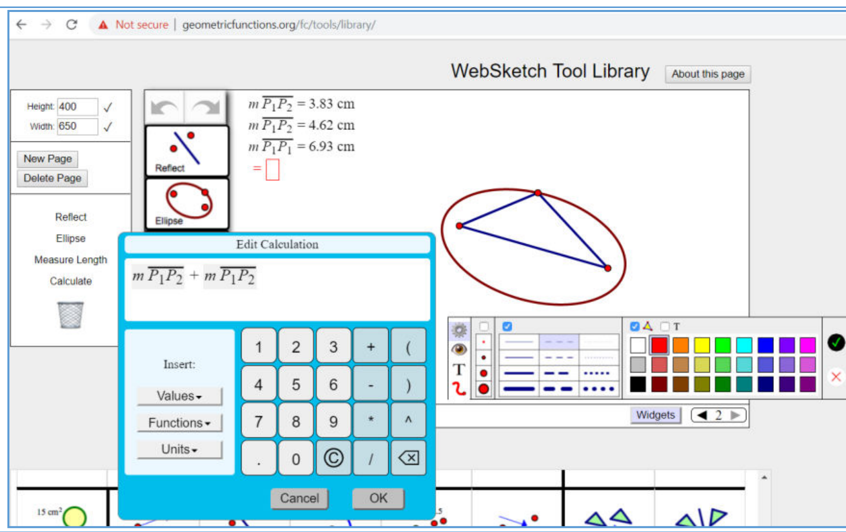


Figure 2.36. Constructing an ellipse using WebSketchpad Tool Library - Changing the visibility of the objects

In my opinion, it is a very powerful tool for developing *strong intuition* with regard to mathematical concepts at all levels (pre-school, Primary or Secondary Education). According to Fischbein (1999) "The *intuitive kind of knowledge* has been a concept in which mainly philosophers have been interested. In the works of Descartes (1967) and Spinoza (1967) intuition is presented as the *genuine source of true knowledge*. Kant (1980) describes intuition as the faculty through which objects are directly known in distinction to understanding which leads to *indirect conceptual knowledge*" (p.11)

Web Sketchpad dynamic tool has *multiple components*, including the Tool Library (from which you can select the tools you need for your construction), the viewer, and the desktop Sketchpad (the web page where you can construct your websketches). According to Scher "To provide a convenient starter set of tools that can be used across a wide variety of activities, Web Sketchpad includes a "Tool Library" with over 60 tools that can be added to a websketch. Accompanying the tool library is a viewer page, where one or more websketches can be uploaded simultaneously for review by the teacher or for class presentations” (personal e-mail communication on July, 23, 2019).

Widgets also provide several benefits to users (for example, a student can change the visibility (showing or hidden) of any object, can drag sketch objects even when style or visibility widgets are active, etc.). Widgets also give the advantage of being able to change the colours of the graphs, the grid or the shapes, which affects students emotionally, encouraging them to “love” mathematics (**Figures 2.36, 2,37a, b, c, d**).

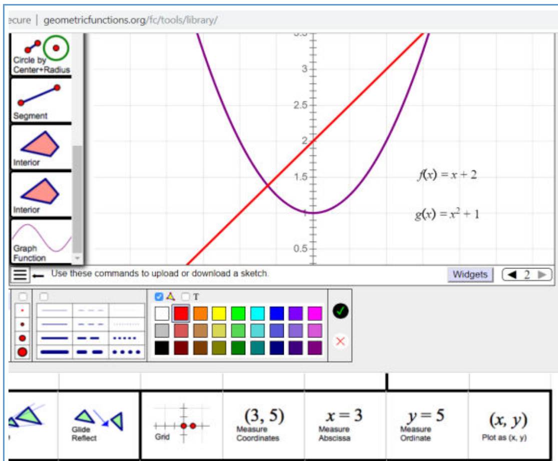


Figure 2.37a. Constructing graphs using WebSketchpad Tool Library

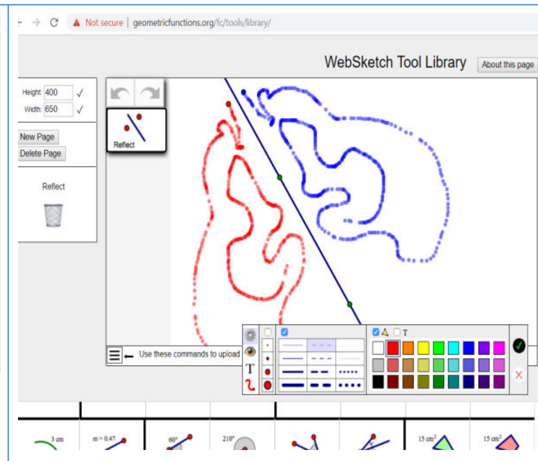
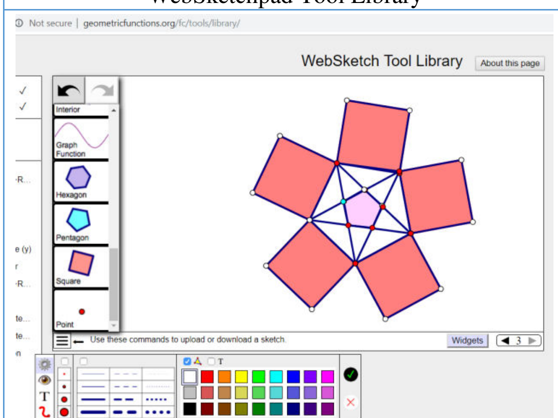
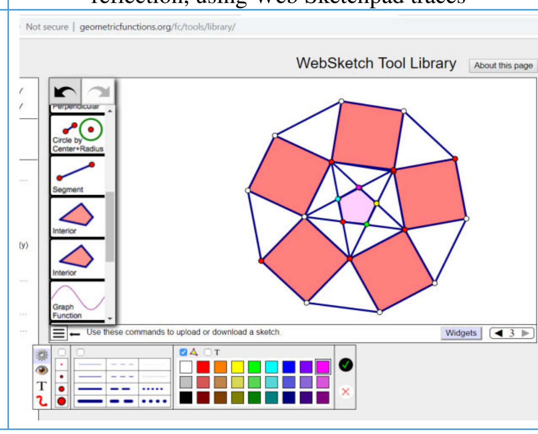


Figure 2.37b. Visualizing the properties of reflection, using Web Sketchpad traces





Figures 2.37c, d. Constructing tessellations, playing with colors and “loving” mathematics with Web Sketchpad (from the first years in school)

The most important thing is that *no one has to remember how to use the tools*, which is something I love also about Sketchpad. In the **Figures 2.36 and 2.37a, b, c, d,** I constructed graphs, reflection of points through symmetry lines and their traces, regular polygons and tessellations. They are “easily” generated on screen. At the following page (Wepage [19]) Daniel Scher (2019) writes:

> “Constructing a square requires tools, and Web Sketchpad features a particularly innovative tool interface […] there’s no need for the student to remember or figure out what objects to click, in what order, to use the tool successfully. This overview of the entire tool gives the student an opportunity to consider what objects the tool is going to create and plan how to integrate these tool objects into the existing sketch.[…]
> The Web Sketchpad tool interface was designed with student tasks in mind.[…]. Students can be encouraged to be more self-reliant and self-directed, concentrating on the mathematics of the task rather than following directions from a worksheet or from the teacher”.

Researchers have investigated the way that Web Sketchpad can be used in class:
Steketee & Scher (2018) in their study “*Enacting Functions from Geometry to Algebra*” argue that

> "Web Sketchpad supports a constructionist approach to students’ activities of creating, manipulating, and investigating mathematical objects, thus linking their sensorimotor activity to their conceptual

understanding. The software provides a simple interface with no menus, based on dragging and on using a small set of tools designed by the activity author "(p. 59).

Using Web –based Sketchpad (Webpage [20]) for her research Sinclair (2018) argues that:

> "Web Sketchpad is multi-touch, which means that users can drag multiple objects at the same time (e.g., three children can each drag all the vertices of a triangle as they cooperate ..many children can interact simultaneously , each potentially using more than one finger. Moulti-touch dynamic geometry thus offers both mathematical and pedagogical opportunities that have only recently been pursued (Jackiw, 2013). (Sinclair, 2018, p. 209)

What is very crucial for students of any age is *to love mathematics*, to enter their mathematics class *without fear*, and to have in mind that mathematics *can be touched on screen, can be colored, can be understood, and can be built*. This can be achieved if mathematics is presented to students in class through gaming in computer environments from their first years at school (Patsiomitou, 2016c).

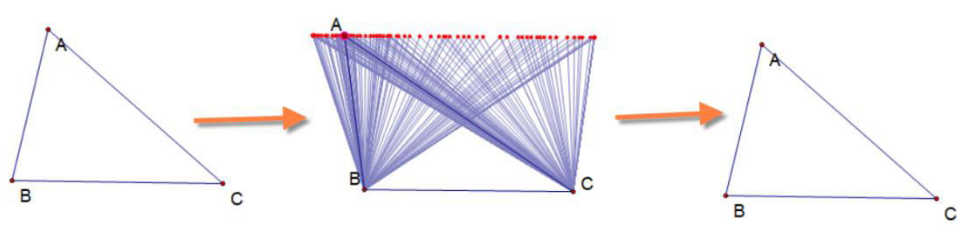
Chapter III.

3.1. Dynamic Geometry Software: An 'Alive' Microworld

Dynamic geometry systems (DGS) are microworlds designed to facilitate the teaching and learning of Euclidean geometry, Algebra and Calculus. Microworlds have been described (Edwards, 1998, p. 74) "as 'embodiments' of mathematical or scientific ideas" that, in the words of Sinclair, & Jackiw (2007) "are extensible (so that the tools and objects of the environment can be built to create new ones), transparent (so that its inner workings are visible) and rich in representations." (p.1). Dynamic geometry software has been used broadly in research regarding the teaching and learning process of geometry over the past several decades (see for example the articles written in *Educational Studies in Mathematics* and *International Journal of Computers for Mathematical Learning)* (Leung & Or, 2007, p. 177).

Such research with dynamic geometry has verified that the software is useful in provoking *cognitive conflicts* (e.g, Hadas, Hershkowitz, & Schwarz, 2000; Giraldo, Belfort & Carvalho, 2004), developing students' *deductive reasoning* (e.g, Hollebrands & Smith, 2009; Hollebrands, Connor & Smith, 2010; Patsiomitou, 2008a, 2011, 2012a), and *developing students' geometrical thinking* (e.g, Yousef, 1997; Sinclair, 2001; Patsiomitou, 2008a, 2012a,b, 2018b), according to the theory of van Hiele. A DGS microworld can play a fruitful and crucial role in the process of creating and evaluating conjectures which promote student creativity, and in so doing greatly contribute to developing mathematical reasoning. There are 2-dimensional DGS packages, such as the Geometer's Sketchpad (Jackiw, 1991/2001), Cabri II (Laborde, Baulac, & Bellemain, 1988), Geogebra (Hohenwarter, 2001, 2002), Cinderella (Richter-Gebert & Kortenkamp, 1999) etc. as well as 3-dimensional DGS packages, such as Cabri 3D (Laborde, 2004), etc.. El-Demerdash (2010, pp. 23-26) reports and clarifies many purposes and functions of a DGS software which are briefly reported here: (a*) as a construction tool provides* "an accurate constructor for creating geometric configurations and has the ability to automatically adjust and preserve the variant and invariant properties of constructed geometric configurations under dragging in a visual, efficient, and dynamic manner" (El-Demerdash, 2010, p. 23), (b) *as a visualization tool* (e.g., Straesser, 2002, 2003; Christou et al., 2005), (c) *as a modeling tool* (Oldknow, 2003), (d) *as a tool for experimentation, exploration and discovery* (e.g., Clements & Battista, 1992; Hollebrands, 2002, 2003; Kortenkamp, 2004), (e) *as a tool for problem solving and problem posing* (e.g., Christou et al., 2005), (f) *as a tool for teaching geometry with the utilization of transformations and the construction of proof* (e.g., Hollebrands, 2003, 2007; Haj-Yahya, & Hershkowitz, 2013).

The diagrams that are provided to the students in a DGS environment are important spatiovisual representations that facilitate understanding of the problem's information as well as the conceptualization of the problem's structure. In other words the 'dynamic' diagrams support visual reasoning, which aids translation from visual to verbal representations and the construction of meaning. Finzer and Jackiw (1998, cited in Scher, 2002) propose three attributes related with dragging as characteristic features of any "dynamic geometry" software program (Scher, 2002, p.72):



Figures 3.1 a, b, c. Visualizing the effects of dragging and tracing.

1. *Manipulation is direct*. When users drag point A, they do not think to themselves that they are dragging the mouse, which in turn moves point A. Rather, they sense that they are dragging point A itself.
2. *Motion is continuous*. As point A of moves (**Figure 3.1a, b, c**), it does so without any discernible jumps or gaps in its movement. Motion flows like film animation.
3. *The environment is immersive*. The behavior of circles, squares, and other onscreen objects seems as real as their physical counterparts" (Finzer & Jackiw, 1998, cited in Scher, 2002, p.72).

Ruthven (2003) in her study "*Linking algebraic and geometric reasoning with dynamic geometry software: Final report to the Qualifications and Curriculum Authority*" reports the basic features of the DGS in detail, giving examples for c*lassical constructions, transformational constructions, coordinate constructions, function graphing, measuration and calculation.* She points out that "Dynamic geometry software is best known as a means of constructing and manipulating dynamic representations of geometrical objects in the plane. It provides tools supporting classical, transformational and coordinate methods of construction. Rather than creating a single static example of a generic geometrical object, the software makes it possible to create *a dynamic construction* which retains its defined characteristics but changes its visible form on the computer screen under manipulation" (p. 9).

"Hot-spots" is a *dynamic notion* introduced by Hegedus (2005) in his study "*Dynamic representations: a new perspective on instrumental genesis*". Hegedus with the notion of "hot-spots" denotes the dynamic "points" or "dots", namely the dynamic objects of a DGS environment which are "actually instantiated at an infrastructural level and are a product of new, dynamic medium" (p.7), reporting also Kaput (2000). According to Hegedus (2005):

> "The "hot-spot" in our chosen software environments is not an artifact of the environment but *an axiomatic part of the system* that allows "true" mathematical figures to be built. Dragging a "hot-spot" is not the same as "using a hammer to try to hit a nail" – note the verb use. A hot-spot will always be used well for dragging, a hammer will not always be used for hitting well. A hotspot will always be dragged and a hammer is never hit but instead used to hit. Will they ever be the same? Well, the hammer is still as effective as the hitter. The hitter hits a particular point. The action is directed by the actor. The local environment does not help with the accuracy or efficiency of the tool use, it resides with the user and practice. In addition, the action of dragging a hot-spot leads to the software environment reacting in some way" (p.2-3)[…] "Here is the critical point: the hot spot is no longer directly owned by the user. It is an infrastructural piece of the environment from which the user is now receiving feedback" (p. 5).

Hegedus concludes that dynamic representations scaffold students thinking

> "[…] grounded in the mathematical structure (axiom, definitions, rules) that are efficiently preserved when the representations are executed. The student as user has the support of rigorous scaffolding deep in the infrastructure that is extremely difficult to replicate in static, inert media. Mathematical constructions in algebra and geometry become more dynamic, motion based events, with explorations, conjectures and reasoning based around the aggregation of mathematical objects or co-actions of students and software environment" (Hegedus, 2005, p.9).

Jackiw, & Sinclair (2009) in their study "*Sounds and pictures: dynamism and dualism in Dynamic Geometry*""examine and evaluate several *new mathematical representations* developed by "The Geometer's Sketchpad v5 (GSP5)" from the perspective of their dynamic mathematical and pedagogic utility or expressibility". Jackiw, & Sinclair claim the primary contributions of Dynamic Geometry's principle of dynamism to the emerging concept of *''Dynamic Mathematics''* to be twofold:

- first, the powerful, temporalized representation of continuity and continuous change (dynamism's mathematical aspect), and
- second, the sensory immediacy of direct interaction with mathematical representations (dynamism's pedagogic aspect)" (p. 413).

Jackiw, & Sinclair characterize the new ways in which simple *pictures and sounds* can play /take different roles in the GSP5 environment by activating several *new mathematical representations*: *"pictures as pure ornament, pictures as integrated illustration, pictures as modeling scaffolds, pictures as geometric objects to construct with, pictures as geometric objects to construct, sounds as special effects, sounds to inspire mathematical precocity, sounds to build with and sounds as objects to build*" (p.420-423). Jackiw, & Sinclair support that "sensory interaction with […] novel dynamic representations in GSP5 affect mathematical modeling opportunities, student activity and engagement (p. 413). Üstün & Ubuz (2004) also consider that "the Geometer's Sketchpad is an important vehicle of technological chance in geometry classroom. […] The shapes are first created and then they are explored, manipulated and transformed to ideal concept". Olkun, Sinoplu & Deryakulu (2005) also argue that "the Geometer's Sketchpad is a suitable dynamic environment in which students can explore geometry according to their van Hiele levels" (p.3).

In my study "*An 'alive' DGS tool for students' cognitive development*." (Patsiomitou, 2018b) I report the following effects on students' thinking in relation to DGS software.

A. **A first and very important effect on students' thinking** stems from the Sketchpad software allowing the user to create sequential linking pages so that the whole Sketchpad file becomes an "*alive book*" (Patsiomitou, 2005a, p. 63, in Greek; Patsiomitou, 2014). The *"alive digital representations*" (Patsiomitou, 2005a, p. 67) function, which makes the whole figural diagram "*alive*", giving the students *the potential to focus their attention on simultaneous modifications (and transformations) of objects on the screen* (Patsiomitou, 2005a, p. 68), also yielded important results during my investigations. According to Sketchpad Help system "Over time, you may want to add additional pages to a document. For example, you may want to organize a series of sketches that develop an argument; you may want to present an activity that has several parts; or you may want to explore a conjecture in more depth than would be possible in a single sketch".
B. **A second important effect on students' thinking** stems from the *dynamic transformations in a DGS environment,* a way of modifying an object on screen. We can change a figure's orientation, a figure's size or we can reconfigure it from its parts (Duval, 1995b, 1999). Translations, rotations, and reflections are the kind of transformations that preserve the size and shape of a figure. Any transformation (i.e. rotation, translation, reflection) of an object on screen produces a similar or congruent object image on screen. If we drag any point of the object the same transformation occurs to the image object that means that the image object (or reversely) follows the dragging results that refer to the object (e.g., Patsiomitou, 2009).
C. **A third important effect on students' thinking** occurs from dynamic constructions, that are the constructions created in a DGS environment. Daniel Scher (2002) in his study describes the characteristics of a traditional static construction in contradiction to a dynamic construction. The static constructions possess two characteristics as Scher (2002, p. 1) states: "they are static and particular". In Scher's (2002) words "*the dynamic objects can be moved and reshaped interactively […and] a single on screen image represents a whole class of geometric objects*" (p.2).
D. **A fourth important effect on students' thinking** occurs from the construction of custom tools /scripts (e.g., Patsiomitou, 2005, 2006 a, b, 2007, 2008d, 2012a, b, 2014). As Straesser (2001) supports:
   "Apart from practical considerations (like exactness and ease), DGS-use can be structured according to conceptual units by means of macro-constructions. DGS-constructions are not bound to follow the small units of traditional drawing practice. Offering new tools that are unavailable in paper and pencil geometry, DGS-use widens the range of accessible geometrical constructions and solutions. If these tools become everyday instruments in the hands and minds of the user" (p.332).
   During the construction of a custom tool a user determines the order the dynamic objects have to be created. This is in accordance with what Balachef & Kaput (1997) support:
   "The order in which actions take place could become arbitrary in the eyes of users, which can have significant consequences. [...] This demonstrates the impact of the orientation of the plan which is in general forgotten in elementary geometry, but is recalled to the user as a result of the sequencing of actions (Payan 1992)". (p.13)
   I shall further discuss the meaning of custom tools in the next section.
E. **The fifth [and most] important effect on student's thinking** stems from the DGS software's *dragging facilities*. Sketchpad's dragging behavior transforms an object on screen moving that object on the screen.
   According to Laborde (1994, cited in Scher, 2000, p. 43)
   "*The idea of movement in geometry is not new—the Greek geometers devised various instruments to describe mechanically defined curves—but the use of movement was nonetheless 'prohibited in strict geometric reasoning' for reasons that were more metaphysical than scientific. The 17th century marked a break with Greek tradition, and the use of movement to establish a geometric property or carry out a geometric construction became explicit. One can find numerous examples starting then […] This idea was first expressed in school geometry by the replacement of the geometry of Euclid's Elements by the geometry of transformations (which continues to be the only kind of geometry taught in some countries)—quite some time, one must point out, after the characterization of geometry as the study of the invariants of transformation groups, and also quite some years after a daring proposition made in France by Meray (Nouveaux éléments de géométrie, first edition 1874) […] Meray's idea was to teach geometry through movement: translational movement allowed for the introduction of the notion of parallelism; rotational movement led to perpendicularity*. (pp. 61-62, French original, Scher, 2000, p. 43)

For example, if we create a triangle on screen it can be dragged and transformed into an infinite number of figural-triangles that determine the concept of triangle in every change of orientation and shape. Hölzl (1996) investigated how students used the heuristic of drag & link to manipulate a dynamic diagram and discover properties. Arzarello, Olivero, Paola & Robutti (2002) in their study "*A cognitive analysis of dragging practices in dragging environments*" introduced a hierarchy suitable for classifying the different functions of dragging in Cabri in order to describe some of their cognitive features in learning processes (p.66), "developing Hölzl's (1995, 1996) research" (p.67). They identified five different modalities which students use according to their purposes during the solution process of open problems (Olivero, 2003):

- *"Wandering dragging*: moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities.
- *Bound dragging*: moving a semi-dragable9 point, which is already linked to an object.
- *Guided dragging*: dragging the basic points of a figure in order to give it a particular shape.
- *Lieu muet dragging*: moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path (lieu muet), even without being aware of this.
- *Line dragging*: drawing new points on the ones that keep the regularity of the figure.
- *Linked dragging*: linking a point to an object and moving it onto that object.
- *Dragging test*: moving dragable or semi-dragable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have" (p.66)

Students using dragging are led "*to understand how a geometric construction can be defined by a system of dependencies*" (Jackiw and Finzer, 1993). Dragging preserves the properties of geometrical objects constructed in the DGS environment. According to Mariotti (2000, p.36)

> "the dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities […] it changes its function and becomes a sign referring to a meaning, the meaning of the theoretical correctness of the figure."

Hollebrands (2007) also supports that the students in her study "used reactive or proactive strategies when dragging, either in response to or in anticipation of the effects on dragging" (cited in Gonzalez and Herbst, 2009, p.158-159). Building on the work of previous researchers regarding dragging, I introduced two main diacrises in dragging utilizations with regard to students actions (Patsiomitou, 2011, p. 362): (a) *the theoretical dragging* in which the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties and (b) the *experimental dragging* in which the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure. Dragging an object in a DGS environment leads to the transformation of the object.

- The object (e.g., a rectangle constructed in a theoretical way) remains unaltered in terms of its structural characteristics, but the length of a side on screen is transformed due to the manner of its construction (a 'visual way' transformation, in the words of Duval). The object's orientation also can be transformed in what Duval (1995b) calls '*a place way' transformation*.
- The object is messed up as a result of the non-theoretical way in which it has been constructed (its construction depends on the student-user's conceptual understanding).
- The object is restructured, remaining an invariant construction on screen, because it has been constructed in a theoretical manner (*a mereological way* of shapes' reconfiguration).
- The object is unaltered as it is dragged on screen from a point. It appears as a static object, but it remains intrinsically dynamic due to the dependence of the aforementioned point's parent objects. In my opinion, it is a *hybrid object* (Patsiomitou, 2019a, b), which transforms the whole diagram to a *hybrid-dynamic representation*.

The transformation of an object on screen using dragging can be combined with other techniques to cause a combination of transformations on screen (e.g., Patsiomitou, 2008b, c, 2010, 2012a, b): (a) dragging and tracing objects (b) dragging and measuring objects (c) dragging and animating objects (d) dragging a transformed object or its image (by rotation, translation or reflection) or more complex such as (a) dragging, tracing and animation and (b) dragging, measuring and rotating etc.

It is not within the scope of this section to discuss the dragging facilities in any more detail, but like to Goldenberg & Cuoco (1996) I would argue that

"Dynamic Geometry *needs its own axiomatic foundation* to define the objects and postulates of its environment. (In particular, such a foundation would describe and, following Poncelet, properly mathematize the dragging transformation)" (cited in Jackiw, & Sinclair, 2009, p.415).

Generally speaking, a computer learning environment such the Geometer's Sketchpad scaffolds students' co-building of the meanings introduced in the teaching and learning activity. The design of activities in the learning environment (the software) as a part of the instruction thus has a crucial role to play in the comprehension of mathematical meanings. Jackiw, & Sinclair (2009) also argue that:

"[…] A Dynamic Geometry [object] is not an illustration, in other words— not an example of some more abstract, general, or encompassing idea—it is that idea and fully manifests its extent. At the same time, the dragged [object] implies a dragging intelligence. And this hidden actor, in whose hands the [object] *comes alive*, is the other focus of research attention" (p.414).

Over the 14 years I have been using various software environments I have employed them for many different purposes and functions, which I have published in papers, that have been uploaded onto my ResearchGate or Academia profile, and which I shall briefly report here:

- For generating and investigating accurate constructions of 2D or 3D geometrical objects (e.g., Patsiomitou, 2005a,b, 2007b, d, 2008a, b, 2009 b, c, g);
- For interpreting Euclid's *"Elements"* (e.g., Patsiomitou, 2006f, 2007e, 2008c, e, 2009a);
- For connecting History of Mathematics with technology (e.g., Patsiomitou, 2007c, 2008f);
- For generating and investigating spiral constructions (e.g., Patsiomitou, 2007b, 2008g);
- For generating a library of custom tools and their use in the research process (e.g., Patsiomitou, 2006d, 2006g, 2008d, 2009 b, c, 2018b);
- For constructing meanings in geometry, algebra or calculus, using Construction or Transform menu (and using these menus in the research process) (e.g., Patsiomitou, 2005a, b, 2006g, 2007a, 2012a, c);
- For blending DGS with web and whiteboards as part of the teaching process (e.g., Patsiomitou, 2006b, 2012c, 2018a);
- For applying instructional design processes (e.g., Patsiomitou, 2006c, 2007a, 2007b, 2007c, 2007d, 2008a, b, 2009f, 2010, 2018a);
- For blending DGS with CAS in order to co-construct the concept of mathematical meanings (e.g., Patsiomitou, 2007d, 2015d);
- Blending several DGS software for the construction of definitions (e.g., Patsiomitou, 2006e, 2015d);
- For developing students' abilities at conjecturing, arguing, proving, and constructing proofs in or out of class (e.g., Patsiomitou, 1999; 2006e, 2008a, b, c, d, e, h, 2009e, h, 2010, 2012a, b, 2014);
- For developing affective approaches which engender the love of mathematics (Patsiomitou, 2006e, 2007a, b, d, 2009d, h, 2010);
- For investigating, verifying and discovering relations (Patsiomitou, 2006g, 2007a, b, d, 2009d, h);
- As a modelling tool for the modelling process of real-world problems and using them in the research process (Patsiomitou, 2008a, b, 2012a, b, d, 2013b);
- For the modelling process of algebraic identities, using algebra tiles as structural algebraic units and implementing them in the research process (Patsiomitou, 2007e, 2008c, 2009a, 2010);
- For investigating the development of correlations between the dynamic tools use and the construction of meanings (Patsiomitou, 2009b,c, d, g, 2011a, b);
- For developing visual proofs and digital proofs (Patsiomitou, 2006e, d, 2009e, 2010);
- For generating numbers (for example, (φ) fi, (π) pi) through the development of iteration processes (Patsiomitou, 2006f, g, 2007c, 2016a, b, 2018a);
- For problem solving and problem prosing (Patsiomitou, 2006c, e, f, 2008a,b, 2012a, 2019a, b);
- For introducing and developing the notion of "Linking Visual Active Representations" and investigating the implementation of LVARs in the teaching process in multiple studies (Patsiomitou, 2008a, b, 2009b,c, 2010, 2011a,b, 2012a, b, d, 2015a, 2016a, b, 2019a);
- For introducing and developing the notion of "instrumental decoding" and investigating through several studies (Patsiomitou, 2011a, b, 2012a, b, 2015c);
- For developing dynamic propositions (Patsiomitou, 2011a, b, 2016a, b);

- For developing the notion of dynamic hypothetical learning trajectories. progressions (Patsiomitou, 2006f, 2007b, c, d, e, 2008a, 2012a, b, 2018a);
- For developing the notion of Dynamic Teaching Cycle (2012a, b, 2014);
- For developing an empirical classification model for sequential instructional problems in geometry (Patsiomitou, 2008a, 2019a);
- For introducing other notions (Patsiomitou, 2006e, 2008a, b, 2011a, b, 2019a, b);
- For enriching the mathematics curriculum by enhancing it with digital resources. (Patsiomitou, 2006b, c, d, e, f, 2007a, b, c, d, e, 2008a, b, c, d, e, f, 2012a, b).

3.2. Dynamic Objects and Instrumental Decoding

Dynamic mathematical objects are a particular kind of mathematical objects, created in a dynamic geometry software (DGS). Generally speaking, microworlds have been created to support abstract thinking through visual representations on computer screen and their transformations. Laborde (2003) in her article "*Technology used as a tool for mediating knowledge in the teaching of mathematics: the case of Cabri-geometry*" stated that:

> "the idea of computer environments as reifying abstract objects and structures originates from the notion of microworld in which it is possible to explore and experiment on representations of abstract objects as if they were material objects" (p.6)

Dynamic geometry environments are defined by Balachef & Kaput (1997) as:

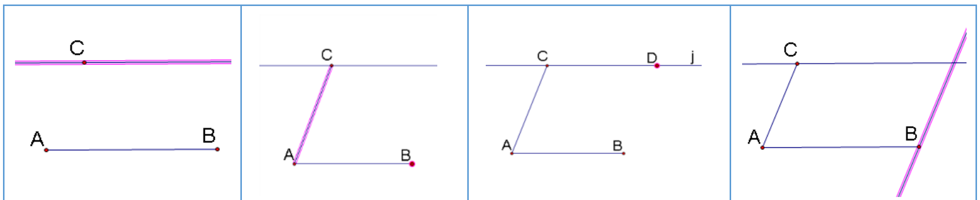
> "(a) a set of primitive objects (point, line, segment, circle, etc.) created by the tools of the software and (b) of elementary actions (for example, commands to draw a perpendicular or a parallel line given a point and a line etc.). (p.8)

Firstly, speaking of a DGS environment, it is important to identify the meanings of geometrical objects in such an environment. I introduce the following notions in my study "*From Vecten's Theorem to Gamow's Problem: Building an Empirical Classification Model for Sequential Instructional Problems in Geometry*" (Patsiomitou, 2019a, p.15):

- A *dynamic geometrical object* (Patsiomitou, 2019a, p. 15) is every object that has been constructed in a dynamic geometry software interface. This object could be a "drawing" or a "figure" which intrinsically has dynamic properties. This definition is complementary to what Gonzalez and Herbst (2009) argue regarding the dynamic diagram as "a diagram made with DGS and that has the potential to be changed in some way by dragging one or more of its parts" (p.154).
- *A dynamic diagram* (Patsiomitou, 2019a, p. 15) is an external representation composed out of a set of rationally related dynamic objects in a DGS environment. A dynamic diagram can be a simulation of a problem modelled in the DGS environment, which includes many geometric objects and combinations of interaction techniques implemented in these objects.
- A *dynamic section* (Patsiomitou, 2019a, p. 15) is a set of dynamic diagrams that are linked to each other procedurally and conceptually, even if they may differ structurally. A dynamic section contains meanings belonging to the same class that are united or joined into a whole, which in the concrete situation symbolically means they exist in one ["alive" book] section or they are dynamically linked.

In the Geometer's Sketchpad environment (or the Web Sketchpad) anyone can create a dynamic section by linking pages in the same file. In this way, a solution to a problem can be separated into sequential componential steps that help a student to create linking mental representations in his/her mind (Patsiomitou, 2008b, c, d, 2009 a, b, 2010, 2011, 2012a, b, 2013, 2014, 2018a, b, 2019a, b).

I support the following from the empirical results of my investigations (e.g., Patsiomitou, 2011, 2012a): The construction of a dynamic diagram in a DGS environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("construct a parallelogram" for example) into a mental image, which is to say an internal representation recalling a *prototype image* (e.g., Hershkovitz, 1990, Presmeg, 1992) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. This process requires the student to decode their actions using software primitives, functions etc. In order to accomplish a construction in the software the student must acquire the competence for *instrumental decoding* (Patsiomitou, 2011, p. 362) meaning the competence to transform his/her mental images to actions in the software. Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, based on Duval's (1995a, b) semiotic analysis of students' apprehension of a geometric figure. As I mentioned before, Duval has distinguished three kinds of operations, one of which is the place way, meaning an operation which changes a figure's orientation. During the development of a construction, I think that the student has to develop three kinds of apprehension when selecting software objects which accord with the types of cognitive apprehension outlined by Duval (1995b, pp.145-147) namely *perceptual, sequential, discursive, and operative apprehension*. In concrete terms, the *competence of instrumental decoding* in the software's constructions depends on: a) the *sequential apprehension* of the tools selection (i.e. s/he has to select point C and segment AB and then the command (fig. 1) meaning that s/he has to follow a predetermined order); b) the *verbal apprehension* of the tools selection which means the student has to verbalize this process, (i.e. s/he says "I am going to select point C and the segment AB") and c) *a place way type of elements operation* on the figure (i.e. when s/he transforms the orientation of the elements to apply the command selecting point B and the opposite side AC, for example in Figure 3.2d) due to his/her perceptual apprehension (Figure 3.2.b, c). Then s/he has constructed the operative apprehension of the figure's elements for the construction, meaning the competence to operate the construction. The figures below (**Figures 3.2 a, b, c, d**) illustrate the linking visual active representations (e.g., Patsiomitou, 2008a) of the steps in the students' construction of the parallelogram.



Figures 3. 2a, b, c, d: Sequential steps fro the **c**onstruction of a parallelogram

In other words, the notion of *instrumental decoding* explains a student's competence to transform his/her mental images to actions in the software, using the software's tools and commands.
The basic tools of a dynamic geometry environment are a) Circle (equivalent to Compass) b) Segment/Ray/Line (equivalent to Unmarked Straight Edge) c) Point (which simply enables us to place one of the fundamental 'objects' of Euclidean geometry) d) Pointer (which crucially enables us to drag objects). (Lopez-Real, & Leung, 2004, p.5) When these tools are combined with the software's options menu, they allow the user to produce constructions which must conform with the principles of Euclidean geometry if they are to function and pass the dragging test.
Hollebrands, Laborde and Straeser (2008, p.165) described the distinction between the three different kinds of points in a DGS environment: (a) a free point "can be directly dragged anywhere in the plane (degree of freedom 2)", (b) a point on an object "can be dragged only on this object (degree of freedom 1)" and (c) a constructed point "cannot be grasped and dragged (degree of freedom 0) but moves only if an element of which it is dependent is dragged".
This means that the student has to know the theory of geometry if s/he is to generate a correct geometric construction (or *robust construction* in the words of Laborde (2005)). And while we have explained that, in the software, the constructions can contain the same mathematical logic as the constructions on paper, there are substantial differences in the manner in which the tools are used. For example, we can construct a rectangle using (Patsiomitou, 2006c, 2019b, p, 41):

- Segments (tools) and perpendicular or parallel lines (commands) from the *Construct menu* of the DGS environment;
- Segments (tools) and transformational processes from the *Transform menu*;
- *Parameters* to represent its sides and its angles from the Graph menu.

Moreover, we can construct a custom tool/script (or macros in Cabri) to repeat a construction of a rectangle, which we have previously constructed. Straesser (2002, p.65) supports that "even if the DGS programs differ in their conceptual and ergonomic design, they share […] the ability to group a sequence of construction commands into a new command (macro-constructions". Kadunz (2002) in his study "Macros and Modules in Geometry" also argues that "Literature from mathematics education research offers three characteristic features to make a certain software for elementary Euclidean geometry a "DGS":

- "dragmode" as dynamical modeling of traditional tools from Euclidean geometry,
- "macros" to condense a series of constructions steps into one software command,

- "locus of points" to show the path of one or more points when dragging another point (cf. Graumann et al. 1996, p. 197) (Kadunz, 2002, p. 73).

Kadunz (2002, p. 73) considers that among other characteristic features in a DGS environment are "macros". According to Kadunz (2002):

> “If users and/or developers condense a sequence of commands which is often used into one unit, one command, they define a "macro". It will be labeled by a clear name (a signifier) and can be used by this throughout the whole consecutive work. Internally and hidden from the user, a "macro-expander" will substitute the signifier by the initial sequence of commands” (p. 73).

Researchers in cognitive psychology (e.g., Dörfler, 1991; Dubinsky, 1988; Frick, 1989) report that chunking information facilitates memory and retrieval. In a chunk, knowledge is condensed “into a unit available to the learner as a whole” (Kadunz, 2002, p.73). Weibell (2011) also states:

> “One effective strategy that can be used to extend [or increase] the amount of information held in working memory is chunking (Miller, 1956). Chunking is a process of recoding multiple bits of information into a meaningful representation that contains the same amount of information, but takes up fewer slots in memory” (p.110).

Chunking “supports and facilitates cognitive processes involved in encoding, extracting, remembering, and understanding information” (Winn, 1993; Gobet et al., 2001 quoted in Sedig, & Sumner, 2006). According to Straesser (2001, 2002, 2003) macros [/custom tools] “can help to structure a geometrical construction by condensing a complicated sequence of construction steps into one single command”. This is in other words “a chunk of knowledge”, as Simon (1980) points out: “A chunk is any perceptual configuration [...] familiar and recognizable” (Simon, 1980, p. 83) that helps the students to *reverse their thoughts* (e.g., Patsiomitou, 2012a).

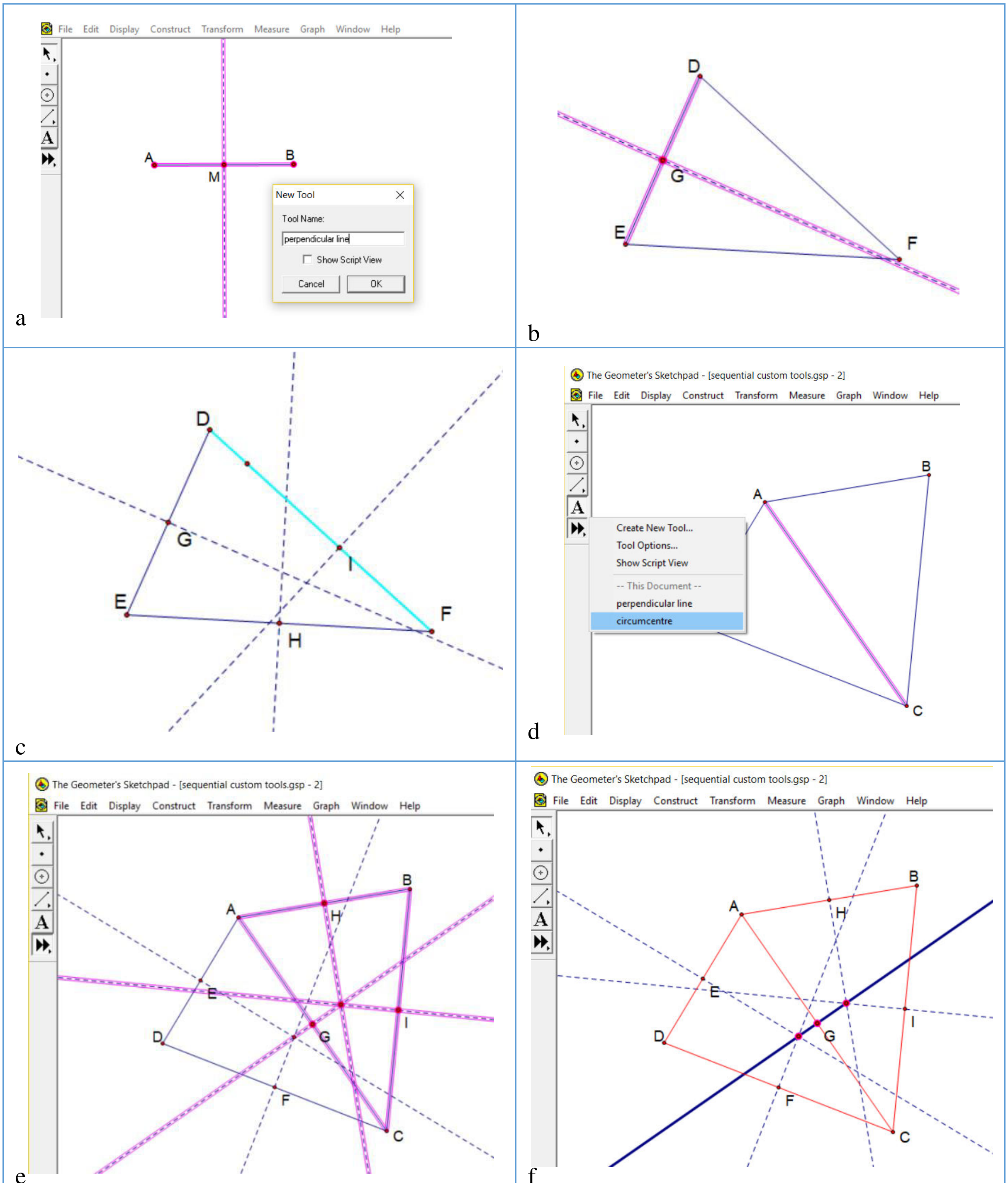
In my opinion, a custom tool *is an encapsulation of a sequence of primitive objects and construction commands into a new tool, combining information of the construction in a consequential mode*.

The idea of scripting/constructing custom tools was to create “personal tools”, or tools that a student could use for his her needs. According to Scher (2000, p.45) “Jackiw viewed the scripting feature of Sketchpad as a way for students to start from the “atoms” and gradually build their own collection of reusable, multi-step constructions”. Kadunz (2002) also states that “to the user, the macro function is a black box producing defined output from defined input” (p. 74).

A script /custom tool combines in a concrete and sequential order the steps that have been used to accomplish the construction. For example, if we construct a square, we can save the concrete construction in a custom tool which can repeat the construction in the concrete way used by the creator of the custom tool, meaning that is processes the objects in the same sequence. The dragging of the custom tool constructed on screen follows the rules that refer to the primitives and commands incorporated into the custom tool (i.e. if we have measured angles or segments, or calculated a ratio, during our construction of the tool, then the concrete measures and calculations are repeated any time we implement the custom tool). If we drag the tool, the measures follow the increasing or decreasing of the length of the segments and angles (e.g., Patsiomitou, 2005a, p.83).

By constructing a custom tool, we can help students to extend the capacity of their working memory, since the knowledge the student must retain is reduced. Nonetheless, the basic underlying notion is that a student is able to codify a construction and the concrete codification shape what the student can do when s/he will encounter a new situation related to the concrete that has been abstracted and codified with the use of custom tool.

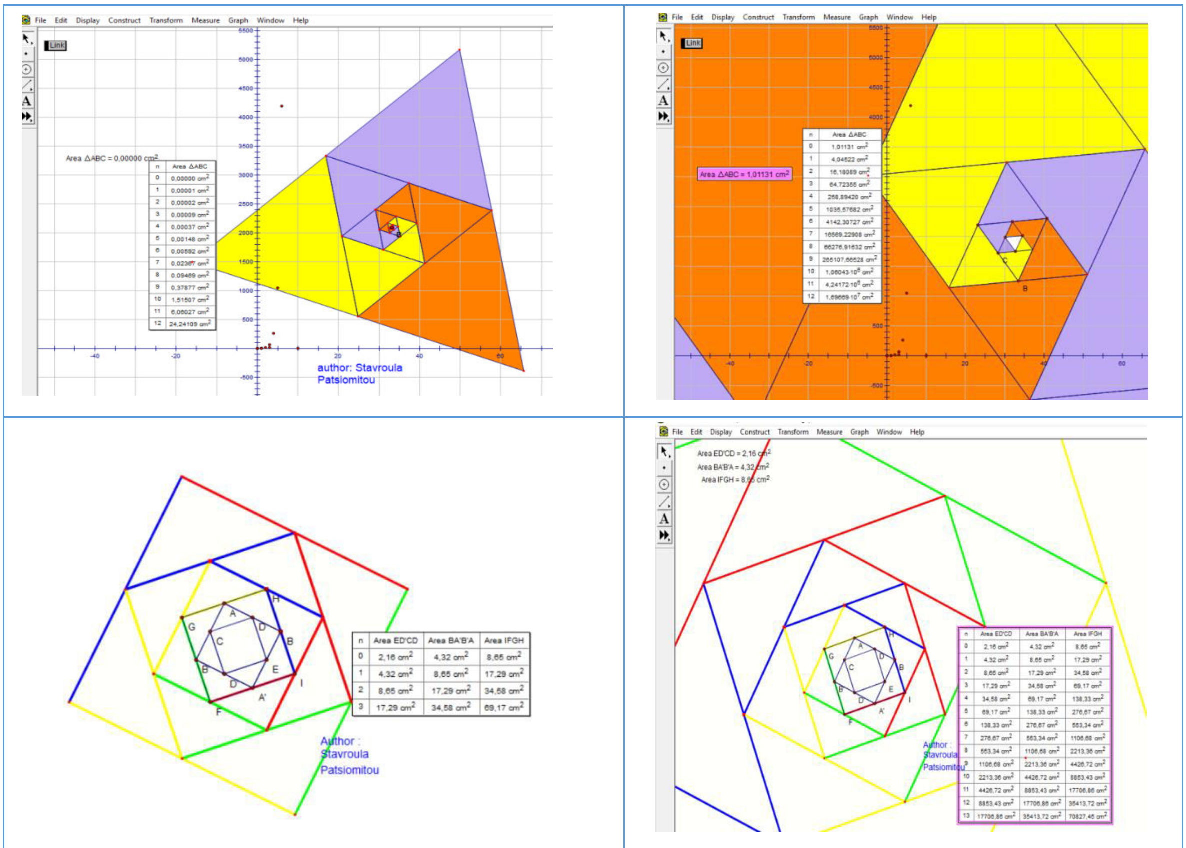
I shall provide an example to illustrate it: Suppose we need to construct the lines perpendicular to every side of a triangle in order to prove that they all coincide at the same point --the circumcentre. Then we need to construct the circumcenter of the triangles ABC and ADC (the triangles are formed when we draw the diagonal of the quadrilateral ABCD). A simple way to do this is to construct the midpoints on each of the triangle’s sides and then to construct the perpendiculars, repeating the process three times. Again, the same process will be repeated to construct the circumcentre of the triangles ABC, ADC. Another way would be to construct and implement sequential custom tools: (a) a custom tool for constructing a line perpendicular to a segment (“perpendicular line” custom tool) (b) a custom tool for constructing the circumcentre of a triangle (“circumcentre” custom tool). (**Figures 3.3.a, b, c, d, e, f**). The actions we need to accomplish the whole process are the following: (a) constructing the custom tool “perpendicular line” (b) implementing the custom tool to the sides of the triangle (c) constructing the custom tool “circumcentre”, *encapsulating the previous construction* (d) implementing the “circumcentre” custom tool to the triangles in the quadrilateral.



Figures 3.3 a, b, c, d, e, f. Construction and implementation of sequential conceptually-linked custom tools

This way of construction is in a more abstract level than the previous way, as the student is pushed through the process *to a reification of sequential nested objects.*

This action has a presupposition: the students to know in advance that a side of triangle is a segment or to understand the double role of the objects (van Hiele level 3). Moreover, the orientation of the sides may generate a cognitive obstacle, especially for students at van Hiele levels 1 or 2. This is because students very often fail to recognize the modification of the orientation of tools due to a lack of *place way apprehension* during the *instrumental decoding* process. The custom tools help them to simplify the construction process.

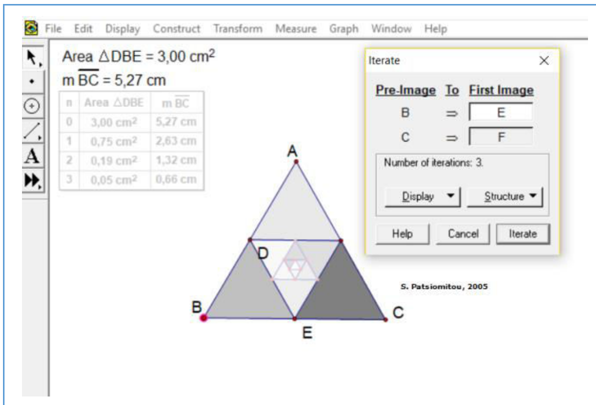


Figures 3.4 a, b, c, d. Constructing Baravelle spirals using the iteration process (Patsiomitou, 2005a, b, 2006g, 2007b, e)

As a result of the construction and application of a custom tool the direct perception of the user is attained with regards to the steps in the development of the construction pertaining to (see) (e.g., Patsiomitou, 2007a, 2014, 2018a, b, 2019a): 1) the repetitions in the measurements or calculations of the areas of initial shapes 2) the developmental way of the construction of the figure and 3) its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file. If we have constructed a custom tool which incorporates the use of iteration processes, in the case of Geometer's Sketchpad the application of the custom tool will include the iteration at every new step during every new application of the custom tool (**Figures 3.4 a, b, c, d and Figure 3.5a, b, c**). **Figures 3.5a, b** present the sequential steps of a construction of a Baravelle spiral which has been introduced by Chopin (1994). Mariotti (2000) declares that in a construction generated using dynamic geometry software "[…] the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality" (p.27). This is in accordance with what Jones (2000, p.56) points out that "dynamic geometry systems (DGS) would seem to have the potential to provide students with direct experience of geometrical theory, and thereby break down what can be an unfortunate separation between geometrical construction and deduction". The following statement is something I strongly support as complementary to something I stated in a previous study (Patsiomitou, 2008d, Patsiomitou, 2018b, p.51):

> *Custom tools are 'alive' encapsulated objects created in a DGS environment that operate as a referent point for organizing, retrieving and reversing information, and thus facilitating the anticipation and manipulation of the instrumented action schemes during an instrumental genesis process. A custom tool can become a medium for students' cognitive development and to develop their abstract thought.*

In order to comprehend the advantages (and disadvantages) of the construction mode in the dynamic geometry software, it is necessary to examine the differences between it and the mode of construction using static means. This will allow us to compare the two modes. For instance, in using a straightedge with measurements, the mode of constructing a figure in the software (e.g a square of side a) could be different from the mode students use to construct it on paper.



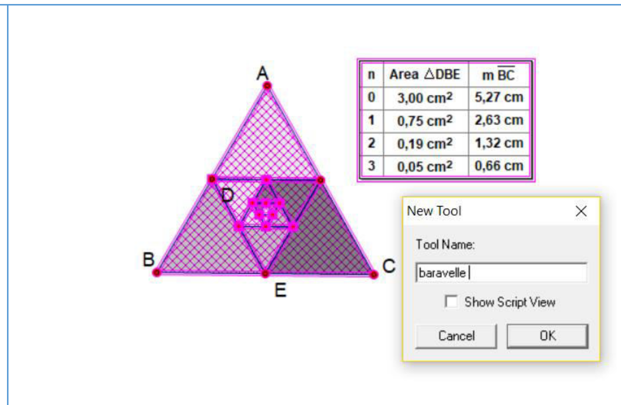


Figure 3.5a. Constructing a Baravelle spiral (Patsiomitou, 2005a, 2007b, 2009b, in Greek) **Figure 3.5b.** Constructing a custom tool "Baravelle" (Patsiomitou, 2005a, 2007b, 2009b, in Greek)

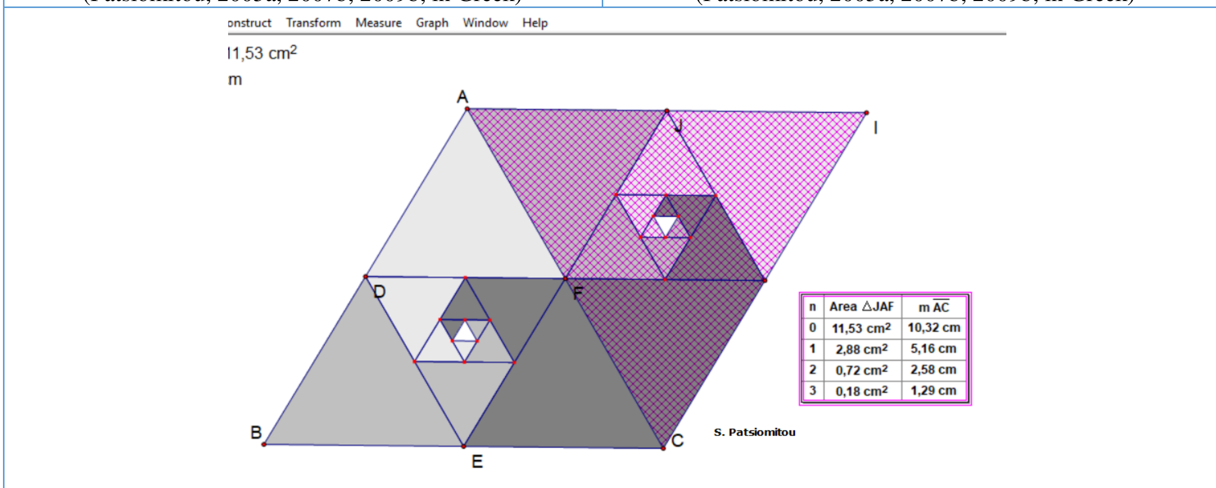
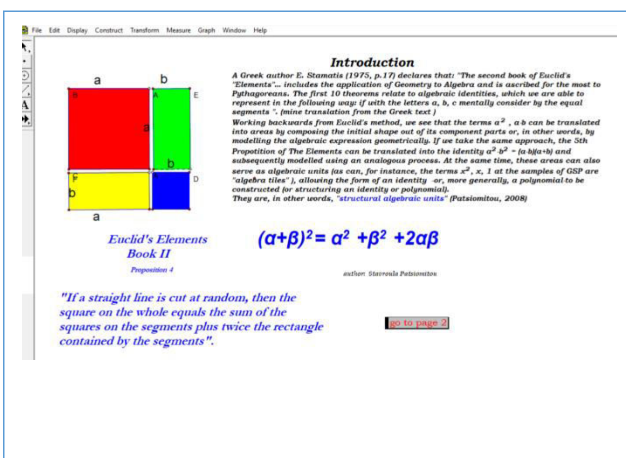


Figure 3.5c. Implementing the custom tool "Baravelle" to a side of the triangle ABC in order to construct a more complex structure (Patsiomitou, 2007b, 2009b, in Greek)

Duval (1999) argues that "[…] Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization." (p.21). By forcing students to think of ways of constructing an equal segment, this methodological weakness can thus provoke a cognitive conflict in students, and in so doing raise the level of difficulty.



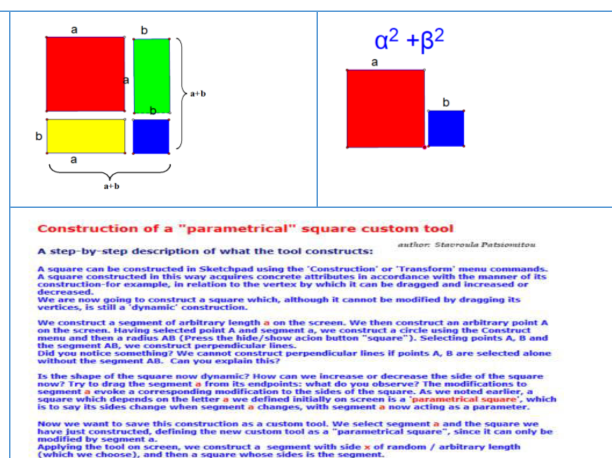


Figure 3.6a. Proposition 4 (Euclid's Elements, Book II)(Patsiomitou, 2008c, p.198, 2009a) **Figure 3.6b.** Construction of a parametrical square and a parametrical rectangle custom tool)(Patsiomitou, 2008c, p.198, 2009a)

One such way would be to define side ‘a’ as an arbitrary segment on the screen and then use it as a radius of a circle in the construction. This construction method induces a different mental perception in the students with regard to construction in the software. In this way, the sides of the square cannot be modified from the vertices of the shape using the dragging modality. Instead, they depend on the modification of the initially defined segment a. The arbitrary segment ‘a’ could thus be confined as a non-collapsible compass to either the square or any shape whose a side is equal to ‘a’. This construction procedure depends on the students’ level of conceptual knowledge and cognitive abilities. As a consequence, the construction of the shapes depends either on segments a & b--both of which are arbitrary defined --and the relationship among them, or on the students’ geometrical knowledge of the relationships between and properties of shapes.

Another important point about this construction is that students can use it to verify and to construct arguments, and in so doing overcome an *epistemological obstacle* which one frequently finds obscuring comprehension in students’ tests using static means (e.g., they mistakenly write that $(a+b)^2 = a^2+b^2$) (**Figures 3.6 a, b**).

During the process of learning mathematics, students face:

1. *Obstacles whose nature is cognitive* and relates to the Geometry itself as a subject. For example

- the student does not recognize the basic components of the figure, or does not know how the figure’s elements are defined
- the student does not know what the sequence of actions is s/he has to follow to construct a figure
- the student cannot formulate the sequence of actions that s/he has to follow to construct a figure

In one way or another, the above relates to the student’s competence at translating between different kinds of representation (verbal, graphic, symbolic, etc).

2. *Obstacles whose nature relates to the technology* used by the students. For example:

- The student has not constructed *usage schemes* for the tools, namely how to use the tools
- The student has memorized the way in which tools can be used, which leads him/her to take meaningless actions in the sense that their actions have no connection to logical reasoning.

As Mesquita (1998) reports

> “[…] the nature of the illustration is the main obstacle in the problem […]. Even if children are accustomed to other kinds of representations, at least the ones associated with perspectives, textbooks almost exclusively use “objects” as external representations. In fact, the analysis of the pupils’ answers in our study suggested that once the obstacle created by the nature of the external representation was overcome, pupils made the necessary substitutions to solve the problem” […] For this reason, the nature of the external representation may become an obstacle to pupils understanding.” (p. 193-194).

Obstacles can be seen as an opportunity for students to reflect on their own learning rather than allow this to be a barrier to achieve understanding of mathematical ideas. In my PME35 study “*Theoretical dragging: a non-linguistic warrant leading to “dynamic” propositions*”, I introduced the notion of *instrumental obstacle* (Patsiomitou, 2011, p. 365): “I distinguished a few types of instrumental obstacles due to student lack of competence in *instrumental decoding*. I am going to describe two of them including snapshots of the research process (Patsiomitou 2011, p. 365).

A. The students (mentioned in my study as M2, M3, M8, and M14) tried to construct a parallelogram using the Geometer’s Sketchpad. Most students at van Hiele level 1 were unable to understand the *sequential apprehension* of the tools selection, because they were unable to understand the logic of the sequence of actions or unable to link this logic with the theory of geometry. For example M14 (van Hiele level 1 at the pre-test) faced an instrumental obstacle which depended on her sequential apprehension of the objects to be used for the construction. She tried to construct a parallel line by selecting the line alone and then the menu command, which is to say she followed an irrational sequence of actions. At this point, she faced an *instrumental obstacle* and commended in an informal way on the non-activation of the software’s command (saying “[the command] is not illuminated again”). Subsequently, her interaction with the software, led to a *cognitive conflict* which helped her to apprehend the sequence of actions. Students of van Hiele level 2 developed the three kinds of apprehension along with the other members of the group: verbal apprehension emerged as a result of the previous action in the software, namely as a result of the interaction with the tools. For example, as a result of the previous action M2 (van Hiele level 2 at the pre-test) states: “this will be a line parallel to segment AB”.

B. The utilization of Euclidean definition of a segment presented level-2 students with instrumental obstacles in the DG environment. Thus: the group prompted student M8 (van Hiele level 2 at the pre-test) to select the segment in order to construct a perpendicular line. Among the definitions he knew was the definition of a segment mentioned above. He therefore followed the definition of the textbook, decoding the verbal expression by selecting the segment and its endpoints. This action results in the command not being activated on screen, so he was unable to continue the process. This is to say a cognitive conflict occurred between what the students knew from the Euclidean geometry definitions they had learned and what they encountered in the DGS environment. Exactly the same thing happened to student M2 when she tried to select a segment to construct its midpoint. This action led the students to apply new rules inductively and to understand empirically something that we could define by answering the question "what is a 'dynamic' segment?" *The 'dynamic' segment is a portion of a straight line which does not consist of points. Dynamic points can be placed independedly on the dynamic segment and move free with one degree of freedom on the path to which they belong. This means that a point placed on a segment has its two degrees of freedom transforming into one degree of freedom.* In a second example, student M3 tried to select a point on the straight perpendicular line intersecting with the segment AB in order to construct the sides of an isosceles triangle. Trying to decode the verbal formulation "select a point on the straight line" in the DGS environment they were unable to do it on the dynamic line (or the dynamic segment) they had constructed. Student M3 thus faced a cognitive conflict which led him to understand that he had to select an independed point and put it on the line. This is exactly the time in which student set a new rule something we could define: *the selection of a segment in a DGS environment occurs with the selection of its internal alone, which represents the set of points in the Euclidean definition*". Tools in a DGS environment can be transformed into *psychological tools* as Mariotti (2000) states:

> "Tools have a twofold function, the former, *externally oriented*, is aimed at accomplishing an action; the latter, *internally oriented*, is aimed at controlling the action" […] The process of *internalisation* as described by Vygotskij may transform tools into *psychological tools*: when internally oriented a 'psychological tool' will shape new meanings, thus functioning as semiotic mediator" (p. 35)

3.3. Artifacts, Tools and Instruments

Every tool used in a DGS environment is a digital artefact. According to Cerulli (2004) "An artefact, for us, will be an object which has been in some way produced by humans. As a consequence every artefact for us is an object, but not all the objects are artefacts; for instance, a stone, in general, is an object but not an artifact" (p. 7). According to Norman (1991) "A *cognitive artefact* is an artificial device designed to maintain, display, or operate upon information in order to serve a representational function". (p. 17). Kaptelinin (2003) states that cognitive artifacts (a) emphasize the cognitive, rational, information processing functions served by technologies used by human beings[…] (b) are intended for individual, rather than collective use […] and (c) do not change individuals' capabilities […] " (p.831). Bartolini Bussi, Mariotti & Ferri (2003) in their article "*Semiotic mediation in the primary school*" discuss the *primary, secondary and tertiary artefacts* introduced by Wartofsky (1979).

> "[…] *Primary artifacts* are those directly used in this production; *secondary artifacts* are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artifacts are therefore representations of such modes of actions" (Wartofsky 1979, cited in Bartolini Bussi et al, 2003, p. 78)"

Mariotti (2000) argues that "the functioning of an artefact in the development of meaning can be described taking into account the process of *semiotic mediation* which develops at different levels:

- The pupil uses the artefact, according to certain utilisation schemes, in order to accomplish the goal assigned by the task; in so doing the artefact may function as a semiotic mediator where meaning emerges from the subject's involvement in the activity.
- The teacher uses the artefact according to specific utilisation schemes related to the educational motive. In this case, […] the utilisation schemes may consist in particular communication strategies centred on the artifact" (p. 36).

Vygotsky distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*) and offers a list of examples (Bartolini Bussi et al, 2003, p. 78): "language, various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps, and mechanical drawings, all sorts of conventional signs and so on (Vygotsky, 1974, p.227, cited in Bartolini Bussi et al, 2003, p. 78)

An 'artefact', or a tool with which the interaction takes place during the mathematical activity, is transformed into an '*instrument*', according to the theory of *instrumental genesis* (Verillon & Rabardel, 1995). Many researchers (Guin, & Trouche, 1999; L. Artigue, 2000, 2002; Trouche, 2003, 2004; Trouche, & Drijvers, 2014; Drijvers, 1999, 2003; Drijvers, & Trouche, 2008; Drijvers, Godino, Font, and Trouche, 2013; Patsiomitou, 2008a, 2012a) have reported on the dual interactive process involved in *instrumental genesis* (Verillon & Rabardel, 1995), which is a theoretical framework appropriate to describing the interactions occurring from the integration of technological tools into mathematics education. Firstly, it is essential to distinguish the notion of '*artefact or artifact*" from the notion of "*instrument*" (Rabardel, 1995, 2002). According to Drijvers, Godino, Font, and Trouche (2013):

> "An *artefact* is an–often but not necessarily physical–object that is used to achieve a given task. It is a product of human activity, incorporating both cultural and social experience. Think of a hammer, a piano, a calculator, or a dynamic geometry system on your PC. What exactly is the artefact in a given situation is not always clear: for example, in the case of dynamic geometry software, it is a matter of granularity if one considers the software as one single artefact, or if one sees it as a collection of artefacts, such as the construction artefact, the measurement artefact, the dragging artefact, and so on (Leung, 2008)"[...] Following Rabardel (2002), we speak of an instrument if a meaningful relationship exists between the artefact and the user for a specific type of task. The in many cases ongoing, nontrivial and time-consuming process of an artefact becoming part of an *instrument* in the hands of a user is called *instrumental genesis*."(p.26)

Instrumental genesis also takes place in a class of students who share the same objective. It is distinguished in two distinct processes the '*instrumentation process*' and the '*instrumentalization process'*. Concretely Artigue (2000) in her study "*Instrumentation issues and the integration of computer technologies into secondary mathematics teaching*" states that instrumental genesis is directed towards:

a) "the artefact, loading it progressively with potentialities, and eventually transforming it for specific uses" (it is called the *instrumentalization process* of the artefact)
b) "the subject, and leads to the development or appropriation of schemes of instrumented action which progressively constitute into techniques which allow us to solve given tasks efficiently" (it is called *the instrumentation process*)" (p. 10)

This dynamic active functionality of the tool presupposes the student to act on the tool (external use of the construction) thus the tool is shaped by the user during the *instrumentalization process* while the artefact simultaneously acts upon the subject (internal use of the structure) and the tool affects and shapes the users' thought during the *instrumentation process* (e.g., Guin, & Trouche, 1999; Artigue, 2000; Trouche, 2004; Drijvers & Trouche, 2008; Patsiomitou, 2008a, b, c, d). Consequently, the student creates an accommodation of his older scheme about a concept while s/he accommodates a tool to investigate the concept through the use of the tool (Patsiomitou, 2008a, d). Rabardel (1995, 2002) calls the schemes "linked to the utilization of an artifact, *utilization schemes"* (p. 82). The need to use a tool leads the student during the instrumental genesis process to the development or appropriation of *usage schemes* and *schemes of instrumented action*. Rabardel defined two levels of schemes within utilization schemes:

- "*Usage schemes* are "related to 'secondary tasks' […] corresponding to the specific actions and activities directly related to the artifact" (p.83)
- "*Instrument-mediated action schemes* (or s*chemes of instrumented action)* are related to 'primary tasks' […] aiming at operating transformations on the objects of activity" (p.83).

Moreover Rabardel reports the *"instrument-mediated collective activity schemes,* which "concern the specification of the types of action or activity, of the types of acceptable results etc. when the group shares a same instrument or works with a same class of instruments" (p.84).

Through the instrumented action schemes, mathematical knowledge and knowledge of the tool are combined. As Trouche (2004, p. 286) notes: "A scheme has thus three main functions:

- *a pragmatic function* (it allows the agent to do something),
- *a heuristic function* (it allows the agent to anticipate and plan actions)
- *and an epistemic function* (it allows the agent to understand something)."

From Trouche's point of view, "instrumental geneses are individual processes, developing inside and outside classrooms, but including of course social aspects" (**Figure 3.7**) (personal e-mail correspondence with Professor Trouche on April 4, 2008).

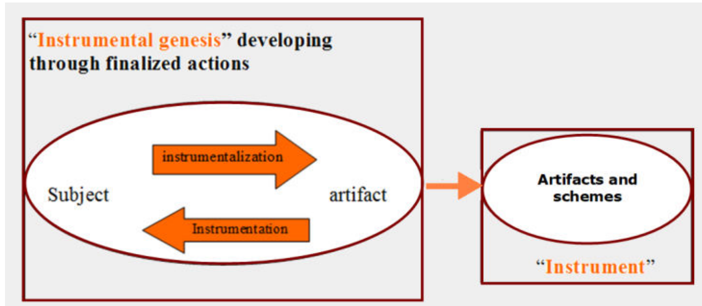


Figure 3.7. The schema of instrumental approach (Trouche, & Patsiomitou, cited in Patsiomitou, 2008, p. 362)

In the Figure 3.7 a schema of *instrumental approach* is depicted which was constructed in cooperation with Prof. Trouche (personal e-mail correspondence with Professor Trouche on April 2, 2008, based on Trouche's (2006) schema of instrumental approach) (Patsiomitou, 2008, p. 362). Trouche supports that "an artefact is transformed thus through instrumental geneses, oriented by finalized actions, assisted by instrumental orchestrations, into *an instrument*". According to Artigue (2000),

> "*An instrument is thus seen as a mixed entity, constituted on the one hand of an artefact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appropriation of socially pre-existing schemes*."(p.10)

An instrument (Rabardel, 1995) combines both an *artefactual, material structure* (external result) and *a psychological schematic structure* (internal result) directly linked to the use of the artifact (e.g., Artigue, 2000; Trouche, 2003, 2004). This is in accordance with what Beguin & Rabardel (2000) state with regard to structures an instrument is made:

> "- *psychological structures*, which organize the activity;
> - *artifact structures*, which […] are the signs and symbols in the code used to think of and express solutions, along with the paper, pencils, erasers, and so on, that serve to produce and modify the diagrams" (p.179). (Figure 3.8).

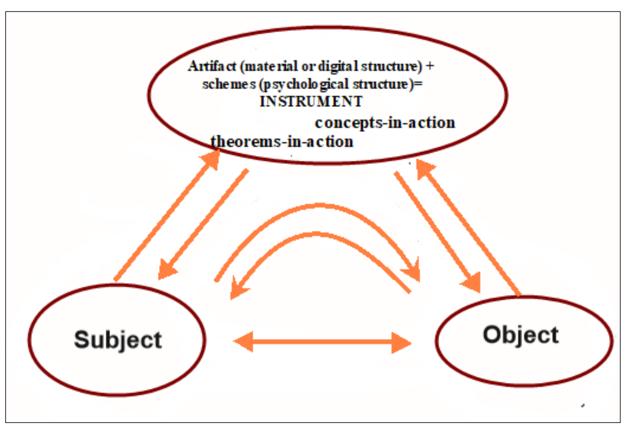


Figure 3.8. The mediating instrument (Beguin & Rabardel, 2000, p. 179) (an adaptation for the current study).

During the learning process, students discuss their ideas and make inferences in relation to the diagrams' dynamic transformations. The construction of schemes during the instrumental genesis process is what researchers consider when studying long-term uses of technology. According to Trouche (2003, 2004) a scheme of instrumented action constructed during the instrumental genesis process incorporates *operational invariants* (namely *theorems-in-action* and *concepts-in-action*) (Vergnaud, 1998). The notions of scheme, *theorem-in action* and *concept-in-action are defined by* Vergnaud (2009) as follows:

- "A *scheme* is the invariant organization of behavior for a certain class of situations.
- A *theorem-in-action* is a proposition which is held to be true;
- A *concept-in-action* is an object, a predicate, or a category which is held to be relevant ('concepts implicitly believed to be relevant')" (p. 168).

Rabardel (2005) mentions Vergnaud (1996, 1998, 2009) and his *theory of conceptual fields. A* scheme comprises four different kinds of ingredients:

-"*anticipations of the goal* to be reached, expected effects and possible intermediary stages;

-*rules of action* along the lines of "if-then" which allow the sequencing of subjects' actions to be generated;

-*inferences (reasoning)* that allow the subject to calculate rules and anticipations based on information and the operational invariants system he/she disposes of;

-*operational invariants* that pilot the subject's recognition of elements pertinent to the situation and information gathering on the situation to be dealt with" (Rabardel, 2005, p.79).

Docq and Daele (2001, p.200) point out, the two principles identified by Rabardel, which are linked to the production by the subject of his/her own using schemes for a new tool:

- The '*economy principle'* where the subject tends to choose the most familiar or the most available tool and to use it for as much actions as possible and
- The '*search for efficiency*' where the subject tends either to choose another tool or to use the proposed tool but in a way designers of the tool had not anticipated (informal use, or 'catachreses' according to Rabardel).

This means that students many times use a tool in *an economical mode* or a *catachresis mode*. An *economical mode of the tool* is determined when a student tends to use a tool that previously has been used for a first task "to carry out a new task" (Rabardel, 1995, p.96). In other words s/he makes economy of the use of tools. The idea of 'catacresis' in the words of Beguin & Rabardel (2000)

> "is employed in the field of instrumentation to refer to the use of one tool in place of another, or to using tools to carry out tasks for which they were not designed" […] catacresis [is]an indicator of the user's contribution to the development and use of an instrument. The existence of catacreses reveals that the subject creates means more suited to the ends he or she is striving to achieve, and constructs instruments to be incorporated into the activity in accordance with his or her goals" (p.180).

According to Martinez-Maldonado, Carvalho, and Goodyear (2018, p.5) "the theory of *instrumental genesis* has been built on *activity theory* (e.g., Leontiev, 1978; Engeström, 1987, 1990, 1999; Nardi, 1996) and the theory of *situated cognition* (Brown, Collins and Duguid, 1989, cited in Martinez-Maldonado, Carvalho, and Goodyear, 2018, p. 5)". Similarly, Kaptelinin (2003) states that instrumental genesis "is based on activity theory, which deals with purposeful interactions of active subjects with the objective world (Leontiev, 1978). These interactions, or activities, are understood as social, hierarchically organized, developing, and mediated by tools" (p. 832).

Activity theory is a psychological theory that has been developed from the work of Soviet cognitive psychologists (e.g., Vygotsky, 1978; Luria, 1928; Kuutti, 1978; Leontiev, 1978). Engeström (1987, 1990, 1999) developed a version composed of the following interacting components: mediating artefacts or tools, subject, object, community, division of labour, and rules.

> "Activity theory: Constructs from Activity Theory are used in a number of papers, largely as an analytical tool. For example, community is a key element in Engeström's (e.g., 1999) third generation framework in Activity Theory, in which he presents his "Expanded Mediational Triangle" deriving from first and second generation versions of Vygotsky's meditational triangle. Here some subject achieves an object or goal through the mediation of an instrument or artifact (or tool). As well as the mediation of artefacts (in our studies, such as text books, on-line systems or mathematical symbols), Engeström suggests that Rules, Community and Division of Labour are also important mediators in an activity system. Thus, in taking activity theory as a basis for research into mathematics teachers' learning through collaboration, the idea of the community in which learning occurs is central to the concept of mediation. Several studies use an Activity Theory frame through which to address the situative aspects of the study. The frame is in some cases Engeström's triangle; in others it is a three-layer framework attributed to Leont'ev consisting of Activity related to Motive, Actions related to Goals, and Operations related to Conditions. According to Leont'ev, Activity is always motivated, although the motive might not be explicit. Within motive we have actions which are always explicitly goal related. Action and goals depend upon operations and conditions within activity."
>
> **(Robutti , Cusi, Clark- Wilson, Jaworski, Chapman, Esteley, Goos, Isoda, Joubert, 2016, p. 671)**

A short description of these components has been given by Jonassen et al. (1999, p.161, cited in FitzSimons, 2005, p.770):

- “The *subject* of any activity is the individual or group of actors engaged in the activity.
- The *object* of the activity is the physical or mental product that is transformed.
- *Tools [or mediating artefacts]* can be anything used in the transformation process. [...] The use of culture-specific tools shapes the way people act and think. [...] Tools alter the activity and are, in turn altered by the activity.
- The activity consists of the goal-directed actions that are used to accomplish the object—the tasks, actions, and operations that transform the object” (**Figure 3.9**).

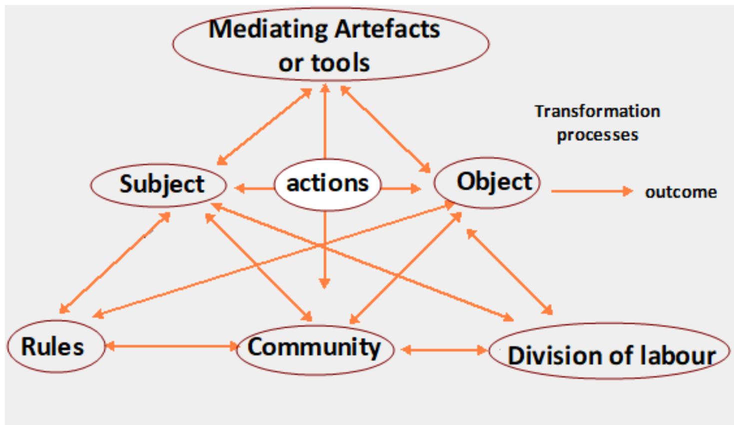


Figure 3.9. Engeström's model (1987) of activity theory (cited in FitzSimons, 2005, p.770) (an adaptation for the current study)

Activity theory has been used in numerous papers as analytical tool. The theory focuses on how subjects transform objects and the mediation processes (Robutti et al, 2016, p.671). According to Nardi (1996) in her study “*Activity Theory and Human-Computer Interaction*”:

> “Activity theory proposes that activity cannot be understood without understanding the role of *artifacts* in everyday existence, especially the way artifacts are integrated into *social practice* (which thus contrasts with Gibson's notion of affordances). Cognitive science has concentrated on information, its representation and propagation; activity theory is concerned with practice, that is, doing and activity, which significantly involve ``the mastery of ... external devices and tools of labor activity" (Zinchenko 1986)”.

3.4. DGS Transformations in Geometry – “A Metamorphosis”

A student can construct “dynamic” representations using the facilities offered by a DGS software. As I mentioned before, this means that the student can use transformation tools like rotation or reflection in addition to the Compass and Straightedge tool. Rotation, reflection, translation, dilation are isometries.

> “The first component of the word *isometry* is from the Greek word isos (isos means “equal”). The second is from the Greek work *metron* (metron means “a measure”) (Schwartzman, 1994, cited in (Webpage [21]). *An isometry* is a mathematical transformation that retains certain measurements: most importantly, it retains the distances between particular points. Any isometry f is a function 1-1 correspondence and, as such has an inverse f^{-1}, which is also an isometry” (e.g., Coxeter, 1961; Yaglom, 1962 cited in Webpage [21]).

The focus on transformations is in accordance to Coxford & Usiskin (1975), who report that, “the use of different types of transformations in the curriculum simplifies the mathematical development (for example, the definitions of congruence and similarity cover all figures). Therefore, the proofs of many theorems are simpler and more accessible to all students” (Coxford & Usiskin, 1975, Preface, p.v). Furthermore, Coxford & Usiskin argue that “transformations are used because

- They can be understood by students of widely varying abilities
- They give a unifying concept to the geometry course
- They provide assistance for future work in algebra and calculus” (Preface, p.vi)

De Villiers (1997) in his study “*The Future of Secondary School Geometry*” discusses “Klein's famous Erlangen-program (1872) which described geometry as the study of those geometric properties which remain invariant (unchanged) under the various groups of transformations” (p. 3). According to De Villiers (1996) geometry could be classified according to this view as follows: "(a) *isometries* -[transformations of plane figures which preserve all distances and angles (congruency)] (b) *similarities* -[transformations of plane figures where shape (similarity) is preserved] (c) *affinities* –[transformations of plane figures where parallelism is preserved] (d) *projectivities* –[transformations of plane figures which preserve the collinearity of points and the concurrency of lines] and (e) *topologies* –[transformations of plane figures which preserve closure and orientability] " (p.3).

Whiteley (1999) in his study "*The Decline and Rise of Geometry in 20th Century North America*" argues that "Transformations' are the key concept of geometry. Reasoning with transformations should be a central theme of our learning of geometry (Yaglom, 1968) […] Transformations and change within geometry are central to understanding geometry" (p.15)

Transformations used by the students in the DGS environment can be distinguished through the following (Patsiomitou, 2014, p.30) (Figures 3.10a, b, c, d, e):

- Transformation generated from the *reflection, dilation, rotation, or translation* of the object. Dragging on rotated (dilated, reflected, or translated) objects maintain the congruency and structural relationship between the elements of the construction.
- Transformations generated from the utilization of the *action buttons tools* (for example, the hide/show action button, the link button, the movement button, or animation).
- Transformations generated from the *annotation of the dynamic diagram* (for example, use of colours, formulations, and the trace tool). Moreover, the combination of transformations (e.g., the trace tool and dragging tool, the calculations and the dragging of the geometrical object's points).
- Transformations generated from the *application of the custom tools*. The application of custom tools reorganizes the external representation. The application of a custom tool (or the repetition of the application of a custom tool) is accomplished in a sequence of steps directly perceived by the user. Consequently, custom tools operate as a referent point for *organizing, pursuing, and retrieving information.*
- Transformations generated from the *synthesis of the dynamic diagram.*
- Transformations generated from the *reconfiguration of the dynamic representation.*
- Combinations of transformations due to the synthesis *of the software's interaction techniques* (Sedig & Sumner, 2006).
- Complex transformations of the *LVAR dynamic representations* (Patsiomitou, 2008a, b).

The diagrams' reconfiguration through the complex synthesis of combinations of transformations can lead to a continuous interaction of *discursive, visual and operational apprehension* (e.g., Patsiomitou, 2008b, c, 2010, 2011a, b, 2012a, b, 2013, 2014, 2018b). In the words of Dina van Hiele (1984) the diagram goes through a *metamorphosis* as a result of the manipulations of reconfigurations "*followed by a phenomenological analysis and an explicating of its properties: it becomes what we call a [dynamic] geometric symbol*" (Dina van Hiele in Fuys et al., 1984, p.221; Patsiomitou, 2018b). Transformations on prototype elements (e.g., points, line segments) lead the students to (1) visualize the objects that are constructed in the first phase of the process and (2) perceive a few properties of the figure's symmetry initially at the visual level. It is observed that students connect, in their minds, representations that help them to respond to the next level, according to the theory of van Hiele. Therefore, **dynamic geometric transformations** are defined (Patsiomitou, 2014, p. 31):

> *as the modification of the diagram on screen that result in the modification in one or more incorporated geometric objects. This could be an elicitation from the addition, cancelation of the diagram's elements that cause the rearrangement of the diagram, its anasynthesis, its metamorphosis or even the modification of any object's size or orientation.*

Moreover, *a metamorphosis could be seen as we apply one or more interaction techniques, or their combination, on the diagram's objects.* The difficulty of students to imagine transformations on geometric figures during problem solving situations is based in the nature of *geometrical concepts* which Fischbein (1993) defined as an amalgam of: "*abstract ideas* on one hand and *sensory representations* reflecting some concrete operations on the other" (p. 14). In this point we are limited to refer the effects of the construction through rotation in a DGS environment.

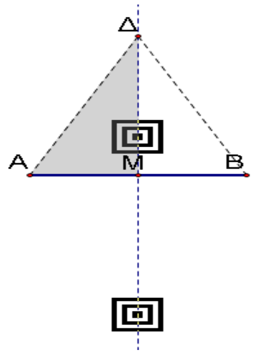


Figure 3.10a. Reflection of a segment for the construction of an isosceles triangle (Patsiomitou, 2009b, d, in Greek)

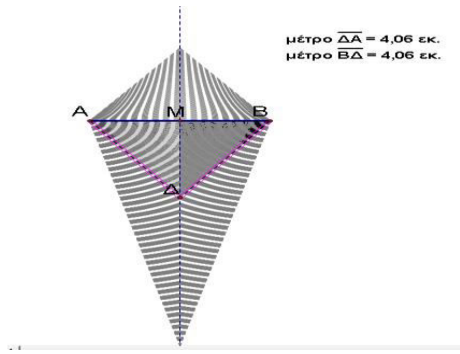


Figure 3.10b. Reflection and trace of a segment for the construction of a rhombus (Patsiomitou, 2009b, d, in Greek)

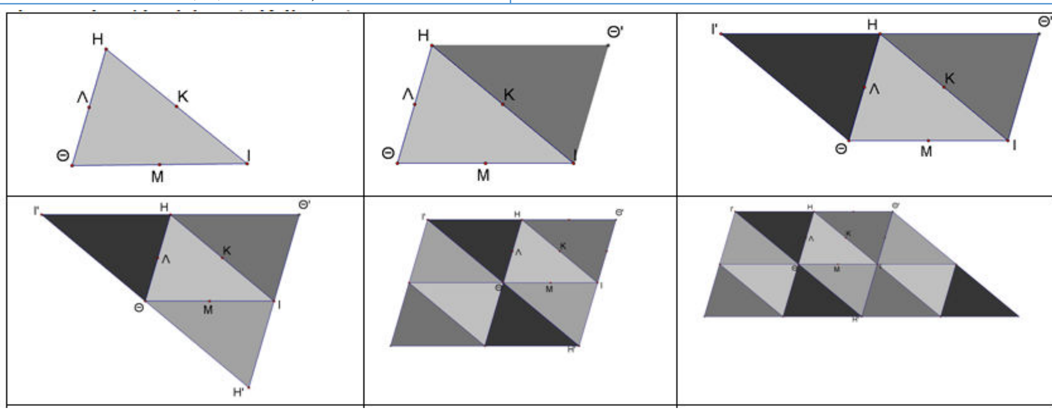


Figure 3.10c. Synthesis of more complex figures through rotation aiming to introduce similarity theorems (Patsiomitou, 2009b, d, g, in Greek)

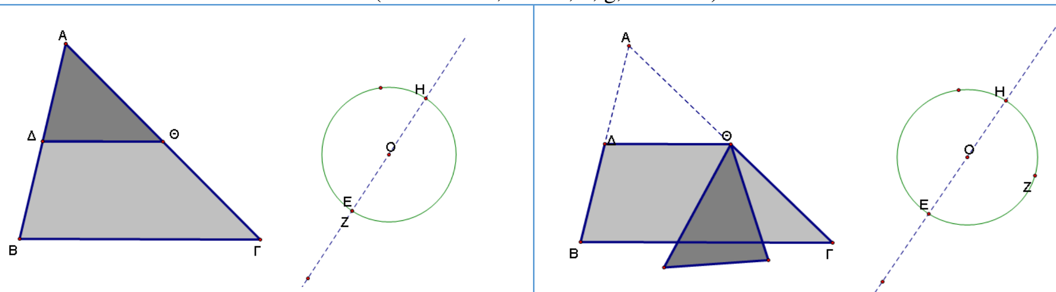


Figure 3.10d. Visual proof through reconfiguartion of the diagram (Patsiomitou, 2009b, e, in Greek)

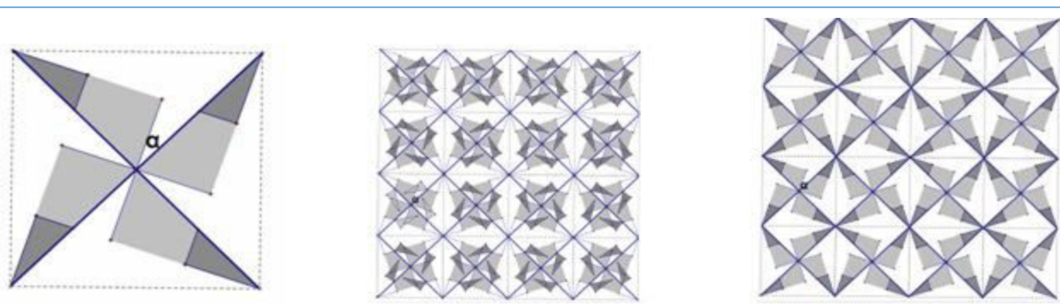


Figure 3.10e. Transformation as a synthesis of action buttons in the animated tesselation- a metamorphosis (Patsiomitou, 2009b, f, h, in Greek)

We follow these next steps to create a rotation of an object in Sketchpad v4 (e.g., Patsiomitou, 2008a): to begin with, we select the point which will act as the center for rotation and define it on the transform menu as ‘mark center’. Then we select the object we would like to rotate based on an angle, choosing the specified/fixed angle (for example 90^o). When the command runs, a new object is created which is a rotated image of the original object. The rotation of the object for 90 degrees in the software leads the students to conceptually grasp the meaning of a) perpendicularity/a right angle; b) congruent shapes. This transformation has a significant impact: during *the instrumental approach*, the student structures a *utilization scheme* of the tool, and consequently a mental image of the functional/operational process of rotation, since any modification/ transformation of the initial figure (input) results in the modification/transformation of the final figure (output).
As mentioned above, the transformation of an object on screen using dragging can be combined with other techniques to cause a combination of transformations on screen (e.g., Patsiomitou, 2008b, c, 2010, 2012a, b): (a) dragging and tracing objects (b) dragging and measuring objects (c) dragging and animating objects (d) dragging a transformed object or its image (by rotation, translation or reflection) or more complex such as (a) dragging, tracing and animation and (b) dragging, measuring and rotating etc. I will discuss the different kinds of transformations and *transformational results* that ensue from implementing dragging on screen (Patsiomitou, 2019b, p. 43-44):

- *Dragging and tracing of a geometric object (for example a point, segment or line)*

Dragging a point on screen results in the transformation of its position and the simultaneous appearance of traces on screen tracking the path the point has followed or the tracks that a line passes due to dragging transformations. This action reveals in the determination of a basic property of the diagram that cannot be directly perceived from the diagram in its hybrid form, or a property of the diagram that remain stable and unaltered.

- *Dragging and measuring (or calculations) the geometric object.*

Dragging a point on screen leads to a change in the measurements of the object, which we have chosen to display and in its calculations. In this case, the measurements change, but the calculations may do one of two things: they may remain unchanged, indicating a stability that demonstrates the validity of a theorem or general theoretical approach (a proposal or a confirmed porisma--meaning a conclusion or an inference) or they may change, allowing the user to observe and draw conclusions from empirical results.

- *Dragging and animating, or dragging, animating and tracing objects*

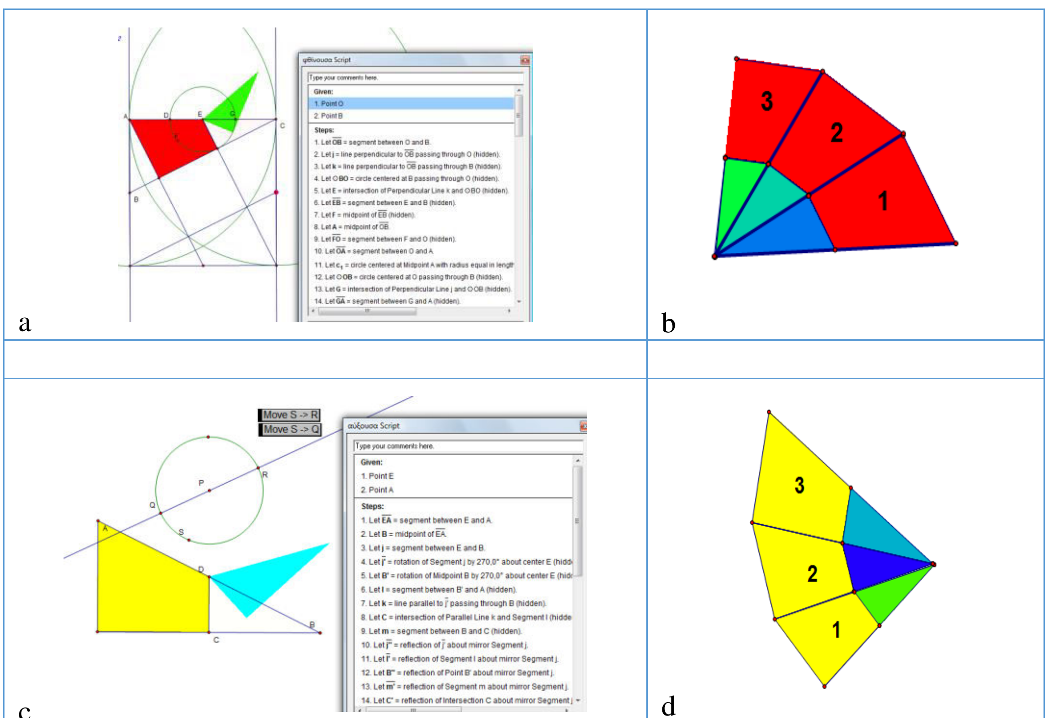
A point on an object is dragged--for example, the vertex point of a triangle to which a point on one side is connected with motion. The animation of the diagram and the simultaneous dragging allow us to understand a condition which is not defined during the diagram’s structuring process. For example, it may make us aware of a theoretical limitation that has not been determined or established before, but which appears on the diagram when it is dragged. This condition leads into an investigation of the validity of a theorem or proposal.
Transformations in geometry are mentioned by many researchers as ‘*geometric functions*’ (e.g., Hollebrands, 2003, p.57; Steketee & Scher, 2016, p.450; Patsiomitou, 2006c, p.1072, 2019, p.16). Hollebrands (2003) defined transformations as follows:

> “Transformations are special functions because they are both one-to-one and onto. Understanding that a transformation is one-to-one involves knowing that if you have two different elements in the domain (two points A and B such that $A \neq B$) then the output for A under the transformation will be different from the output of B under that same transformation ($T(A) \neq T(B)$ where T represents a transformation). Understanding that a transformation is onto involves knowing that every element in the range (every point Q in the plane) has a corresponding element in the domain (a point P in the plane) such that $T(P) = Q$”. (p. 57)

Steketee & Scher (2016) also report *dependent* and *independent* variables, denoting the geometric transformations of objects in a DGS as “*geometric functions*” and arguing that:

> “Cognitive scientists tell us that students build abstract mathematical concepts by connecting those concepts to the physical world through *conceptual metaphors* (Lakoff and Núñez 2000; Radford 2012), such as the metaphor that numbers are points on a line. Geometric functions are based on a similar metaphor—that geometric variables are movable points. […] This metaphor enables students to use dynamic software to create a point (*the independent variable*), construct another point (*the dependent variable*) that depends on the first, and drag to observe the resulting covariation and relative rate of change. In other words, a geometric function relates the preimage point—the independent variable x—with its image—the dependent variable that is a function of x.” (p. 450)

The iteration facility in Geometer's Sketchpad environment is a transformation process very crucial for the construction of recursive processes. In many previous studies, I have reported ways of constructing fractals using the iteration transformation. For example, for the needs of my study "*DGS 'custom tools/scripts' as building blocks for the formulation of theorems-in-action, leading to the proving process*" (Patsiomitou, 2006d, in Greek) I created two custom tools which combined "beauty" with iteration processes, using the Geometer's Sketchpad software. *The result on screen was "beautiful" and "alive".*
The Ancient Greeks, particularly the Pythagoreans, believed in an affinity between mathematics and beauty, as described by Aristotle "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful" (Sinclair, 2004). According to Sinclair (2004, p.262) many "mathematicians (e.g., Hadamard, 1945; Penrose, 1974; Poincaré, 1913), as well as mathematics educators (e.g., Brown, 1973; Higginson, 2000) have drawn attention to some more process-oriented, personal, psychological, cognitive and even sociocultural roles that the aesthetic plays in the development of mathematical knowledge". Sinclair (ibid.) declares that "they associate the aesthetic *with mathematical interest, pleasure, and insight*, *and thus with important affective structures*…".
In my ATCM study "*Custom tools and the iteration process as the referent point for the construction of meanings in a DGS environment*" (Patsiomitou, 2008d), I have done a detailed description of the design process of the custom tools used for the construction of activities in the linked multiple pages facilitated by Geometer's Sketchpad v4 software. My aim was to increase my students' aesthetic perception and sensibility, in parallel with the construction of mathematical meanings. The resulting successive pages could be compared with *an alive, vivid, section of a textbook (Patsiomitou, 2005a, 2018b, 2019a, b)*. The first pupils which played with the spirals and investigated their properties were my children.

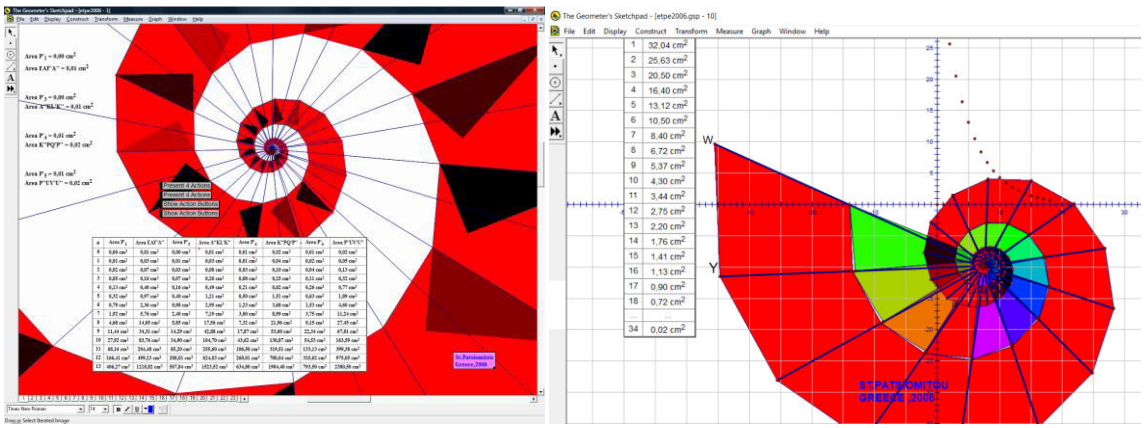


Figures 3.11a, b, c, d. Construction and implementation of the custom tools (Patsiomitou, 2006d, e, ;2008d, p.182, 2009)

The rearrangement demonstration occurs on the right triangle whose vertical sides are proportional to the original right triangles' sides in a ratio of 2:1. Rearranging the construction, students could be helped as new information is highlighted otherwise difficult to understand. Prior to constructing the tool, I also measured/calculated the areas and lengths of the sides of the initial construction. Although the final result of the two methods for constructing the initial right triangle including the rearrangement appear identical, they lead to ways of constructing a custom tool whose application provides different results in both computational and constructional (scheme) terms.

For example applying the tools three times in succession produces the results in **Figures 3.11 b, d**. This means that as we can see in the illustration, the areas of the shapes steadily decrease (**Figure 3.11b**) or increase (**Figure 3.11d**). Concretely, applying the tool using the appropriate method for constructing it, we take different constructional, representational results:

- In method A, the longer vertical side of the initial triangle becomes the hypotenuse of the next right triangle in the sequence. Meaning the sequence of the measurements and calculations that emerges is descending.
- In method B, the hypotenuse of the initial triangle becomes the longer vertical side of the next right triangle in the sequence. Meaning the sequence of measurements and calculations that emerges is ascending.



Figures 3.12a, b. The Al-Lu-The[1] spiral (Patsiomitou, 2006d, e, 2008d, p.182-185)

If we iterate the initial points of the construction of the tool we can take different results relating to the construction the measurements and the calculations. As it is well known for someone who uses the Sketchpad software the result of the process of iteration (Steketee, 2002, 2004; Jackiw, & Sinclair, 2004) can be accompanied with the construction of the tables that repeat the process of initial measurements and calculations in dynamic linking with the diagram, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table, the sequence of the natural numbers is presented (e.g., Patsiomitou, 2005a, 2007a). In that way through this operation, the environment of the software promotes the exploration of the sequences. The iteration process by functioning thus has integrated or embodied the meaning of sequence while there is a direct connection between the user's perception and the abstract mathematical meaning. As a result of the construction and application of the custom tool as much as the process of *iteration* the direct perception of the user is attained in regard to the steps in the development of the construction pertaining to (Patsiomitou, 2007a):

- the repetitions in the measurements or calculations of the areas of initial shapes
- the developmental way of the construction of the shape and
- its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file.

The process of animation can produce the changes in the tabulated measurements (calculations) that allow the user to examine the dynamic process. **Figures 3.12a, b** illustrate the construction of the tables that repeat the process of initial measurements and calculations of the ascending (or descending) sequence in dynamic connection with the shape. In the software, via the process of iteration we have the potential of the constructions, thus becoming more complex being in theory rendered inductively to infinity. This function of the software also constitutes a certain crucial and essential particularity, while the construction with a compass and a straightedge as static tools of geometry has a beginning and an end.

[1] "Al-Lu-The" is an abbreviation generated from the names of my children (**Al**exandros-**Lou**kia-**The**ano)

3.5. Hybrid-Dynamic Objects

Students face difficulties when they explore mathematical objects, no matter if they are in a static or dynamic environment. They have to mentally operate on the abstract object, even if it is visually supported by a computing environment. This is what Laborde (2003) investigates, interrogates or (probably) asks herself: "but if the thought experiments on abstract objects are not available (as it is often the case for learners), a crucial question about learning is whether such environments could favour *an internalization process of the external actions in the environment*". In my studies "*From Vecten's Theorem to Gamow's Problem: Building an Empirical Classification Model for Sequential Instructional Problems in Geometry"* (Patsiomitou, 2019a) and "*Hybrid-dynamic objects: DGS environments and conceptual transformations*" (Patsiomitou, 2019b) I present a new kind of objects in DGS environments the "*hybrid-dynamic objects*".

A. To explain my thoughts I presented a few examples form Algebra, Calculus and Geometry which indicate how the term "hybrid" is reported in the international literature. Many researchers use the word "hybrid" to denote something that does not obviously belong in a given class of objects, or a mixed entity composed of different elements. Kaput (1991) for example revisits the problem that Gauss phased to sum the integers from 1 to 100, "exploiting a convention for expressing generality in mixed numerical and algebraic notation" (p.68). Kaput mentions a "hybrid sum" (numeric and algebraic) which is illustrated using the powerful mode of another "hybrid sum" (figurative and symbolic) (**Figure 3.13a, b**).

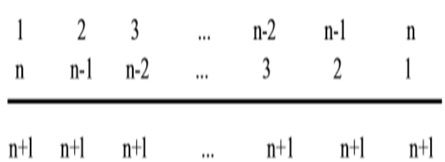


Figure 3.13a: A convention for expressing generality in a "hybrid sum" (a mixed numerical and algebraic notation) (Kaput, 1991, p.68)

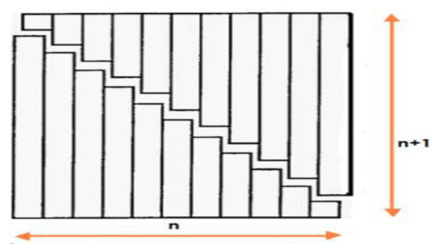


Figure 3.13b: A convention for expressing generality in a "hybrid sum" (a mixed figurative and symbolic notation with an array of rods) (Kaput, 1991, p.65)

Verillon & Andreucci (2006) in their study "Artefacts and cognitive development: how do psychogenetic theories of intelligence help in understanding the influence of technical environments on the development of thought?" report Rabardel (1995) who argued that during instrumental genesis "the resulted instruments are actually *hybrid entities*, on the one part are psychological and on the other part artefactual" (p.12). Morgan et al. also mention the representational hybrid nature of the Turtleworlds environment, because it behaves like a hybrid between Logo and Dynamic Manipulation systems due to the 'variation tool' (Morgan et al. https://www.itd.cnr.it/telma/docs/Rep_Del_Draft3.pdf, p.7). Cerulli (2004) also mention "a hybrid language to be used to bridge the natural language with the mathematical one" (p.36). As Cerulli states "the evolution of meanings is based on the idea of deriving, from a used instrument, hybrid signs which refer both to the practice with the instrument and to the sphere of theory of mathematical knowledge" (p. 142).

B. Why did I term these objects "hybrid-dynamic"?

If we use a parameter "a" to define a function y=ax (or the function $y=ax^2$ etc.) and represent it in a Dynamic Geometry System (DGS), the family of representations we take as we animate the parameter could result in the perception of an empirical generalization of the concept of function. The traces of the object $y=ax^2$ as we animate the parameter "a" provide the path through which the function is transformed (**Figures 3.14 a, b, c, d**). Then we can transform the parameter, but the result of the parameter's alterations affects the linked graphic representations, providing a family of objects with the same properties, which can help students, achieve a deeper understanding.

These traces are not a static mathematical object. They are not dynamics, as they cannot be dragged, but neither are they static. So what kind of object are the lines the traces leave on screen? Traces play an important role in helping students understand the transformations of parameters and their impact on the graphic representations. I have denoted them as *hybrid objects* (Patsiomitou, 2019a, p. 15).

For this I introduced the meaning of

• *hybrid object* (Patsiomitou, 2019a, p. 15) to denote an on-screen geometric object that is intrinsically dynamic, but remains untransformed /unaltered on screen, even though dynamic dragging is applied or implemented on it. This situation comes about because of the hybrid object’s dependence from its parent objects. Briefly, a hybrid dynamic object is something that does not obviously belong to either the static or dynamic world. It is an object created in a DGS by means of complex transformations (or on which complex transformations can be performed); something between a static and a dynamic object; an object that is intrinsically dynamic, signifying a static behavior which is rendered dynamic by to the users’ actions.

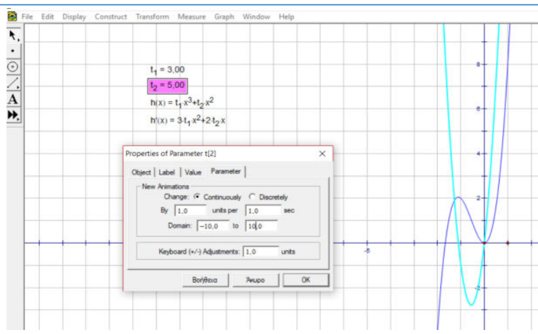


Figure 3.14a. Creating a function, also its derivative using parameters and plotting them

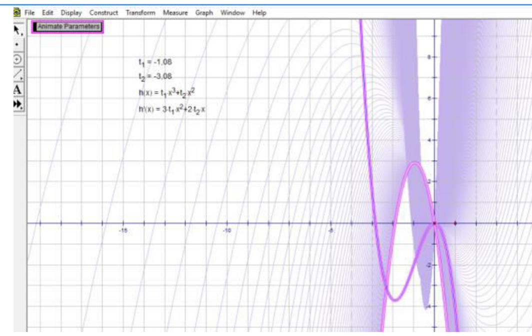
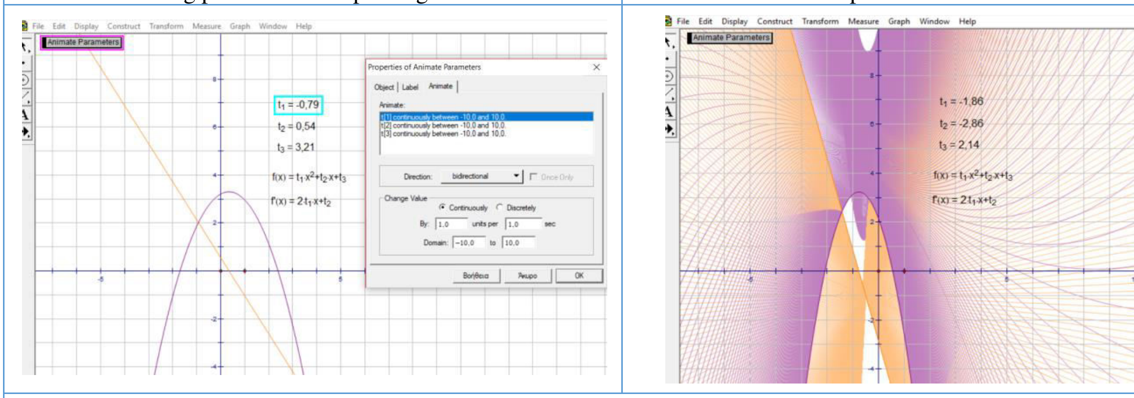


Figure 3.14b. Tracing objects by animating parameters



Figures 3.14 c, d. Snapshots of families of functions using the animation of parameters (e.g., Patsiomitou, 2009b, 2019b, p. 34)

In other words a hybrid object is the result of an effect on a dynamic object on screen. As a consequence it is loaded with intensive interactive features.

• *hybrid diagram* (Patsiomitou, 2019a, p. 15) in the DGS environment to denote the untransformed on-screen diagram, which has been created to stay hybrid and become dynamic if we implement a transformation on its parents. The diagram is intrinsically dynamic, but a user could use it as an image or a static diagram, if s/he does not know how to make it dynamic. It is important to point out at this point that: the transformation of objects in a DGS environment is dependent on whether these objects have been defined, as hybrid objects or not.

C. How did I conceive the notion?

I became aware of the notion of hybrid-dynamic objects since 2005, when I started experimenting with parameters and parametrical constructions in Sketchpad. For the needs of my study “*Transformations on mathematical objects through animation and trace of their dynamic parameters*” (Patsiomitou, 2006a, in Greek), I instrumentally decoded Vecten’s theorem (**Figures 3.18 a, b**) using parameters (Patsiomitou, 2006a, in Greek, pp. 1270-1273). I have considered Vecten’s theorem to be particularly interesting since 1985, when I investigated (in paper-pencil environment) all the sub-problems (reported in “Jesuit Geometry” a translation in Greek, p.774, published in *Annales De Gergonne,* 1816, vol.VII, p.322) with great interest.

In my study "F*rom Vecten's Theorem to Gamow's Problem: Building an Empirical Classification Model for Sequential Instructional Problems in Geometry*" I describe a few sub-problems of the Vecten's theorem and their solution (Patsiomitou, 2019a, b, p.12-14) which I also report in the current study.

Vecten's Theorem: Construct a triangle ABC. Construct two squares ABDE, ACIT, externally on the sides AB, AC of the triangle ABC respectively. Prove that

I. If M is the midpoint of the side BC then AM= ET/2 (Figure **3.15a**)
II. AM is perpendicular to ET. (Figure **3.15a**)
III. If O is the midpoint of ET then AO=BC/2. (Figure **3.15b**)
IV. AO is perpendicular to BC. (Figure **3.15b**)
V. If S is the fourth vertex of the parallelogram EATS then the sides CD and BI are congruent and perpendicular to BS and CS respectively. (Figure **3.15c**)
VI. If G is the midpoint of the segment DI, then the BGC triangle is a right and isosceles triangle. (Figure **3.15d**)

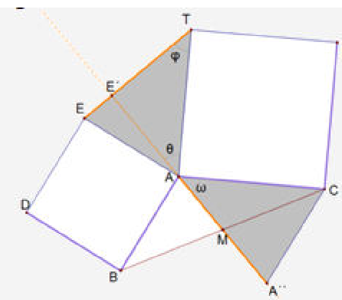


Figure **3.15a.** Sub-problem I, II (Patsiomitou, 2019a, p. 13)

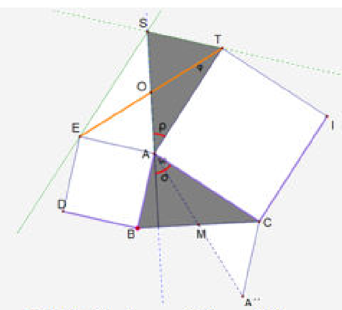


Figure **3.15b.** Sub-problem III, IV (Patsiomitou, 2019a, p. 13).

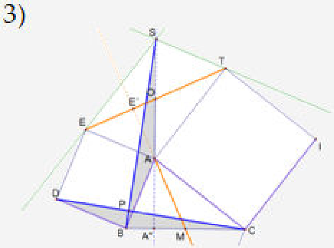


Figure **3.15c.** Sub-problem V (Patsiomitou, 2019a, p. 13).

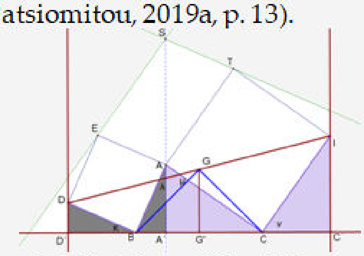
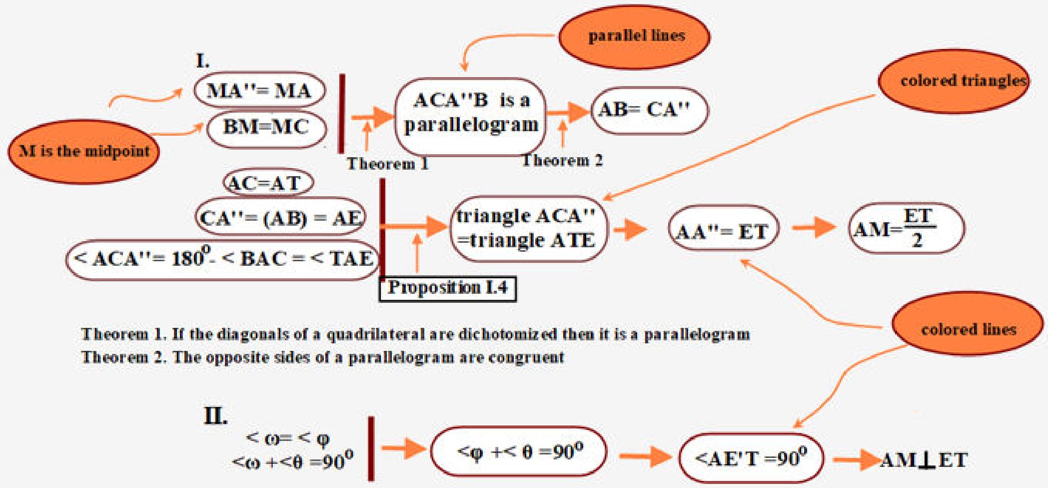
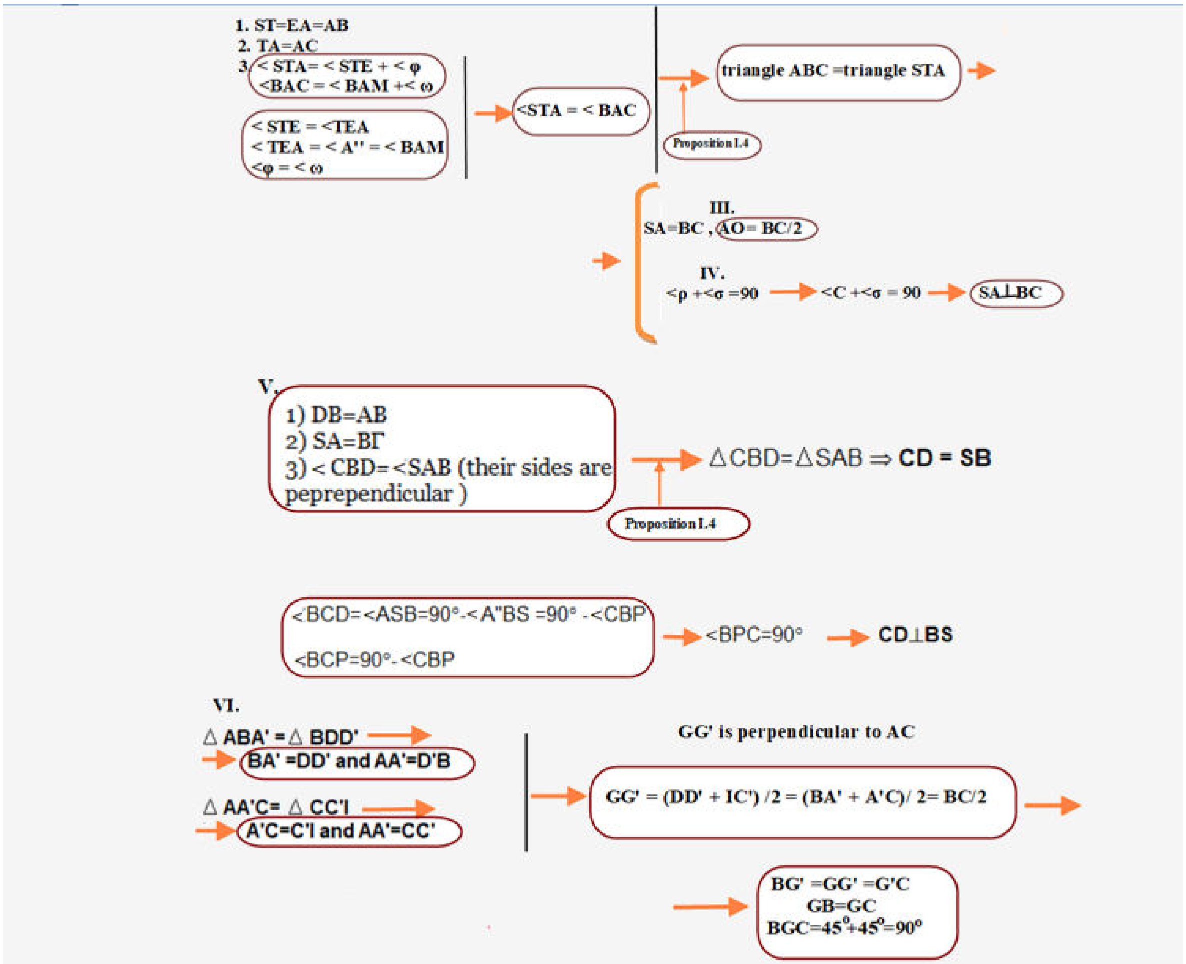


Figure **3.15d.** Sub-problem VI (Patsiomitou, 2019a, p. 13).





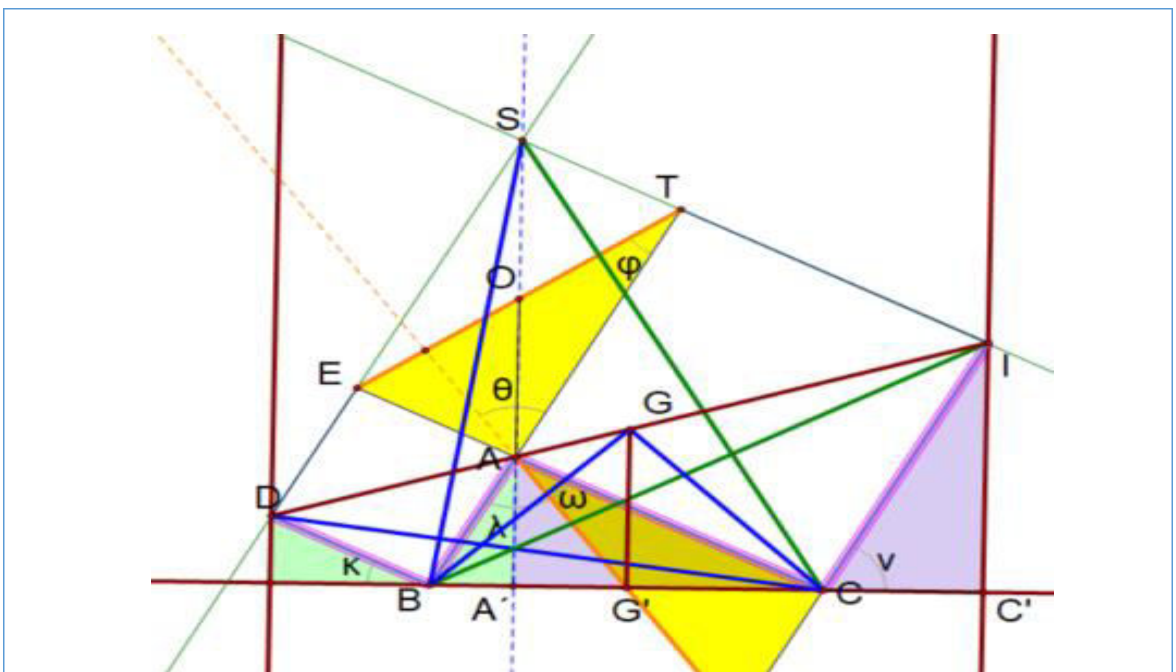
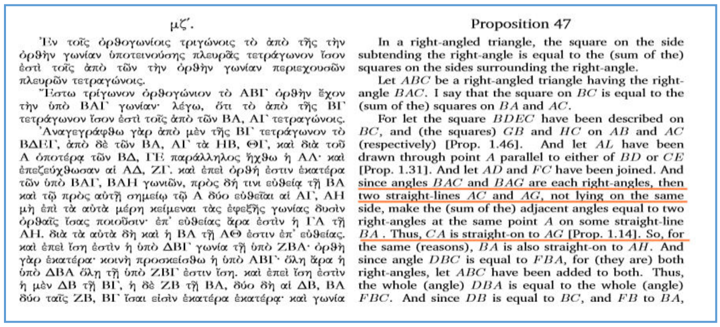


Figure 3.16. A diagram for the Vecten's sub-problems mentioned above (ABC is a right triangle) (Patsiomitou, 2019a, p. 14).

If we drag the lines AB, AC until they become perpendicular (**Figure 3.16**) then a student has to prove that the lines AE, AC belong to the same line, something that is omitted /or dismissed by the students. This part of the proof is highlighted in Euclid "Elements" (e.g., Proposition I.47) (see for example Fitzpatrick, 2007, p. 46).



μζ΄.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις.

Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθείᾳ τῇ ΒΑ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῇ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΑ τῇ ΑΘ ἐστιν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῇ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΓ, ἡ δὲ ΖΒ τῇ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία

Proposition 47

In a right-angled triangle, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-angle.

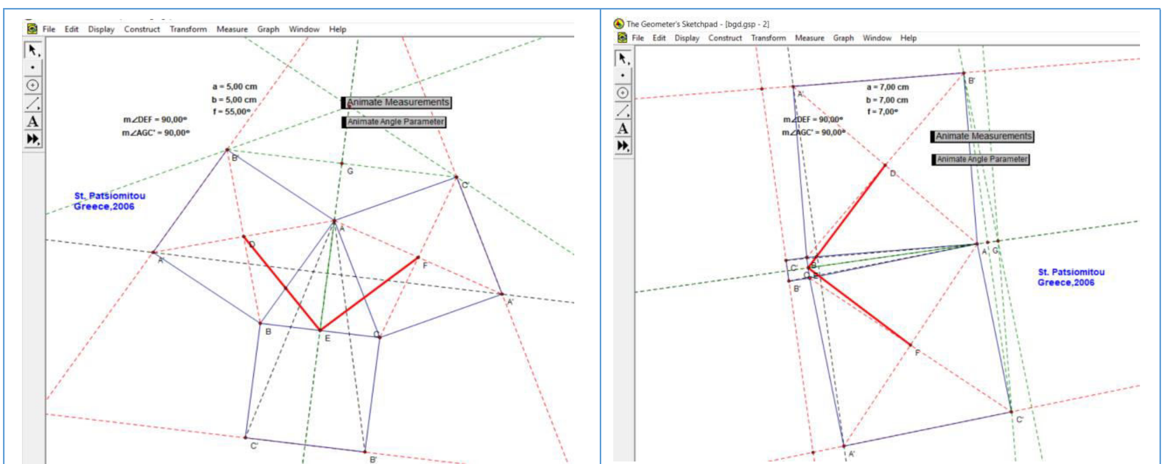
Let ABC be a right-angled triangle having the right-angle BAC. I say that the square on BC is equal to the (sum of the) squares on BA and AC.

For let the square $BDEC$ have been described on BC, and (the squares) GB and HC on AB and AC (respectively) [Prop. 1.46]. And let AL have been drawn through point A parallel to either of BD or CE [Prop. 1.31]. And let AD and FC have been joined. And since angles BAC and BAG are each right-angles, then two straight-lines AC and AG, not lying on the same side, make the (sum of the) adjacent angles equal to two right-angles at the same point A on some straight-line BA . Thus, CA is straight-on to AG [Prop. 1.14]. So, for the same (reasons), BA is also straight-on to AH. And since angle DBC is equal to FBA, for (they are) both right-angles, let ABC have been added to both. Thus, the whole (angle) DBA is equal to the whole (angle) FBC. And since DB is equal to BC, and FB to BA,

Figure 3.17. Screenshot from the Proposition I.47 (Fitzpatrick, 2007, p. 46)

In the **Figures 3.18 a, b**, I have constructed the sides AB=a, AC=b as well as the angle <BAC=f by using parameters in order to investigate more deeply the properties of Vecten's theorem (Patsiomitou, 2006, in Greek, pp. 1270-1273; Patsiomitou, 2019 a, b). The animation of all parameters is a direct object manipulation which transforms every part of the object. This leads to a kind of *algebraic geometry*, which takes the parametric sides and angles as input and provides a continuous transformation of the diagram as output (Patsiomitou, 2006a, pp.1270-1273, in Greek). According to Leron & Paz (2006) in their work "The slippery road from actions on objects to functions and variables"

> "*to be specific, the metaphorical mapping would map action to function, object (or the state of the object) to variable, and the initial and final state of the transformed object to the function's input and output*." (p. 128)



Figures 3.18 a, b: Screenshots of the sequential diagrams of Vecten's theorem in Sketchpad produced by animating the parameter of angle (Patsiomitou, 2006a, p. 1270-1273, in Greek; Patsiomitou, 2019a, p. 15-16)

A student's action on parameters leads to a transformation of objects. The students can also investigate a concrete situation of the hybrid-dynamic representations, choosing to assign concrete magnitudes to the parameters (**Figure 3.18 a, b**). Moreover, the user can directly perceive *infinite alterations of the same figure on screen* (Patsiomitou, 2006, p. 1273, in Greek) and conceive of an abstract mathematical object. This mode of construction is completely different from the simple construction mode which uses dynamic tools, because the student consciously perceives the modification of the dynamic objects on screen. We can thus speak about *functional geometry* and through the conservation of figures' properties about the concept *of geometric function* (Patsiomitou, 2006, p. 1273, in Greek).

In the **Figures 3.18 a, b** the whole representation *is a hybrid diagram*, meaning it is completely determined by its parameters and cannot be moved if we drag any point on it. The diagram has intrinsically dynamic properties, but is different from a dynamic diagram created using the 'Construct' or 'Transform' menu in that. It can only be altered if we animate its parameters, supporting a visualization of infinite occasions of dynamic objects which maintain the same structure but they are modified in a mereologic, optic and place way in the words of Duval (1999).

D. Is segments' addition a hybrid-dynamic object in DGS using parameters**?**

In my study "*Hybrid-dynamic objects: DGS environments and conceptual transformations*" (Patsiomitou, 2019b) I explain through examples how the addition of segments in a DGS environment is a hybrid-dynamic object. Concretely I report the following:

A segment (or a line) in the Euclidean geometry is a geometrical object. We can create segments in a DGS environment, then measure their length and calculate their sum. We can also use the symbol "+" to represent the process of segments' addition, leading to the concept of segments' sum in geometry, in a similar way that Davis et al. (1997, p.134) report its pivotal role in algebra. Davis et al. mention that

> *"The symbol 4+2 occupies a pivotal role, as the process of addition (by a variety of procedures) and as the concept of sum. Soon the cognitive structure grows to encompass the fact that 4+2, 2+4, 3+3, 2 times 3, are all essentially the same mental object" (Davis et al., 1997, p.134).*

In a previous study I defined the meaning of dynamic segment as follows (Patsiomitou, 2011):

> *"The 'dynamic' segment is a portion of a straight line which does not consist of points. Dynamic points can be placed independently on the dynamic segment and move free with one degree of freedom on the path to which they belong. This means that a point placed on a segment that intrinsically is designed with two degrees of freedom is transformed to a segment object with one degree of freedom" (p. 365).*

All geometrical or algebraic objects in the Geometer's Sketchpad environment operate in "a dependency diagram, a directed acyclic graph" (Jackiw & Finzer, 1993, p.295): The 'given' objects in a construction are the 'parents' and they are free to move on the screen, in contradiction to *dependent* objects which are the 'children' of the objects on which they depend in some fashion, that are constrained. According to Sketchpad Help System:

> "The objects you can create in Sketchpad fit into several general categories. Some of the objects are purely geometric entities—points, lines, rays, segments, circles, arcs, interiors, loci, and some iterations. Other objects are either numeric or algebraic entities—measurements, parameters, coordinate systems, calculations, and functions. And finally, some objects in Sketchpad—captions and action buttons—are primarily used in descriptions, explanations, and presentations".

One way to analyzing students' formulations during their interaction with dynamic geometry transformations on dynamic or dynamic-hybrid objects is to consider those formulations through the *Action-Process-Object-Schema* (APOS) theory lenses, a theory developed from Dubinsky and his colleagues (e.g., Dubinsky, 1988, 1991a,b; Dubinsky & McDonald, 2001), based on the theory of reflective abstraction (Piaget, 1970). Concretely, according to APOS theory (Cottrill et al., 1996; Dubinsky & McDonald, 2001) when a student constructs mental Actions, Processes and Objects, then s/he organizes them to mental Schemas to understand a mathematical concept and solve the problems (APOS theory). According to APOS theory, in order to understand a mathematical concept a student must manipulate physically or mentally a transformation on mental or physical objects, in other words an "Action" on objects, as a reaction to stimuli perceived from the external environment, focusing on the way that a procedure thus could be used as an input to another procedure; actions on objects then can be interiorized to become a Process, which accordingly can be encapsulated to become Objects and then can be organized to become Schemas. According to Cottrill et al. (1996):

> "*An action* is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or an act of recalling some fact from memory. It may also be a multi-step response, by then it has the characteristic that at each step, the next step is triggered by what has come before. When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes *a process*" (Cottrill, et al, 1996, p. 171, in Davis, Tall and Thomas, 1997, p. 133). [authors italics…]

Making a review on the briefly reported studies it is obvious that many researchers have mentioned the meanings of Action-Object-Process-Schema, to describe the phenomena observed in the area of Algebra and Calculus. Can these meanings be implemented in the mathematical area of Euclidean or Dynamic geometry? What is their impact in *the reification process*? Hollebrands (2003) investigated the nature of students' understandings of geometric transformations in the context of "The Geometer's Sketchpad" environment and she analyzed students' conceptions of transformations as functions, using APOS theory. Hollebrands (2007) also addressed the way students interpret objects created with the use of the dynamic program when they are learning about geometric

transformations. As Hollebrands argued "*the nature of the abstractions that students made as they worked with technology seemed to be related to their understanding of transformations and the tool*" (2007, p. 190). Generally speaking, when we solve a problem in geometry, we construct a figure in a few steps and in such a way that a procedure can be used as an input to the next--and almost always sequential--procedure. Students construct mental actions, as they engage in problem solving, performing transformations on objects either explicitly or from memory. The student or the teacher can perform an operation mentally and execute it on the computer screen. This process creates objects which "is based in a reification of mathematical objects and relations that students can use to act more directly on these objects […] a new experiential mathematical realism" (Balacheff & Kaput, 1997, p. 469-470).
As I write in my study "*Hybrid-dynamic objects: DGS environments and conceptual transformations.*" (Patsiomitou, 2019b) "The case of the addition of two segments in geometry represented by two separate objects identified by two letters, one for each edge of each segment (for example segments AB, CD) is more complex, because it includes both a figural and an algebraic entity. The figure of the segment which represents a concrete real "thing" is the *figural part*; the number which is the measure of the segments' length (or the distance of the endpoints of the segment) represents the *algebraic part*. In addition, the students have to represent the addition of segments with a concrete segment and then represent this action by means of a *symbolic representation*--namely, the way these segments are defined by letters (AB, CD etc.). The symbol "AB+CD" possesses a central role as the process of segment's addition and as the concept of segments' sum. The cognitive structure encloses the same mental objects (e.g. CD+AB= FG+EF if FG=CD and EF=AB). As a result, the construction, measurement and calculation of segments in a DGS environment differ from the same process in a static environment. Then, we can define an *elementary geometrical procept* (**Figure 3.19a, b, c**).
It is thus clear that the sum of the segments as an object derived from calculations in a DGS environment is an algebraic, geometric and "dynamic" entity. I shall break down the process of adding two segments in the DGS environment into three phases:

Phase A. If we create two segments in the Geometer's Sketchpad and then measure and calculate their sum, the actions on mathematical entities at one level become mathematical objects in their own right at another level (Piaget, 1972a, b).

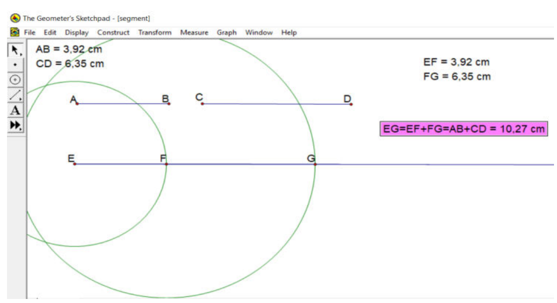
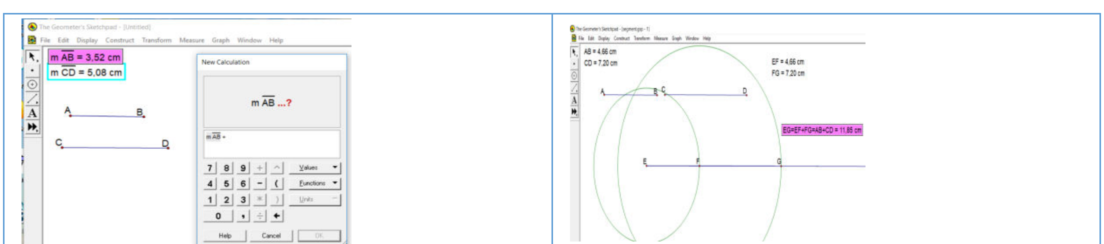


Figure 3.19a. The addition of two segments in a DGS (Patsiomitou, 2019b, p. 38)

The calculation of segments is a process becoming reified as an object, which includes a few procedures, in the words of Gray & Tall (1991, 1994) who distinguished between "*the specific procedure as an explicit sequence of steps and the input-output process where different procedures can have the same input-output*". Selecting the calculation command displays the calculator with which we can sum the segments by selecting the measurements of each, as illustrated in Figure 8a below.



Figures 3.19b, c. The concept of sum of two segments in a DGS. (Patsiomitou, 2019b, p. 39)

To construct objects in a DGS environment, we can use first-order parental objects, second-order child geometrical objects, and auxiliary objects. I shall try to list in the table below all the actions and symbols involved in the process of adding the segments, the sequence of actions and objects involved. I shall also report the theoretical construct and try to anticipate how students will understand and conceive of the process and the answers they will produce. Generally speaking, if we construct a segment using the tools provided by the DGS software, this concrete segment is the *parent object* and the measurement the *child object*. In the previous example, points F, G cannot be altered by dragging due to their dependence on their parent objects. Dragging points A, B affects the position of point F (just as dragging points C, D affects the position of point G). Students can understand that "*if we modify segment AB, segment EF will be modified also*".
In the **Table 3.1** I have done a description with regard to the objects and the actions. The anticipated answers of students during the interaction with the process lead to the following result: The transformation of all the objects mentioned above, leads the students to conceive the *unaltered properties of the mixed entity*. They can express a *concept-in-action or theorem-in-action, through* the reification of mathematical objects and the interiorization of the process of dynamic movement, counting and dragging the segments: this is a *procept-in-action*, meaning a process which leads to a concept-in-action or theorem-in-action.

Table 3.1. Actions and symbols involved in the process of adding the segments (Patsiomitou, 2019b, p. 39)

Actions	**Objects**	**A theoretical view in Euclidean Geometry**	**An anticipation of a student's answer in interaction with the dynamic diagram**
Construction of the segments (a procedure which produces the figural part of the objects).	Inputting two segments of unequal (or equal) length.	P1 P2	-The segments can become almost equal if we drag them. -We can change segment's orientation on screen.
Definition of segments using letters (for example AB, CD)	Symbols used to define the objects (an elementary procept to the concept of equal or unequal segments)	P3 P4 P5	-The distance between the endpoints of a segment is affected if we drag them.
Construction of a line Ex	An auxiliary geometrical object (an elementary procept)	P6 P7	-We can change the orientation of the line and we can place multiple points on it which can be dragged in two directions.
Construction of two sequential segments EF, FG on the line Ex, by constructing circles (E, AB) and (F, CD) with centers E, F and radius AB, CD, respectively.	• first order parental geometrical objects (A, B, C, D, AB, CD) • auxiliary geometrical objects (line Ex and the circles) • second order child	P8 P9 P10 P11	-If we drag the endpoints of the segment AB, then segment EF will also be transformed. -If we drag the endpoints of the segment CD, then segment FG will also be transformed. -If we try to drag the point F (or G) it cannot be dragged.

	geometrical objects (EF, FG, EG) (Figure 8b)		
Measuring the segments AB, CD, EF, FG, EG.	Realizing the measurements are *algebraic objects linked to the geometric objects* mentioned above.	P12	-We can measure the lengths of the segments - If we drag the segment AB, the measurement of the segment AB will be transformed as well as the measurement of the segment EF. -The measurements of the segments AB, EF are congruent.
Calculating the sum of the segments EF, FG.	*A complex dynamic object*	P13	-We can calculate the sum of the segments using a calculator which results to an accurate calculation.
A symbolic expression of the segments' sum (for example EG=EF+FG=AB+CD=10,27 or 11,85)	A mixed entity: *a hybrid object with numeric, algebraic, geometric, figurative, symbolic and dynamic characteristics.*	P14: Betweenness theorem P15: (Addition theorem)	-As we modify the parental objects, the segments' length, measurements, and dependent calculations are modified, too, while the calculated result of the segments' addition changes accordingly.

The theoretical answers of Euclidean Geometry mentioned in the Table 1 are the following (Coxford & Usiskin, 1975):

P1: If two distinct points are in a plane, the line determined by these points is a subset of the plane. (p. 20)
P2: Two points determine a line. (p. 21)
P3: To each pair of points there corresponds a unique real number called the distance between the points. (p. 22)
P4: Suppose A and B are points, then: (a) AB≥0, (b) AB=0 if and only if A=B and (c) AB is also the distance between B and A, that is AB=BA. (p. 24)
P5: The segment with endpoints A and B is denoted by AB and is the set whose elements are distinct points A, B and all points between A and B. (p.26)
P6: A line is an infinite set of points (p. 22)
P7: A line is a set of points and contains at least two distinct points. (p. 18)
P8: A circle is the set of all points in a plane at a fixed distance (the radius) from a fixed point (the center). (p.180)
P9: Two radiuses of the same circle are congruent segments.
P10: Congruent radiuses determine congruent circles.
P11: Points E, F, G are collinear since they are all on line Ex. (p.19)
P12: The midpoint of a segment AB is the point M in AB with AM=MB (p. 30)
P13: The length of a segment is the distance between its endpoints. (p.26)
P14: (Betweenness theorem). If a point B is between A and C, then AB+BC=AC. (p.26)
P15: (Addition theorem) If B is on AC, then AC=AB+BC (p.375)

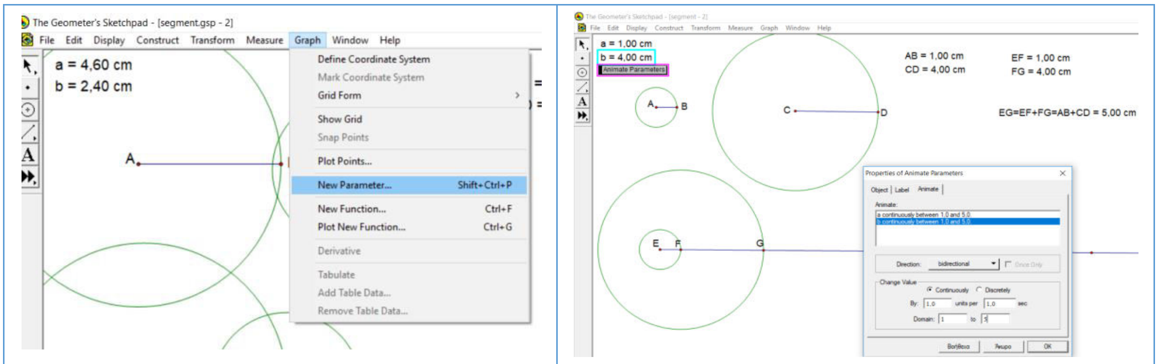


Figure 3.20a, b. The addition of two segments in a DGS, using parameters (Patsiomitou, 2019b, p. 40)

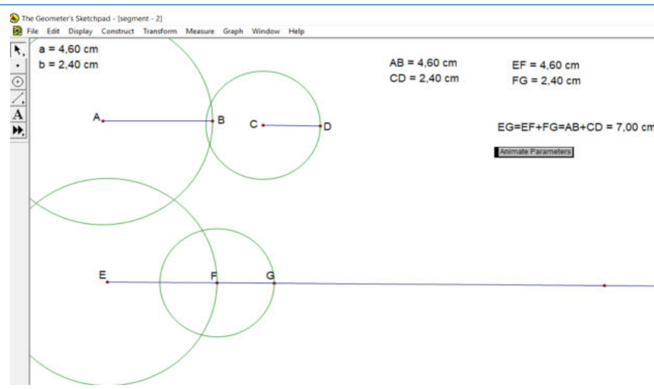


Figure 3.20c. Visualizing the concept of the sum of two segments in a DGS, using parameters (Patsiomitou, 2019b, p. 40)

Phase B. If we create the segments' addition, by defining the segments AB, CD using the parameters a, b (meaning, by setting a corresponding parameter to each segment, the parameter "a" for the segment AB and the parameter "b" for the segment CD) then we have created concrete invariant objects in a DGS environment. In order to create the parameters we can use the "create a new parameter" command from the Menu, Graph (**Figures 3.20a**). According to Sketchpad Help system *"Parameters are simple given numeric values. Unlike measurements and calculations, they do not depend on other objects for their value. A parameter is defined by a single number and an optional unit*". We can choose to construct a segment for example with length equal to 2cm, or with such a length as we wish. These parametrical segments can be transformed dynamically by transforming (e.g., by using animation) the parameters with which they have been created, meaning the parental objects in a continuous/or not process (**Figure 3.20b, c**).
Firstly, the animation on parameters turns the dynamic diagram to a more detailed and complex representation than the one we have created using the tools (e.g. segments, lines and circles). Points B, D have only one degree of freedom and can be dragged only on the path they belong. The figures can become larger or narrower, but it is not easy to change their orientation (for example, if the circle-path to which they belong becomes hidden). We can change the value of the parameter or define the domain values between which the parameter takes on values, meaning that *the geometrical object depends on the values given to an algebraic object*. The parameter is allowed to range over whatever domain I choose to define, and the *mixed entity has been transformed into a symbolic parametrical and dynamic one* (we can see the "animate parameters" label on screen, which allows parameters to be altered with this action affecting the figural part of the object). Secondly, the concept of parameters belongs to algebra. On the other hand, when we create a figure in a static environment, we never use a parameter to create the figure, just as we never define a segment as a parameter for use in our construction. Moreover, animating the parameters transforms the synthesis of the diagram into an "infinite" number of snapshots, which the user would probably not consider manipulating by her/himself. For the segments' addition I can summarize the following:
In general, a concrete parameter defines the particular member of a function family. As the parameter changes the transformations of segments, as well as the transformations of the diagram's synthesis appear on screen.
In the examples mentioned above the segments AB=a, CD=b under the transformation T of the dynamic parameters will become the corresponding elements T (a), T (b). The dynamic objects created using parameters

play a pivotal role in fostering/scaffolding understanding. Are these objects *dynamic*, or have we created “static” objects in a DGS environment? What is their “static” role in a DGS environment? What are the transformations the concrete dynamic diagram and the objects created in this representation perform? Moreover, can we make a “construct” that appears invariant, even if we drag its visible points on screen? Does this diagram have the same properties? In this case, we have created a “different” hybrid diagram.

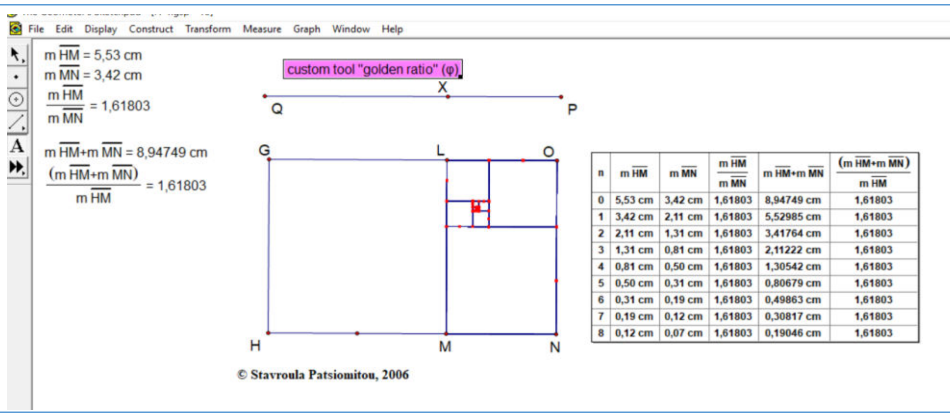


Figure 3.21a. Creating a golden rectangle using a custom tool (Patsiomitou, 2006g, p. 61, in Greek; Patsiomitou, 2019b, p. 43)

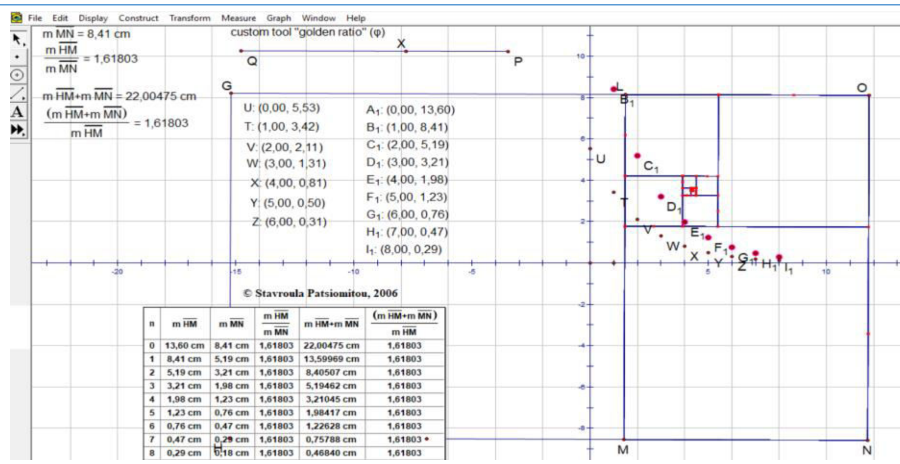


Figure 3.21b. Dynamic linking of the tabulated measurements with the plotted points (Patsiomitou, 2019b, p. 43)

Phase C. In the **Figures 3.21a, b,** I have constructed a golden rectangle using two important procedures (Patsiomitou, 2006g, p. 61): “creating a *custom tool* that repeats the ratio 1, 61803 (=number φ), and the *iteration process* that repeats the whole procedure and the measurements and calculations displayed in the table”. In this construction, we can view algebraic objects, diagrammatic objects and tabular representations, along with parametrical objects used *operationally and structurally*, and dynamic or hybrid objects. In the tabular representation, we can view the results of measurements and calculations repeated thanks to the iteration process, which generates final for initial objects on a one-to-one basis. According to Patsiomitou (2005a, 2006a, g, 2007a, 2008d, 2014, 2018a, b): Through the application of the custom tool the possibility is given to the user to acquire an inductive way of thinking for the finite steps of the construction but the generalisation with regard to the constructional result can be achieved from the process of iteration which inductively renders the construction theoretically to infinity. This function of the software also constitutes a certain crucial and essential particularity, while the construction with a compass and a ruler as formal tools of static geometry has a beginning and an end. In the software, via the process of iteration we have the potential of the constructions thus becoming more complex being in theory rendered inductively to infinity. The result of the process of iteration is the construction of the tables that repeat the process of initial measurements and calculations in dynamic connection with the shape, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table the sequence of the natural numbers is presented. In that way through this operation, the environment of the software promotes the investigation of the sequences. The iteration process by functioning thus has integrated or embodied the meaning of sequence while there is a direct connection between the user's perception and the abstract mathematical meaning. The process of animation can produce the changes in the tabulated measurements (calculations) that allow the user to examine the dynamic process. These changes come as result of the fluctuations in the size of an artefact-fractal which have the possibility of increasing (decreasing) and altering orientation".

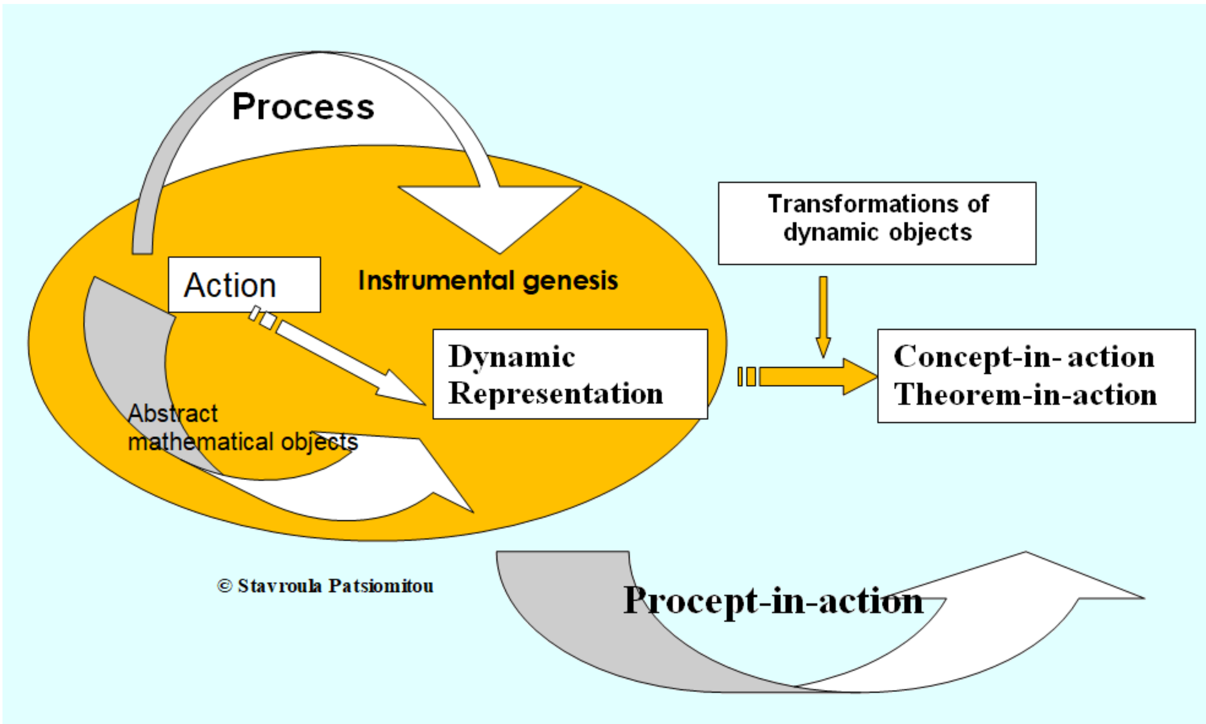


Figure 3.22. A procept-in-action during instrumental genesis (Patsiomitou, 2019b, p. 44) (modified)

The dynamic linking of the tabulated measurements from the first two columns results in the plotted points illustrated in **Figure 3.21b.** The plotted points are dynamically linked to both the figural object and the tabular representation, but cannot be moved or dragged, and are left unaffected if we drag point G (a DGS object with two degrees of freedom), even if the measurements in the tabular representation are affected. The plotted points are dynamic-hybrid objects. In other words, it is a geometric function which repeats one-to-one transformations on algebraic, geometric and dynamic objects. The concepts-in-action (and theorems-in-action) which occur during the procedure are the results of dynamic elementary *procepts-in-action*. They are intrinsically dynamic and their impact on students' understanding of the meaning of sequence is crucial (Patsiomitou, 2005a, in Greek). For example, as I mentioned in previous works (e.g., Patsiomitou, 2005a, 2007a, 2019b) "The surprise was made by a female-student who, while passively watching and not participating in the duration of the process she comprehended that "*as N increases (natural numbers), E (the area) is continuously reduced*" a fact which she expressed verbally and repeated it in writing. From this, we may conclude that she momentarily overcame her fear of mathematics, after she had a verbal interaction with the remaining members of the team and was led towards the comprehension of the meaning of limit only by the representations and the reaction towards the computer software".
Building on the above, I think there is a continuous process ongoing in students' mind as they create a concept. The meaning of 'procept' is thus dynamic in a DGS environment; adapting its meaning to a *'procept-in-action'* (**Figure 3.22**) for the DGS environment could thus support the appearance of *operational invariants* (Vergnaud, 1998, 2009) during the problem-solving situation and the students' actions on a dynamic object or a dynamic representation/diagram.

Chapter IV.

4.1. How do Students Learn in a Constructivist Framework?

Students' cognitive growth is a major aim of mathematics education. Researchers have interpreted it in different ways, such as that cognitive growth can occur between others, through *developmental stages* (e.g., Piaget, 1937/1971; van Hiele, 1986), as development *of proof schemes* (e.g., Balacheff, 1987, 1988, 1991, 1998, 1999, 2008, 2010; Harel & Sowder, 1996, 1998, 2007, 2009; Harel, 2001, 2008) or as dynamical development of students' *mental representations* (e.g., Cifarelli, 1998) when students confront problem-solving situations. Pegg & Tall (2005) identify two main categories of theories to explain and predict students' cognitive development, (or cognitive growth or conceptual deveopment):

- "*global theories of long-term growth* of the individual, such as the stage theory of Piaget (e.g., Piaget & Garcia, 1983), or the van Hiele theory (e.g., van Hiele, 1986; Fuys et al., 1984);
- *local theories of conceptual growth* such as the action-process-object-schema theory of Dubinsky (Czarnocha et al., 1999; Dubinsky, & McDonald, 2001) or the unistructural multistructural-relational-extended abstract sequence of SOLO Model (Structure of Observed Learning Outcomes, Biggs & Collis, 1982, 1991; Pegg, 2003)" (p.188).

The difficulties which arise when a student studies geometry begin with the way s/he perceives a shape. The perceptual competence of a student to 'see' a figure's properties depends on his/her development of *cognitive structures* and ability to think *abstractly*. The development of a student's *cognitive structures* makes him/her able to perform the "*hypothetical representation of his/her internalized organization of the concepts in long-term memory*" (McDonald, 1989, p.426). Skemp's view of the abstraction process is that "a concept is the end product of […] an activity by which we become aware of similarities […] among our experiences" (Skemp, 1986, p.21 in White & Mitchelmore, 2010, p.206). Moreover, Schwartz, Herschkowitz & Dreyfus (2001) argue that

> "[…] *Abstraction* is not an objective, universal process but depends strongly on context, on the history of the participants in the activity of abstraction and on artifacts available to the participants. Artifacts are outcomes of human activity that can be used in further activities. They include material objects and tools, such as computerized ones, as well as mental ones including language and procedures; in particular, they can be ideas or other outcomes of previous actions" (p.82).

Stein et al. (2000) proposed a cognitive demand frame, which separates tasks into low-level and high level depending on the cognitive demands they place on the student. Tchoshanov, Lesser and Salazar (2008) presented a modified version of this cognitive demand model which includes three levels: (1) facts and procedures; (2) concepts and connections; and (3) models and generalizations (Tchoshanov, 2013, p. 67).

- *At the first level* Tchoshanov et al. refer to level descriptors including a student's competence at "recalling facts, recognizing basic terminology, stating definitions, naming properties and rules, conducting measurements, solving routine problems", etc.
- *At the second level*, Tchoshanov et al. refer to level descriptors including a student's competence at "selecting and using appropriate representations, translating between multiple representations, transforming within the same representation, explaining and justifying solutions to the problems, solving non-routine problems", etc.
- *At the third level* Tchoshanov et al. refer to level descriptors including a student's competence at "generalizing patterns, generating mathematical statements, deriving mathematical formulas, proving statements and theorems", etc.

A constructivist view of learning considers the student as an *active participant* and learning as an *active process*. Immanuel Kant (1965), John Dewey (e.g., 1938/1988), Jean Piaget (e.g., 1937/1971, 1970), von Glasersfeld (e.g., 1991, 1995), Vygotsky (e.g., 1934/1962, 1978), Skemp (e.g., 1987) were important philosophers and theorists who gradually changed the traditional "*route by memorization*", the behaviourists' view of learning mathematics, to a sociocultural-constructivist view of learning mathematics. From an epistemological point of view, constructivism emphasizes the construction of meanings in collaboration between the instructor (or /teacher-action researcher) and the student (e.g., Hayes & Oppenheim, 1997). According to O'Toole and Plummer (2004)

> "Taking the view that mathematics is not static but rather humanistic field that is continually growing and reforming, and that children construct their own knowledge (Hersch, 1997), then teaching can no longer be a matter of viewing students' minds as 'empty vessels' ready to adopt internalise and reproduce correct

mathematical knowledge and applications. Rather, we have come to learn that teaching which includes instructional contexts where students are supported to move from their own intuitive mathematical understandings to those of conventional mathematics, produces more profound levels of mathematical understandings (Skemp, 1971)" (p. 3).

Piaget (1937/1971) considered that students' thinking becomes more sophisticated with biological maturity. Students build on their own intellectual structures as they grow up. Piaget introduced the development of student's thinking in stages, based on the process of *equilibration*. Von Glasersfeld (1995, p.68) describes *equilibration* as the process "when a scheme, instead of producing the expected result, leads to a perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium". Consequently, *disequilibration* (Piaget, 1937) situations force students to reorganize their cognitive structures, when a conceptual structure does not act in line with their expectations. The reorganization of the individual's schemata involves the subprocesses or the mechanisms of *accommodation or assimilation* (Piaget, ibid.) which correspond to modifying the pre-existing schemata and building new schemata in the student's mind or interpreting the new information according to pre-existing schemata. Many times students face *misconceptions* (e.g, Nesher, 1987; Swedosh, & Clark, 1998) and *cognitive conflicts* (e.g., Moritz, 1998; Watson & Moritz, 2001). According to Nesher (1987) "*Misconceptions are usually an outgrowth of an already acquired system of concepts and beliefs wrongly applied to an extended domain. They should not be treated as terrible things to be uprooted since this may confuse the learner and shake his confidence in his previous knowledge. Instead, the new knowledge should be connected to the student's previous conceptual framework and put in the right perspective*" (p. 38-39).

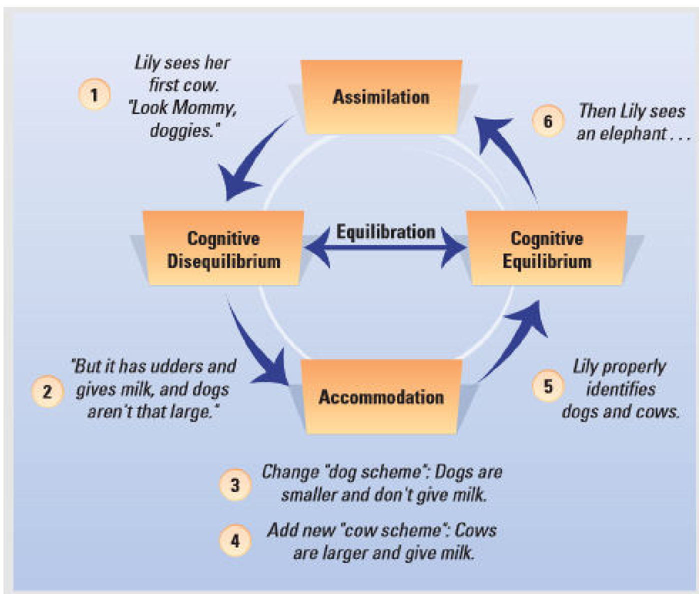


Figure 4.1. The *cycle of equilibration* (Littlefield-Cook, & Cook, 2005, Chapter 5, p.8)

In the last chapter of his work "*The Construction of Reality in the Child*" translated by M. Cook, Piaget (1937/1971) stated that:

"[...] In their initial directions, *assimilation and accommodation* are obviously opposed to one another, since assimilation is conservative and tends to subordinate the environment to the organism as it is, whereas accommodation is the source of changes and bends the organism to the successive constraints of the environment [...] Assimilation and accommodation are therefore the two poles of an interaction between the organism and the environment, which is the condition for all biological and intellectual operation, and such an interaction presupposes from the point of departure an equilibrium between the two tendencies of opposite poles."(pp.2-3)

In other words, Piaget supports that students construct new concepts, 'assimilating' in a conservative way or 'accommodating' in a modifying way their prior knowledge conceptions. In a constructivist approach the reference to *schemes* is essential. Littlefield-Cook, & Cook (2005) support that

"For Piaget, the essential building block for cognition is the *scheme*. A scheme is an organized pattern of action or thought. It is a broad concept and can refer to organized patterns of physical action (such as an infant reaching to grasp an object), or mental action (such as a high school student thinking about how to solve an algebra problem). As children interact with the environment, individual schemes become modified, combined, and reorganized to form more complex cognitive structures" (p.6, in Chapter 5).

Let us look at the way students understand negative numbers and construct the scheme of the "*sum of two numbers*". **Figure 4.2** may be thought of as a *spiral of equilibration*, trying to illustrate how pupils understand and integrate the ways to subtract numbers in several different phases of their learning life, taking into account the "*cycle of equilibration*" mentioned in **Figure 4.1**. In my opinion this process moves like a spiral, starting in the first years of a child's life and continually reiterating the process of assimilation and accommodation for every new concept that is learnt at increasingly abstract levels. The class in the first year of secondary education when teachers are obliged to introduce negative numbers to students is one of the more "difficult" parts of their teaching lives. Students understand how to add and subtract positive numbers and that the signs (+ -) are found between numbers, not in front of them. This is the first point in which "*there is an imbalance between the new experience and the old scheme. Piaget described this imbalance as a state of cognitive disequilibrium. To resolve the disequilibrium, we accommodate, or adjust, our schemes to provide a better fit for the new experience. If we are successful, we achieve cognitive equilibrium. Equilibration therefore is the dynamic process of moving between states of cognitive disequilibrium and equilibrium as we assimilate new experiences and accommodate schemes*" (Littlefield-Cook, & Cook, 2005, p.8, in Chapter 5).

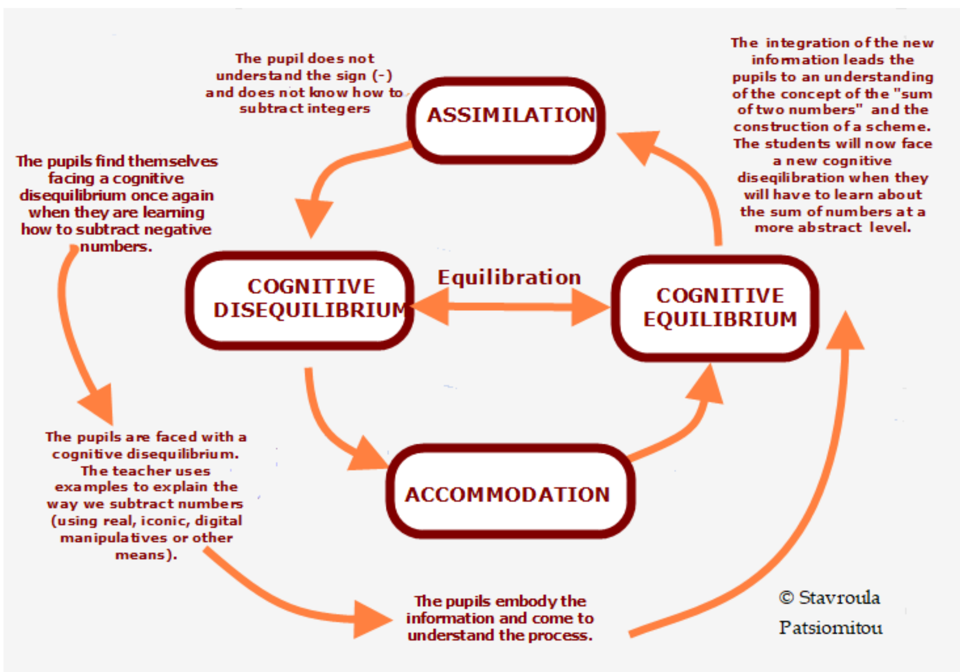


Figure 4.2. My proposal for the "*spiral of equilibration*" students understand the subtraction of numbers, taking into account the "*cycle of equilibration*" mentioned in Figure 4.1.

A very useful method for helping students understand subtraction is the use of coloured manipulatives (**Figure 4.3**). The students learn how to represent integers using color counters. The next step is to experiment with integer subtraction. This is an excellent tool which helps students overcome their *cognitive obstacles*. Sommerville (2005) in her Master thesis describes the difference in learning between a calculator and an abacus. The second is used by Japanese students. According to Sommerville (2005) "[...] in the absence of a real soroban, Japanese students can perform complex arithmentic by creating a mental image of a soroban (i.e., abacus) and imagining the changes in the pattern of the beads in order to complete the task" (p.6)

Since tools exert an influence over the technical and social way in which students conduct an activity, they are considered essential to their cognitive development (see for example, **Figure 4.3**). According to Vygotsky (1978), tools can be considered as external signs and they can become tools of semiotic mediation. He developed the zone of proximal development (ZPD) and defined it as "*the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers*" (p.86).

In Vygotsky's theory, it is taken for granted that less advanced students can learn from their peers who have more competence to solve problems and can interpret a meaning between representational systems.

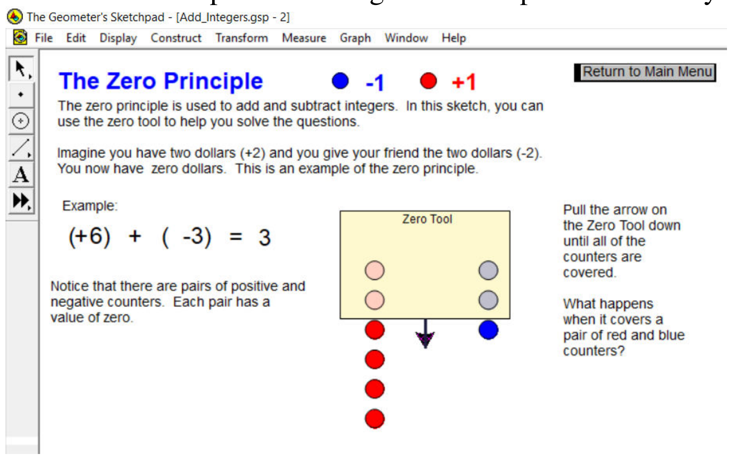


Figure 4.3. The "Zero Principle" Sketchpad file for the understanding of the s subtraction of integers. (Wepages [22, 23])

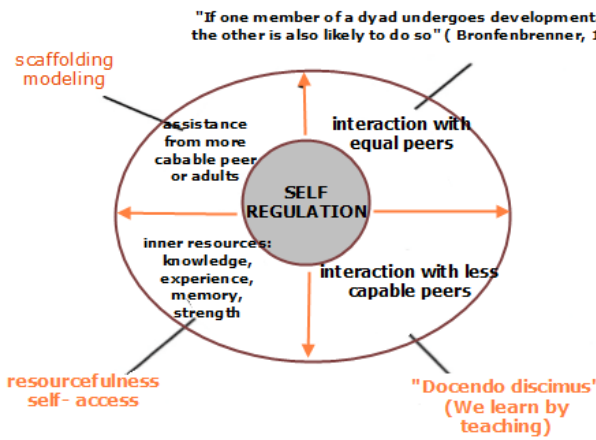


Figure 4.4. An expanded Zone of Proximal Development (Leo van Lier, 2004, chapter 6, p.158) (adapted)

Vygotsky also argues that "if learning can be influenced by social mediation, then conditions can be created in schools than can help students learn" (Vygotsky, 1978 p. 86). Vygotsky's theory in educational research led to studies of how children learned through collaborative interaction with adults, and it became common to use the term "scaffolding" to describe the interaction between adult and child (e.g., Rogoff & Wertsch, 1984). Leo van Lier (2004) expanded the notion of ZPD as a multidimensional activity space within which learners learn also when they themselves act as 'experts' or 'teachers' to each other. According to Leo van Lier (2004)

> "In the next quadrant of the diagram (Figure 4.4), I suggest that learners learn also when they themselves act as 'experts' or 'teachers' to each other. By explaining or illustrating difficulties or skills to a less accomplished peer, students clarify and hone their own abilities in the process. Such peer teaching is a special case of what Swain has called pushed output (Swain, 2000). In creating a joint ZPD, both the instructing learner and the instructed learner make their ideas clearer, sometimes by trial and error, always by orienting towards mutual comprehension, and by pushing towards clarity of expression. […] In all then, it seems eminently justifiable to see the ZPD in an expanded sense, not just as an unequal encounter between expert and novice, but also as a multidimensional activity space within which a variety of proximal processes can emerge." (p. 157-158)

The language development is a central idea in the theory of Vygotsky, something that is also common to the theory of van Hiele (Fuys et al., 1984, 1988). Moreover, the mathematical social discourses developed in a small group mediated by cognitive tools enhance the social interactions in class and support the development of students' mathematical communication and understanding of mathematical concepts. As Littlefield-Cook, & Cook (2005) support "it is the language that carries the concepts and cognitive structures to the child, and these concepts become the "*psychological tools*" that the child will use (Vygotsky, 1962)" (p. 26). This is in accordance with the view that learning is an ongoing and evolving importance for students' language development, as well as their development of mathematical terminology and conceptual understanding. Moreover, in the words of Sfard (2001) "[...] *we can define learning as the process of changing one's discursive ways in a certain well-defined manner.*" (p. 3) (see also Sfard, Neshler, Streefland, Cobb, & Mason, 1998; Sinclair & Yurita, 2008; Sinclair, & Crespo, 2006).

Steffe & Tzur (1994) in their article "*Interaction and Children's mathematics*" argued that learning "occurs as a product of interaction [and] the teacher's interventions is essential in children's learning. But in this, we speak in terms of perturbations as well as in terms of provocations, because it is the children who must experience the perturbations" (p. 44). Simon (1995) has developed a view of the teacher's role that includes both the psychological and the social aspects. He supports that "a teacher is directed by his conceptual goals for his students, goals that are constantly being modified" (p.135).

Many teachers try to apply a learning theory's principles to their instruction (though they do not usually achieve the expected results). Others try a combination of theories: drill and practice (a behaviourist view of learning), enquiry and constructivist learning using ICT – in other words, a multiple-theories approach whose results depend on the teacher's different types of knowledge [based on Schulman (1987) and Mishra and Koehler's (2006) framework of Technology, Pedagogy, and Content Knowledge (TPACK)], the students' backgrounds, external resources in the school environment, etc. Critics of the multiple-theories approach to teaching argue that moving back and forth between theories of learning reduces (or eliminates) the coherence, insights and results provided by a single theory, even if this interplay is between theories with complementary perspectives, such as constructivist and sociocultural theories (e.g., Confrey, 1995; Lerman, 1996).

Bransford, Brown & Cocking (2000, p. 22) created an image (**Figure 4.5**) in which they present "how people learn, which teachers can choose more purposefully among techniques to accomplish specific goals". Bransford, Brown & Cocking argue that "With knowledge of how people learn, teachers can choose more purposefully among techniques to accomplish specific goals" (p. 22). I think that learning can occur through interaction, which can be encouraged using a range of techniques. For this, I added arrows, to connect the "lecture based" technique with "skill based" the technique, etc.

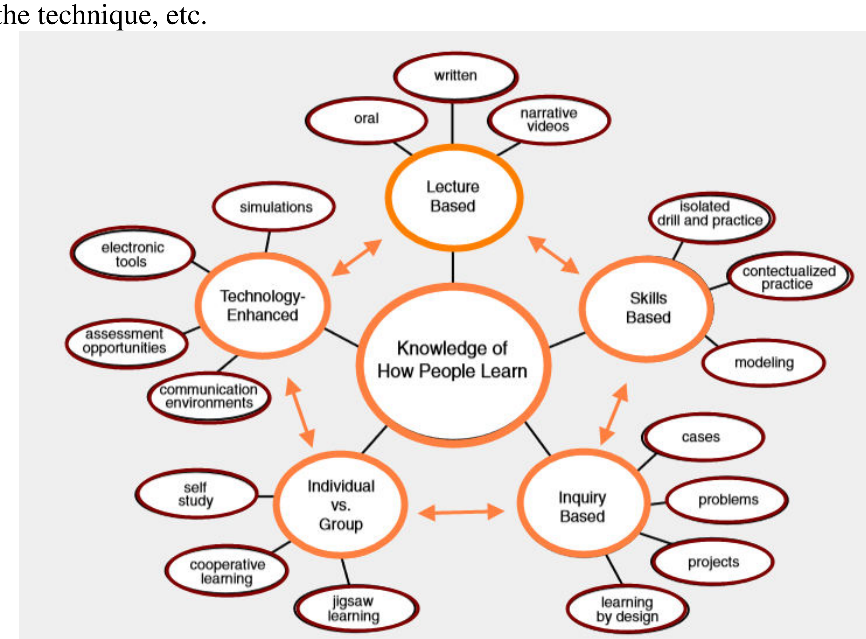


Figure 4.5. "Knowledge of how people learn" (Bransford, Brown & Cocking, 2000, p.22) (an adaptation for the current study)

In my opinion, student learning does not work as a machine into which data, information and the principles of a learning theory are entered and the expected results come out. On the other hand, is the merging of *constructivist and sociocultural perspectives* a theory we can apply to instructional processes and the everyday teaching of mathematics? Can we construct learning paths to apply the principles of constructivism to student's learning? As Fosnot (2003) states

"Although educators now commonly talk about a "constructivist-based" practice as if there is such a thing, in reality constructivism is not a theory of teaching; *it is a theory about learning*. In fact, as we shift our teaching towards trying to support cognitive construction, the field of education has been left without well-articulated theories of teaching. [...] Major questions loom around what should be taught, how we should teach, and how best to educate teachers for this paradigmatic shift. The problem is that all of these pedagogical strategies can be used without the desired learning resulting. This is because constructivism is a theory of learning, not a theory of teaching, and many educators who attempt to use such pedagogical strategies confuse discovery learning and "hands-on" approaches with constructivism".

Bruner (1966) developed an instructional theory. Bruner emphasized the teacher's proper use of language when they introduce a meaning to children. *Discovery learning* was also advocated by Bruner (1961, 1966). He pointed out that discovery learning "increases the interest of students, creates exciting classroom atmosphere, encourages and increases participation, provokes enthusiasm and inquiry, and helps students learn new content" (Bayram, 2004, p.40). Within the theory developed by Bruner (1966) cognitive conflict "occurs when there is a mismatch between information encoded in two of the representational systems, between [...] what one sees and how one says it [...]" (Bruner, Olver, & Greenfield, 1966, p. 11). According to El Rouadi & Al Husni (2014, p. 130) "*Bruner focused on the spiral curriculum which can be explained as follows: learners acquire the basic ideas initially by using their intuition; and after words, the learner builds on them by revisiting these basic ideas as frequent as required until the meaningful understanding is fully achieved*".

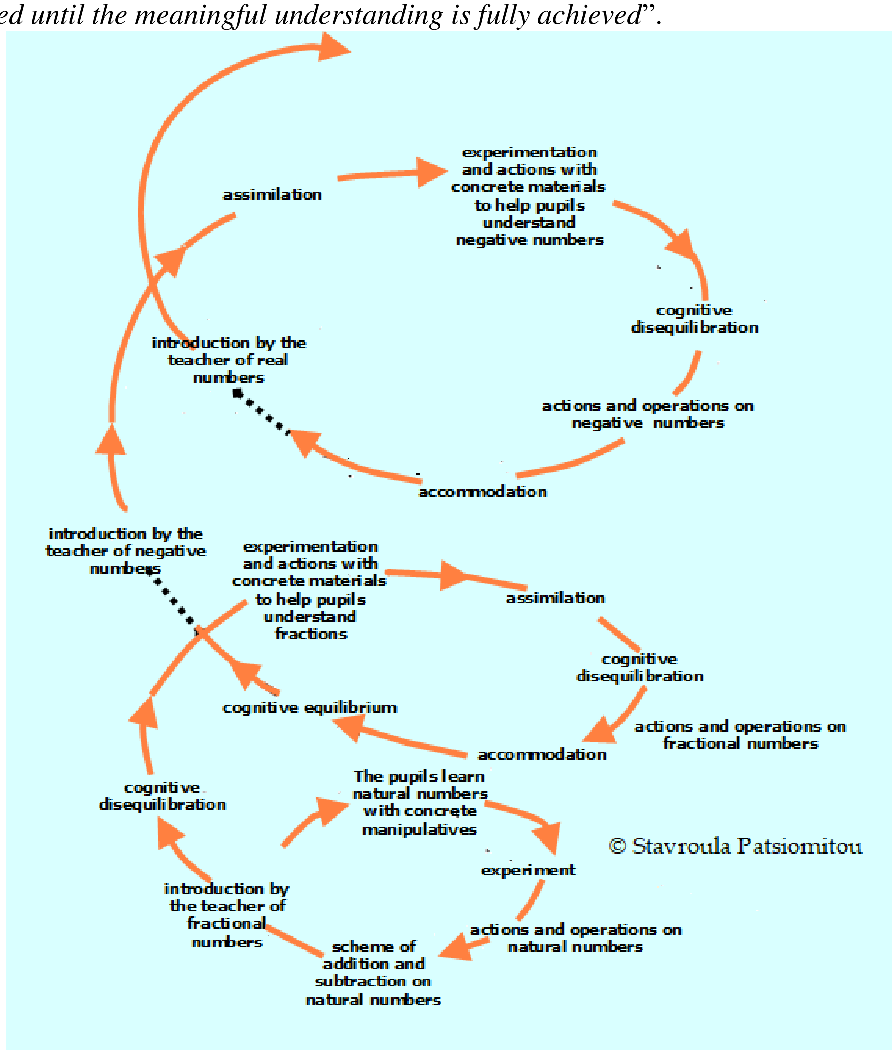


Figure 4.6. My proposal for a spiral curriculum for the learning of numbers, taking into account the aforementioned notions of Piaget and Bruner

Figure 4.6 may be thought/ considered as a spiral curriculum for the learning of numbers, taking into account the aforementioned notions of Piaget and Bruner; how the learning of numbers occurs during the school years from primary to secondary and tertiary education.

Bransford, Brown & Cocking (2000) support that "constructivists assume that all knowledge is constructed from previous knowledge, irrespective of how one is taught (e.g., Cobb, 1994)—even listening to a lecture involves active attempts to construct new knowledge" (p. 11). They point out that "*Like 'Fish is Fish'* everything the children hear [is] incorporated into [their] pre-existing view". 'Fish is Fish' (Lionni, 1970, cited in Bransford et al., 2000) is a tale in which a fish tries to understand how people and cows appear/exist in the external world from the descriptions of a frog that has gone outside to view everything.

Image: The Eric Carle Museum (Webpage [24])

> "The book shows pictures of the fish's representations of each of these descriptions: each is a fish-like form that is slightly adapted to accommodate the frog's descriptions. […] This tale illustrates both the creative opportunities and dangers inherent in the fact that people construct new knowledge based on their current knowledge." (Bransford et al., 2000, p. 11).

In a constructivist frame, cognitive conflict is a basic component in the learning process (Karmiloff-Smith & Inhelder, 1974) and very important for the development of students' geometrical thinking. If the student overcomes this contradiction s/he is able to mental growth. Van Hiele also developed a theoretical model for thought development that can be applied to students' instruction. I shall present their model in the next section.

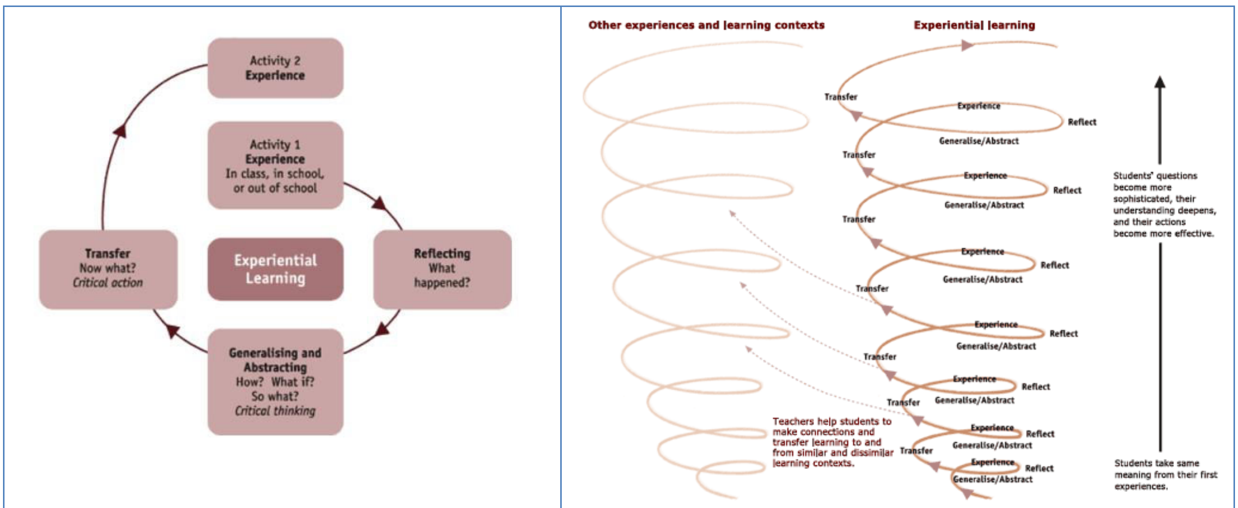
Experiential Learning or learning through experience is a theory developed by David A. Kolb (1984). According to Kolb & Kolb (2013) "The theory, described in detail in Experiential Learning: Experience as the Source of Learning and Development (Kolb, 1984), is built on six propositions that are shared by these scholars:

1. *Learning is best conceived as a process, not in terms of outcomes.[…]*
2. *All learning is re-learning.[…]*
3. *Learning requires the resolution of conflicts between dialectically opposed modes of adaptation to the world.[…]*
4. *Learning is a holistic process of adaptation to the world. […]*
5. *Learning results from synergetic transactions between the person and the environment. […]*
6. *Learning is the process of creating knowledge.* (p.6-7)

According to Kolb & Kolb (2013)

> "The ELT model portrays two dialectically related modes of grasping experience—Concrete Experience (CE) and Abstract Conceptualization (AC)—and two dialectically related modes of transforming experience—Reflective Observation (RO) and Active Experimentation (AE). Learning arises from the resolution of creative tension among these four learning modes. This process is portrayed as an idealized learning cycle or spiral where the learner "touches all the bases"—experiencing (CE), reflecting (RO), thinking (AC), and acting (AE)—in a recursive process that is sensitive to the learning situation and what is being learned. Immediate or concrete experiences are the basis for observations and reflections. These reflections are assimilated and distilled into abstract concepts from which new implications for action can be drawn" (p. 7-8) (**Figure 4.7a**).

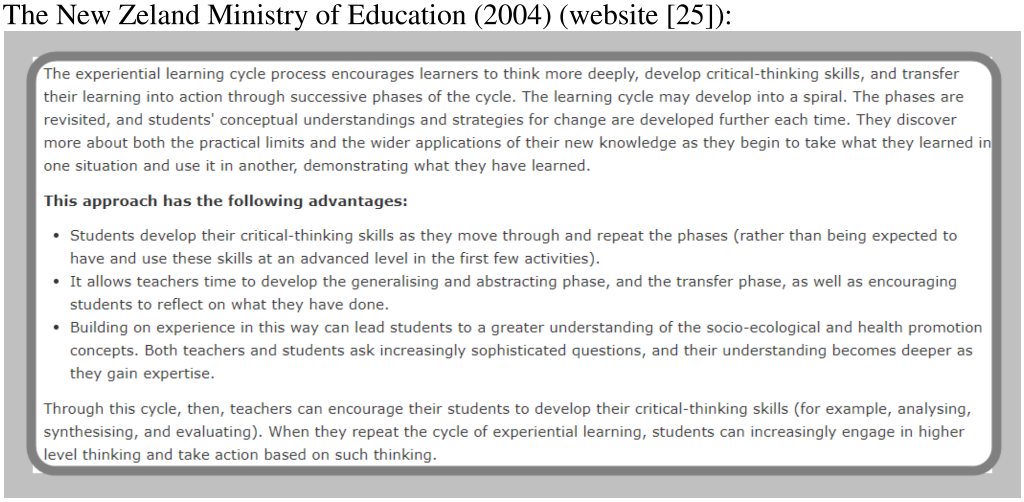
Figure 4.7a. The Experiential Learning Cycle (Kolb, 1984, cited in Kolb & Kolb, 2013, p. 8)



Figures 4.7b, c. The spiraling learning process applied by the New Zealand Ministry of Education (2004) (Website [25])

In the **Figures 4.7b, c**, Kolb & Kolb (2013) depict an amazing idea in a spiral that illustrates the spiraling learning process. As they state "The New Zeland Ministry of Education (2004) has used this spiraling learning process as the framework for the design of middle school curricula. Figures [4.7b, c] describe how teachers use the learning spiral to promote higher level learning and to transfer knowledge to other contexts" (Kolb & Kolb, 2013, p.37). The spiraling learning "begins with activity, moves through reflection, then to generalizing and abstracting and finally to transfer" (Henton, 1996, page 39, cited in website [25]).

According to The New Zeland Ministry of Education (2004) (website [25]):



The experiential learning cycle process encourages learners to think more deeply, develop critical-thinking skills, and transfer their learning into action through successive phases of the cycle. The learning cycle may develop into a spiral. The phases are revisited, and students' conceptual understandings and strategies for change are developed further each time. They discover more about both the practical limits and the wider applications of their new knowledge as they begin to take what they learned in one situation and use it in another, demonstrating what they have learned.

This approach has the following advantages:

- Students develop their critical-thinking skills as they move through and repeat the phases (rather than being expected to have and use these skills at an advanced level in the first few activities).
- It allows teachers time to develop the generalising and abstracting phase, and the transfer phase, as well as encouraging students to reflect on what they have done.
- Building on experience in this way can lead students to a greater understanding of the socio-ecological and health promotion concepts. Both teachers and students ask increasingly sophisticated questions, and their understanding becomes deeper as they gain expertise.

Through this cycle, then, teachers can encourage their students to develop their critical-thinking skills (for example, analysing, synthesising, and evaluating). When they repeat the cycle of experiential learning, students can increasingly engage in higher level thinking and take action based on such thinking.

Summarizing (Patsiomitou, 2014, p. 4-5): Cognitive constructivism is connected with the work of Piaget's (1937/1971) and his views as 'constructivist'. According to Piaget (1937/1971), students' cognitive development depends on their biological maturity. That students' cognitive development depends on the teaching process was argued by Dina van Hiele-Geldof and Pierre van Hiele in their dissertations in 1957 (Fuys, Geddes & Tischler, 1988). Van Hiele theory has its roots in constructivist theories. Bruner's (1961, 1966) proposal of discovery learning [as 'constructionist"] is based on prior knowledge and the understanding of a concept, which [through discovery] grows and deepens. The sociocultural approach has its roots in Vygotsky (1987) who focuses on the acquisition of mathematical understanding as a product of social interactions. Von Glasersfeld (1995) a radical constructivist is differentiated from the work of Piaget as he argues that "knowledge [does not represent an independent world, instead] represents something that […] we can do in our experiental world" (p.6). Building on the concepts mentioned above, the concept of social constructivism is a complex process, while being interactive, constructivist and sociocultural (e.g., Yackel, Cobb, Wood, Wheatley & Merkel 1990; Cobb, Yackel & Wood, 1989, 1992; Yackel, Rasmussen & King 2001; Yackel & Rasmussen 2002; Jaworski, 2003). According to sociocultural and interactive approaches, learning is a part of the culture (Steffe & Gale, 1995) in which the students construct knowledge through their participation in social practices (e.g social class environment) (Cobb & Bauersfeld, 1995, p.4). "A social-constructivist perspective sees discussion, negotiation and argumentation in inquiry and investigation practices to underpin knowledge growth in mathematics, in teaching mathematics and in mathematics teacher education" (e.g., Cobb & Bowers, 1999; Lampert, 1998; Wood, 1999 cited in Jaworski, 2003, p. 17).

Besides, learning is an individual constructive process while knowledge is actively constructed by the student; it depends on the individual's personal work and negotiation of mathematical ideas (e.g., Jaworski, 2003). From the perspective of constructivist theories the process of mathematical knowledge and understanding arises as students try to solve math problems during the classroom (Cobb, Yackel, & Wood, 1992; Simon & Shifter, 1991) and is instigated when students confront problematic situations. Knowing therefore is not taken passively by students but in an active way. Learning thus is characterized in Bauersfeld's interactionism view "by the subjective reconstruction of societal means and models through negotiation of meaning in social intervention" (Bauersfeld, 1992, p.39; Bauersfeld, 1995). Vygotsky (1987) argues that "the child begins to perceive the world not only through his eyes [visually] but also through speech" (p. 32). According to Vygotsky (1987), learning is a complex interplay between scientific and spontaneous use of language. For this, learning is an internalization of social relations and understanding is a result of common negotiation of concepts created by students while interacting with other students in the class (or group) during the mathematical discussions developed (Bartolini Bussi, 1996). "Language is important for cognitive development and learning; without it, an individual lacks [an] efficient system for storing certain types of information that are needed for thinking, reasoning, and concept development" (Westwood, 2004, p.141).

Sfard also defines "learning as the process of changing one's discursive ways in a certain well-defined manner" (Sfard, 2001, p.3). According to Sfard (2001) "thinking is a special case of the activity of communicating" […] "A person who thinks can be seen as communicating with himself/herself, […] whether the thinking is in words, in images or other form of symbols, [..] as our thinking is [an interactive] dialogical endeavour [through which] we argue…" (p.3); with his/her participation the student in a mathematical discussion s/he "learns to think mathematically" (Sfard, ibid., p. 4). Under this approach, the development of thought occurs through dialogue that develops the subject within himself/herself internally (intrapersonally) or in a group in which s/he participates. Moreover, learning is expanding the capacity for dialectical skills and solving problems that could not previously be solved. Furthermore "putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned" (Sfard, 2001, p.1). Consequently, learning is first and foremost the modification / transformation of the ways we think and how we exchange this thought. Moreover, learning is the capacity of dialectical skills and of problem-solving that could not be solved before.

Goos and her colleagues carried out a series of studies --based on sociocultural perspective-- to investigate the teacher's role, the students' discussion in small groups and the use of technology as a tool that mediates teaching and learning interactions (e.g. Goos, 2004, Goos et al, 2002, 2003). If we take the role of teacher seriously as concerns the realisation and planning of activities then, every activity should be based on geometry exactly as Goldenberg (1999) purports it to be –a fundamental principle. Tools like DGS present geometric structures in an environment that emphasizes the continuous nature of Euclidean space, and thus serve as an excellent bridge between geometry and [the other field of mathematics, as well as] analysis. This is very important for the teaching practice because the construction of the meaning can not only be depended or is located in the tool per se, nor uniquely pinpointed in the interaction of student and tool, but it lies in the schemes of use (e.g., Trouche, 2004) of the tool itself.

4.2. The van Hiele Model

4.2.1. Introduction to the Problem

In past decades researchers concluded that high school students fail at Geometry, as it is presented in class through instruction (e.g., Hoffer, 1981; Usiskin, 1982; Van Hiele, 1986; Burger & Shaughnessy, 1986; Crowley, 1987; Fuys, Geddes & Tischler, 1988; Gutierrez, & Jaime, 1987, 1998; Gutierrez, Jaime & Fortuny, 1991; Mason, 1997; Patsiomitou, 2008a, b, 2011a, b, 2012a, b, 2013a, 2018b). They found that students had difficulty developing and structuring the content incorporated in the Geometry Curriculum, as presented in class through instruction, due to an inability to recall linguistic symbols and symbolic representations already known to them, *to release their thinking from a concrete framework* (White & Mitchelmore, 2010, p. 206), and to develop the requisite deductive reasoning (Peirce, 1998/1903) and abstract processes (Skemp, 1986; White & Mitchelmore, 2010).

Pierre van Hiele and his wife Dina van Hiele–Geldof developed a theoretical model of thought development in geometry. The van Hieles distinguished five different levels of thought and how the students progress through levels, during the instruction. Dina van Hiele-Geldof (1957/1984) in her didactic experiments investigated "*the improvement of learning performance by a change in the learning method*" (p.16). She investigated whether it was possible to use instruction as a way of presenting material to participated students, so that the holistic visual thinking of a child can be transformed into concrete abstract thinking in a continuous process, something that is prerequisite for the development of deductive reasoning in geometry.

> "*After observing secondary school' students having great difficulty learning geometry in their classes, Dutch educators Pierre van Hiele and his wife, Dina van Hiele-Geldof developed a theoretical model involving five levels of thought development in geometry. Their work, which focuses' on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next, was first reported in companion dissertations at the University of Utrecht in 1957*." (Fuys et al., 1984, p.6).

Burger & Shaughnessy (1986, p.31) report the descriptions of the five levels that have been identified by Dina van Hiele (1957), as modified by Hoffer (1981):

- "*Level* 0 (visualization): the student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
- *Level* 1 (analysis): the student reasons about geometric concepts by means of informal analysis of component parts and attributes. Necessary properties of the concept are established.
- *Level* 2 (abstraction): the student logically orders the properties of concepts, forms abstract definitions and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.
- *Level* 3 (deduction): the student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions and theorems.
- *Level* 4 (rigor): the student can compare systems based on different axioms and can study various geometries in the absence of concrete models".

A large amount of scholars have been investigated the implications of the theory for the learning of geometry as well as the validation of van Hiele model (e.g., Usiskin, 1982; Mayberry, 1983; Senk, 1985, 1989; Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988; Gutierrez, Jaime, & Fortuny, 1991; Clements & Battista, 1992; Patsiomitou, 2008a, b, 2012a,b, 2011a, b, 2013a, b, 2018b). Research has been conducted, which has set out:

- to *check* the validity of the van Hiele theory and its hypothesis; also to show that the van Hiele level 5 does not appear among high school students [or to show that the incidence of van Hiele level 5 is (close to) zero among high school students] (e.g., Wirszup, 1976; Hoffer, 1981; Mayberry, 1983; Usiskin, 1982; Burger & Shaughnessy, 1986);
- to *identify* the key features of every van Hiele level during the process of recognizing and defining a figure, reporting its basic properties and constructing proof (e.g., Hoffer, 1981; Burger & Shaughnessy, 1986; Gutierrez, Jaime, & Fortuny, 1991);

- to *design* instruction based on the van Hiele model, in order to help students become more effective and acquire competence in the proving process (e.g., Fuys, Geddes, & Tischler, 1988);
- to *examine* if the model can be of use in describing students' thinking during the problem-solving process and their understanding of geometrical or mathematical meanings (e.g., Burger & Shaughnessy, 1986; Clements & Battista, 1992; Fuys, Geddes & Tischler, 1988);
- to *examine* if the van Hiele model can serve as an instrument for predicting the competence of students at geometrical proof (e.g., Usiskin, 1982; Senk, 1989; Usiskin & Senk, 1990).

Many researchers agree that the main reason why students fail at geometry is that they are the recipients of instruction that is at a higher level than they can understand (e.g., Hoffer, 1981; Usiskin, 1982; Burger & Shaughnessy, 1986; Van Hiele, 1986; Crowley, 1987; Fuys, Geddes & Tischler, 1988; Mason, 1997). However, the organization of the instruction, its content and supplementary 'manipulatable' materials [e.g. Dienes cubes (Dienes, 1960), [digital] building blocks (Clements & Sarama, 2002), DGS material as custom tools (Patsiomitou, 2006g, 2012a, 2018b)] have a positive effect on students' cognitive development (e.g., (van Hiele, 1986; Fuys, Geddes & Tischler, 1984; Crowley, 1987; Gutierrez, Jaime & Fortuny, 1991; Clements & Battista, 1992; Patsiomitou, 2012a).

Clements & Battista (1992) argue that the constructivist approach forms the basis of the theory underpinning the use of such a digital environment in the teaching and learning of geometry. Researchers also consider van Hiele's theory to comprise one of the best frameworks within which to study, teach and learn geometrical processes (Atebe, 2008, p.3). Moreover, van Hiele's theory provides a framework for validating the design of instructional sequences in school geometry, *as was recognized in the NCTM's Curriculum and Evaluation Standards for School Mathematics* (Jaime & Gutierrez, 1995, p. 592). Many teachers, educators and researchers have developed and applied activities in DGS software environments, in order to incorporate new technologies into the teaching of geometry in class, just as Cabri (Laborde, J, M., Baulac, Y., & Bellemain, F., 1988), or The Geometer's Sketchpad (Jackiw, 1991) (e.g., Hölzl, 1996, 2001; Laborde, 1998; Hoyles & Healy, 1999; Clement & Battista, 1992; De Villiers 1998; Yerushalmy & Chasan 1993; Oldknow, 1995, 2003; Sanchez & Sacristan, 2003; Hollebrands, 2003, 2004, 2006, 2007; Christou, Mousoulides, Pittalis and Pitta, 2004a,b, 2005; Patsiomitou, 2008a, b, 2012a, b).

The five levels of thinking reflect on students' progress and increasing development in the way in which they are able to reason about geometrical objects and their relationships, and focus "on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next" (Fuys et al, 1984, p.6). Freudenthal (1973) argues that

> "good geometry instruction can mean much – learning to organize a subject matter and learning what is organizing, learning to conceptualize and what is conceptualizing, learning to define and what is a definition. It means leading pupils to understand why some organization, some concept, some definition is better than another. Traditional instruction is different… All concepts, definitions and deductions are preconceived by the teacher "(Freudenthal, 1973, p.418).

Dina van Hiele-Geldof (Fuys et al., 1984) also stressed the necessity to arrive to a totally different approach at geometry instruction whereby the students "more adequately experience the build –up of theory" (p.17). The students in the gaps between levels are presented with *disequilibration* situations that force them to re-organize their schemes and cognitive structures. The notion of *cognitive equilibration* is borrowed from Piaget (1937/1954), who used it to refer to an individual re-organizing his/her schemata when his/her experience does not fit within a conceptual structure or does not act in line with his/her expectations. Piaget supports that, to equilibrate, the individual has to modify his/her conceptual structures or schemes in order to better organize his/her experiences. Pierre van Hiele finally, characterized his model in terms of three rather than five levels of thought: visual (level 1), descriptive (level 2) and theoretical (level 3) (van Hiele, 1986 cited in Teppo, 1991, p. 210).

- *Visual* (level 1): *Students recognize shapes globally*. ("[…] There is no why, one just sees it" (p. 83, cited in Teppo, 1991, p. 210).
- *Descriptive* (level 2): *Students distinguish shapes on the basis of their properties.* (Teppo, 1991, p. 211)
- *Theoretical* (level 3): *Students are able to devise a formal geometric proof and to understand the process employed* (p. 86, cited in Teppo, 1991, p. 211):

The language of the theoretical level has a much more abstract character than that of the descriptive level because it is engaged with causal, logical, and other relations of a structure, which at the second level is not *visual. Reasoning about logical relations between theorems in geometry takes place with the language of the third level" (*van Hiele, 1986, cited in Teppo, 1991, p.210).

Many researchers have argued that sequencing instruction that uses consequential activities has positive effects on students' success (e.g., Burger & Shaughnessy, 1986; Battista, 1998; Patsiomitou, 2012a). Battista (1998) developed a sequence of activities with the Shape Maker microworld aiming to encourage students to pass through the first three van Hiele levels. Burger & Shaughnessy (1986) claim that if initial activities are not interesting or are too easy, they might not attract or motivate students to focus on the topic and might not bring with it a sense of success. Fuys et al. (1988), Pierre van Hiele (1959/1984) and others report that progress from one level to the next involves five phases: *information, guided orientation, explicitation, free orientation, and integration* (Fuys et al, 1988, p. 7).

- *"Information* is the phase through which the student is informed about the objects of investigation, "examining examples and counter-examples".
- *Guided orientation* is the phase through which the student is guided to transform the orientation of his/her thinking "doing tasks that involve different relations of the network that is to be formed (e.g., folding, measuring, looking for symmetry)"
- *Explicitation* is the phase through which the student tries to give explanations using his own language. "S/he becomes conscious of the relations, tries to express them in words, and technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures"
- *Free orientation* is the phase through which the student releases his thought "by doing more complex tasks, to find his/her own way in the network of relations (e.g., knowing properties of one kind of shape, investigates these properties for a new shape, such as kites)".
- *Integration* is the phase through which the student integrates his knowledge. "S/he summarizes all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g., properties of a figure are summarized".

4.2.2. The Characteristics/Indicators of the van Hiele levels

Table 4.1 Burger & Shaugnessy's (1986) van Hiele levels' indicators

Burger & Shaughnessy (1986) in their study provided the following van Hiele level indicators and descriptions as response to clinical interview tasks concerning triangles and quadrilaterals (p.43-45):

Level 0: 1. *Use of imprecise properties* (qualities) to compare drawing and to identify, characterize, and sort shapes. 2. References to *visual prototypes* to characterize shapes. 3. Inclusion of *irrelevant attributes* when identifying and describing shapes, such as orientation of the figure on the page. 4. *Inability to conceive of an infinite* variety of types of shapes. 5. Inconsistent sorting; that is, sorting by properties not shared by the sorted shapes. 6. Inability to use properties as necessary conditions to determine a shape [...]	**Level 1** 1. *Comparing shapes* explicitly by means of properties of their components. 2. *Prohibiting class inclusions* among general types of shapes, such as quadrilaterals. 3. *Sorting by single attributes*, such as properties of sides, while neglecting angles, symmetry, and so forth. 4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape. 5. Descriptions ot types of shapes by explicit use of their properties, rather than by type names, even if known. [...] 6. Explicit rejection of textbook definitions of shapes in favour of personal characterizations. 7. Treating geometry as physics when testing the validity of a proposition [...]
Level 2 1. Formation of complete definitions of types of shapes. 2. Ability to modify definitions and immediately accept and use definitions of new concepts. 3. Explicit references to definitions. 4. Ability to accept equivalent forms of definitions. 5. Ability to sot shapes according to a variete of mathematical precise attributes. 6. Acceptance of logical partial ordering among types of shapes, including class inclusions. 7. Explicit use of "if-then" statements. 8. Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if o implies q and q implies r, then p implies r) and the law of detachment (modus ponens). 9. Confusion between the roles of axiom and theorem.	**Level 3** 1. Clarification of ambiguous questions and rephrasing of problem tasks into precise language. 2. Frequent conjecturing and attempts to verify conjectures deductively. 3. Reliance on proof at the final authority in deciding the truth of a mathematical proposition. 4. Understanding of the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, proof. 5. Implicit acceptance of the postulates of Euclidean geometry.

Table 4.2. Mason's (1998) van Hiele levels' indicators

Mason (1998) also describes the levels of geometry understanding as follows (p.4):	
Level 1 (Visualization): 1. Students recognize figures by appearance alone, often by comparing them to a known prototype. 2. The properties of a figure are not perceived. 3. Students make decisions based on perception, not reasoning.	**Level 2 (Analysis):** 1. Students see figures as collections of properties. 2. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. 3. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.
Level 3 (Abstraction) 1. Students perceive relationships between properties and between figures. 2. Students can create meaningful definitions and give informal arguments to justify their reasoning. 3. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. 4. The role and significance of formal deduction, however, is not understood.	**Level 4 (Deduction):** 1. Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. 2. Students should be able to construct proofs such as those typically found in a high school geometry class. **Level 5 (Rigor):** Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems […].

Many researchers elaborated on van Hiele levels and described the characteristics of every level (e.g., Burger & Shaugnessy, 1986; Pierre van Hiele, 1986; Crowley, 1987; Mason, 1998; Battista, 2007; 2008, 2011; Patsiomitou, 2012a). They applied the van Hiele model to their investigations, determining the levels of thought and their characteristics and modifying the prototype version introduced by Van Hieles. Fuys et al. (1988) consider language to be a crucial factor in moving students through the hierarchy of van Hiele levels. They conclude that each van Hiele level defines its own language (symbols) with their own network of relations. Only when students have realized the interrelations and connections between the structures can they progress up the levels. Mason (1998) mentions also Clements and Battista (1992) who proposed the Level 0 (pre-recognition) (p.5). According to them:

> *"Students at this level notice only a subset of the visual characteristics of a shape, resulting in an inability to distinguish between figures. For example, they may distinguish between triangles and quadrilaterals, but may not be able to distinguish between a rhombus and a parallelogram"* (cited in Mason, 1998, p.5).

According to Battista (2011) *"Some studies indicate that people exhibit behaviors indicative of different van Hiele levels on different subtopics of geometry, or even on different kinds of tasks (Clements & Battista, 2001)".* Battista (2007) "has elaborated the original van Hiele levels to carefully trace students' progress in moving from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians" (p.851). This is a "totally different approach to assessing van Hiele levels" (Battista, 2011, p. 523). Battista's (2007) first three levels which are the most usual to high school students are described below.

TABLE 4.3. Battista's (2007) first three van Hiele levels' indicators

Level 1 (Visual-Holistic Reasoning): *"Students* identify, describe, and reason about shapes and other geometric configurations according to their appearance as visual wholes. They may refer to visual prototypes, […].Orientation on figures may strongly affect Level 1 students' shape identifications.[…]" (p.851).	
Level 2 (Analytic-Componential Reasoning): "Students [acquire through instruction] a) an increasing ability and inclination to account for the spatial structure of shapes by analyzing their parts and how their parts are related and b) an increasing ability to understand and apply formal geometric concepts in analyzing relationships between parts of shapes". (pp.851-852). Battista (2007) identified three sublevels between levels 2 and 3:	***2.1. Visual-informal componential reasoning:*** Students describe parts and properties of shapes informally and imprecisely [and] students' informal language ranges greatly in precision and coherence […].
	2.2. Informal and insufficient-formal componential reasoning: Students begin to acquire formal conceptualizations [but] their reasoning is still visually based, and most of their descriptions and conceptualizations still seem to occur extemporaneously as they are inspecting shapes[…].
	2.3. Sufficient formal property-based reasoning. Students explicitly and exclusively use formal geometric concepts and language to describe and conceptualize shapes [but] their definitions are not minimal […]
Level 3 (*Relational –Inferential Property-Based Reasoning*): Students explicitly interrelate and make inferences about geometric properties of shapes.[…] The verbally-stated properties themselves are interiorized so that they can be meaningfully decomposed, analyzed, and applied to various shapes". This level incorporates "*empirical relations, componential analysis, logical inference, hierarchical shape, classification based on logical inference*" (pp.852-853). Battista (2007) identified four sublevels between levels 3 and 4:	***3.1. Empirical relations***. Students use empirical evidence to conclude that if a shape has one property, it has another […]
	3.2. Componential analysis. By analyzing how types of shapes can be built one-component-at-a-time, students conclude that when one property occurs, another property must occur.[…]
	3.3. Logical inference. Students make logical inferences about properties: they mentally operate on property statements […]
	3.4. Hierarchical shape, classification based on logical inference. Students use logical inference to reorganize their classification of shapes into a logical hierarchy. […]

Battista expands significantly on the van Hiele levels in two places: in the development of thinking based on properties, and in the development of inferences about those properties.
In my opinion, van Hiele's description of level 2 corresponds to Battista's description of level 2.1; Battista's description of level 2.3 relates to Mason's level 3 and both relate to the development of students' ability to define geometric objects. Moreover, there is no stability in the process, but this depends on the geometry activities the student participates in, and on the teacher's instructions that lead to the evolution of each individual student's level. According to Fuys et al. (1988, p.8) "the major characteristics of the van Hiele "levels" are the following:

"(a) The levels are sequential.
(b) Each level has its own language, set of symbols, and network of relations.
(c) What is implicit at one level becomes explicit at the next level.
(d) Material taught to students above their level is subject to reduction of level.
(e) Progress from one level to the next is more dependent on instructional experience than on age or maturation.
f) One goes through various "phases" in proceeding from one level to the next".

4.2.3. The Symbol and Signal Character in the van Hiele Model

The meanings of *symbol* and *signal* are very important in the van Hiele model. Skemp (1987) defines a symbol as "a sound, or something visible, mentally connected to an idea" (p. 47). Piaget (1952/1977) in his work "*The origins of intelligence in children*" (translated by Cook, M.) also, states:

> "The "symbol" and the "sign" are the signifiers of abstract meanings, such as those which involve representation. A "symbol*" is an image evoked mentally or a material object intentionally chosen to designate a class of actions or objects*. So it is the mental image of a tree that symbolizes in the mind trees in general, a particular tree which the individual remembers, or a certain action pertaining to trees, etc. Hence the symbol presupposes representation […]. *Symbol and sign are only the two poles, individual and social, of the same elaboration of meanings*" (p.191).

Dina van Hiele also supports that (Fuys et al, 1984, p.215)

> "The word 'symbol' should here be interpreted as meaning *'a mental substitute for a complex of undifferentiated relations that is subsequently elaborated in the pupil's mind.'* The rhomb, for instance, is a symbol of the following characteristics: it has four equal sides, equal opposite angles, diagonals that bisect the angles and are perpendicular to each other".

What is important is the students' competence when it occurs to identifying a figure's properties (*symbol character*) and to gradually identifying a concrete figure from a set of properties (*signal character*): for example, when a student observes an equilateral triangle in his textbook, being able to identify the figure's congruent sides and angles. The equal sides and angles are the main characteristic of a triangle; this is *a symbol* for the equilateral triangle**.** Then s/he can identify additional properties (for example, "every angle of an equilateral triangle is equal to 60 degrees"). All these properties are interrelated and can become a concept for the concrete mathematical object (i.e. the equilateral triangle mentioned above). Subsequently, the student can use a combination of properties to construct the equilateral triangle. In other words, the student now possesses the concept of the triangle: an abstract idea conceived in her/his mind. This is *a signal* for the concrete *figural concept*.
Generally, in my opinion, a symbol is a mental image of a class of objects with concrete characteristics and properties. A sign is the social aspect of the symbol which was previously created in an individual's mind.
Van Hieles described periods between levels. In these periods the students have characteristics of both levels. For example, during the first period (between the first and the second levels) the students' perceptual competence in relation to a geometrical object gradually transforms from a global perception of the object to the perception of an object with concrete characteristics and properties. During the second period students focus less on the symbol, and the figure is replaced by a list of properties which identify the symbol. The figure now gets the *signal character*. The next period connects the second and third level. This is the period in which students identify the common properties of a class of figures and categorize the figures as inclusions of other figures in accordance with their additional properties. Pierre van Hiele writes (1986, p. 168)

> "when after some time, the concepts are sufficiently clear, pupils can begin to describe them. With this the properties possessed by the geometric figures that have been dealt with are successively mentioned and so become explicit. The figure becomes the representative of all these properties: It gets what we call the "*symbol character*". In this stage the comprehension of the figure means the knowledge of all these

properties as a unity.[…].When the symbol character of many geometric figures have become sufficiently clear to the pupils, the possibility is born that they also get a *signal character*". This means that the symbols can be anticipated.[…]. When this orientation has been sufficiently developed, when the figures sufficiently act as signals, then, for the fisrt time geometry can be practiced as a logical topic"

Building on van Hiele's ideas Choi-Koh (1999) supports that:

> Many symbols begin with an image onto which observed properties and relationships are temporarily projected. After those properties and relationships are explained by analysis or discussion, however, the symbol loses the characteristic of an image acquires verbal content and thus becomes more useful for operations of thought. That is symbols have properties that a geometric figure has and symbols are compared and recognized by those properties. […]When symbols influence orientation of thought they act as signals.[…] If the symbol and signal properties of a figure are sufficiently developed, then the implicit meaning of the figure is understood. After students have learned that it is possible to give relationships an imlicitatory character, they deduce that it is possible for that character to sometimes exist in only one direction" (p.302).

Cannizzaro & Menghini (2003, p.2) have also clarified the meanings of symbol and signal, supporting that

- "Van Hiele's *symbol* (1958, 1974) represents a first level of *perception* at which pupils condense the properties of a known geometrical figure.
- Van Hiele's *signal* represents a second level of *description* or *analysis* at which perceptions are translated into descriptions, though without specific linguistic properties—of which the *significant signal* is most significant in the description.
- At the third level--*definition*--the student starts to observe relations logically, assigning significance to implication, and therefore definition, in terms of geometrical relations. This, according to van Hiele, is the *essence of geometry"*.

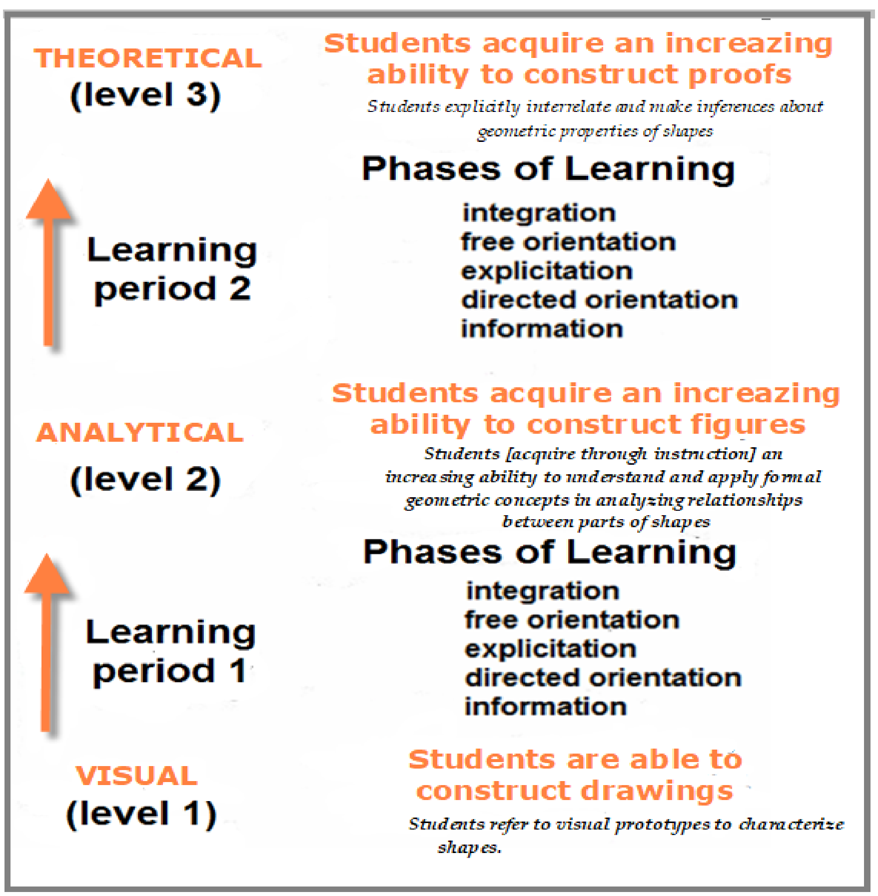


Figure 4.8. An adaptation on Teppo's diagram (1991, p.210) taking into account Battista's (2007) elaboration of the van Hiele levels (Patsiomitou, 2012a)

Teppo (1991) also supports that "students progress from one level to the next as the result of purposeful instruction organized into five phases of sequenced activities that emphasize exploration, discussion, and integration" (p. 212). Teppo has constructed a diagram, in which she explains the learning periods through which students are able to progress, given appropriate instruction, to the next van Hiele level. According to Teppo (1991) the first period connects the first level with the second and the second period connects the second and third levels. The aim of this first period is to transform the way students perceive geometric objects (for example Teppo, 1991; Pusey, 2003; Genz, 2006). This means *transforming the visual image (Mariotti, 1997) or drawing (Parsysz, 1988) they perceive, into a figure with concrete properties*. The figure then becomes a symbol or acquires the *symbol character*. I created an adaptation to the diagram constructed by Teppo during the writing of my PhD thesis, to cover the results of my study (e.g., Patsiomitou, 2012a). In Figure 4.8, I have incorporated into the diagram the period at which students acquire *an increasing ability to construct proofs*.
The diagram takes into account Battista's elaboration aforementioned in this section. The diagram also incorporates the diacrises in the meanings of drawing and figure, which are referred to by many researchers.
Classroom studies have shown that a van Hiele's level one (or two) student "often fails in the construction of a geometric configuration which is essential for the solution of the underlying geometric problem" (Schumann & Green, 1994, p.204). This happens because at the lower levels students are able to perceive the diagrams holistically, "they [also] recognize shapes in objects" (Gawlick, 2005, p.370). In Level 2, students are also becoming able to (or acquire an increasing ability to) "construct figures" (Gawlick, 2005, p. 370). Students' conceptual understanding has to do with their understanding of abstract ideas (Rittle-Johnson and Schneider, 2014). Pieron (1957, cited in Fischbein, 1993, p. 139) defines concepts as "*symbolic representations (almost always verbal) used in the process of abstract thinking* […]". As a student's mind moves forward to van Hiele levels, s/he is able to interlink concepts to produce a meaning. As Fischbein (1993) points out:

> "What characterizes a concept is the fact that it expresses an idea, a general, ideal representation of a class of objects, based on their common features. (p. 139) […] When you draw a certain triangle ABC on a sheet of paper in order to check some of its' properties […] you do not refer to the respective particular drawing but to a certain shape which may be the shape of an infinite class of objects (p. 141) […] all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties" (p.142).

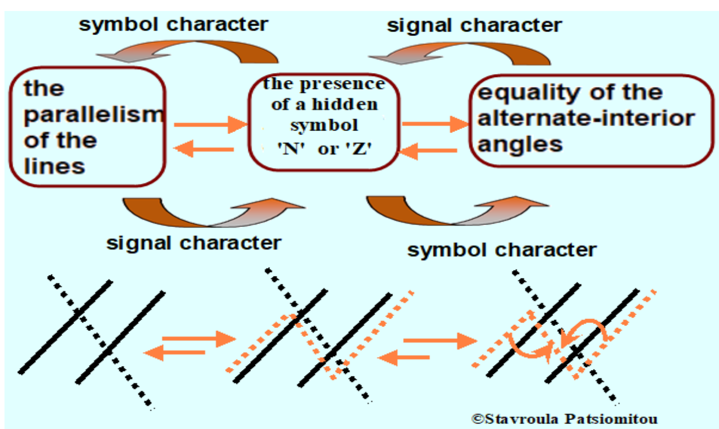


Figure 4.9. An example of a diagrammatic illustration of students' interplay between symbol and signal character (Patsiomitou, 2018b, p. 39) (modified)

Dina van Hiele (Fuys et al. 1984) explains the meanings symbol-signal with the following example: "the parallelism of the lines implies (according to their signal character) the presence of a saw, and therefore (according to their symbolic character) equality of the alternate-interior angles" (p.218).
Alternatively, the acquisition of students' signal character can be seen as their competency to *reverse reasoning in their thinking* (Patsiomitou, 2012 a, b). My students, for example, identify the letter "Z" or "N" (a hidden symbol) when they try to prove the equality of the alternate–interior angles (**Figure 4.9**). If the students have the competency to reverse their reasoning, then they have also acquired the competency to form a proof, as they have the competency to order logically their utterances (Patsiomitou, 2012a, b).

According to Vygotsky (1987), learning is a complex interplay between *scientific and spontaneous use of language*. Vygotsky (1987) argues that "the child begins to perceive the world not only through his eyes [visually] but also through speech" (p. 32). As it is mentioned by many scholars, the students during the process change the way they define the objects. For this, learning is an *internalization* of social relations and understanding is a result of common negotiation of concepts created by students while interacting with other students in the class (or group) during the mathematical discussions developed (Bartolini Bussi, 1996). Subsequently, a definition that a student formulates is an indication of his/her van Hiele level. According to Dina van Hiele (Fuys et al., 1984)

> "On reaching this third level of thinking, which we call insight into the theory of geometry, we can start studying a deductive system of propositions […]. Definitions and propositions now come within the pupils' intellectual horizon" (p.219).

Gutierrez and Jaime (1998) in their study "*On the assessment of the van Hiele levels of reasoning*" summarize "the main characteristics of the processes used to distinguish among students at the different van Hiele levels" (p.31) in the following **Figure 4.10.**

TABLE 1
Distinctive Attributes of the Processes of Reasoning in each van Hiele Level

	Level 1	Level 2	Level 3	Level 4
Recognition	Physical attributes	Mathematical properties	---------	---------
Use of definitions	---------	Only definitions with simple structure	Any definition	Accept several equivalent definitions
Formulation of definitions	List of physical properties	List of mathematical properties	Set of necessary and sufficient properties	Can prove the equivalence of definitions
Classification	Exclusive, based on physical attributes	Exclusive, based on mathematical attributes	Can move among inclusive and exclusive	---------
Proof	---------	Verification with examples	Informal logical proofs	Formal mathematical proofs

Figure 4.10. The main characteristics of students belonging at different van Hiele levels (Gutierrez and Jaime, 1998, p.31) (adapted)

In terms of geometrical figures "students can be thought of as having their own *concept images* and their personal concept *definitions* of [these] figures" (Fujita & Jones, 2007, p. 6).
Tall and Vinner (1981, p. 152) defined a *concept definition* as "a form of words used to specify that concept" and *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process" (cited in, Fujita & Jones, 2007, p. 6).
Govender & De Villiers (2003) argue that "definitions do not exist independently of human experience in some "ideal" Platonistic world, so that all we can do is to "discover" them. The fact that definitions are not discoveries, but human "inventions" for the main purpose of accurate mathematical communication is therefore not addressed" (p.42).
Govender & De Villiers (e.g., 2003, p. 46) clarified students' definitions as follows:

- *"Arbitrary definition*: a different, alternative but correct definition for the same concept.
- *Necessary and Sufficient definition*: It contains enough information […] and only those elements of the set we want to define.
- *Correct definitions:* A description (definition) which contains conditions (properties) that are sufficient is said to be correct.[…]
- *Incorrect definitions:* A definition is incorrect if it contains an incorrect property or if it contains insufficient properties.
- *Incomplete* definitions: It contains insufficient and incorrect properties

- *Economical* definitions (and *uneconomical definitions*): It has only necessary and sufficient properties.

For the use of my study I defined two more kinds of definitions students use (Patsiomitou, 2012a, 2013a, p.802):

- *Arbitrary and economical definition:* is a definition which is a synthesis of arbitrary (a different, alternative but correct definition for the same concept) and simultaneously it has only necessary and sufficient properties.
- *Dynamic perceptual definition:* refers to the term by which the student informally 'defines' a geometrical object by using the tools of the software. The use of computer software can effectively support the student's progression through van Hiele levels.

The introduction of DGS and computers generally into the teaching and learning of geometry has led researchers, educators and psychologists to incorporate these tools into their investigations in order to examine how they can support reasoning and raise a student's van Hiele level.

Olive (2000) emphasizes the need to use DGS in the teaching of secondary mathematics:

> "*At the secondary level dynamic geometry environments can (and should) completely transform the teaching and learning of mathematics. Dynamic geometry turns mathematics into a laboratory science rather than the game of mental gymnastics, dominated by computation and symbolic manipulation, that it has become in many of our secondary schools. As a laboratory science, mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanations for the phenomena*." (p. 17)

Gawlick (2005) similarly, argues that "there is a need to further develop these levels – and to utilize DGS for this." (p. 361). Gawlick (2005) has conducted investigations using DGS. He introduced through his experiments a correspondence among the use of the DGS tools and the development of students' van Hiele level. According to Gawlick (2005):

1. "The *drag mode* is a key tool to advance from level 1 to level 2".
2. "*Macros and loci* suit to support the step from level 2 to level 3".
3. "*Families of loci* can be used to progress from level 3 to level 4". (p. 365).

According to Gawlick (2005) the characteristics of the five van Hiele levels are the following (p. 362):

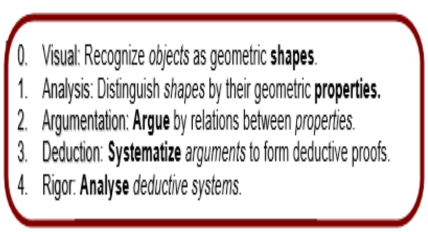


Figure 4.11a. The characteristics of the five van Hiele levels (Gawlick, 2005, p.362)

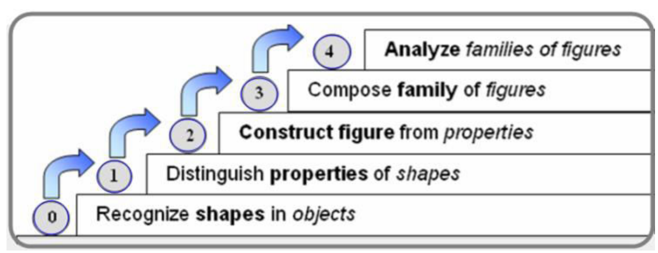


Figure 4.11b. Gawlick's interpretation of van Hiele levels (2005, p.370)

Gawlick depicted his reinterpretation of van Hiele levels, in which a student who receives scaffolding instruction moves to the next step up. He argues that "dynamic manipulations help students to transit from the first to the second van Hiele level" (p.361). Gawlick adopted Freudenthal's (1973) view of geometry who "viewed progressive mathematization as the main goal of school mathematics. For this ongoing task, he provided a framework by recursively defined levels: The activity of the lower level, that is the organizing activity by the means of this level, becomes an object of analysis on the higher level" (p. 362). As Gawlick supports

> "*Progression through these levels will not occur all by itself, but needs to be triggered by giving the students suitable tasks that really afford the building of new concepts*" (p. 362).

Gawlick (2005, p.370) argues that a *dynamic* approach is better suited to developing thinking at an advanced level on two counts: Firstly, tasks prepared for lower levels can be continued at higher levels, which helps familiarize students to the habit of 'discovery'. Secondly, it provides a solid basis for the van Hiele *phases of learning* to come, since it allows students to *explore* the topic in a directed orientation phase and then use their *existing knowledge* to *build the new concepts* for themselves. Level 3 (deduction) is identified "as the level at which the students construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions" (Gawlick, 2005, p.370).

The reversion of thinking is developed and facilitated from the use of DGS tools (Patsiomitou, 2012a, b; 2018b). In order to emphasize my argument, I shall incorporate in the paragraphs that follow, an experiment that is described at length in my study "*An 'alive' DGS tool for students' cognitive development*" (Patsiomitou, 2018b). The excerpt relates to an episode midway through the second phase, at which time students were working on tasks involving symmetry and transformations. What is described here lasted almost 30 minutes. I have reported the importance of the use of the custom tools in many previous studies (e.g., Patsiomitou, 2005a, 2006d, e, g (in Greek), 2007a, b, 2008a, d, 2012a, b, 2014). For my study, I used a custom tool I had previously created to help students visualize the meaning of central symmetry in correlation with the meaning of a segment's midpoint. This was very crucial for the evolution of the construction of a parallelogram through its diagonals.

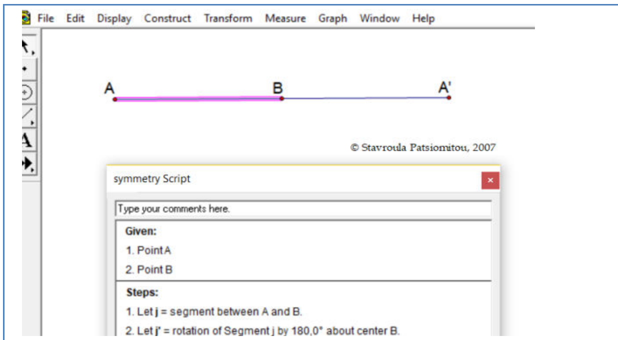


Figure 4.12a. Implementing the custom tool "symmetry" (Patsiomitou, 2018b, p.44)

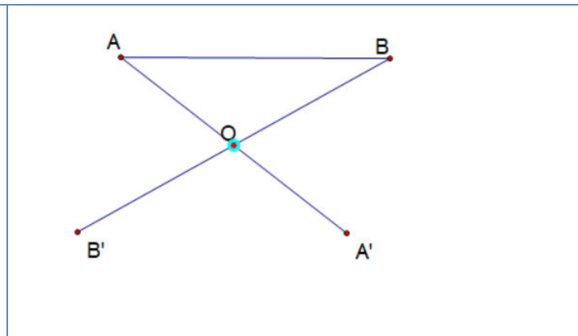


Figure 4.12b. The "X" utilization scheme of the custom tool (Patsiomitou, 2018b, p.45)

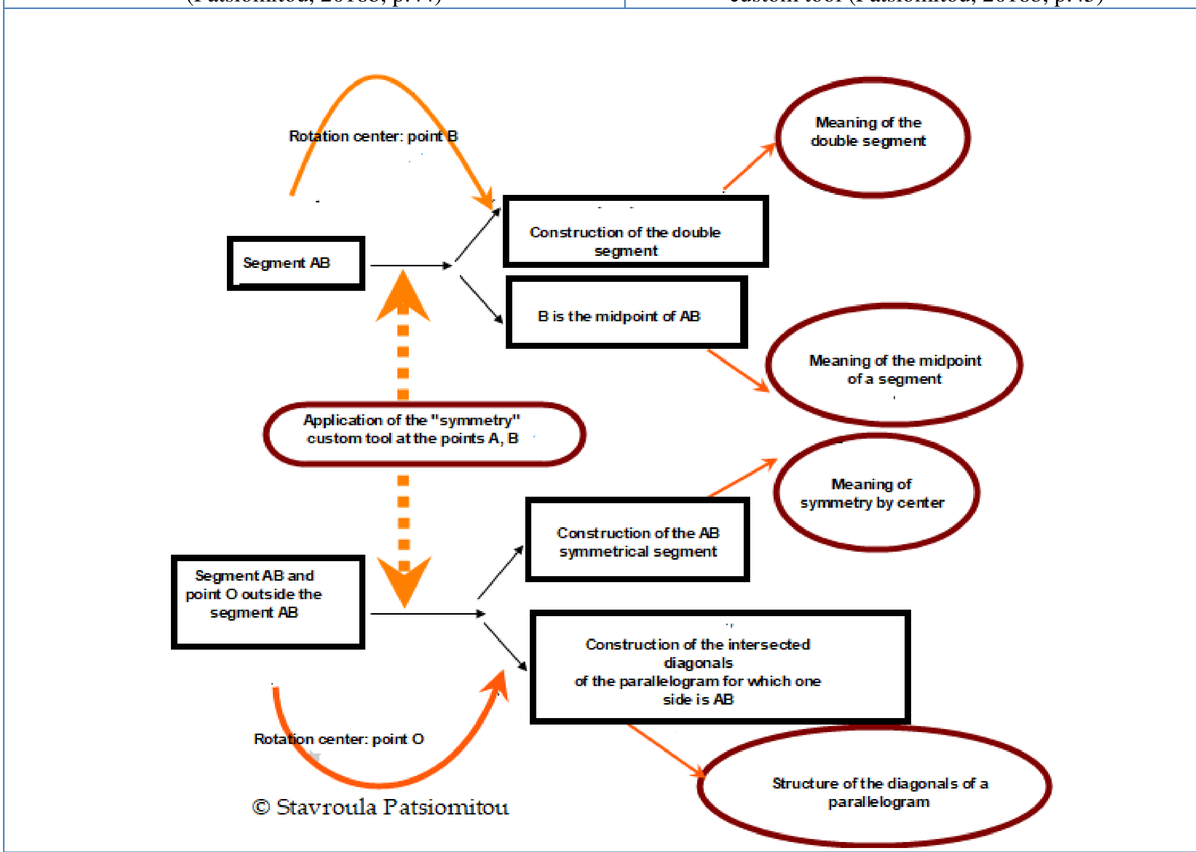


Figure 4.12c. Analysis of the use of the custom tool "symmetry" (Patsiomitou, 2014, p. 20)

The construction of the tool is very simple (**Figure 4.12a**), and has crucial effect on the development of the students' thinking (Patsiomitou, 2012a, b). The idea of creating the concrete custom tool occurred after creating a similar tool to construct the "golden ratio" (Patsiomitou, 2006g, p.61, in Greek). The problem I posed the students was this: *Can you construct a rectangle using the properties of its diagonals?*

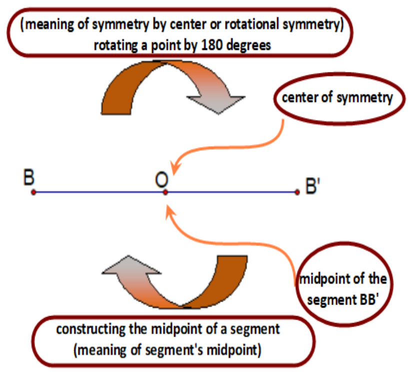


Figure 4.13. A dual role: midpoint and/or symmetry by center (Patsiomitou, 2012a, b, 2018b)

If my students implemented it on screen, they could view a segment with its midpoint. This tool helped them connect the meaning of symmetry by center with the meaning of a segment's midpoint. If they applied it twice on a point F, they visualized an "X" symbol which students view when constructing *the diagonals of a parallelogram*. **Figure 4.12a** illustrates an implementation of the custom tool once on screen. In **Figure 4.12b**, I implemented the tool twice on point F. **Figure 4.13c** illustrates an analysis of the use of the custom tool "symmetry" for the construction of meanings.
The modification of the angle between the segments (e.g., AA΄ and BB΄) as well as the lengths of the segments determines the kind of parallelogram which is produced/ generated (e.g., Patsiomitou, 2012a, b, p.72) (**Figure 4.14**).

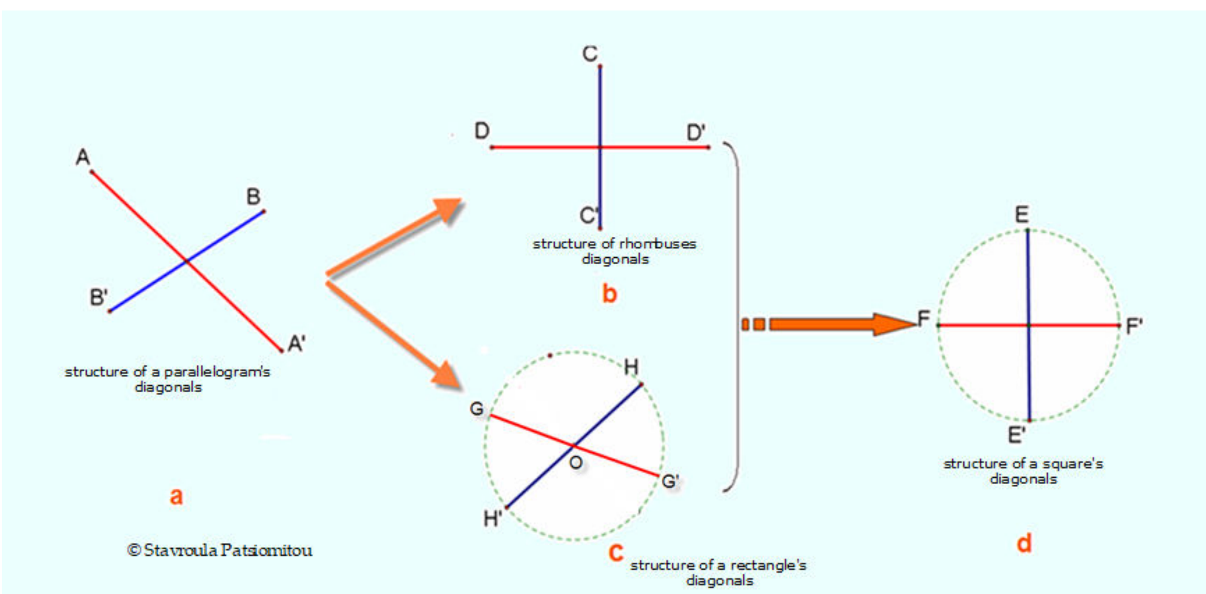


Figure 4.14: Structure of parallelogram's diagonals (modified from Patsiomitou, 2012 a, b, p. 72)

Firstly, a student-user assimilates the meaning incorporated in the use of the tool into his preexisting knowledge (for example s/he connects the meaning of the symmetry by center with the meaning of the segment's midpoint). S/he may then face an obstacle (an *instrumental obstacle*) (Patsiomitou, 2011a, p. 362) with regard to the use of the tool*s,* due to student lack of competence in *instrumental decoding*. For example, the tool cannot be applied on a segment to find its midpoint. This occurs because I created the tool with concrete properties (**Figure 4.12c**) *to incorporate the meaning of rotating a point by 180 degrees*. This assumption generates a *cognitive conflict* in the student. On the other hand the student discovers new ways to use the tool according to his/her thought development. This in accordance with what Steffe & Olive (1996), Olive (1999), Olive & Steffe (2002), Olive et al. (2010) state: the mathematical knowledge which children build up during their engagement in a mathematical activity, is distinguished among others to

‘children’s mathematics – the mathematics that children […] construct for themselves and is available to them as they engage in mathematical activity’;
‘mathematics for children – the mathematical activities that curriculum developers/writers and teachers design to engage students in meaningful mathematical activity’ (Olive & Makar, 2010, p.136)

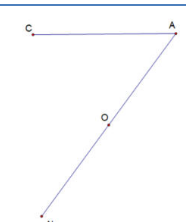
4.2.4 Are Custom Tools a Means for the Development of Students’ Thinking?

In my study “*An ‘alive’ DGS tool for students’ cognitive development”* (Patsiomitou, 2018b), I present the impact of the custom tool on students' thinking, as well as the development of their abstract thinking, the recognition of instrumented action schemes through the emergence of theorems and concepts-in-action and the verbalizing of concepts during the process. In the field notes mentioned below the investigation process is described (Patsiomitou, 2018b, p.45-48).

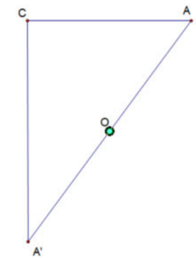
Fieldnote 1: The students [M15 is a male student (van Hiele level: 2) and M16 is a female student (van Hiele level: 1)] constructed a parallelogram using the scaffolding effect provided by the tool. This point in the research is quite similar to other situations I faced in my previous studies with different pairs of students. The students faced a cognitive conflict because they could not use the terminology accurately. Most of them confused the meaning of angle bisector (‘dichotomos’ in Greek) with the meaning of ‘diagonal’. This confusion did not help them when they had to solve a problem, because, while the diagonals do also dichotomize the angles of the vertexes in a few quadrilaterals (i.e. rhombus, square), this is not the case in other quadrilaterals (i.e. parallelogram, rectangle, and trapezium). This confusion grew during the construction of a figure -parallelogram. Moreover, the students have to differentiate the angle bisector of an angle from an angle bisector of a triangle (to an angle bisector of a parallelogram). M15 can recognize and name properties of the parallelogram, but he still does not see relationships between these properties (Mason, 1998, level 2). In the concrete case M15 defines the object with a *dynamic and economical definition*. This is a sign that the student is moving to the van Hiele level 3. M16 makes decision based on perception. She recalls the structure of a parallelogram’s diagonals. M16 recognizes a property of the parallelogram from the ‘alive’ [*active*] *representation* on screen. M16’s pretest level was 1; this is clear from her answers, as she makes decisions based on perception.

[1]M16: *A parallelogram .What kind of parallelogram?* M16 constructed two intersected segments using the custom tool. *Then we shall join these. [sides] …but,… it is not a parallelogram!*
[2] Researcher: *What are the prerequisites for a quadrilateral to be a parallelogram?*
[3] M15: *The opposite sides must be congruent; the diagonals must be dichotomized…..*
[4] Researcher: *What can you view in the current situation? Do these segments dichotomize each other?*
[5] M15: *Yes!*
[6] M16: *They are congruent! (She moves the figure using dragging.)*
[7] M16: *They are congruent! It is a parallelogram! (She meant the half segments of a parallelogram’s diagonal).*
[8] Researcher: *ok…it is a parallelogram …Can you construct a rectangle?*
[9] M15: *Well,…an angle bisector … (pointing to a diagonal)*
[10] Researcher: *Diagonal, you mean!*
[11] M15*: Yes …they must be dichotomized and…. they must be congruent.*
[12] Researcher: *Correct both! Can you construct it?*

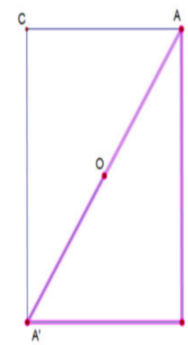
[13] M15 constructed a segment with the custom tool trying to visualize as a diagonal of a rectangle …He stopped and looked at it on screen.
[14] M15: *I shall construct it as we constructed the parallelogram.*
[15] Researcher: *What should the rectangle’s diagonals be?*
[16] M15: *Congruent …I shall construct a segment with the tool...*
[17] Researcher: *So, how can you construct a diagonal equal to this one?*



[18] M15: *I shall rotate it.*
[19] M16**:** *Construct a point …not on the segment! …. Choose it and rotate the segment…*
[20] M16: *We should have 90 degrees…*
[21] M15: *Yes! I got it!*
[22] M16: *Let's draw a straight line.*
[23] M15: *We can construct a straight line …we shall construct its midpoint (it looks like he wants to apply the custom tool to find the midpoint of the segment).*
24] M15 selects the segment and its endpoints and tries to construct the midpoint from the menu.
[25] M16: *Why are you doing this? The tool (meaning the custom tool) can construct the midpoint.*



[26] M15: *Eureka! I shall construct parallels from these points*
[27]M15: *I shall join these two points.*
[28] M15: *Then I shall construct the symmetrical triangle by 180 degrees (Figure 4.13 a, b, c)*
[29] M15 selected the midpoint and constructed a rotational symmetry of the triangle.
[30] M15: *Ok! It is readyyyy!*
[31] M16: *Is it a rectangle? Drag this point.*
[32] M16: *Choose a vertex to drag!*
[33] M16: *It is a very nice parallelogram! (laughing) …but you went to Trikala and back when you were constructing it (a Greek expression for when a person follows a less than easy and obvious path when carrying out a task).*



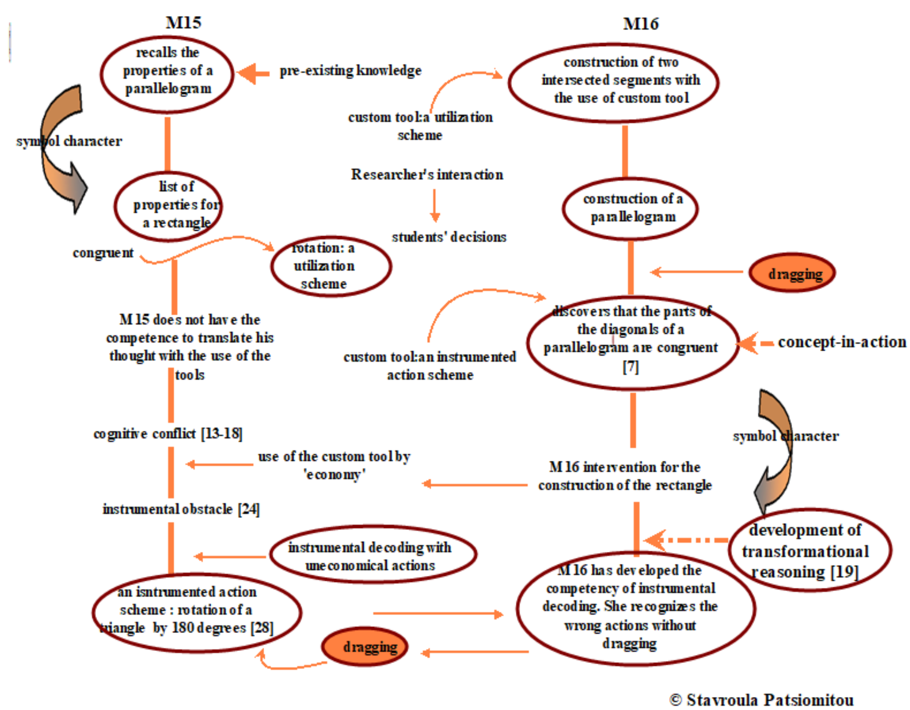
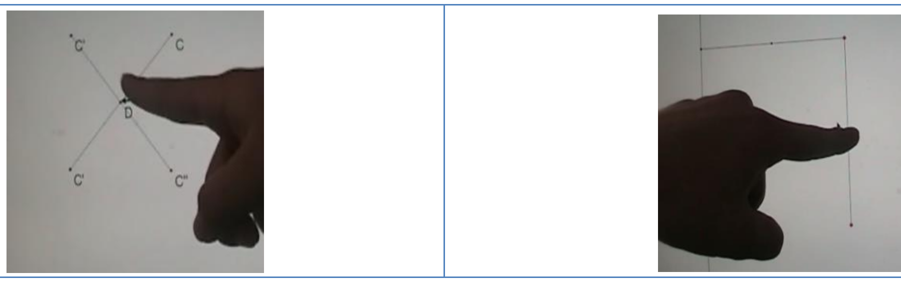


Figure 4.15. Analysis of students' thought through the use of the tools, in a pseudo-Toulmin diagram (Patsiomitou, 2018b, p.46)

Fieldnote 2: M15 started with the construction of a segment using the custom tool. Then he constructed a segment AC and joined the point C with the point A΄. He tried CA΄ to seem vertical to CA. Then he rotated the triangle CAA΄ by 180 degrees. His construction of the parallelogram is complex (Figure 4.15). M15 knows the properties of the figure "rectangle", but cannot implement them to construct it. He cannot "*instrumentally decode his words to a figure on screen*" (Patsiomitou, 2011a, b, 2012a, b). He had to bring a perpendicular line down to the segment CA. He was familiar with the procedure for constructing a perpendicular line to a point on a segment, but he did not use it. On the other hand, he constructs a "parallelogram" figure using a *reconfiguration* of a triangle. The rotation of the triangle by 180 degrees could be the definition of a parallelogram when we use rotational transformation. M15 uses a combination of informal and formal descriptions of shapes (Level 2.2. according to Battista's classification). He knows that the rotated segments are congruent [point of the dialogue 18]. M15 is beginning to acquire formal conceptualizations that can be used to "see" and describe spatial relationships between parts of shapes. M16 is trying to use the tool in a *catachresis mode*, as she has extended the properties of the tool in her mind. The [alive] tool has affected her thoughts, as she has constructed an *instrumented action scheme* [point of the dialogue 25] (although she is trying to use the tool with catachresis of its use).

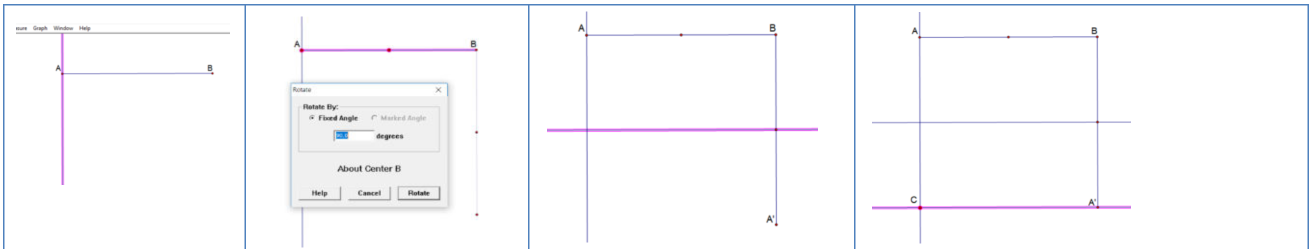
[34] M16: *Giiive me the mouse (laughing)…we shall construct a line …**we shall rotate this point by 180 degrees**.*
[35] M15: *This is a parallelogram again.*
[36] M16: *But …its diagonals are congruent!*
[37] M15: *Why? …you can measure them …drag them now…*
[38] Researcher: *How can you construct a segment congruent to this one?*
[39] Both: *we can rotate it …or reflect it …They will be symmetrical.*
[40] M16: *Eureka! We can do it!*
[41] M16: *We shall select this point (means the midpoint) …we shall select this endpoint and we shall rotate it by 90 degrees.*



Figures 4.16a, b. Students' gestures during the research process (capturing images from the video) (Patsiomitou, 2018b, p.46)

Fieldnote 3: M16 rotates point A through 90 degrees. She then uses the custom tool, applying it to points A´ and O. She insisted that the diagonals are congruent (point [45]), but as M15 was not convinced by the dragging facility, she measured the segments and dragged them again using a combination of transformations. She ended up constructing a square when trying to construct a rectangle, as during the *instrumental decoding* she constructed a point A´ in a concrete position (A´O = OA and A´O is perpendicular to AO). The most important conceptual event occurs ([49]) when she expresses a logical hierarchy regarding the inclusion of the rectangle and the square (**Figures 4.16, a, b**).
[42] M16: *Now I shall do it with the ease way …*
[43] M16 selects the custom tool and applies it to the point and to the midpoint.
[44] M15: *Againnnn, it is a parallelogram!*
[45] M16: *Why? Its diagonals are congruent!*
[46] They select them and measure them.
[47] M16: *It is a rectangle!*
[48] Researcher: *What is it? Drag all the vertexes!*
[49] M16: *…may be it is a square … but the square is also a rectangle...so it is ok! I constructed it!*
[50] M15: ***The square is a rectangle**??? What does she say?*
[51] I did not explain or mention why the square is also a rectangle, but posed one more question.
[52] Researcher*: Can you construct a rectangle? Not a square.*
[53] M15: *I can do it!*
[54] M15 constructs a segment AB. [55] M15: *Now we shall construct a perpendicular to this point (point A).*
[56] M15 then constructs the midpoint of the segment. *…I shall rotate only the half segment by 90 degrees… Oh, eureka!!*
M15 rotates the whole segment AB about center B by 90 degrees.

[57] M16: *You have constructed a square again!*
[58] M15: *No!*
[59] M16: *Yesss! This segment is congruent to this segment!*
[60] M15: *Ok! We shall construct a parallel line from this point (A΄). This will be a rectangle…*
[61] M16: *This is a square as all its sides are congruent and perpendicular (she means to one another)* (**Figures 4.17a, b, c, d, 4.18**)



Figures 4.17a, b, c, d. Sequential steps of the construction

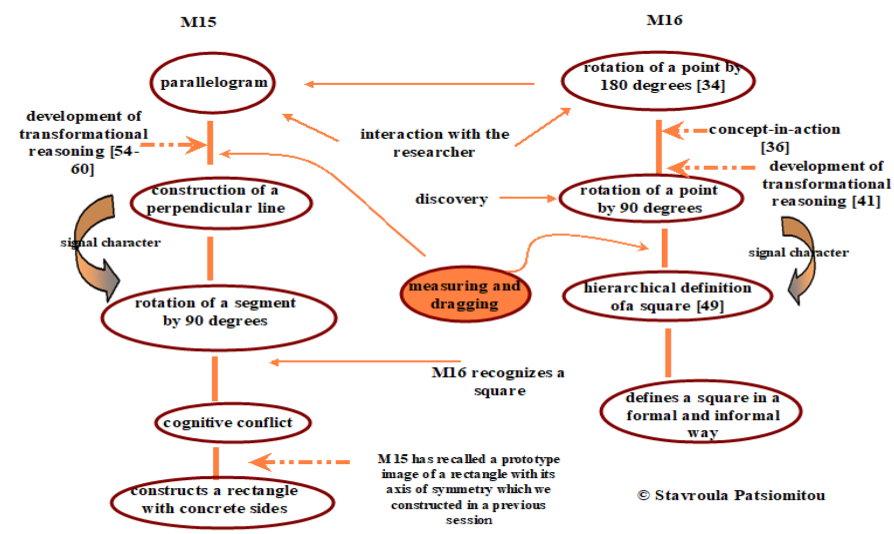


Figure 4.18. An illustration of the use of the tools in a pseudo-Toulmin diagram (Patsiomitou, 2018b, p.48)

Fieldnote 4: M15 tried to construct a rectangle. He has recalled a prototype image of a rectangle with its axis of symmetry which we constructed in a previous session. He ultimately constructed a rectangle whose side is half the length of the side of the square ABA΄C (**Figure 4.18**). He is in transition to Level 3, but still lacks the competency to *instrumentally decode* a figure. M16 did not delete all the lines. She had something in mind while M15 constructs his specialized kind of rectangle. She was not sure about the next step, but no one could take the mouse from her hand. She implied that BC΄ is a perpendicular line, as she constructed point C΄ by rotating point C, and she implied that CC΄΄ is perpendicular to CA. She did not prove the sequential steps using deductive reasoning, but the construction steps she follows is an indicative of the development of abstract thinking. M16 developed what Simon (1996) calls *transformational reasoning*. What is transformational reasoning? In the words of Simon (1996):

> "*Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generate*" (p. 201)

Fieldnote 5: M16's conception of the meaning of the rectangle [62-67 of the dialogue] and the rectangle's *instrumental decoding* was ***the most incredible I have ever seen*** a student display when using the concrete tool (**Figure 4.19**). While a concept is an idea shared and accepted by the mathematical community, a student's conception refers to a student's explanation of a concrete concept. In other words, it relates to with the way the student shapes the idea in his/her mind. M16 made many transformations in her mind in order to construct the rectangle. She constructed an arbitrary point C and she rotated it by 180 degrees. She implied the congruence of the triangles CAM, MBC′, and subsequently the congruency of the segments C′B, CA. In order to construct a segment equal to the segment C′B, she used the tool. In other words, she constructed a conceptual object in her mind in which she encapsulated the properties of the tool. The implementation of the tool once again to construct the diagonal C′′C′′′is a strong indication that she was absolutely sure the diagonals of the rectangle would be congruent. She used the tool appropriately and efficiently (not with economy or catachrese). Moreover, she displays sequential place-way and verbal competency when using the tools. All these are strong indications that she has developed abstract thinking.

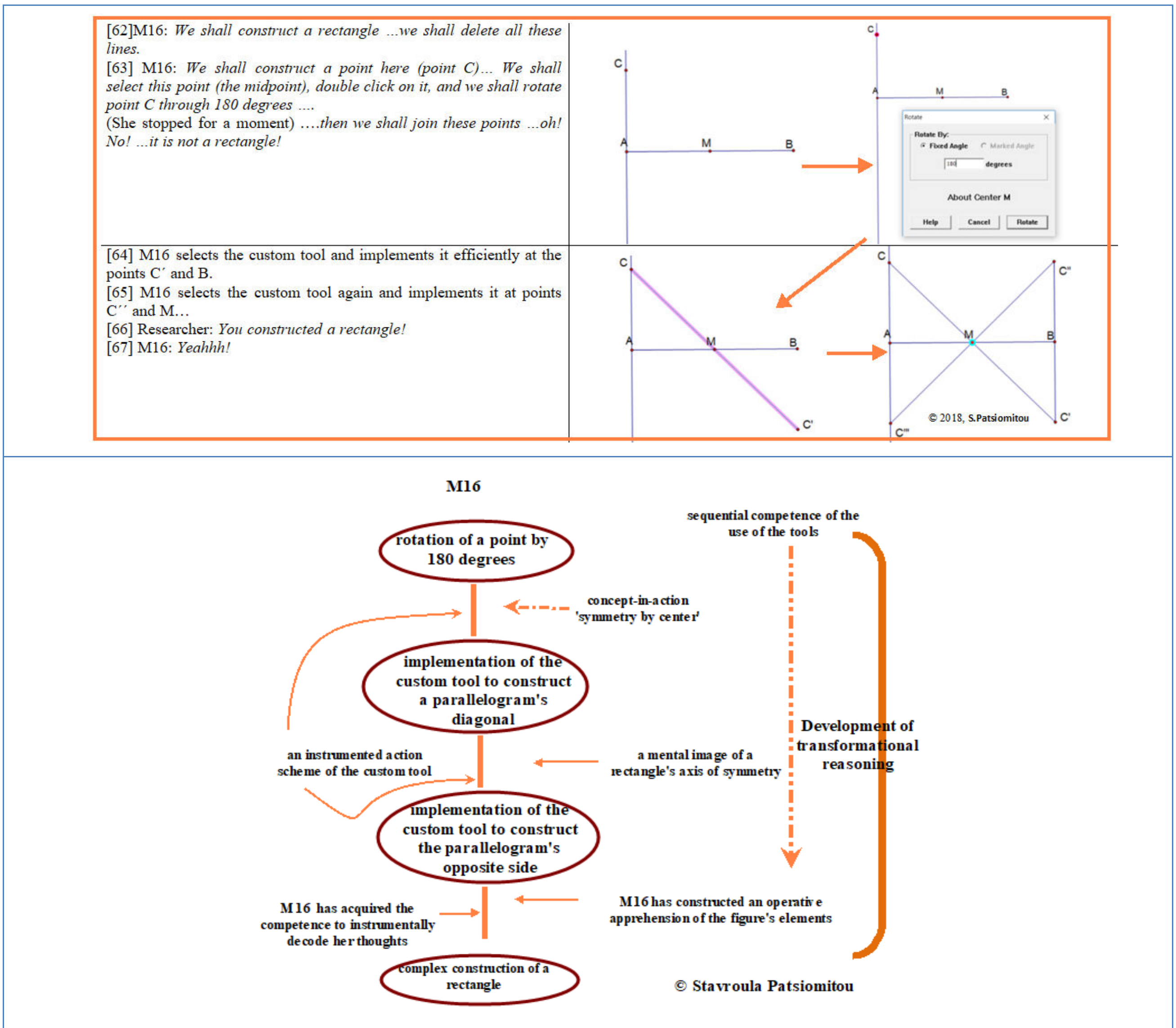


Figure 4.19. M16's analysis of the construction in in a pseudo-Toulmin diagram (Patsiomitou, 2018b, p.48)

Regarding my interaction with the students, I think it was the necessary for the students to move on during the process. As Burkhardt (1988) notes,

> "[…] the teachers must perceive the implications of the students' different approaches, whether they may be fruitful and, if not, what might make them so. pedagogically [also] the teacher must decide when to intervene, and what suggestions will help the students while leaving the solution essentially in their hands, and carry this through for each student, or group of students, in the class" (Burkhardt, 1988, p. 18).

When analyzing the students' dialogues, I used the meanings I introduced in my description of the theoretical underpinning. Duval's (1999) theory views students' perceptual apprehension as complementary to Vergnaud's

(1998) theory of operational invariants in the context of a process of instrumental genesis. This is in accordance with what Fou-Lai Lin & Kai-Lin Yang (2002) support:

> "While Duval's *cognitive architecture*, an organization of several systems, put emphasis on multifunctional registers, Vergnaud's cognitive theory of practice put emphasis on the mechanism of *conceptual field*. Their perspectives on cognition seemed complementary for analyzing how subjects developed definitions and propositions of geometrical figures. Duval supported us a framework of perceptual categories to describe conversion and coordination between different registers, and Vergnaud supported us a framework of mental organization to explain cognitive mechanisms" (p. 20).

During the research process, students applied the tools and constructed what Rabardel (1995) calls *utilization schemes* of the tool/artefact. This process led to the development of *schemes of instrumented action*. The students assimilate the figures properties and are in the stage to perceive and accommodate the interrelationships between the properties of the figures. The next step is to use deductive processes and understand class inclusions. This result will occur when the students have transformed the figures' symbol character into figures' signal character—a transformation that corresponds to the third level of geometric reasoning. Moreover, the students used the tools efficiently or in an economical (/ catachresis) mode. They constructed schemes of instrumented action as a result of the efficiently use of the custom tool or its use in an economical mode. They extended its use in a *catachresis mode* to construct the midpoint of a segment. This served for the construction of meaningful mental schemes to solve the problem. Consequently, the 'instrumented action' scheme, which is based on the construction and use of the custom tool, led students to construct mental objects. In the sequence of mental activities the students followed, mathematical knowledge and knowledge of the tool were combined. They constructed a *first order instrumented action scheme* and shaped the meaning "symmetry of point by 180 degrees", then *a second order instrumented action scheme* and shaped the meaning "the diagonals of parallelogram are dichotomized". According to Drijvers & Trouche (2008)

> The difference between *elementary usage schemes* and *higher–order instrumented action schemes* is not always obvious. Sometimes, it is merely a matter of the level of the user and the level of observation: what at first may seem an instrumented action scheme for a particular user, may later act as a building block in the genesis of a higher-order scheme. […] a utilization scheme involves an interplay between acting and thinking, and that it integrates machine techniques and mental concepts […] the conceptual part of utilization schemes, includes both mathematical objects and insight into the 'mathematics of the machine'(p. 372)

The use of a combination of transformations using dragging (and measuring or the rotating and/or implementing custom tools) helped them to shape the figural concept first of "the parallelogram", then of "the square", and finally of "the rectangle". The implementation of the custom tool helped students to shape a schematic entity in terms of their perceptions, and then led them through various stages to more abstract levels of cognitive perception. This also agrees with Edelman's viewpoint (1989/1992): "in forming concepts,...the brain areas responsible for concept formation contain structures that categorize, discriminate, and recombine the various brain activities occurring in different kinds of global mappings" (quoted in Davis & Tall, 2002).

This means that custom tools can serve as *structural units of knowledge*, as *conceptual objects* and hence as '*schemes'*, too, including the structure and function of the encapsulated objects (e.g., Patsiomitou, 2008d). In my study "*An 'alive' DGS tool for students' cognitive development"* (Patsiomitou, 2018b), I conclude that the participated students M15, M16 developed efficient strategies to use the DGS tools. M15's actions are the reverse of the actions he used to construct the axis of symmetry of a rectangle. He then constructed the symbol character of the rectangle. He did not make a rectangle with arbitrary sides, but rather a concrete rectangle. This is an indication that he is in transition to level 3, as he had not constructed the signal character of the rectangle. M16 has developed the competency to reverse her thoughts through the competency to make complex use of the tools to instrumentally decode the properties of the figures. The symbol character of the figures reflects in her thought. She has constructed the interrelationship between the meanings of the "parallel line in the middle of the distance of two parallel lines" with the meaning of axis symmetry and the meaning of the congruency of the diagonals of a rectangle. She does not express her thoughts in words, but she has been sufficiently developed the rectangle's and square's signal character. In the current study, the participating students constructed: (a) the utilization scheme of the symmetry by center in correspondence with the midpoint of a segment; (b) the "X" utilization scheme of the custom tool, which was very important for the construction of a broader scheme, namely the instrumented action scheme of "the diagonals of a parallelogram". In Gawlick's opinion (2005) in a dynamic approach "the students can explore the topic in a directed orientation phase and then build the new concepts for themselves, drawing on their previous knowledge"[...] "so students get accustomed to the tools as well as to a "discoverer's" habit of mind"(p.370). As Pierre van Hiele writes (1986, p. 168) a figure gets the "symbol character" when it becomes the representative of its properties as a unity. In my opinion, when the student is able to reverse his/her thoughts and to anticipate the symbol of the figure, then the figure has received its signal character. The student can now list the similarities and differences between figures. S/he can also explain why a characteristic is not included in figures' characteristics

Many researchers (for example Goos, Galbraith & Renshaw, 2002; Dekker & Elshout-Mohr, 2004) recognise the "potential of working in small groups" (Dekker & Elshout-Mohr, 2004, p. 39). Moreover, the mathematical discourses developed in a small group mediated by cognitive tools such as the Geometer's Sketchpad enhance the social interactions and students' mathematical communication. According to Sfard (2001) "most of our *learning is nothing else than a special kind of social interaction aimed at modification of other social interactions*. […] Thus, whatever the topic of learning, the teacher's task is to modify and exchange the existing discourse rather than to create a new one form scratch. If so, we can define learning as the *process of changing one's discursive ways in a certain well-defined manner*." (p.3)

Sang Sook Choi-Koh (1999) investigated the development of students' thinking, using The Geometer's Sketchpad software. In his PhD thesis he identified four learning stages in terms of symbol, signal and "implicatory" properties. He also used "active visualization", meaning "the process of forming and interpreting geometric, dynamic representations within a computer environment" (1999, p. 302). **Figure 4.20** depicts Choi-Koh's van Hiele visual model of instruction.

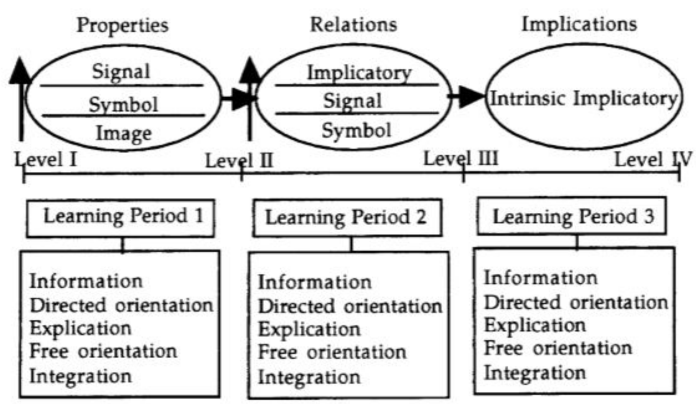


Figure 4.20. Choi-Koh's (1999, p.302) van Hiele visual model of instruction

Figure 4.21 illustrates an adaptation of the van Hiele model, which I created in relation to Choi-Koh's (2001) and Battista's (2007) levels of thinking, through the use of "*active, alive tools*" (Patsiomitou, 2018b). To clarify, when a student interacts with figural materials (for example a digital figure in a DGS environment), s/he interacts with the figure's characteristics: the equality of a square's sides and angles, the perpendicularity of a kite's diagonals, etc. Now s/he has in his/her mind which of these characteristics determine the concrete figure. During the second period of instruction s/he acquires a gradual competency to construct figures and during the third period of instruction the students are able to gradually construct proofs. In other words, this will be a change in a student's informal discursive way to express his or her thoughts in formal language.

In such a discursive process the students play the role of the 'actor' in the activity of the mathematical discussion and the teacher the role of the participated 'observer", who frequently intervenes with crucial questions designed to prompt mathematical discussion. Freudenthal (1991) "criticized the constructivist epistemology from an observer's point of view" [and] "saw mathematics from an actor's point of view" (Gravemeijer & Terwel, 2000, p.785). Which is to say, constructing meaningful activities for the students by imagining how the students might interact with the instructional materials, what obstacles they had to overcome, the possible (or multiple) solutions they could find, how their thinking could be raised due to the evolution of mathematical discussions they participate in. This is in accordance with what Freudenthal argues that "*doing mathematics is more important than mathematics as a ready-made product*" (Gravemeijer & Terwel, 2000, p.780) Building on a theoretical perspective of learning, Bowers & Stephens (2011) support that first, if learning is viewed as a socially situated practice, then (a) teaching can be seen as the practice of *orchestrating mathematical discourses* and (b) learning can be seen as the ways in which students engage in these discourses. In short, the role of any teacher (or teacher educator) can be seen as negotiating the emergence of conceptual discourse that involves the use of appropriate tools [...] The role of the student is also intricately related to his or her participation in the discourse with a focus on the ways in which tools mediate the discussions and acceptable ways of proffering and debating mathematical ideas. (p. 287)

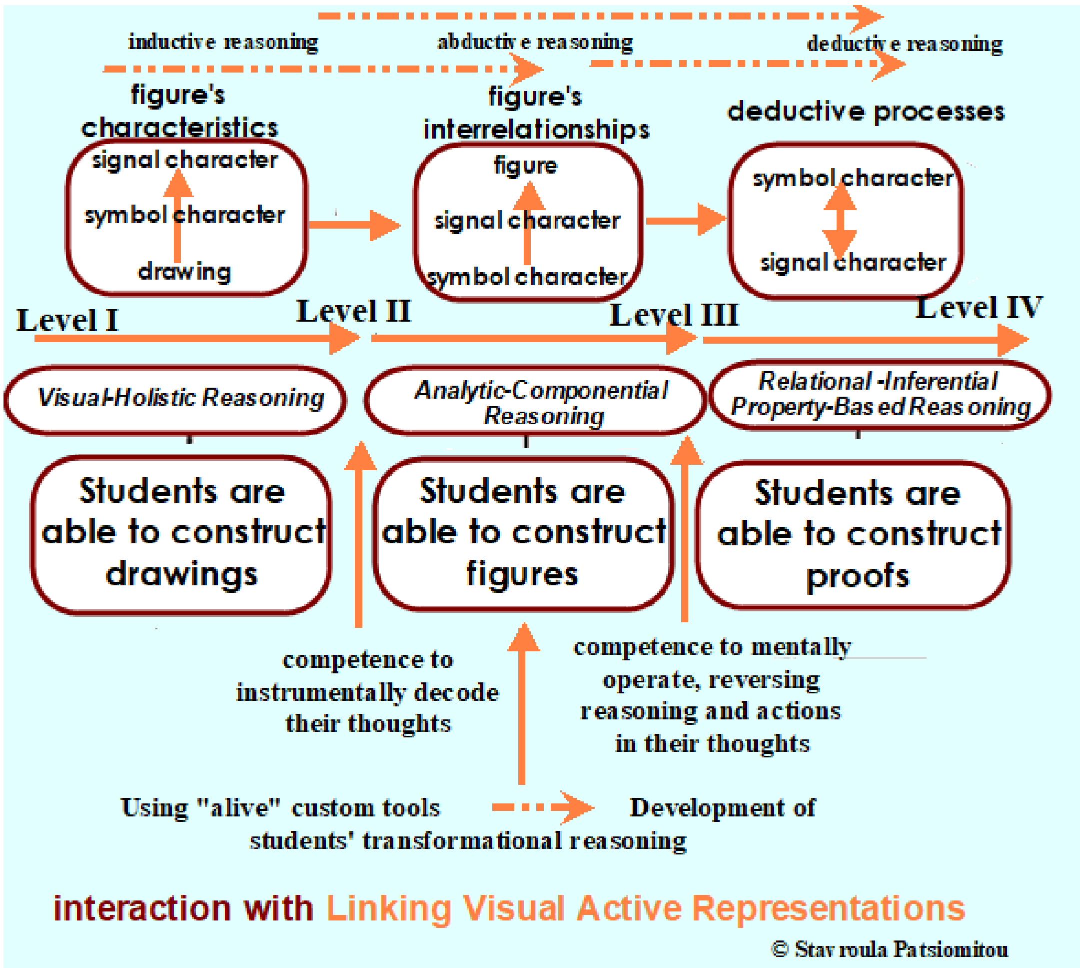


Figure 4.21. An adaptation of the van Hiele model (Patsiomitou, 2018b, p.50) (modified)

Building on the ideas mentioned above I think that *dynamic reinvention of knowledge* is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment, “knowledge for which they themselves are responsible” (Gravemeijer & Terwel, ibid.)

4.3. The Development of Student’s Mathematical Competencies

Another point of view suggests that the development of student’s geometrical thinking results from the development of their skills (Hoffer, 1981) or competencies in mathematical thinking and reasoning, argumentation, modeling e.g., (Niss, 1999) etc. Hoffer (1981) proposed the following types of skills, reported by Morris (1986, p. 162-163) and Abdefatah (2010, p.46). (**Figures 4.22, 4.23**)

- *"Visual skills*, including the ability to: recognize various plane and space figures; observe parts of a given figure and their interrelations; identify centres, axes, and planes of symmetry of a given figure; classify given figures by their observable characteristics; deduce further information from visual observations; and visualize the geometric representations (models), or counter-examples, which are implied by given data in a given deductive mathematical system.
- *Verbal skills*, including the ability to: identify various figures by name; visualize figures from verbal descriptions of them; describe given figures and their properties; formulate proper definitions of the words used; describe relationships among given figures; recognize the logical structure of verbal problems; and formulate statements of generalizations and of abstractions; correct use of terminology and accurate communication in describing spatial concepts and relationships.
- *Drawing skills*, including the ability to: sketch given figures and label specified points; sketch figures from their verbal descriptions; draw or construct figures with given properties; construct figures having a specified relation to given figures; sketch plane sections and intersections of given figures; add useful auxiliary elements to a figure; recognize the role (and limitations) of sketches and constructed figures; and sketch or construct geometric models or counter-examples; communicating through drawing, ability to represent geometric shapes in 2-d and 3-d, to make scale diagrams, sketch isometric figures.
- *Logical skills*, including the ability to: recognize differences and similarities among given figures; recognize that figures can be classified by their properties; determine whether or not a given figure belongs to a specified class; understand and apply the desirable properties of definitions; identify the logical consequences of given data; develop logical proofs; and recognize the role and limitations of deductive methods; classification, recognition of essential properties as criteria, discerning patterns, formulating and testing hypotheses, making inferences, using counter-examples.
- *Applied skills*, including the ability to: recognize physical models of geometric figures; sketch or construct geometric models of physical objects; use properties of geometric models to conjecture properties of physical objects or sets of physical objects; recognize the usefulness of geometric models for physical objects or situations; develop geometric models for natural phenomena, sets of elements in the physical sciences and sets of elements in the social sciences; and use geometric models in problem solving; real-life applications using geometric results learnt and real uses of geometry e.g. for designing packages etc". (Hoffer, 1981 cited in Robert Morris, 1986, p. 162-163)

Figure 4.22. Hoffer's (1981) types of skills (Morris, 1986, p. 162-163) (adapted)

Hoffer's (1981) matrix of geometric thinking levels and geometric skills

LEVELS / SKILLS	RECOGNITION	ANALYSIS	DEDUCTION
VISUAL	Recognize geometric constructions by its picture [...]	Recognize the relationship between different kinds of geometric constructions	Uses information about a geometric construction to deduce more information
DESCRIPTIVE	Naming a geometric construction and [...] describe geometric construction	Defines geometric concepts clearly. Describes the relationships between geometric constructions.	Understand the difference between the definition, postulate and theorem.
LOGICAL	Understand the meaning of construction reservation in different situation	Uses the properties of geometric constructions to identify the subset relations	Uses logic to prove and being able to deduce new knowledge from given facts.

Figure 4.23. Hoffer's (1981, p.15) matrix of geometric thinking levels and geometric skills (cited in Abdefatah, 2010, p.46)(adapted)

Therefore, if the teaching process of students is aimed to develop these skills then it leads to the development of their geometrical thinking. Niss (1999) and his colleagues proposed the following competencies that can be described as an individual student's ability to (e.g., Niss, 1999, 2003; Neubrand et al. 2001):

Mathematical thinking and reasoning:[…] mastering mathematical modes of thought; posing questions characteristic of mathematics; knowing the kind of answers that mathematics offers, distinguishing among different kinds of statements; understanding and handling the extent and limits of mathematical concepts; generalizing results to larger classes of objects.

Mathematical reasoning and argumentation: […]knowing what proofs are; knowing how proofs differ from other forms of mathematical reasoning; following and assessing chains of arguments; having a feel for heuristics; creating and expressing mathematical arguments; devising formal and informal mathematical arguments, and transforming heuristic arguments to valid proofs, i.e. proving statements.

Mathematical communication: [**…**] being able to communicate, in, with, and about mathematics; expressing oneself in a variety of ways in oral, written, and other visual form; understanding someone else's work.

Modelling competency: […] being able to analyse and build mathematical models concerning other subjects or practice areas; structuring the field to be modeled; translating reality into mathematical structures; interpreting mathematical models in terms of context or reality; working with models; validating models; reflecting, analyzing, and offering critiques of models or solutions; reflecting on the modeling process; communicating about the model and its results; monitoring and controlling the entire modeling process.

Problem posing and handling competency: […] problem identifying, posing, specifying; solving different kinds of mathematical problems.

Representation competency: […] being able to handle different representations of mathematical entities; decoding, encoding, translating, distinguishing between, and interpreting different forms of representations of mathematical objects and situations as well as understanding the relationship among different representations; choosing and switching between representations.

Symbol and formalism competency: […] decoding and interpreting symbolic and formal mathematical language, and understanding its relations to natural language; understanding the nature and rules of formal mathematical systems (both syntax and semantics); translating from natural language to formal/symbolic language; handling and manipulating statements and expressions containing symbols and formulae.

Communicating in, with, and about mathematics competency: […] *understanding others'* written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content; *expressing oneself*, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters.

Aids and tools competency: […] being able to make use of and relate to the aids and tools of mathematics, including technology when appropriate.

The visualization competency and the *competency of students to develop recursive processes* conceptually and structurally (e.g., for the construction of fractal objects in a DGS) is also very important for the solution of problems with fractal constructions (Patsiomitou, 2005a, 2014).

Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, namely the *instrumental decoding competence* (Patsiomitou, 2011a, b), based on Duval's (1995a,b) semiotic analysis of students' *apprehension of a geometric figure*.

4.4. Proof and Proving, Argumentation and Deductive Reasoning

The tenet of proof has been analyzed from a range of pedagogical, historical, and cognitive viewpoints. Olivero (2003, pp.10-11) in her remarkable PhD thesis reports the frameworks in which proof has been discussed:

- *"Historical and epistemological studies concern the evolution of the notion of proof over time* (see e.g. Barbin, 1988; Arsac, 1999b)
- *The status of mathematical objects, properties and relations involved in the teaching and learning of proof* (see e.g. Balacheff, 1987; Thurston, 1995; Hanna, 1996; Lolli, 1999; Rav, 1999)
- *Students' cognitive processes when constructing or understanding proofs* (see e.g. Duval, 1991; Harel & Sowder, 1996; Sowder & Harel, 1998; Garuti, Boero, & Lemut, 1998; Bartolini Bussi, 2000; Healy, 2000a; 2000b; Maher & Kiczek, 2000; Simon, 2000; Küchemann & Hoyles, 2001)
- *The role of proof in the mathematics curriculum* (see e.g. Hanna, 2000; Knuth, 2000)
- *Possible ways of working with proof in the teaching and learning context* (see e.g. Hoyles, 1998; Sekigushi, 2000)" (cited in Olivero, 2003, p. 10-11).

De Villiers (1999b) argues that one of the biggest problems identified by researchers is how to teach geometrical proof to students, which is an indispensable ingredient in their cognitive development.

In recent decades, issues regarding formal proofs, argumentation, conjecturing and reasoning has been thoroughly investigated by the mathematical community with regard to mathematics instruction (e.g. Hanna, 1983a, b, 1989a, b, 1995, 1996, 1998, 2000a, b, 2001; Duval, 1991, 1996; De Villiers, 1990; Mason & Pimm, 1984; Semadeni, 1984; Markman, 1991; Boero et al, 1995; Chazan, 1993; Pedemonte, 2001, 2002, 2007; Furinghetti et

al., 2001; Mariotti, Bartolini Bussi, Boero, Ferri & Garuti, 1997; Arzarello, Micheletti, Olivero, Paola & Robutti, 1998; Balacheff, 1999; Rav, 1999; Rodd, 2000; Hanna &Janke, 1993, 1996, 1999, 2002; Forman et al, 1998a, b; Harel & Sowder, 1998, 2007, 2009; Harel & Tall, 1991; Hoyles & Kuchemann, 2002; Sacristán, & Sánchez, 2002; Chi Ming, 2005; Yang, & Lin, 2008; Patsiomitou, 2012a, b). A few researchers have also suggested changes to the way in which geometry it taught and to the geometry curriculum (e.g., McDonald, 1989, p.425).
Others, such as Harell (2008) argue that a Geometry curriculum is neither appropriate nor convenient if its main aim is not to encourage students' competence in deductive reasoning. Harell argues that instruction must lead to students developing ways of understanding and thinking (Harell, 2008, p. 487). Similarly, Healy & Hoyles (1998), state that proof lies at the heart of mathematical thinking and that it is the deductive reasoning that supports the process of presenting proofs, which distinguishes mathematics from the empirical sciences.

> "*Proof is the heart of mathematical thinking, and deductive reasoning, which underpins the process of proving, exemplifies the distinction between mathematics and the empirical sciences*" (Healy and Hoyles, 1998 p.1).

In Greek secondary-level schools, students are taught Euclidean Geometry. Jones (2002) in his study "*Issues in the teaching and learning of geometry*" states:

> "Around 300 BCE much of the accumulated knowledge of geometry was codified in a text that became known as Euclid's Elements. In the 13 books that comprise the Elements, and on the basis of 10 axioms and postulates, several hundred theorems were proved by deductive logic. The Elements came to epitomise the axiomaticdeductive method for many centuries. It is likely that no other works, except perhaps the Christian Bible and the Muslim Koran, have been more widely used, edited, or studied, and probably no other work has exercised a greater influence on scientific thinking. While some parchments do exist from the 9th century, it is said that over a thousand editions of Euclid's Elements have appeared since the first printed edition in 1482, and for more than two millennia this work dominated all aspects of geometry, including its teaching" (p.123).

Secondary-level students in Greece face many difficulties trying to learn the definitions and theorems in the geometry textbook and applying them to their geometrical constructions. For example, in the early years of Greek secondary school, the students are taught what kinds of quadrilateral there are; the focus is firstly on the main properties of quadrilaterals, with regard to its sides and angles, which they memorize. As a result, students do not remember them in subsequent years. They only remember very basic notions regarding perpendicularity and parallelism of the sides of quadrilaterals. Furthermore, construction of parallel and perpendicular lines is taught in the first year of secondary school and is performed by the students with a ruler and a compass; nevertheless, with the use of static means, the students are usually satisfied with producing 'soft constructions' which fulfill visual criteria. In the first year of secondary-level school, the meanings related to quadrilaterals are introduced in class in a strict form, with emphasis on the relations of inclusion and categorization, which the students do not comprehend when these meanings are introduced in a static environment. This becomes obvious when the students are asked to list common and non-common properties of quadrilaterals, (e.g. the square and the rectangle). The notion of symmetry and relative constructions are included in the Mathematics' textbook used by first-year secondary-school students. However, insufficient time is devoted to understanding them, as the Geometry textbook includes a large number of geometrical notions which have to be taught. Jones (2002) self responding to his question "why include geometry in the school mathematics curriculum" gives the following answer:

> "The study of geometry contributes to helping students develop the skills of visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof. Geometric representations can be used to help students make sense of other areas of mathematics: fractions and multiplication in arithmetic, the relationships between the graphs of functions (of both two and three variables), and graphical representations of data in statistics. Spatial reasoning is important in other curriculum areas as well as mathematics: science, geography, art, design and technology" (p. 125).

Usiskin, 1982, Senk, 1989 and other scholars have conducted studies using van Hiele levels as a possible predictor of success in proof writing. According to Usiskin (1982, p. 87)

> "*about 70% of the students who studied proof could do simple proofs requiring only one deduction beyond those made from the given. Thus about 30% cannot do even the simplest proofs. About half of the students who study proof can do proofs requiring longer chains of reasoning*".

Research also validates the difficulty of the Geometry content. Jones (2002) argues that "[…] proof had to be reproduced by students exactly in the form given in Euclid (including in the order the proof occurred in Euclid).

For very many pupils their experience of geometry was far from positive" (p. 127). The above research results encouraged researchers to discover ways of introducing formal proofs into geometry/maths instruction inductively, using computer software. For example, mathematical microworlds (e.g. Logo, DGS software) increase the chances of students becoming able to construct geometrical meanings, an area in which school textbooks have proved unsuccessful (Clements, Battista & Sarama, 2001, p.6). Govender and De Villiers (2003) argue:

> "[…] the dynamic nature of the rhombi constructed in Sketchpad seemed to make the acceptance of the hierarchical classification of a square as a special rhombus far easier than in a traditional non-dynamic environment, as the student teachers could easily drag the constructed rhombus until it became a square.[…]" (p. 57).

The transition from the traditional teaching of Euclidean proof to new trends driven by the availability of microworlds for teaching mathematics has increased the interplay between, on the one hand, what is referred to as investigation or exploratory experimentation using computer software and, on the other, conjecturing, convincing with argumentation and proving.

Researchers has studied the impact DGS software environments has had to the development of arguments and the construction of meanings in Geometry (e.g., Arzarello, Micheletti, Olivero & Robutti, 1998; Laborde, 1998; Christou, Mousoulides, Pittalis and Pitta, 2004a,b, 2005; Patsiomitou, 2008a, b, 2010, 2012a, b, 2018b). Proof and proving have been conducted, also using DGS environments or other software, to bridge the gap between the empirical–experimental and theoretical parts. As a teacher of mathematics I am constantly aware that we have to differentiate the teaching of proof (e.g., Ball et al., 2002) from what is the product of proof and what is the process used to arrive at the product of proof (Ferrando, 2005, p. 37). My thorough study on proof and proving was influenced by the studies of numerous researchers. In the current section, I shall try to briefly report the most important parts of the research studies mentioned below:

- Toulmin's (1958) model for the analysis of argumentation;
- Peirce's (1960) kinds of reasoning, for example *inductive, abductive, deductive;*
- Simon's (1996) introduction of *transformational reasoning;*
- Hanna's (2000, 2001) functions of proof;
- Bell's (1976) identification of students' justifications;
- Balacheff's (1998) justifications of students to "pragmatic" justifications and "intellectual" justifications and the complexity of students' way of proving;
- Harel & Sowder's (1998) proof schemes classification;
- Duval's (1991) structure of proof or reasoning by the triad: *entry proposition* or *given statement, rule of inference, and conclusion;*
- Duval's (1998, 1999) cognitive analysis of argumentation and mathematical proof ;
- De Villier's (1999b) study "Rethinking Proof with The Geometer's Sketchpad" and his expansion on Bell's work: *Proof as Explanation, Proof as Discovery, Proof as Verification, Proof as Challenge, Proof as Systemization.*

For a teacher struggling to teach Euclidean proof in a high school geometry class, the proving process--including the students' exploratory experimentation--is more important than its product, which is the rigorous-formal proof. In my opinion, there is some confusion, even among teachers of mathematics, about the meanings of *justifying, conjecturing, arguing, explaining* etc.

The topic of proof is discussed extendedly in the study of Hanna (1983) "*Rigorous proof in mathematics education*" (Hanna, 1983b). Hanna (1989b) highlights the importance of distinguishing between "*proofs that prove"* and "*proofs that explain*". Hanna (2000b) also reports a list of the functions of proof and proving (Bell, 1976b; de Villiers, 1990, 1999; Hanna and Jahnke, 1996):

- *"verification"* : concerned with the truth of a statement
- *explanation* : providing insight into why it is true
- *systematisation* : the organisation of various results into a deductive system of axioms, major concepts and theorems
- *discovery:* the discovery or invention of new results
- *communication*: the transmission of mathematical knowledge
- *construction*: of an empirical theory
- *exploration :* of the meaning of a definition or the consequences of an assumption

- *incorporation:* of a well-known fact into a new framework and thus viewing it from a fresh perspective"(Hanna, 2000b, p. 8).

De Villiers (1999b) in his study "*Rethinking Proof with Geometer's Sketchpad*" states that "discovery, intellectual challenge, verification, systematisation", are a range of functions of proof, that have to be communicated to students in a meaningful way, following the sequence shown in the **Figure 4.24** (Jones, 2002, p. 132).

Figure 4.24. A learning sequence of functions of mathematical proof (De Villiers, 1999b, cited in Jones, 2002, p. 132)

De Villiers (2004b) also, in his study "*The role and function of quasi-experimental methods in mathematics"* reports the methods "*that refer to all non deductive methods in mathematics involving experimental, intuitive, inductive or analogical reasoning*" (p. 398). These are the following:

- *Conjecturing:* looking for an inductive pattern, generalization etc.;
- *Verification*: obtaining certainty about the truth or validity of a statement or conjecture;
- *Global refutation*: disproving a false statement by generating a counter-example;
- *Heuristic refutation*: reformulating, refining or polishing a true statement by means of local counter-examples;
- *Understanding*: the meaning of a proposition, concept or definition or assisting with the discovery of a proof;

In my opinion, before students try to prove formally, or a teacher teaches them Euclidean proofs, it is important for their understanding that they are given the opportunity to discover, explain and experiment with regard to the correctness of a statement, proposition or theorem using a computer environment. The important thing is that students become able to engage in "analysis" and "synthesis", as they interact with the software (or paper-pencil) environment. In general, *analysis* is a Greek work which has been used since antiquity to denote a process of breaking down an intellectual or substantial whole into its component parts; in contrast, *synthesis* denotes the combination of separate elements or components with the aim of forming a coherent whole.

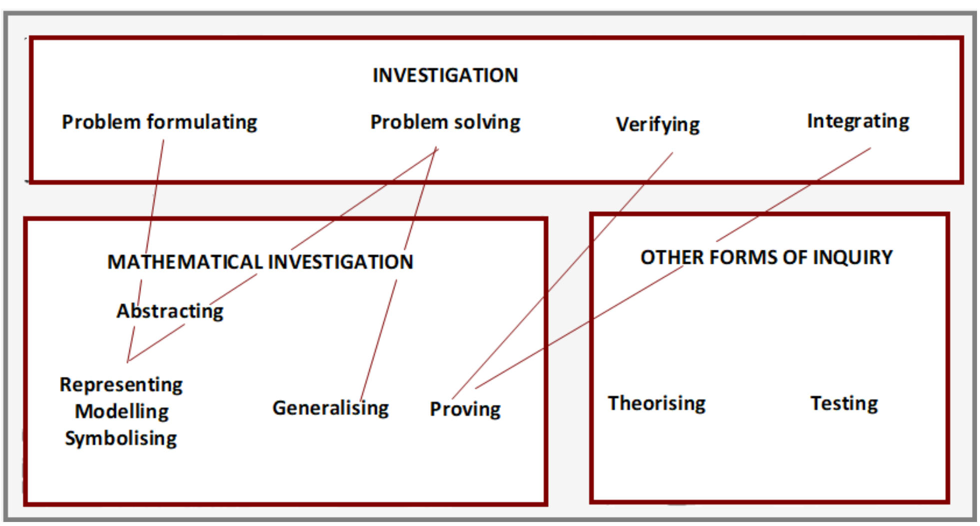


Figure 4.25. Components of the process of investigation (Bell, 1979, p. 362) (adapted)

Bell (1976) distinguishes students' justifications into two categories: "*empirical justifications*" (characterized by the use of examples to convince someone), and "*deductive justifications*" (characterized of the use of deduction to connect data with conclusions) (cited in Marrades & Gutierrez, 2000, p.90). In the **Figure 4.25,** Bell (1979) illustrates the relations among the notions of (a) investigation and problem solving, (b) Proof, and (c) Representation, generalisation and abstraction. As Bell (1979) states:

“Representation, generalisation and abstraction are certainly all important aspects of mathematical activity but the whole is greater than its parts, and the term *'mathematisation'* has been used to denote the particular combination of these activities in the way we recognise as mathematics” (p. 372)[...] “If generalisation is the characteristic pure mathematical process, that of applied mathematics is modelling, that is the representation of some situation via a diagram, a symbolic expression or some other form of analogy” (p. 376).

Duval (1991) expressed the triad of proof or deductive reasoning with three elements: *Entry propositions (or given statements or data), Rules of inference, and New propositions (or conclusion)* as it is pictured in the **Figure 4.26**. The "inference" step is the passage from an hypothesis (or en entry proposition, or given data) to a Sumperasma (conclusion or a new proposition), thanks to a given rule.

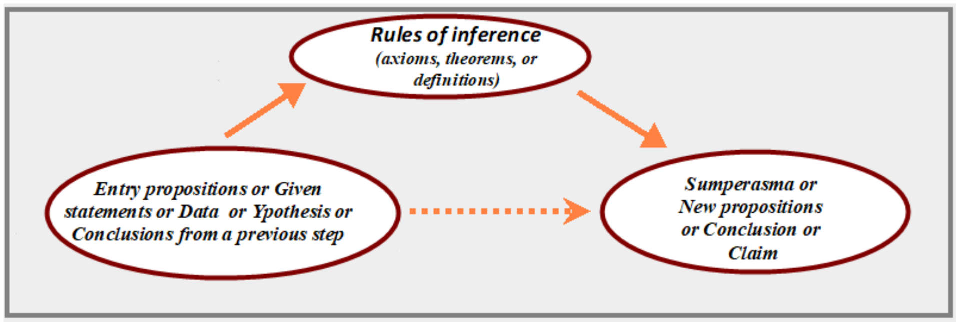


Figure 4.26. Duval’s (1991, 1996) structure of deductive reasoning (Olivero, 2003, p.36; Miyakawa, 2004, p.337) (an adaptation for the current study)

According to Ferrando (2005) the notions of argumentation and proof are different for Duval (1991):

“Duval (1991) makes a clear distinction between argumentation and deductive reasoning. Argumentation is based on the structure of the language and on the listener’s representations; therefore the semantic content of the propositions is fundamental. Deductive reasoning is characterized by an “operational status” (statut opératoire) given by: 1) *Entry propositions* (propositions données), which are hypotheses or conclusions of a previous step; 2) *Rules of inference* (régles d’inférence), which are axioms, theorems, and definitions; 3) *New propositions* (obtenues) which are the result of the inference. In a deductive step the propositions are not related to each other for their semantic value, but only by virtue of their operational status. According to Duval a proof can be so defined only if it is a logical-formal derivation, there is no concern for its semantic value but only for the syntactic value” (p. 44)

Pedemonte (2002) in her study “*Relation between argumentation and proof in mathematics: cognitive unity or break?*” states:

“Differences between *argumentation* and *proof* have been deeply analysed in the work of R. Duval: despite the use very similar linguistic forms and propositions connectives, *there is a gap between the two processes.* According to Duval (1991), the structure of a proof may be described by a ternary diagram: data, claim and inference rules (axioms, theorems, or definitions). Within proofs, the steps are connected *by a recycling process* (Duval, 1992, 1993) the conclusion of a step serves as an input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. In other words the connection between two propositions is an intrinsic connection (Duval, 1992-1993): the statement is considered and re-interpreted from different points of view. For these reasons the distance between proof and argumentation is not only logic but is also cognitive: in a proof, the epistemic value depends on the theoretical status whereas in argumentation it depends completely on the content. Then it is easy to observe the cognitive distance between the two processes” (p. 72-73)

Miyakawa (2004) argues also, that “As the rule of inference connects two statements, it can be expressed in the form of an implication “If A then B”..[...]” (p. 337).

In this context, a proof step means the application of a theorem the student knows. Moreover, if we investigate *students’ competency to geometric proofs* in the lower secondary level (Ufer & Heinze, 2008, p.1) we can see that usually it consists of one, two or three “proof steps”. As Ufer and Heinze (2008) argue, it is unusual for multi-step proofs to be constructed stepwise, but building a plan for the proof will require ideas for all or most of the steps, which must be looked at all together. To do this, the students need to be able to understand that

statements' status can differ/change (e.g. the hypothesis for the first step in a proof is also the premise for a second step) and to be able to use this understanding to form chains of deductive arguments (Duval, 1991).
Balacheff (1988) divides justification of students to "*pragmatic"* justifications and "*intellectual"* justifications. Balacheff defines as

- "*Pragmatic proofs*" those proofs which rely upon action (p.2).
- "*Intellectual proofs*" those proofs which use verbalizations of the properties of objects and of their relationships (p.2)

Balacheff further divides the *pragmatic justification* into

- *Naive empiricism* : justification by a few random examples, affected by prototypes (p.5)
- *Crucial experiment*: justification by carefully selected examples; it identifies awareness of the problem of the validity of a mathematical statement, taking into account the problem of generalization (p.6).
- *Generic example*: justification by an example representing salient characteristics of a whole class of cases (p.7).

Balacheff further divides the *intellectual justification* into

- *Thought experiment*: the justification is disassociated from specific examples, eliminating the particular
- *Symbolic calculations*: the justification is based solely on transformations of symbols or symbolic expressions

Balacheff (1988) pinpoints that

> "The passage from pragmatic proofs to intellectual proofs requires a cognitive and linguistic base. Our disr egard of the complexity of this passage could be one of the main reasons for the failure of the teaching of mathematical proof, since this passage is very often considered only at the logical level" (p. 10).

Sacristan & Sanchez (2002, p. 170) in their study "*Processes of proof with the use of technology: discovery, generalization and validation in a computer microworld*" give emphasis to the role of language for the transition from a pragmatic proof to an intellectual proof, as a pragmatic proof is "*based on effective actions carried out on the representations of mathematical objects*". Rather, intellectual proof requires the use of language to formulate the properties of and relations between mathematical objects; intellectual proof is detached from the actions on objects, as these actions have been interiorized. Language facilitates communication between the students in a group, allowing them to describe, clarify and discuss the structures they observe and re-conceptualize identified misconceptions. Students discuss how to solve problems and learning occurs in a context of collaborative, social interactions that leads to understanding (Roehler & Cantlon, 1997).
Marrades & Gutierrez (2001) in their study "Proofs produced by secondary school students learning geometry in a dynamic computer environment" present an analytic framework to describe and analyze students' answers to proof problems (p. 87). According to Marrades & Gutierrez (ibid.) "a complete assessment of students' justification skills has to take into consideration both products (i.e., justifications produced by students) and processes (i.e., the ways in which students produce their justifications" (p. 88). In the following **Figure 4.27** Marrades & Gutierrez (ibid.) summarize the types of justifications which have previously reported in details in their study.

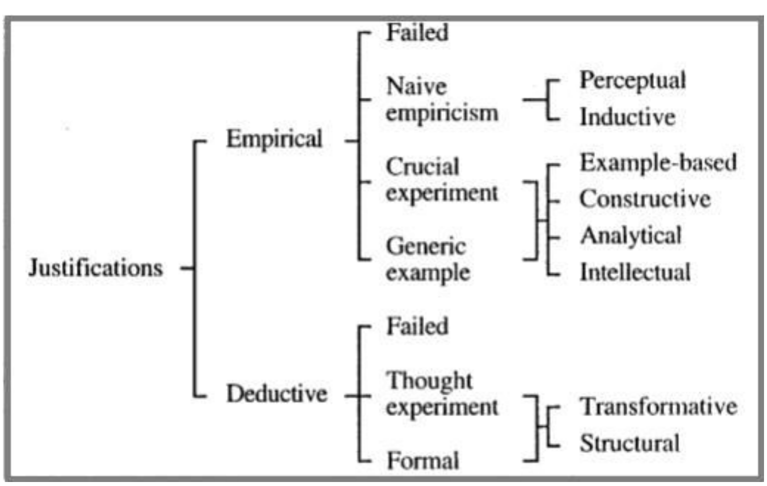


Figure 4.27. Types of justification (Marrades & Gutierrez, 2001, p. 94)

Harel & Sowder (1996) in their study "*Towards Comprehensive Perspectives on the Learning and Teaching of Proof*" (also Sowder & Harel (1998) define proving as

> "*the process employed by an individual to remove or create doubts about the truth of an observation*" (Harel & Sowder, 1996, p.6; Harel, 2001).

They argue that proving process is divided in two sub-processes:

- *ascertaining*, is the process employed by an individual to remove /or eliminate his/her doubts about the truth /validity of an assertion and it is directed internally
- *persuading*, is the process employed by an individual or a community to eliminate other people's doubts about the truth /validity of an assertion and it is directed externally (Harel & Sowder, 1996, p. 6).

Harel & Sowder (1996) define "proof schemes" as a combination of the following three definitions, (p. 6)

1. *Conjecture versus fact*: an assertion can be conceived by an individual either as a conjecture or as a fact (a conjecture is an assertion made by an individual who is uncertain of its truth) [...].
2. *Proving (*as mentioned above)[...]
3. *Ascertaining versus persuading* (as mentioned above).

Harel & Sowder further consider that a taxonomy of proof schemes consists of three classes: (a) The external conviction proof schemes class, (b) The empirical proof schemes and (c) The deductive proof schemes class (**Figures 4.28a, b**).

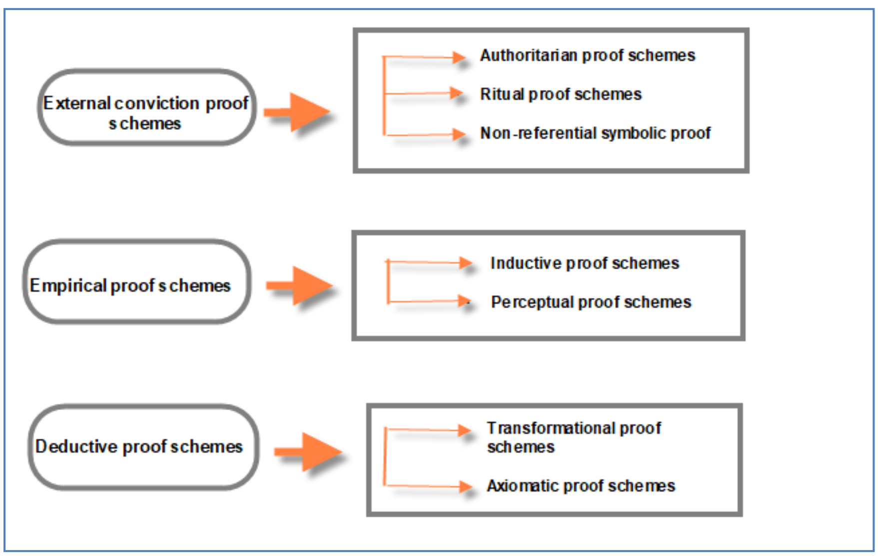


Figure 4.28a. Proof schemes (Harel & Sodwer, 1996; Harel, 2001, p. 41) (an adaptation for the current study)

The external conviction proof scheme class is distinguished among three proof schemes:

- *Authoritarian proof scheme*
- *Ritual proof scheme*
- *Non-referential symbolic proof scheme*

The empirical proof schemes class is distinguished between two proof schemes

- *Inductive proof schemes*
- *Perceptual proof schemes*

The deductive proof schemes class is distinguished between two proof schemes

- *Transformational proof schemes*
- *Axiomatic proof schemes*

Harel (2001) in his study "*The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction*" offers a taxonomy of deductive proof schemes (transformational proof schemes and axiomatic proof schemes) consisting of more subcategories as it is illustrated in the **Figure 4.28b.**

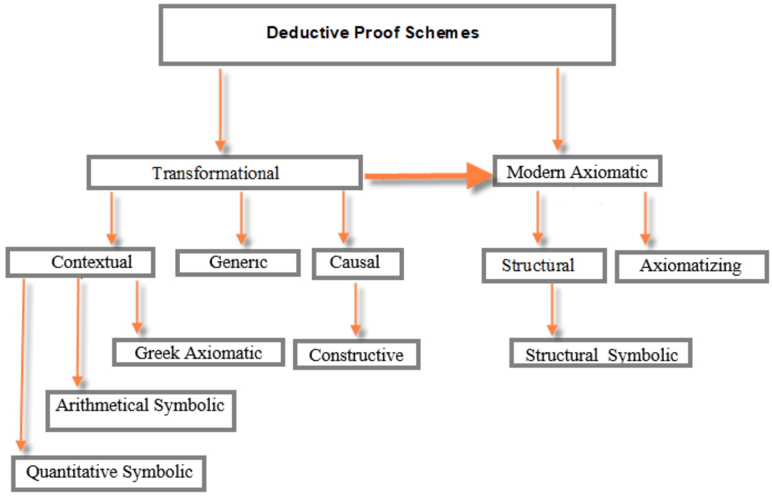


Figure 4.28b. Deductive proof schemes (Harel, 2001, p.41) (adapted)

Harel (2008) also in his study "A DNR Perspective on Mathematics Curriculum and Instruction Part I: Focus on Proving" defines "proof" and "proof schemes" as follows:

> "*A proof* is the particular argument one produces to ascertain for oneself or to convince others that an assertion is true, whereas *a proof scheme* is a collective cognitive characteristic of the proofs one produces" (p.489).

Harel (2008) gives several examples to explain the difference regarding his classification of proof schemes. Furthermore, according to Harel "A proof is a cognitive product of the proving act, and proof scheme is a cognitive characteristic of that act" (p.489). Moreover, "a proof is a way of understanding, whereas a proof scheme is a way of thinking" (p. 490) (**Figures 4.28c, d**).

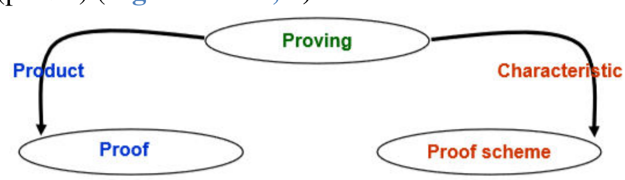


Figure 4.28c. The triad of proving, proof, and proof scheme: a proof scheme is a common characteristic of proofs—the products of one's mental act of proving (Harel, 2008, p. 490).

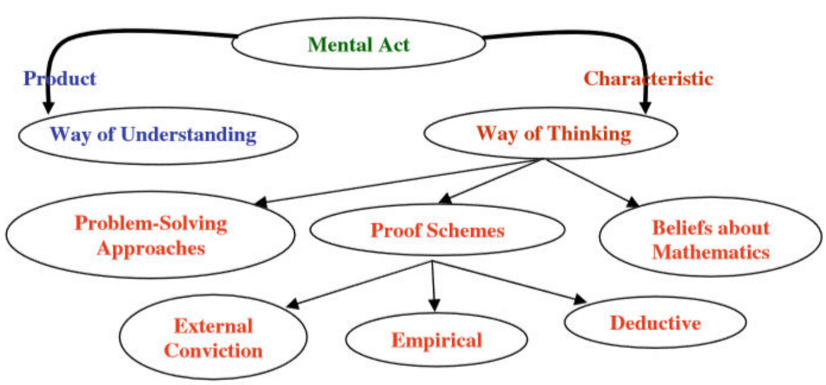


Figure 4.28d. The triad, mental act, way of understanding, and way of thinking (Harel, 2008, p.493)

In the **Figure 4.28d**, Harel (2008) depicts "the three categories, problem-solving approaches, proof schemes, and beliefs about mathematics, comprising ways of thinking; and the three categories, external conviction, empirical, and deductive, comprising proof schemes" (p.493).

Furthermore, Harel (2008) suggests that

> "[…] given the focus on proof and argumentation in current documents […] there is a need for teachers to understand the difference between ''argumentation'' and ''mathematical proof;'' without it, teachers would likely be advancing argumentation skills and little or no deductive reasoning" (p. 499).

Proof and proving process can be achieved by a student if s/he has developed his/her thinking. The development of a student's thinking has to do with the development of his/her competence on *deductive reasoning*. During the problem-solving process, students develop different kinds of reasoning including *inductive, abductive, plausible and transformational reasoning* (Harel & Sowder, 1998; Peirce, 1992; Simon, 1996). For this reason, it is crucial to investigate how students' reasoning at different levels develop during the problem-solving process--as the students shift from the particular to the general aspect of figures and become able to produce deductive reasoning--and what steps the students follow when they develop a proof as a product.

Peirce (1992, p.189) classifies different types of inference thus: "[…] Deductive or Analytic, [and the] Synthetic [as] Induction and Hypothesis [or Abduction]". Deduction starts with a general rule and arrives at a conclusion—put otherwise, it refers to conclusions that are reached on the basis of a logical chain of reasoning whose every step necessarily follows on from the step before (Ennis, 1969, p. 7 quoted in Simon, 1996, p.197). Inductive reasoning works in the other direction, starting with specifics/particulars and inferring a general rule(s). Peirce described the terms *deduction, abduction and induction* in terms of *rules, cases and results* as it is described in the **Figure 4.29** below:

Deduction:

Rule – All the beans from this bag are white

Case – These beans are from this bag ∴

Result – These beans are white

Induction:

Case – These beans are from this bag

Result – These beans are white ∴

Rule – All the beans from this bag are white

Hypothesis:

Rule – All the beans from this bag are white

Result – These beans are white ∴

Case – These beans are from this bag.

Figure 4.29. Peirce's (1878) descriptions od deduction, induction and abduction in terms of rules, cases and results (CP 2.623, cited in Reid, 2003, p.2; Ferrando, 2005, p.9)

Ferrando (2005) opines that "abduction is the only logical operation that introduces new ideas, deduction explicates and proves that something must be; induction evaluates and shows that something actually is operative" (p. 17). Similarly, Baccaglini-Frank, & Mariotti, (2009) argue that: "[…] abduction marks the transition from the conjecturing to the proving phase […]. Abduction guides the transition, in that it seems to be key in allowing solvers to write conjectures in a logical 'if…then' form, a statement which is now ready to be proved" (Baccaglini-Frank, & Mariotti, 2009, p. 233).

- "*Deductive reasoning* is the process of inferring conclusions from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiment" (Ayalon, & Even, 2008, p.235)
- "*Induction* is where we generalize from a number of cases of which something is true, and infer that the same thing is true of the whole class. As, where we find a certain thing to be true of a certain proportion of cases and infer that it is true of the same proportion of the whole class". (CP, 2.624 cited in Ferrando, 2005, p.9).

• "*Abduction* consists in studying facts and devising a theory to explain them" (5.145); *Abduction* "consists in examining a mass of facts and in allowing these facts to suggest a theory" (CP, 8.209, cited in Ferrando, 2005, p.15).

• *Abduction* is where we find some curious circumstances, which would be explained by the supposition that it was a case of a certain rule, and thereupon adopt the supposition […] (CP, 2.624, cited in Ferrando, 2005, p. 80)

According to Simon (1996) "transformational reasoning in many cases overlaps with both inductive and deductive reasoning" (p.204). Simon (1996) also defines *transformational reasoning* as follows (p. 201):

"*Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated*" (Italics by the author) (p.201).

With the notion of "mental enactment" Simon "refers to operations carried out in mental images" (p. 201). Also Simon points out that "*Transformational reasoning involves not just the ability to carry out a particular mental or physical enactment, but also the realization of the appropriateness of that process to a particular mathematical situation* (Italics by the author)*" (p. 203).*

Toulmin's (1958) model of argumentation is a model which relates the involved elements: claims, data, warrants, backings, qualifiers and rebuttals in the argument formulated by an individual (or a group of students that participate) (**Figure 4.30a**).

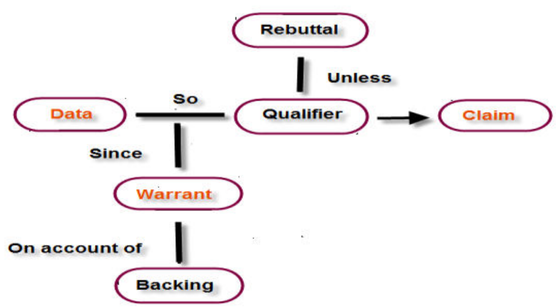


Figure 4.30a. Toulmin's (1958) model of argumentation (adapted).

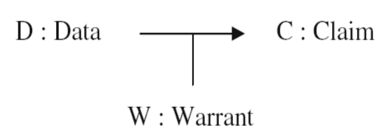


Figure 4.30b. Toulmin's (1958) basic structure of an argument (Pedemonte, 2007, p.28).

According to Inglis, Mejia-Ramos & Simpson (2007) "Toulmin's (1958) scheme has six basic types of statement, each of which plays a different role in an argument.

- The conclusion (C) is the statement of which the arguer wishes to convince their audience.
- The data (D) is the foundations on which the argument is based […].
- The warrant (W) justifies the connection between data and conclusion (e.g. with a rule, a definition or a theorem)
- The backing (B) supports the warrant […]
- The modal qualifier (Q) qualifies the conclusion by expressing degrees of confidence
- The rebuttal (R) potentially refutes the conclusion by stating the conditions under which it would not hold.[…]." (p.4)

These elements are represented in the **Figure 4.30a** in which the relationships between them are expressed in sequential order. In other words, Toulmin's model consists of the elements described above, which are explicit or implicit. Several times an argument does not include qualifiers and rebuttals. Krummheuer (1995) suggested and applied a reduced model of the original scheme, consisting of claims, data, and warrants of arguments "to examine the learning of mathematics in the context of collective argumentation" (p.11). As suggested by Krummheuer (ibid.), during a classroom activity (or during group cooperation) one or more students could be contributing towards the formulation of the argument, attempting to convince the other participants of the group, including the class teacher (or the researcher). Pedemonte (2007, p.28) has presented Toulmin's (1958) basic structure of an argument constructing a figure with the three basic elements mentioned above (**Figure 4.30b**).

Pedemonte (2003) in her study "*What kind of proof can be constructed following an abductive argumentation?"* describes the basic structural elements involved in the Toulmin's model as follows:

> "In any argumentation the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the *claim.* The second step consists of the production of *data* supporting it. It is important to provide the justification or *warrant* for using the data concerned as support for the data-claim relationships. The warrant can be expressed as a principle, a rule and the like. The warrant acts as a bridge between the data and the claim" (p. 3).

For the representation of a theoretical diagram using tools and theoretical constructs I introduced a pseudo-Toulmin's model (Patsiomitou, 2011a, 2012a, b) --based on Toulmin's model (1958) -- in which: (1) the data could be the dynamic diagram, or an object and (2) a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation). The **Figure 4.31** presents a pseudo-Toulmin's model through example.

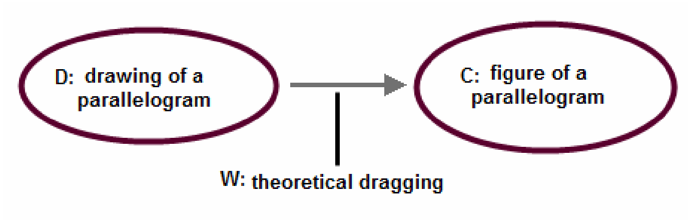


Figure 4.31. An example of a reduced pseudo-Toulmin's model (Patsiomitou, 2012b, p. 57)

In the Figure 4.31, a drawing of a parallelogram is the data (D), the theoretical dragging is the warrant (W), and the figure of the parallelogram is the claim (C). This means that a student can theoretically drag a point-vertex of a drawing-parallelogram and transform it into a figure-parallelogram, trying to acquire additional properties.

Also, I have expanded the pseudo-Toulmin's model in order to express a relationship between the figures or a sequence of diagrams and students' cognitive analysis as they use the tools.

4.4.1. Indicative Examples of Students' Argumentation and Proving

Argumentation of students can be represented using Toulmin's model. A very interesting problem which attracts students to investigate it, is Varignon's problem (reported in the study of Oliver, 2001).

Varignon (1654-1722) proved that "*a parallelogram is formed when the midpoints of the sides of a convex quadrilateral are joined in order*". Varignon's proof was published in 1731 in "*Elemens de Mathematique"* (Oliver, 2001, p.316). I shall report here a few indicative examples of students' argumentation, aiming to explain the different kinds of reasoning. Complementary to this, a deductive system of axioms, theorems and propositions as well as concepts and definitions can help the student to organize the proving process.

A. The following excerpt belongs to the third phase, when the students M7, M8 and M13 investigated several instances of Varignon's theorem occurring from the use of dragging (Patsiomitou, 2012a). My aim was for the students to understand the hierarchy of quadrilaterals and how we can construct a classification of them. This is in accordance with what Dina van Hiele argues: "A classification made by the students is to be considered by the teacher as proof that the subject matter has been assimilated, that associations have been formed, that the subject matter can be handled independently" (Dina van Hiele in Fuys et al., 1984, p.170).

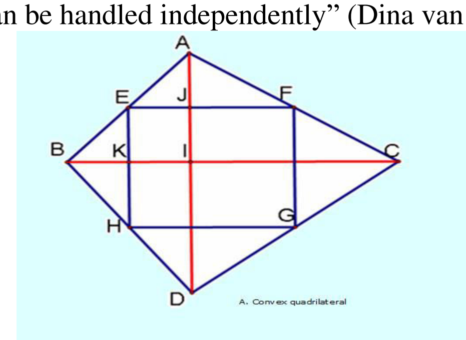


Figure 4.32a. Implementation of Varignon's theorem to a convex quadrilateral

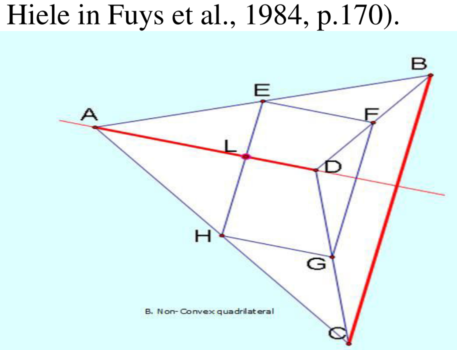


Figure 4.32b. Implementation of Varignon's theorem to a non-convex quadrilateral

R: *You mean that "If the diagonals are vertical lines, then the shape EFGH is a rectangle"?*
M_{13}: *Is n't this a right angle?* (Pointing to angle AIB) (**Figure 4.32a**)
M_{13}: *Can we prove that this small shape (he means EJIK) is a rectangle?... It has a right angle (points to the angle KIJ of the diagonals), and that its sides are parallel and congruent. (Points to the parallelism of the segments)… EJ//KI and EK//AI, therefore it is a right angle.*

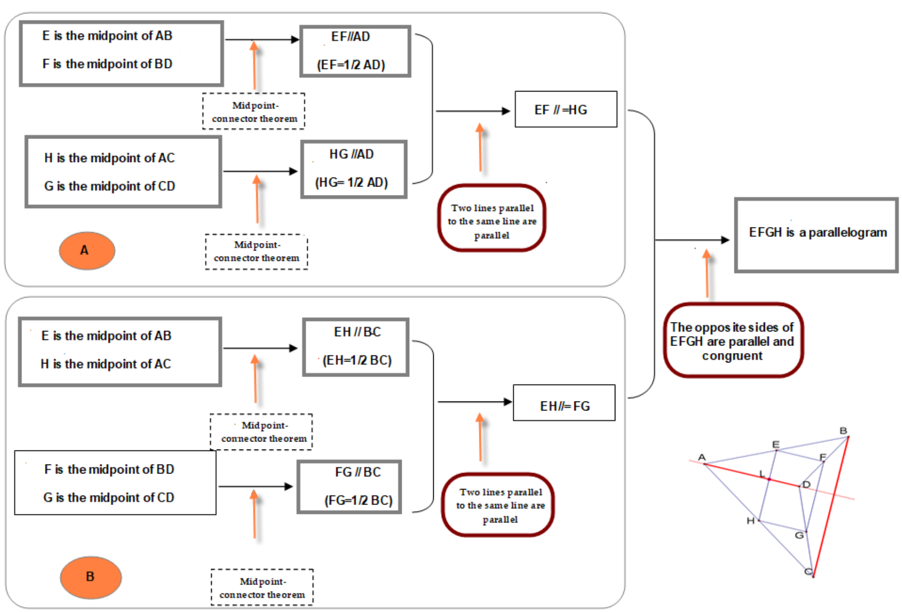
This is an important point in M_{13}'s development of thinking as he recalled the midpoint-connector theorem (i.e. "The segment connecting the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side" reported in Coxford & Usiskin, 1975, p. 273) as well as an *economic definition* of rectangles. He combines them both and uses deductive reasoning to support his argument. The student uses Peirce's case, rule and result as follows:

Case A: Its sides are parallel and congruent (*EJ//KI and EK//AI)* and [therefore] angle KIJ is a right angle.
Rule B: If a quadrilateral has [opposite] parallel sides and one of its angles is a right angle, then […].
Result C: The quadrilateral is a rectangle.

In addition, investigating the case of a non-convex quadrilateral with the same group of students led to similar results. An excerpt from the students' discussion follows below:

R: *Why is it a rectangle?* (**Figure 4.32b**)
M_{13}: *Because GF//=BC/2, EH//=BC/2 and so is GF//=EH*
R: *This is true for a parallelogram.*
M_7, M_{13}: *But again they intersect vertically (and both point at ALE angle).*
M_{13}: This is a right angle (points to ALH angle), so the vertical is right angled (points to ELD)
R: *Why are they vertically intersected? Why EH is vertical to AD?*
M_8: ***Because** in the triangle ACB, EH//=1/2 CB, and therefore EH//CB.*
M_{13}: *Then H is a right angle (LHG) because it is an alternate interior angle.*

The students "see" figures in the whole diagram and use deductive arguments to support their thinking. For example M13 answers: *"Because GF//=BC/2, EH//=BC/2 and so is GF//=EH"*. This argument is more complex than it seems. For the analysis, I shall use the reduced Toulmin's model of argumentation (e.g., Krummheuer, 1995; Pedemonte, 2007). M13 is absolutely right, as a student could use either A or B paths to support his thinking.



B. The following illustration (**Figure 4.33a**) is an example I used to explain how students developed different kinds of reasoning, a topic I discuss in my study "*Theoretical dragging: A non-linguistic warrant leading to 'dynamic' propositions*" (Patsiomitou, 2011a, p.366). As I have written (Patsiomitou, 2011a, p. 366): "Students M9, M10, M14 (van Hiele level 1) tried to construct a parallelogram. They constructed a segment AB and a point C, then a parallel line j from the point C. They were unable to understand how to continue the process by constructing a parallel line from point B, which is to say they lacked competence in the *place way operation* of the tools. M14 constructed a point D on line j and began with repeated *experimental dragging* on point D. Visually she understood that point D had to preserve its congruency with the opposite segment AB if the figure were to remain a parallelogram. So she dragged point D again, stating "the dot (i.e point D) must be almost here in order to become congruent to the segment AB". She used informal language saying "it must be "the same" [distance]". She then used *theoretical dragging* to turn her drawing into a figure of a parallelogram.

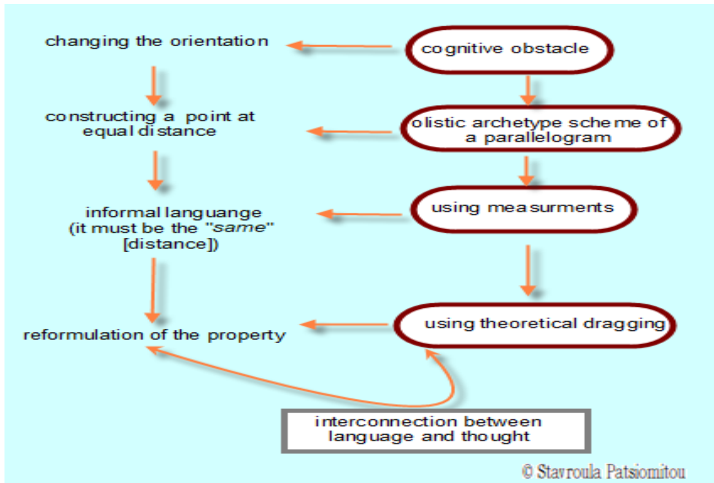


Figure 4.33a. A cognitive analysis of the use of the tools

By virtue of *instrumental genesis* the student has constructed an *instrument* which includes the *utilization scheme* of dragging and the meaning of the congruent opposite sides of a parallelogram. Seen in this light, the software's primitives are non linguistic visual data and the tool is the warrant for the construction of a dynamic proposition which is empowered in a dynamic geometry environment. This is what I identify as an interconnection between language and thought in student's mind. Moreover, the first part concerns students' *procedural knowledge* and the second on the right students' *conceptual knowledge* (**Figure 4.33b**). The transformation also of formulations is a result of the transformation of the dynamic diagram. (**Figure 4.33c**).

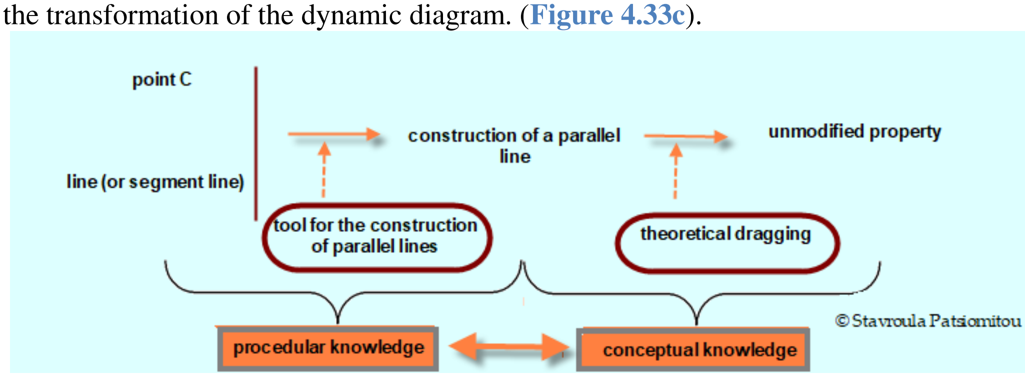


Figure 4.33b. Interplay /interconnection between *procedural and conceptual knowledge* represented in a pseudo-Toulmin model

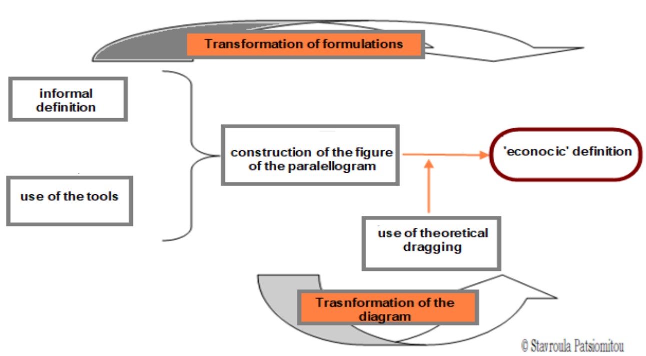


Figure 4.33c. Transformations of formulations due to the transformation of the diagram

C. The diagram (Figure 4.34) is an adaptation of Toulmin's model with tool use (Patsiomitou, 2011a, p. 367). Points C, D are the *data* D1, D2 for the actions that follow.

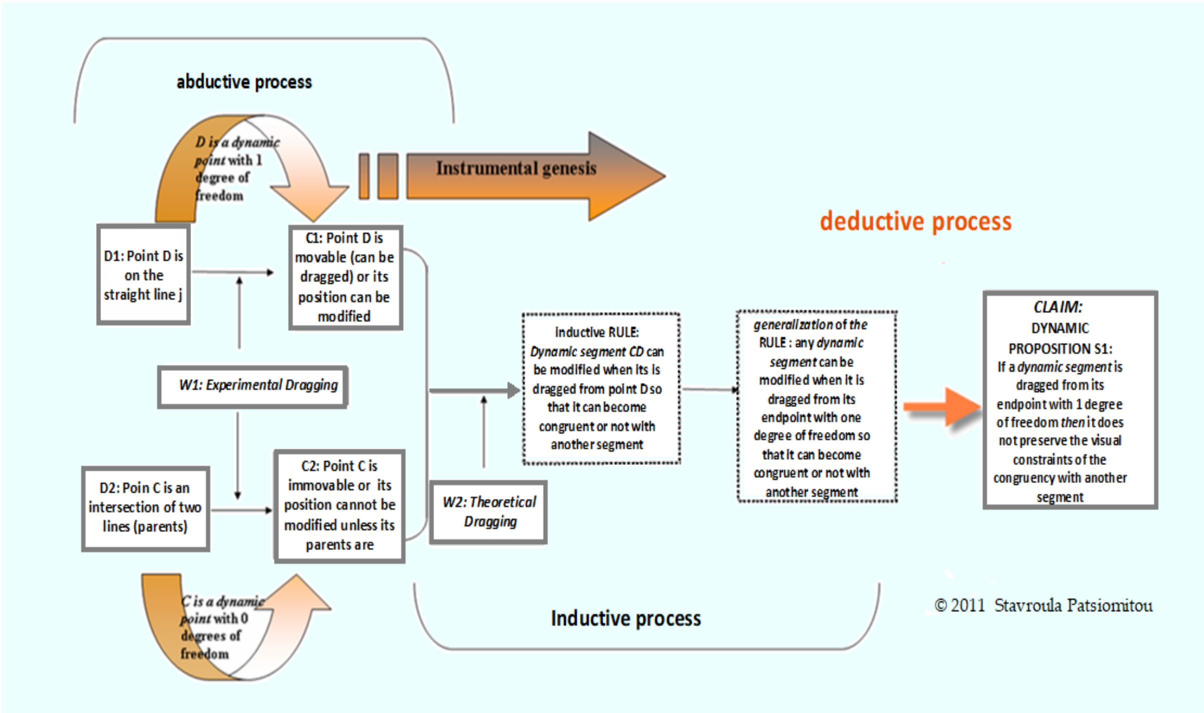


Figure 4.34. A pseudo-Toulmin model for the dynamic proposition through the use of the tools (Patsiomitou, 2011a, p. 367)

The experimental dragging tool operates as non-linguistic *warrant in* Toulmin's model for the students' understanding of both the stability of point C and of the modification of point D and, hence, of segment CD. The construction of the *claims* C1, C2 begins with "*an observed fact*" (Pedemonte, 2007, p.29). Through *instrumental genesis,* the tool affects the students' understanding that opposite sides of a concrete parallelogram should be congruent, making this *the abductive part* of the process. *Theoretical dragging* affects the construction of an intrinsic *inductive rule* (i.e. The dynamic segment CD can be modified when it is dragged from point D so that it becomes congruent or not with another segment) which leads to a *generalization of the rule* for any dynamic segment. The transformation of the position of the point through dragging leads to the transformation of the segment, which leads in turn to the comprehension of the *dynamic proposition* S1 which is the *deductive claim* (i.e. If a dynamic segment is dragged from its endpoint with one degree of freedom then it will not preserve the visual constraints of congruency with another segment).

D. In the next Figure 4.35 I present an example, which shows the role of the DGS tools for the development of students' deductive reasoning, during the fourth phase of the research process. The Figure 4.35 is a pseudo-Toulmin model describing the structure of the following argument:
M2:"*IPQG is a trapezium because PI and QG are perpendiculars to IG as we concluded from the rotation for 90^o. ...we must prove that X is the midpoint of any segment that can be. ..These (pointing to PQ, HL) seem to be diagonals but where is the quadrilateral ...If we prove that PQ and HL are the diagonals of a parallelogram then the diagonals are dichotomized"*. M2 tried to prove that HPQL is parallelogram.
Detailed analysis of the topic is incorporated in my study "*Students learning paths as 'dynamic encephalographs' of their cognitive development"* (Patsiomitou, 2013a, p. 805).

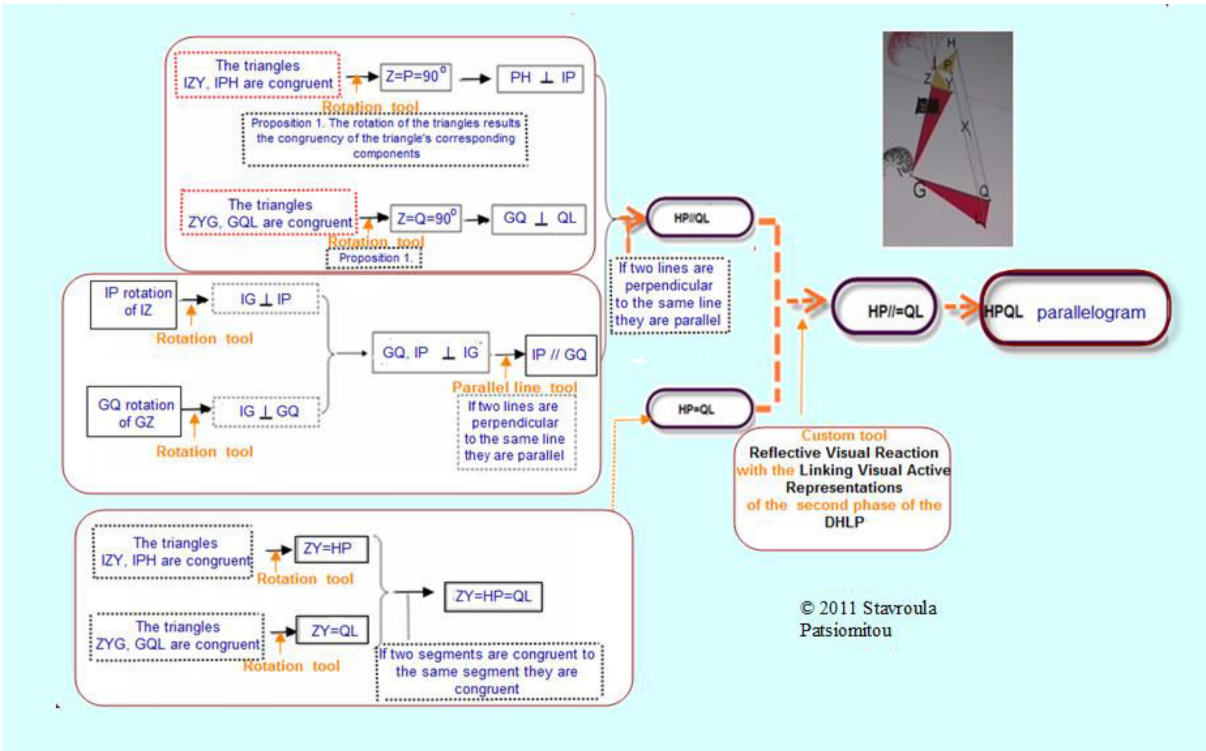


Figure 4.35. Students' deductive argumentation during the fourth phase of the DHLP (Patsiomitou, 2012a, b; Patsiomitou, 2013a, p. 805)

CHAPTER V.

5.1. Problem, Problem Solving and Problem Based Learning in Mathematics Education

The word "problem" is derived from the Greek word "provlema" with etymology from the verb "provalein", whose meaning covers "projecting, showing, revealing, displaying, presenting": i.e. 'provalein' refers to a goal presented in a question. (See also, https://etymonline.com).

The word "problem" is defined as:

- "[…] An inquiry starting from given conditions to investigate or demonstrate a fact, result, or law". (Webpage [26]);
- "[…] Something that causes difficulty […and especially a mathematics problem] is a question to be answered or solved by reasoning or calculations". (Webpage [27]);
- "[…] A question raised for inquiry, consideration or solution". (Webpage [28]).

Charles & Lester (1982) define a problem as a task for which "the person confronting it wants or needs to find a solution, has no readily available procedure for finding the solution and must make an attempt to find a solution." (Charles & Lester, 1982, p. 5, in Nunokawa & Fukuzawa, 2002). Newell & Simon (1972) write that "a person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it" (p. 72). The definition of the word problem especially in mathematics or physics has to do with a proposition or an inquiry stating something to be proved. In order to answer this inquiry we must combine data and information, and then we can derive a solution following logical inferences and deductive reasoning. In my opinion, mathematical problem solving is a process which satisfies the following presuppositions: (a) 'input' in the form of the verbal description of a mathematical problem which includes general information; (b) 'input' in the form of mathematical statements that constitute the problems' hypotheses; (c) 'a goal' expressed in a statement; (d) concrete preexisting knowledge (i.e. axioms, theorems, proofs, concepts, definitions, formulas and methods) and appropriate heuristic skills; (e) appropriate logical inferences and reasoning (e.g., deductive, inductive, abductive, transformational). According to Mayer (1983) a problem consists of *givens*, *goals* and *obstacles*, as described in the following **Figure 5.1**. The problem solving process derives abstractions and infers consequences and other findings from input data and information to produce a solution that addresses the task and leads to a "Sumperasma" (a Greek word whose meaning encompasses both "a logical conclusion" and "a summary in a few words").

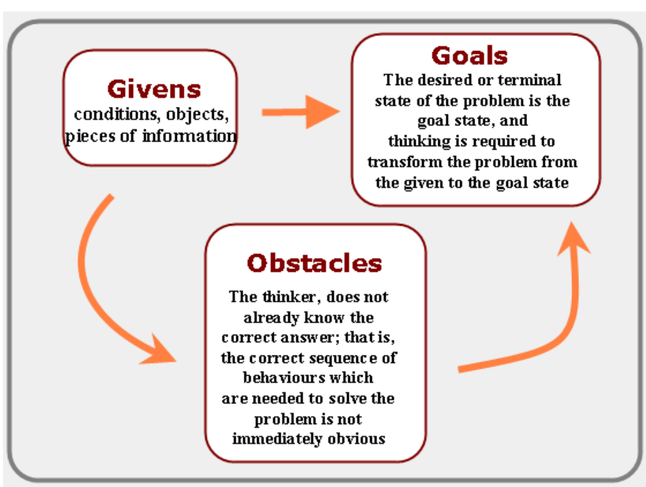


Figure 5.1. Defining problem (Mayer, 1983, p.4 in Stoyanova, 1997, p.2) (an adaptation for the current study)

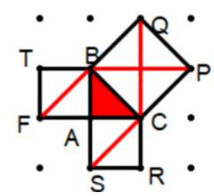
Aamodt (1991) also states that "A mathematical problem may be structured [or divided] in sub-problems, in which case the problem solving process may be correspondingly split into sub-processes" (Aamodt, 1991, p.31). As a teacher of mathematics, I have often asked myself the following questions:

- Are students able to build a reasonable and meaningful representation of a problem by means of a conscious and intentional process?

- Do students connect the process of representing the problem with preexisting knowledge that can be brought to bear on the problem?
- During the problem-solving process, do students demonstrate significant, meaningful and appropriately organized connections between pieces of information in their statement of the problem?
- Do students construct a logical correspondence between the structure of the verbal expression of the problem and the structure of its solution?
- Can we identify different levels of investigation in problem-solving in order to enhance the abstract thinking of our students?
- What conceptual considerations need to be taken into account when designing problems in a dynamic geometry environment? How do these conceptual considerations impact on our students' learning and understanding of mathematics?

Learning through problem solving can be addressed by both open-ended complex geometric problems and non-open strict geometric problems, presented in a static or dynamic environment. In order to distinguish open from non-open problems, I will quote the following example from my introduction to the Pythagorean Theorem:

Table 5.1. Examples of Open and Non-Open Problems

Non-open problem	Open problem
Given a right triangle prove that (a) $a^2=b^2+c^2$ (a is the hypotenuse of the right triangle) and (b) the concrete relation (Pythagorean theorem) is satisfied only in right-angled triangles.	a) If the area of the square TBAB is equal to $1cm^2$, can you calculate the areas of the squares constructed outside the triangle ABC? What do you observe? Repeat your experiments doubling the side of the square TBAB in your dot paper and write down the new results. Continue and formulate a rule for this situation.  b) Consider the squares constructed externally on the sides a, b, c of a right triangle. If a=5cm, b=3cm, c=4cm can you calculate the areas of the squares? Calculate a^2, then b^2+c^2. What do you observe? Does this occur to every right triangle? Can you formulate a rule for this phenomenon? Does this rule holds true for all right triangle regardless of the lengths of their sides? Does this Pythagorean relation characterize only right-angled triangles?

Students must be encouraged to solve their own problems that mirror real life situations. The open problem can be solved using different approaches and in multiple ways, encourages and stimulates discovery, prompts students to generate conjunctures and most students can get involved as Arsac et al. (1988) mention: "The statement of the problem […] fosters discovery […] creates a situation stimulating the production of conjectures.[…]" (Arsac, Germain & Mante, 1988 in Furinghetti & Paola, 2003, p.398).

The solution to an open problem cannot be reduced to a routine problem that requires a technique the student has probably memorized; instead it provides the student with the freedom to generate conjectures. Conjectures are the first step for the students to formulate logical inferences and then deductive argumentation, depended on their level of understanding or on their van Hiele level. In his book "*How to Solve It*", Polya (1957/1966) based on his experience as a teacher of mathematics suggested four problem-solving phases, pointing out the cognitive actions linked to the process of problem-solving (**Figure 5.2**). George Pólya (1966) addressed also the difference between "tasks" and "mathematical problems". He also distinguished routine from non-routine problems, from a teacher's point of view. As he states:

> "[…] The nonroutine problem demands some degree of creativity and originality from the student, the routine problem does not. […] I shall not explain what is a nonroutine mathematical problem: If you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed such tension and triumph in one of your students, look for another job and stop teaching mathematics". (Pólya, 1966, pp. 126–127, reported in Szabo, 2017, p.40)

UNDERSTANDING THE PROBLEM

First. You have to *understand* the problem.	*What is the unknown? What are the data? What is the condition?* Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory? Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.	Have you seen it before? Or have you seen the same problem in a slightly different form? *Do you know a related problem?* Do you know a theorem that could be useful? *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown. *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible? Could you restate the problem? Could you restate it still differently? Go back to definitions.

CARRYING OUT THE PLAN

Third. *Carry out* your plan.	Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth. *Examine* the solution obtained.	Can you *check the result?* Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method, for some other problem?

Figure 5.2. The four problem-solving phases (Polya, 1957) (an excerpt from the manuscript (Webpage [29])

The problem-solving process, including diagram construction, can be experienced using a "*brainstorming technique*" session, which is regarded as the most effective tools we know about creative problem-solving (e.g., Osborn, 1953). In a "brainstorming technique" session, students express/formulate what they know with the teacher helping them by introducing the concepts through essential questions, writing their ideas on the board and organizing them into a "concept map" (e.g., Novak, 1990), using also an approach inspired by history or a historical contextualization of the meanings included in the problem. The book "History of Mathematics" (Exarhakos, 1997, vol. A, in Greek) includes an extended report of problems, also incorporating the Babylonian writings on display in the British Museum. It is very interesting to look at the Babylonian solution to the following verbal problem "The area minus a side of a square is 870; find the side of the square". Problems of this kind probably appeared in Babylon *circa* 1830 BC. An adaptation of this problem is reported in Patsiomitou (1999, Euclid A, p. 34, in Greek): "The area minus a side of a square is 6; find the side of the square" (the numbers are in base ten). First of all, the problem is translated into its symbolic form as follows: x^2-x=6. The coefficients used in the equation are 1, -1, and -6. The Babylonian mathematicians didn't use minus or negative numbers. For this the coefficients used were 1, 1, 6. Firstly they followed the steps (Patsiomitou, 1999):

- Calculate the half of the coefficient of x to result ½
- Multiply ½ by itself to get ¼
- Calculate the product ¼ to 6 to result 25/4
- Calculate the square root of 25/4 to get 5/2
- Calculate the result 5/2 to the half of the coefficient of x, ½ to get 6/2=3
- This number is the solution to the problem

This was a Babylonian approach that could help us to formulate a model for solving any quadratic equation of the form x^2+bx=c (**Figure 5.3**) As I have noted in previous studies (e.g., Patsiomitou, 1999, 2014) quadratic equations in the ancient history of mathematics are reported in Babylonians and Chinese, as well as in ancient Greek mathematics. The ancient Egyptians, Chinese and Indians also used practical arithmetic to solve quadratic equations when they appeared in a real-life problem, which is to say they discovered a practical solution to meet their every day needs. This kind of solution was empirical in other ways, an inductive way of procedural thinking that helped them to proceed on to calculations (e.g., to construct something or to educate their students).

- Calculate the half of the coefficient of x to result $\frac{b}{2}$
- Multiply $\frac{b}{2}$ by itself to get $\frac{b^2}{4}$
- Calculate the product $\frac{b^2}{4}$ to c to result $\frac{b^2+4c}{4}$
- Calculate the square root of $\frac{b^2+4c}{4}$ to get $\sqrt{\frac{b^2+4c}{4}}$
- Calculate the result $\sqrt{\frac{b^2+4c}{4}}$ to the half of the coefficient of x, $\frac{b}{2}$ to get

  $x=\sqrt{\frac{b^2+4c}{4}}+\frac{b}{2}=\frac{b+\sqrt{b^2+4c}}{2}$ (1)
- This was a solution to the [real] problem.

Figure 5.3. A generalization of the Babylonian solution to the problem (Patsiomitou, 1999)

If we try to apply the formula (1) (**Figure 5.3**) to solve the quadratic equation x^2-5x =+6, we can accept the positive solution, which is the number 6, but dismiss the negative solution, which is -1.
The reported Babylonian mathematical problems include a large number in which the goal is to calculate an area or a dimension of a square or rectangle. There are many problems in which the area of a rectangle is given along with one of its dimensions, and the goal is to find the other dimension. The way ancient peoples calculated the solutions to these problems is of great interest (Exarhakos, 1997, p. 187, in Greek). In the following **Figure 5.4**, I present a *concept mapping* for the concept of quadratic equations.

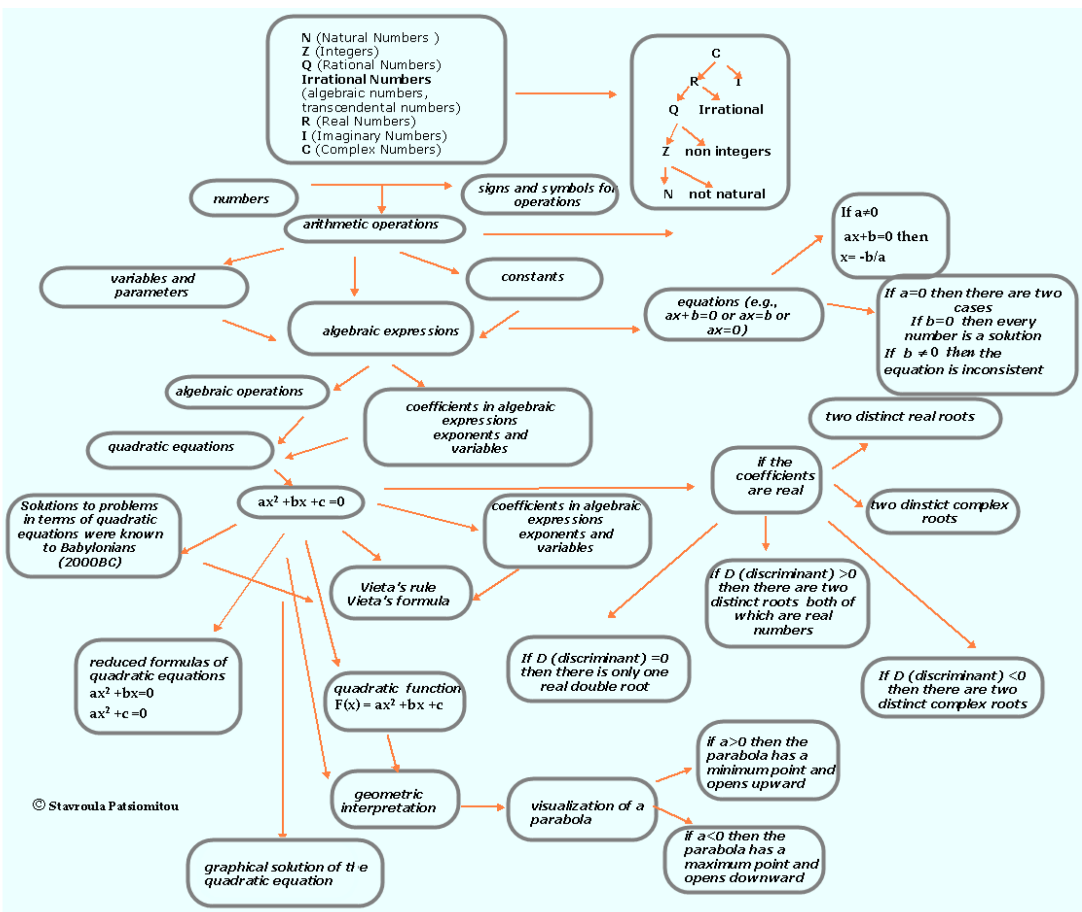


Figure 5.4. My proposal for a concept mapping for the understanding of the concept of quadratic equations

A similar problem has been posed to my secondary and tertiary level students, and I found that they worked in groups and found solutions to problems more easily with the concept maps than without them. According to Novak & Ganas (2007) "concept maps were developed in 1972. […] Out of the necessity to find a better way to represent children's *conceptual understanding* emerged the idea of representing children's knowledge in the form of a *concept map*. Thus, was born a new tool not only for use in research, but also for many other uses" (p. 29).

Brainstorming technique depends on the students' thinking to create connections among meanings (e.g. when a student hears the meaning of the Pythagorean theorem, his/her brain automatically associates it with the meaning of square as well as with a formula connecting the sides of the right triangle). Researchers (e.g., Iraksen, 1998) have found that brainstorming is an effective technique for students to develop their cognitive skills by generating and organizing their ideas. The whole process can enhance cooperative learning as well as encourage student engagement in the learning process by dealing effectively with students' cognitive conflicts and improving their critical thinking skills. Many students are not able to translate the verbal representation of a geometrical problem into an iconic representation during the problem-solving process. And even if the students overcome this obstacle with the help of the teacher, many do not know how to continue the process, especially in the case of geometrical problems. Cognitive conflicts and cognitive obstacles, "aha" phenomena and enthusiasm occur many times over during the problem-solving process as a student works individually or in cooperation/interaction with other students and the teacher. In other words, the problem-solving process combines characteristics from the theoretical background of constructivist learning, of discovery learning, and of learning through social interaction. Mathematical problem solving process concepts can also be introduced informally and subsequently connected formally to the theory.
Clements (2000) reports the characteristics that have good mathematics problems for students (adapted from Russell, Magdalene, & Rubin, 1989; Wheatley, 1991, cited in Clements, 2000):

- "are meaningful to the students;
- stimulate curiosity about a mathematical or nonmathematical domain, not just an answer;
- engage knowledge that students already have, about mathematics or about the world, but challenges them to think harder or differently about what they know;
- encourage students to devise solutions;
- invite students to make decisions;
- lead to mathematical theories about (a) how the real world works or (b) how mathematical relationships work;
- open discussion to multiple ideas and participants; there is not a single correct response or only one thing to say;
- are amenable to continuing investigation, and generation of new problems and questions." (p. 12)

Schmidt (1983, 1993) based on empirical studies, proposed that problem based learning (PBL) has the following effects on student learning (see also Tomaz , van der Molen , and Mamede, 2013, p. 12): "(a) Activation of prior knowledge (b) Elaboration on prior knowledge through small-group discussion (c) Restructuring of knowledge in order to fit the problem presented (d) Learning in context and (d) Motivation to learning"
Schmidt (1989) in his study "*The rationale behind problem –based learning*" suggests a sequence of actions and processes that are involved in problem –based learning. These are included in the following **Table 5.2**:

Table 5.2: Steps involved in problem-based learning (Schmidt, 1989, p. 107)

Step 1:	Clarify terms and concepts not readily comprehensible.
Step 2:	Define the problem.
Step 3:	Analyze the problem.
Step 4:	Draw a systematic inventory of the explanations inferred from step 3.
Step 5:	Formulate learning goals.
Step 6:	Collect additional information outside the group.
Step 7:	Synthesize and test the newly acquired information.

Tomaz, van der Molen, and Mamede (2013) suggest PBL and explain how it works in the following excerpt:

> "Summarizing, PBL works as follows: from analysis and reflection of a problem situation presented, the participants in small groups (tutorial groups) identify their key knowledge gaps and establish what they need to learn (learning goals) to solve the problem (Schmidt, 1983). During the study of the problem, participants have to rely on literature research, personal study, consultations with specialists, if necessary, and other sources of information, in order to achieve the learning objectives, and at the end of cycle, solve the problem.[…]" (p. 12)

Isaksen, Dorval, and Treffinger (2000) also suggest a plan for the *creative problem solving*, consisted of four components (**Figure 5.5**): (a) Understanding the problem/the challenge (b) Generating ideas (c) Preparing for action and (d) Planning the approach, everything of which is subdivided in other stages as is reported in the figure above (cited in Hwang, Chen, Dung, Yang, 2007, p. 194).

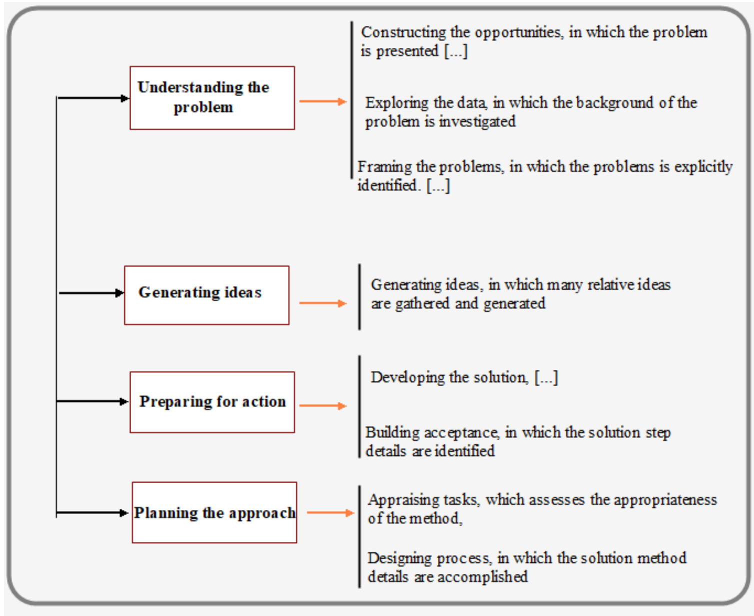


Figure 5.5. Creative problem solving (Isaksen, Dorval, and Treffinger, 2000, cited in Hwang, Chen, Dung, Yang, 2007, p. 194): (an adaptation for the current study)

Jonassen (2004) in his monograph "*Learning to solve problems*" also argues that

> "*Successful problem solving* requires that learners actively manipulate and test their models. Thinking is internalized activity (Jonassen, 2002), especially when solving problems, so knowledge and activity are reciprocal, interdependent processes (Fishbein and others, 1990). We know what we do, and we do what we know. Successful problem solving requires that learners generate and try out solutions in their minds (mental models or problem spaces) before trying them out in the physical world" (p.7).

Stoyanova (1997) identified (a) free situations, (b) semi-structured situations and (c) structured situations to improve students' *problem posing and problem solving* in a range of classroom contexts. As she states (p.63-69):

- "*In free problem-posing situations*, students are asked to generate a problem from a given, contrived or a naturalistic situation. (e.g. describe some problems which relate to the right angled triangle).
- *In semi-structured problem posing situations*, students are given a situation in which they are invited to explore and formulate a problem which would draw on the knowledge, skills, concepts and patterns gained from their previous mathematical experiences.
- *In structured problem-posing situations*, problem-posing activities are based on a specific problem or a written solution".

Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman (2005) in their work "*An Empirical Taxonomy of Problem Posing Processes*" also identified a theoretical model of problem posing as follows: "editing quantitative information, their meanings or relationships, selecting quantitative information, comprehending and organizing quantitative information by giving it meaning or creating relations between provided information, and translating quantitative information from one form to another" (p. 149).

A student can develop *successful problem solving* if s/he fulfills the following factors involved in the problem solving process (Stacey, 2005, p.342): Students must have as prerequisite deep mathematical knowledge and general reasoning abilities, as well as the ability to implement heuristic strategies for solving non-routine problems. It is also necessary to have "helpful beliefs (e.g. orientation to ask questions)" and "personal attributes (e.g., confidence, persistence, organization) for putting in order their thoughts, organizing and managing their actions. Also, students should develop their communication skills and the ability to work with other students effectively in cooperation.

Stacey (2005) created a diagram to present all the factors involved to successful problem solving process (**Figure 5.6**)

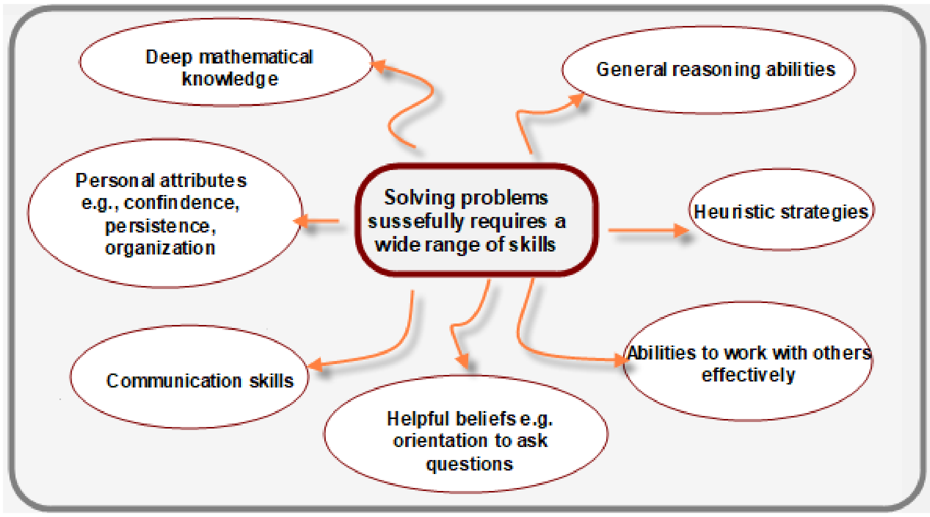


Figure 5.6. Factors involved to successful problem solving process (Stacey, 2005, p.342, cited in Anderson, 2008, p. 2) (an adaptation for the current study)

In my study "*From Vecten's Theorem to Gamow's problem: building an empirical classification model for sequential instructional problems in geometry*" (Patsiomitou, 2019a), I present an empirical classification model for sequential instructional problems in geometry, concerning the importance of students building a representation of a problem, the role which modeling a real-world problem plays in the students' gradual investigation of a problem. My work with students at the secondary and tertiary levels leads me to identify *five types of geometrical problems* (Patsiomitou, 2019a, p.3):

- *Dynamic geometrical problems with non-given answers* (abbreviated as **DGNA**) which the students investigate in a DGS environment using linking visual active representations (LVARs) (e.g., Patsiomitou, 2008a, b, 2012a, b). Such problems improve motivation and creativity through the use of "why" challenges and "what if" strategies; provoke students' reflecting visual reaction (RVR) (e.g., Patsiomitou, 2008a, b, 2012a, b), by requiring them to employ preexisting theoretical knowledge, perceptual skills, and deductive argumentation.
- *Dynamic geometrical problems with given answers* (abbreviated as **DGGA)** which the students investigate and prove in a DGS environment. Such problems motivate students to create theoretical relationship between information and data which is explicitly provided; to translate this information and data from one form of representation to another and to employ their preexisting theoretical knowledge and deductive reasoning skills.
- *Dynamic geometrical problems modeled in a DGS with hybrid–dynamic geometrical representations* (Patsiomitou, 2018b, p.42) *with non-given answers* (abbreviated as **HGNA**) which the students investigate in a DGS environment. Such problems require the students to interact with a sophisticated level of information and data which is explicitly provided in the DGS environment and to employ advanced theoretical knowledge and abstract thinking.
- *Real world geometrical problems with non-given answers* (abbreviated as **RGNA**) which students investigate in a dynamic or static environment. Such problems relate to 'dynamic' methods in geometry and require students to 'think in motion' in the environment, employing higher order thinking and organizing phenomena by means of progressive mathematization. The benefit of working with real problems in a DGS incorporates the combination of transformations using Linking Visual Active Representations (LVARs).
- *Static geometrical problems with given answers* (abbreviated as **SGGA**) which students solve in a paper-pencil environment. Such problems contain certain information and questions which require students to apply their theoretical knowledge and perceive the structure of the problem and the principles and concepts that could be used to solve it.

The investigational activity of problem solving in a DGS, prompts the students to develop more *reflective ways of thinking* and the teacher to describe the problem in a way, which might be more interesting than in traditional approaches. Moreover, a teacher's intention for his/her students to learn through problem solving investigational process is associated in the words of Tony Brown, (1994) with the "*presupposition about that to be learnt and learning is in a sense revisiting that already presupposed*" (p.148). Tall (2004) used a metaphor of a "traveler" to explain how "different individuals may develop substantially different paths on their own *cognitive journey* of personal mathematical growth". As he argues:

> "As an individual travels […], various obstacles occur on the way that requires earlier ideas to be reconsidered and reconstructed, so that the journey is not the same for each traveler. On the contrary, different individuals handle the various obstacles in different ways that lead to a variety of personal developments, some of which allow the individual to progress through increasing sophistication in a meaningful way while others lead to alternative conceptions, or even failure" (Tall, 2004, p. 286).

Battista (2011) also in his work "*Conceptualizations and Issues related to Learning Progressions, Learning Trajectories, and Levels of Sophistication*" determines differences between the use of terms "*stage*" and "*level*" between researchers. Moreover, he defines the theoretical construct "*a level of sophistication*" in the following paragraph, through which he characterizes students' development of conceptualizations and reasoning:

> "Clements and Battista (1992) described the difference between researchers' use of the terms stage and level as follows. A stage is a substantive period of time in which a particular type of cognition occurs across a variety of domains (as with Piagetian stages of cognitive development). In contrast, a level is a period of time in which a distinct type of cognition occurs for a specific domain (but the size of the domain may be an issue). Battista defines a third construct—*a level of sophistication* in student reasoning as a qualitatively distinct type of cognition that occurs within a hierarchy of cognition levels for a specific domain" (Battista, 2011, p.517).

In my opinion, the teacher's investigational activity in relation to the problem posed has to be implemented at several *levels of sophistication*, if a teacher is to help his/her students to develop deeper understanding and coherent reasoning. Summarizing, I would like to present *five investigational levels of a problem solving process*, synthesizing, elaborating on and addressing conceptual and procedural understanding through feedback provided at every intermediate step in the problem's solution which is designed in the light of the cognitive processes elicited at each level. My aim is to construct a didactic sequence in which the next problem will become the next level in the development of the students' reasoning. Thus, constitutes a *cognitive trajectory* through problem solving for the students' cognitive development (Patsiomitou, 2019, p. 19):

- *The first level of sophistication* is that of open problems using materials (e.g., squared papers, dot papers, or several means, including DGS). This phase can be extended by means of DGNA problems using sequential dynamic LVAR representations. When a student is engaged with the activity of solving a problem modeled by dynamic LVARepresentations s/he connects that activity with both the product and the thought process during investigational process. LVARs scaffold students' mental processes such as perception, information recall and reasoning. Students can also discover the solution through active experimentation.
- *The second level* comes after the introduction of "big ideas" or "core ideas" (Battista, 2011). During this phase, the teacher can use DGGA problems posed for investigation and proof in a DGS environment. The students can mentally combine structural properties of conceived cognitive processes.
- *The third level* is that of real world HGNA problems which are modeled in a DGS environment using dynamic or hybrid-dynamic representations. A teacher can support students' reasoning by giving them other immediate problems which will scaffold the theoretical background required by the problem as they investigate all the possible or multiple solutions to the problem. They can also investigate a concrete situation of the hybrid-dynamic representations, choosing to give to the parameters concrete magnitudes.
- *The forth level* will be that of RGNA problems, accepting a challenge and trying to reinvent the solution. The students at this level must have the conceptual and procedural competence to investigate the problem. At this level, the problem cannot be solved by some routine procedures.
- *The fifth level* will be that of the problem in a SGGA problem in a static environment. This is the level with the higher degree of difficulty. This is why students are not able to solve static geometry problems, when they belong at the lower van Hiele levels.

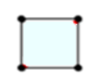
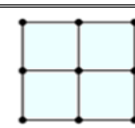
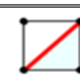
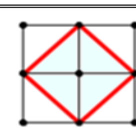
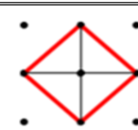
The emerging theoretical construct provides both a methodology for building up the problem-solving process and an approach to addressing difficulties students face in learning geometrical concepts, which uses anticipatory thought experiments in which we envision how we can construct an organizational structure and a learning trajectory through problem solving as the students engage with the process.

5.2. Proof and Proving in the Problem Solving Process

Freudenthal (1971) in his study "*Geometry between the devil and the deep sea*" responding to his own questions, writes:

> "[…] the first piece of education in history we know about, is a lesson of geometry, the Socratic lesson Menon's slave was taught on doubling the square. Socrates taught the slave not the solution of the problem nor solving the problem, but finding the solution by trial and error. He did not teach a readymade solution but the way of reinventing the solution. Two millenia later Comenius said: 'The best way to teach an activity is to show it.'[…]"(p. 414).

This piece of knowledge made me consider a mixed method which my students could use to solve a problem; such a method would require me to design a way for the students to reinvent the solution or discover it using a trial and error method. From a lack of competence my students (13-14 years-old) to composing geometric shapes the "guided" reinvention of doubling the square mentioned in the Socratic lesson, stimulated the use of materials -digital or not- in my class, with which my students could support their reasoning by transforming the shapes, using a trial and error method.

Researcher: *How can you double the area of the square?* **Students:** *We can double the sides of the square!*	
Researcher: *Use this dot paper and construct the new square.* **Students:** *Oh! It is not correct! The square is quadrupled. We have constructed 4 equal squares with the small square.*	
Researcher: *Now, can you find a solution to the problem?*	
……………………………… **Students:** *If we construct the diagonals of the quadrupled square, the new figure inside is also a square …..and it is the double square of the initial square.* **Researcher:** *Can you answer why that happens?*	
Students: *Its sides are congruent and perpendicular to each other.* **Researcher:** *Can you prove that the new square has double area of the initial square?*	

The discussion mentioned above is one I have with my high school students (aged 13-14 years-old) almost every year in class. Only a few students have the competence to answer the last question. This was/is difficult for them, as they did/do not have the competence to transform the right and isosceles triangle in their mind; in other words, they could not generate *mental transformations*. Many students do not have the ability to dynamically visualize and mentally manipulate geometric objects, which is an important skill for solving problems in geometry. Without it, they cannot reflect on or anticipate a possible solution to the problem.

Moreover, according to the van Hiele theory (Fuys et al., 1984) students are not able to formulate *deductive argumentations* as this kind of argumentation occurs when the students have developed their thinking. Freudenthal (1971) supports that

> "In which order, if not in a deductive one, should mathematics be taught? The answer is simple: in that one in which it can be learned, which means, the order in which it could be invented by the student. This is not at all a revolutionary idea. It is the Socratic lesson. In a thought experiment the teacher has been

reinventing the subject matter as though he himself was the student, and this is what he teaches. […] This is a modern reinforcement of the socratic idea" (p.416).

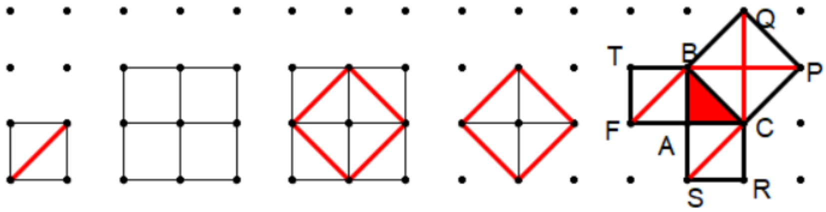


Figure 5.7a. Transforming the shapes using a mixed 'trial and error' and 'guided reinvention" method in my class (Patsiomitou, 2019a, p. 4).

The teachers' task is to design a course "of action that fits anticipated student reactions. More precisely, the idea is that teaching matter is re-invented by students in such interaction" (Gravemeijer & Terwel, 2000, p.786). With regard to the problem mentioned above, firstly, I usually ask my students to experiment using transformations (e,g,. a dot.gsp file or a squared paper) this will help them understand that if they double the side of the square, the area of the square this creates is quadrupled. (**Figure 5.7a**). Freudenthal (1971) supports that

> "[…] transformations in geometry were long ago advocated by F. Klein as a consequence of his so-called Erlanger Programm. The breakthrough of transformations in geometry is of a rather recent date. How to explain this delay, […], where Klein had been the venerated master of a generation of teachers?". [Moreover], "there is not any textbook based on the transformation idea" (p. 433).

Manipulatives constructed from cardboard are an easy and effective way for students to understand a theorem empirically (in a *collaborative learning process*). For example, a student of mine constructed the Pythagorean Theorem using cardboards in two different colours to illustrate it (**Figures 5.7b**).

Figures 5.7b. Transforming the shapes using manipulatives (student's construction, in Patsiomitou, 2012, p.61, in Greek)

The meanings of *collaborative learning and cooperative learning* has been clarified by Kaufman, Sutow, & Dunn (1997) who argue that

> "*Collaborative learning* is a spectrum of instruction that involves small groups of students who have been assigned an academic goal. At one end of the spectrum are transient groups that may be formed to quickly generate some ideas for immediate in-class discussion (e.g., "buzz" groups). *Cooperative learning* is at the other end of the collaborative learning spectrum, since it is a carefully planned learning strategy that involves forming appropriate, sustained learning groups of interdependent members who have been assigned a specific learning goal. Emphasis is placed on student involvement in *active learning* and the development of social skills. Since the outcomes of cooperative learning are strongly dependent on detailed planning and implementation, cooperative learning has become the most operationally well-defined and procedurally structured form of collaborative learning" (p.38)

The use of material figures helped my students gain competence in composing geometric shapes, initially through trial and error and then purposefully find that four congruent isosceles and right triangles can be composed into a square and, ultimately, to intentionally synthesize combinations of shapes into new shapes with a view to reinventing a rule or a theorem. This concrete experimentation on the part of my students is also an excellent mean of incorporating worthwhile ideas and introducing theorems and definitions into my lessons (for example, the Pythagorean Theorem and irrational numbers). Many researchers, mathematicians and mathematics educators (e.g., Bell, 1976b; Hanna, 1983; de Villiers, 1990, 1999; Hanna & Jahnke, 1996; Marrades & Gutierrez, 2000; Varghese, 2017) have recognized different functions of proof and proving as: *verification, justification, explanation, discovery, systemization* etc. because the proving process can provide insight and discovery, justify or verify why a statement is true. Generally speaking, geometric figures or diagrams constitute a unique framework for communicating mathematical ideas, very important for students' development of thinking, especially when technology is incorporated to their construction (**Figure 5.7c**).

Figure 5.7c. Guiding my students to construct figures in different representational environments

Complementary to this, a deductive system of axioms, theorems and propositions as well as concepts and definitions can help the students to organize the proving process.

> "Proving was born as a social act aimed at convincing the listener (Barbin, 1988). The first step consisted of admitting the existence of some initial points, which in Euclid's Elements are named postulates; these are self evident and as such are considered to be true. Proofs for Euclid are chains of propositions derived deductively from initial propositions about primitive objects (postulates). Since the postulates are true, also the other derived propositions are true." (Olivero, 2003, p. 12).

The postulates determined by Euclid in his "Elements" regulate geometrical *deductive reasoning*, formulating the "rules" by which a person can synthesize a proposition in a meaningful and logical manner. According to historians and scholars Euclid's "Elements", was considered to be the most influential textbook. It has been posited that the "Elements" is the second most printed book after the Bible. In the words of Dionysius Lardner (1855) in the preface of his book "*The first six books of the Elements of Euclid*":

> "Two thousand years have now rolled away since Euclid's Elements were first used in the school of Alexandria, and to this day they continue to be esteemed the best introduction to mathematical science. They have been adopted as the basis of geometrical instruction […and] has been adopted as a universal standard".

Evaggelos Stamatis (1957) concretely reports:

> "The first Book of "Elements" includes 23 definitions, 5 postulates, 9 Common Notions and 48 Propositions and problems […]. The first 26 Propositions concern triangles in general […]. The proving methods in "Elements" are four: *synthetic, analytic, proof by contradiction, and proof by induction* […]. Using the synthesis method, when we try to prove a geometric proposition, we proceed from well-known proposals based on definitions and axioms and arrive at the truth of the proposed proposal through a series of appropriate reasoning." (p.17) (my translation of Evaggelos Stamatis' Greek-language manuscript).

The synthetic method *synthesizes* basic objects of Euclidean Geometry (e.g. points, lines) in a formal way using definitions, axioms and propositions. Speaking of logical inferences and deductive argumentation, for me the propositions regarding triangle congruence in Euclid "Elements" are crucial for students to understand and implement in the problem-solving process. Can these fundamental propositions of plane geometry in which triangles are congruent, (included in Book 1 of Euclid's Elements) be transformed in a DGS software? I shall explain their instrumental decoding in Sketchpad in the light of having in mind the following excerpt written by Dina van Hiele (Fuys et al, 1984)

> "[…] the deductive system of Euclid from which a few things have been omitted cannot produce an elementary geometry. In order to be elementary, one will have to start from the world as perceived and as already partially globally known by the children. The objective should be to analyze these phenomena and to establish a logical relationship. Only through an approach modified in that way can geometry evolve that may be called elementary according to psychological principles" (p.24)

This is in accordance with what Furinghetti & Paola (2003) support:

> […] When [Greek geometers] made proofs they were not inside a theory in which axioms were explicitly declared. Initially antique geometry developed in an empirical way, through a naïve phase of trials and errors: it started from a body of conjectures, after there were mental experiments of control and proving experiments (mainly analysis) without any sure axiomatic system. According to Szabo, this is the original concept of proof held by Greeks, called deiknimi. The deiknimi may be developed in two ways, which correspond to analysis and synthesis" (p.398)

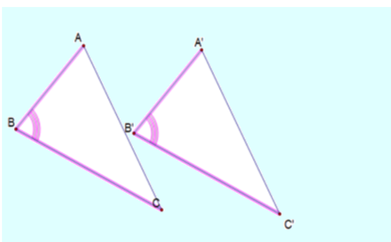


Figure 5.8a. A diagram for Euclid's Proposition I.4 in Sketchpad. (Patsiomitou, 2019a, p. 10).

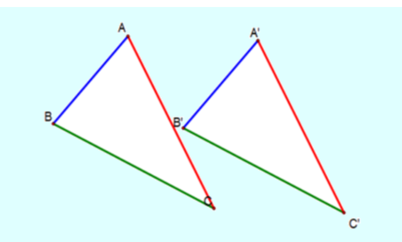


Figure 5.8b. A diagram for Euclid's Proposition I.8 in Sketchpad. (Patsiomitou, 2019a, p. 10).

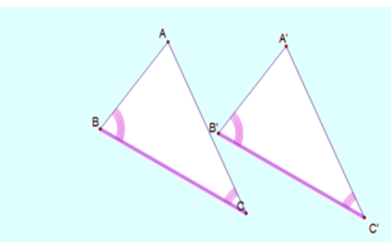
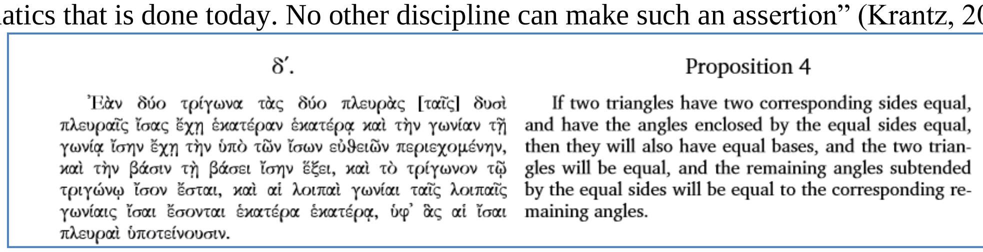


Figure 5.8c. A diagram for Euclid's Proposition I.26 in Sketchpad. (Patsiomitou, 2019a, p. 10).

The three cases in which triangles are congruent are illustrated in the **Figure 5.8a, b, and c**. "Deiknymi" or "apodeiknio" in Greek (translated as "proving" in English) can be represented visually in a dynamic geometry system (DGS) using Linking Visual Active Representations (LVARs) (e.g., Patsiomitou, 2008c, 2009a, b, c). In other words, "deiknimi" can be visualized using Sketchpad' interaction techniques (for example, custom tools, "animating" tools, 'tracing" tools, "hiding and showing" action buttons, and "linking" or "presenting" action buttons, or a combination of interaction techniques in Sketchpad) (e.g., Patsiomitou, 2008a, b; 2010; 2012a, b). The interaction with LVARs has two aspects similar to what Sedig, Rowhani, & Liang (2005, p.422) support regarding VMRs: "the action upon a representation by the user through the intermediary of a human-computer interface, and the representation communicating back through some form of reaction or response." Lopez-Real and Leung (2006) state that DGS including dragging "[…] as a fundamental geometrical object (like that of point, circle)," determines "new 'rules of the game,' or even a new game for geometry'' (p. 676).

> "There is no other scientific or analytical discipline that uses proof as readily and routinely as does mathematics. This is the device that makes theoretical mathematics special: the tightly knit chain of reasoning, following strict logical rules, that leads inexorably to a particular conclusion. It is proof that is our device for establishing the absolute and irrevocable truth of statements in our subject. This is the reason that we can depend on mathematics that was done by Euclid 2300 years ago as readily as we believe in the mathematics that is done today. No other discipline can make such an assertion" (Krantz, 2007, p.1)



δʹ.	Proposition 4
Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἕξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν.	If two triangles have two corresponding sides equal, and have the angles enclosed by the equal sides equal, then they will also have equal bases, and the two triangles will be equal, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.

Figure 5.9. Screenshot of an excerpt included in Euclid' Elements (Fitzpatrick, 2007, p.10)

Proposition I.4 (known with the abbreviation SAS): If two triangles have two corresponding sides congruent and the angles enclosed by the equal sides congruent the two triangles are congruent (SAS) **(Figure 5.9).** We can take it as given that two segments are congruent if they have the same "length" and, similarly, that two angles are congruent if they have the same "angle measure". The method used by Euclid to prove proposition I.4, regarding triangle congruence is a combination of: the method of superposition and the method of proof by contradiction. Initially, the first part of the proposition is proved by moving one of the two triangles so that one of its sides coincides with the other triangle's equal side; it is then proved that the other sides coincide as well. (**Figure 5.10**) (Webpage [30])

Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΔΕ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Ε διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΔΕ· ἐφαρμοσάσης δὴ τῆς ΑΒ ἐπὶ τὴν ΔΕ ἐφαρμόσει καὶ ἡ ΑΓ εὐθεῖα ἐπὶ τὴν ΔΖ διὰ τὸ ἴσην εἶναι τὴν ὑπὸ ΒΑΓ γωνίαν τῇ ὑπὸ ΕΔΖ· ὥστε καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ σημεῖον ἐφαρμόσει διὰ τὸ ἴσην πάλιν εἶναι τὴν ΑΓ τῇ ΔΖ. ἀλλὰ μὴν καὶ τὸ Β ἐπὶ τὸ Ε ἐφηρμόκει· ὥστε βάσις ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει. εἰ γὰρ τοῦ μὲν Β ἐπὶ τὸ Ε ἐφαρμόσαντος τοῦ δὲ Γ ἐπὶ τὸ Ζ ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ οὐκ ἐφαρμόσει, δύο εὐθεῖαι χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. ἐφαρμόσει ἄρα ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ καὶ ἴση αὐτῇ ἔσται· ὥστε καὶ ὅλον τὸ ΑΒΓ τρίγωνον ἐπὶ ὅλον τὸ ΔΕΖ τρίγωνον ἐφαρμόσει καὶ ἴσον αὐτῷ ἔσται,

Figure 5.10. Screenshot of an excerpt of the proof used by Euclid to prove proposition I.4, mentioned in Euclid "Elements" (Mourmouras, 1999) (See Website [31]).

The paragraph mentioned above in Ancient Greek is translated as follows (Fitzpatrick, 2007, p.10):

> "[…] Let the triangle ABC be applied to the triangle A΄B΄C΄, the point A being placed on the point A΄, and the straight-line AB on A΄B΄. The point B will also coincide with B΄, on account of AB being equal to A΄B΄ […]. For, if B coincides with B΄, and C with C΄ and the base BC does not coincide with B΄C΄, then two straight lines will encompass an area. The very thing is impossible. Thus, the base BC will coincide with B΄C΄[…]".

According to Krantz (2007)

> "One of the most important proof techniques in mathematics is "*proof by contradiction*". With this methodology, one assumes in advance that the desired result is false and shows that that leads to an untenable position. But in fact proof by contradiction is nothing other than a reformulation of modus ponendo ponens" (p.6)

Moreover, in the words of Lardner (1855)

> "*Superposition* is the process by which one magnitude may be conceived to be placed upon another, so as exactly to cover it, or so that every part of each shall exactly coincide with every part of the other" (p.5). […] In the superposition of the triangles in this proposition, three things are to be attended to: (a) The vertices of the equal angles are to be placed one on the other. (b) Two equal sides to be placed one on the other. (c) The other two equal sides are to be placed on the same side of those which are laid one upon the other. From this arrangement the coincidence of the triangles is inferred (p. 18).

In the Sketchpad software, this method could be *instrumentally decoded* (Patsiomitou, 2011a, b) by a user using translation transformation, a digital method of "superposition", in which a figure is transferred to another point in space, using a dynamic vector. Concretely, the triangle on the right (A΄B΄C΄) can be produced using the translation transformation, on the triangle ABC (**Figure 5.11a**). A combination of transformations (translation & dragging) also indicates the triangles' congruency by a *superposition method*. The students can also drag the vector and apply the triangle ABC on the triangle A΄B΄C΄ and justify why this is the case. We can also use predesigned movement action buttons to move the triangle ABC onto the triangle A΄B΄C΄ so the two can be superposed confirming the triangles' congruency. We can use Sketchpad's customized 'appearance tools' to indicate the congruent angles, and we can also highlight or color the triangles' corresponding congruent sides in order to point out the congruency. These are the signs that can be visualized by a student during the investigation of the concrete theorem and which indicate congruency. The students can also measure the angles and the sides, and investigate the congruency of the triangles through experimental dragging (e.g., Patsiomitou, 2011a, b; 2012a, b, 2019a) the congruency of the triangles and the power of the theorem. In other words, the experimental dragging leads to a theoretical observation.

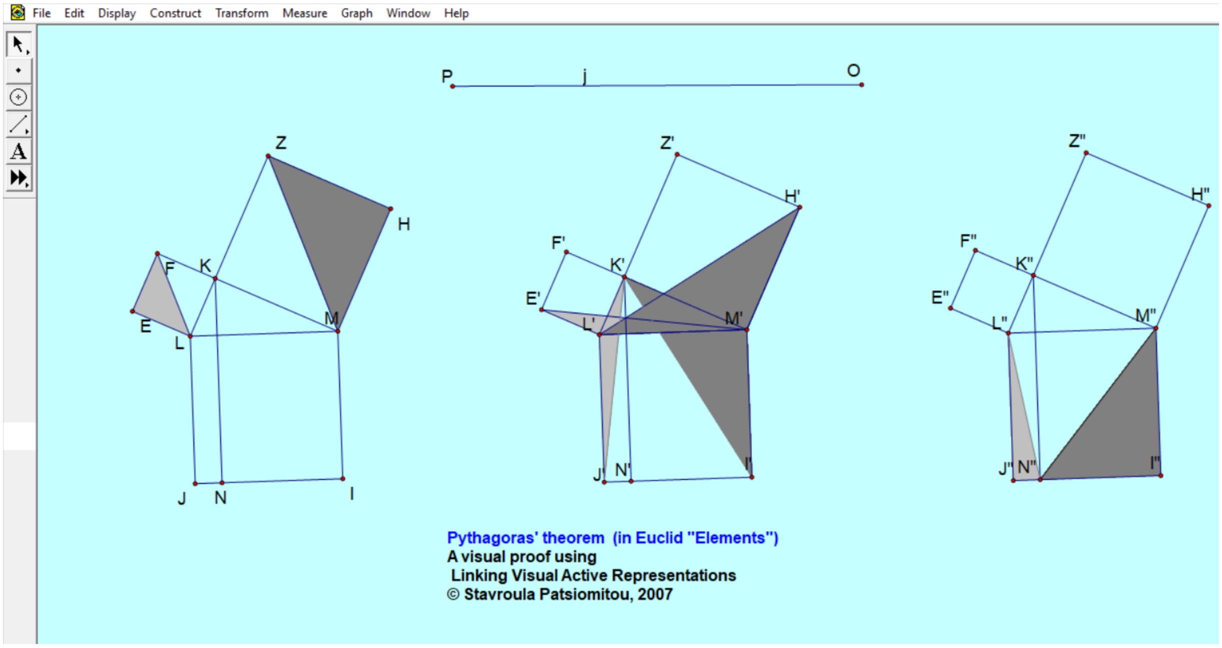


Figure 5.11a. Proposition I.47 using LVARs (Mode 3): A visual proof in three linking diagrams (Patsiomitou, 2019a, p.11) (modified)

Triangles' congruence is used in many proofs. For the Euclidean proof of the Pythagoras' Theorem I have created the three consequential visual representations using the translation transformation (**Figures 5.11a, b**). Every object of the first construction on the left has been translated by the vector j to an image object on the right and the outline figure can be superposed on it. Every representation on the right is more complex and supports the next consequential step on the problem's solution. Nunokawa & Fukuzawa (2002), report Sohma (1997) who stated that "he wanted his students to experience a feeling of 'why?' so that they would be motivated to solve [geometry] problems" (p.31). As Nunokawa & Fukuzawa (2002) argue "the students' feeling of 'why?' was influenced by their understanding of a problem situation" (p. 41). In the current situation the students ask themselves "why is this happening?" at every sequential step. For example, they might ask: why does triangle EFL has an area congruent to the area of the triangle E΄L΄M΄? (: they have the same base EL and the heights of the triangles to the base EL are equal magnitudes). Thiele (2003) explains the meaning of magnitude as follows: "There is no definition of the concept of magnitude (Greek megathos) because there is no superior concept for this fundamental concept. Nevertheless, Euclid is dealing with magnitudes throughout the Elements; […] Magnitudes are generally characterized by the property of being able to increase and decrease" (Thiele, 2003, p. 6). The following questions could also support the structure of the Euclidean proof:

- Why does triangle E΄L΄M΄ has an area congruent to the area of the triangle K΄L΄J? (: they are congruent triangles, so they have congruent areas).
- Why triangle K΄L΄J has an area congruent to the area of the triangle L΄΄J΄΄N΄΄(: the base and the height of the triangles are equal magnitudes).

If we drag any point of the LVARepresentation, the image-points follow the movement also, turning the whole dynamic diagram to an *active alive* one in which we can view sequential transformations that indicate a path for the rigorous proof of the Pythagorean Theorem. The triangle EFL is visually transformed to the triangle E΄L΄M΄, then to the triangle K΄L΄J, and finally to the triangle L΄΄J΄΄N΄΄ (**Figure 5.11a**). Similarly, the triangle ZHM is visually transformed to the triangle L΄M΄H΄, then to the triangle K΄M΄I and finally to the triangle M΄΄N΄΄I΄΄ (**Figure 5.11a**). Consequently, the area of the square FKLE plus the area of the square ZHMK is transformed into the area of the square LMIJ. We can also create an LVARepresentation using more sequential steps, every object on the right side occurs as a translation image of the object on the left side (**Figure 5.11b**). The whole process scaffolds students thinking, given that they cannot visualize / hold all the intermediate steps in their heads for the solution.

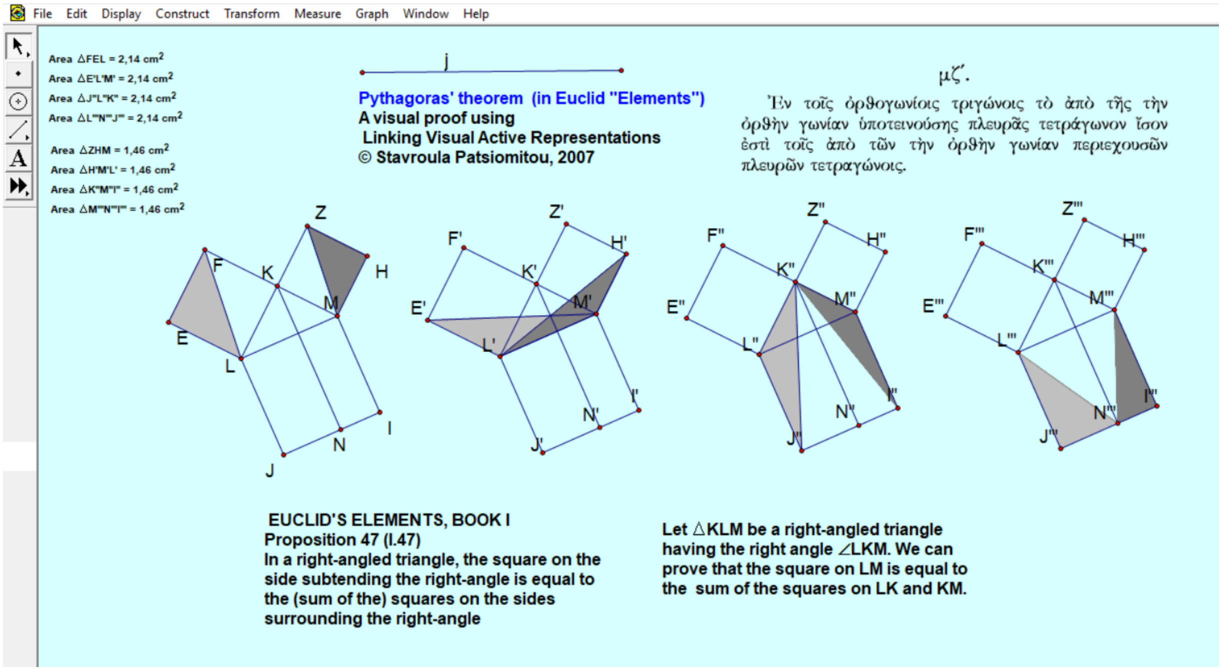


Figure 5.11b. Proposition I.47 using LVARs (Mode 3): A visual proof in four linking diagrams (Patsiomitou, 2019a, p.12) (modified)

If the vector's length is tending to zero, then the vectors' endpoints coincide. This result to the following representation illustrated in **Figure 5.11c** in which we can view the initial triangle EFL transformed to the final triangle LJN, as well as the auxiliary triangles for the visual proof in blue and yellow (i.e., the sequential diagrams have been superposed to the first diagram on the left).

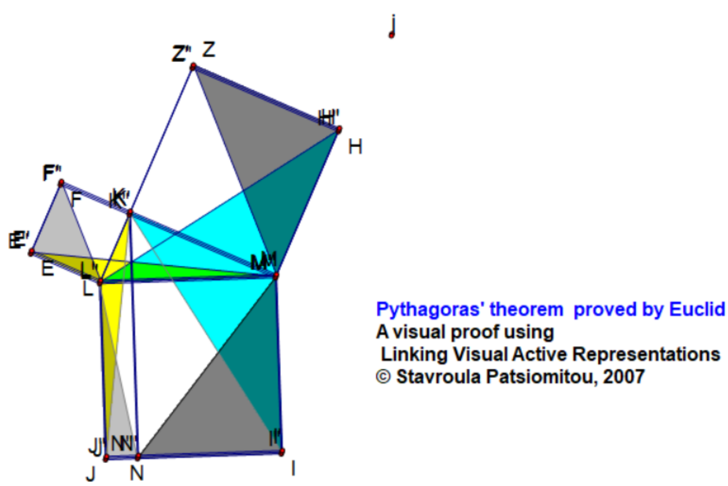


Figure 5.11c. Proposition I.47 using LVAR (Patsiomitou, 2019a, p.12)

5.3. Modeling a real-world problem in a DGS environment

Many researchers (e.g, Burkhardt, 1981; Pierce & Stacey, 2009) have highlighted the idea of solving problems in the real world as essential to understanding and learning mathematics, as well as "a key ability for citizens [who are prepared to make] judgments and decisions" (Stacey, 2012, p.3).
According to De Corte, Verschaffel & Greer (2000), the implementation of the mathematics to solve real world problems can be useful "as a complex process involving a number of phases:

- *understanding* the situation described;
- *constructing a mathematical model* that describes the essence of those elements and relations embedded in the situation that are relevant;

- *working through* the mathematical model to identify what follows from it;
- *interpreting the outcome* of the computational work to arrive at a solution to the practical situation that gave rise to the mathematical model;
- *evaluating that interpreted outcome* in relation to the original situation;
- and *communicating the interpreted results*" (p.1) (**Figure 5.12**).

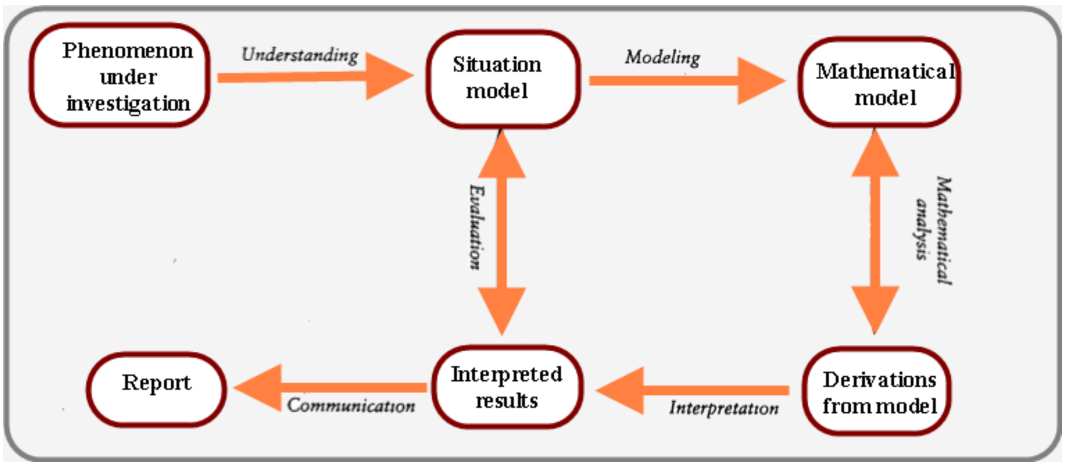


Figure 5.12. Solving real-world problems (De Corte, Verschaffel & Greer, 2000, p.71) (adapted)

De Corte, Verschaffel & Greer (2000) support also that "the […] process of modeling constitutes the bridge between mathematics as a set of tools for describing aspects of the real world, on the one hand, and mathematics as the analysis of abstract structures, on the other" (p.71). If the teaching and learning is based on real –world problem solving modeled in a DGS environment the teacher

> "[…] apart from the aspect of anticipating the mental activities of the students, [...] has to investigate whether the thinking of the students actually evolves as conjectured, and he or she has to revise or adjust the learning trajectory on the basis of his or her findings. In relation to this, Simon (1995) speaks of a mathematical teaching cycle. In a similar manner, Freudenthal (1973) speaks of thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment." (Gravemejer, 2004, p.9).

A real-world word-problem (or an oral mathematical problem) can be illustrated in various types as an image in textbooks or on the board in class (e.g., a picture, a diagram, a table, etc). In this way, a teacher, educator or student can translate a problem's verbal representation into a visual mathematical representation in an effort to convey information and translate from one form of representation to another. In this way, a bridge can be created between the real-world environment, the symbolic representations and the abstract world of a student's thinking, just as Goldin & Janvier (1998) describe/interpret or define the term "*representation*" and "*system of representation*", in connection with mathematics teaching and learning (Goldin & Janvier, 1998, p.1). Most scholars around the world concur in the view that translation and links between mathematical representations are fundamental to understanding how students construct mathematical concepts and solve problems (e.g., Duval, 1993; Eisenberg & Dreyfus, 1990; Janvier, 1987; Kaput, 1994; Presmeg, 1986; Vergnaud, 1987). Kaput et al. (2002) in their paper "*Developing New Notations for a Learnable Mathematics in the Computational Era"* analysed the ways "we use to present and re-present our thoughts to ourselves and to others, (in order) to create and communicate records across space and time, and to support reasoning and computation" (p. 2) namely "how in the evolution of the new representational infrastructures, and the associated artifacts and technologies have, over long periods of time, gradually externalized aspects of knowledge and transformational skill that previously existed only in the minds and practices" (p.33).
Real world images (or digital images) "are potential representations […and] offer the heuristic part of learning" as they "denote something" (Kadunz & Straesser, 2004, p.241, 242). What is important is how the students perceive these potential representations of the environment (natural images or digital), how they use and communicate with each other and how they manage their mental mathematical structures in order to represent the objects in a static or dynamic environment. Mogeta, Olivero & Jones (1999) in their report "*Providing the Motivation to Prove in a Dynamic Geometry Environment*" argue that "setting problem solving within these

environments requires a careful design of activities, which need to take into account the interaction between three elements: the dynamic software, as an instance of the *milieu,* a problem, and a situation, through which the devolution of the problem takes place (Brousseau, 1986)". Most importantly, the diagrams that the students are obliged to translate and the relations that link the objects in the diagram will provide researchers and teachers insights to see their abilities and their weaknesses with respect to the mathematical knowledge that they have structured as a result of the teaching process in class. For this, the verification of students' mistakes and cognitive obstacles during the construction of diagrams will lead us to the reinforcement of the teaching of mathematics in the context of real-world problems.

Goldin (2008) in his study "*Perspectives on representation in mathematical learning and problem solving*" describes what a *model* is, as well as what is *modeling.*

> "a *model* is a specific structure of some kind that embodies features of an object, a situation, or a class of situations or phenomena-that which the model *represents.* The term *modeling* refers to the construction of models-of meaningful structures, within one or more representational systems (possibly mathematical, possibly physical or iconic, possibly digitally-encoded and dynamic). When appropriately interpreted, the model describes some but not other aspects of the relevant situations or phenomena; hence the central importance of the meaningfulness of the representations within which the model has been constructed "(p. 186).

Doerr & Pratt (2008) also in their article "*The Learning of Mathematics and Mathematical Modeling*" state that:

> "A model is a system of objects, relationships, and rules whose behavior resembles that of some other system. Modeling is the activity of mapping from one system to another. This activity is driven by the need to describe, predict, or explain some particular phenomena of interest to the modeler. Elements from the real world of the experienced phenomena are selected, organized, and structured in such a way that they can be mapped onto a model world. This model world necessarily simplifies and distorts some aspects of the real world while maintaining other features and allowing for manipulations of these features (or objects) in accordance with the rules of the model world". (p.261) (**Figure 5.13)**

If the students are engaged in solving a real world problem this process is underlied by the characteristics of the philosophy of *Realistic Mathematics Education* (abbreviated as RME), developed at the Freudenthal Institute and restricted here to the aspect that mathematics should be learned as an activity of progressive mathematization, distinguished to *horizontal mathematization* and *vertical mathematization* (e.g., Treffers,1987; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996; Drijvers, 2003). *Horizontal mathematization* in real world situations refers to the process of modeling from the real world to the model world using mathematical representations. In other words, horizontal mathematization is a process through which a real problem is transformed to a model. *Vertical mathematization* concerns the mathematical abstract process in a higher level of abstraction, connecting concepts and strategies.

Graumann (2005) in his study "*Investigating and ordering quadrilaterals and their analogies in space-problem field with various aspects*" introduces the notion of "a *problem field*". According to Graumann (ibid.) "*a problem field* is a set of problems which are related to each other and have a generating problem (see e.g. Pehkonen 2001). This means that we do not work only with isolated problems (like in mathematical Olympiads) but with a mathematical field or a situation of everyday life where the students besides solving problems also can make investigations, pose problems and find connections, new insights or even mathematical theorems as well as discuss the ways and the limitations of modelling" (p.190).

As I mention in many papers (e.g. Patsiomitou, 2008a, 2014), my further aims, were/are the student's *mathematical literacy* and problem-solving literacy. The latter PISA (*Programme for International Student Assessment*) definition of *mathematical literacy* is as follows (OECD, 2010):

> "*Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens." (p. 4)*

It is very important for students to develop their modeling competency in order to transform real-world problems from the three-dimensional world to the two-dimensional world of the paper and pencil [or DG] environment. Additionally, it is important for them to be able to process in an abstract way. Epigrammatically, the students, through the real world problems can be evaluated with regard to the development of the competencies (OECD, 2006), that have been analyzed from Niss (1999) and his colleagues but similar formulations can be found in the work of many others (e.g., Neubrand et al. 2001) (see Chapter IV, section 4.3). Through the solution of the real world problems, students can be also evaluated regarding their competency for *horizontal and vertical mathematization* (Jupri, Drijvers, & van den Heuvel-Panhuizen, 2012).

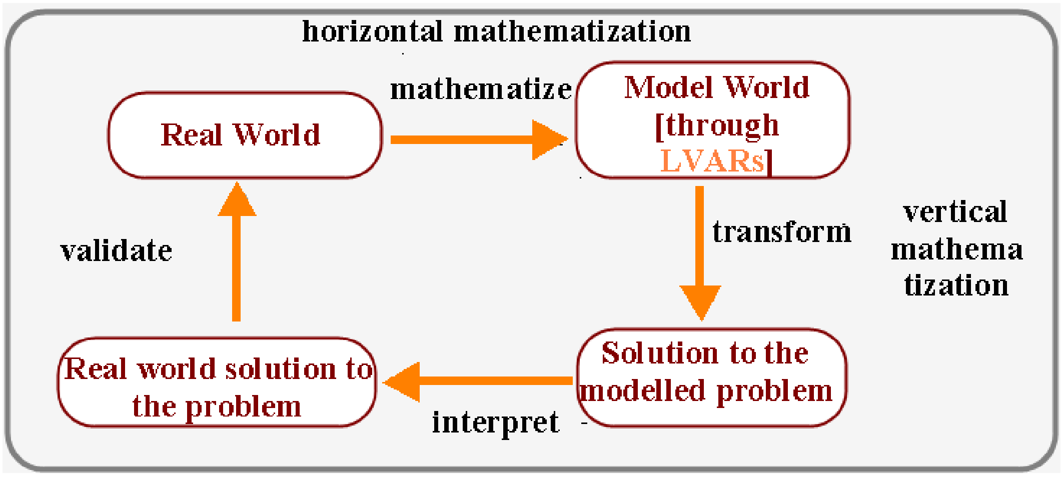


Figure 5.13. Modeling as a cyclic process (Doerr & Pratt , 2008, p. 262) (an adaptation for the current study)

It is difficult for students to move from the real world to the abstract world (or the world of symbols and signs). This difficulty concerns *horizontal mathematization*. Students also face difficulties to deal with the symbols during *vertical mathematization* process (e.g., Treffers, 1987). As Jonassen (2004, p.60) points out:

> "If we want students to be better problem solvers (regardless of problem type), we must teach them to construct problem representations that integrate with domain knowledge. These internal problem representations must be coherent (internally consistent) and should integrate different kinds of representations (qualitative and quantitative, abstract and concrete, visual and verbal)".

When a student understands the problem s/he can creates *meaningful representations* or can create *accurate models*, interpreting the real problem to a mathematical problem. Children have difficulty to perceive the signs of the meanings in the images of the real world. They perceive them as a whole image especially at the lower van Hiele levels. For most researchers, representations can help students to reorganize and translate their ideas using symbols. They are also useful as communication tools (Kaput, 1991) and can function as tools for understanding of concepts, since they help with the communication of ideas and provide a social environment for the development of mathematical discussion. The knowledge of supporting instruments, which are external representational systems for planning activities, allows us to choose between technological tools. The [external] representations facilitate the provision of information about the problem, capture the structure of the problem, and support visual reasoning. On the other hand, the external representations (e.g., formulations or figures) that students construct serve as an indicator of their internal representations, constituting their level of understanding and the developmental level of their geometric thinking. Chinnappan (2006) describes the process of the construction of a representation as a *cyclic event*:

> "The construction of representations is a cyclic event where students continue to refine one representation or change to a different one until the correct match is found between schemas that have been accessed and the goal. The goal could be unknown value that has to be determined or a mathematical result that has to be proved via a chain of reasoning. The above model suggests that instructional methods that would help students decompose problems into sub-problems would benefit them in three ways. *Firstly*, students might be expected to access previously acquired schemas from their memory by examining what is given in the problem. *Secondly*, the accessed schemas could be deployed in solution of sub-problems. *Thirdly*, students could relate the subproblems in ways that would help them reach the problem goal. (p.100)

How does it occur? Information-processing models have been developed to explain inter alia the problem-solving process (e.g., Newell and Simon, 1972; Bower, 1975):"[…] since external stimuli cannot get inside an organism, the representation of them […] and their interaction is what we call "information" […].' (Bower, 1975, p.33). Massaro & Cowan (1993) report that "information refers to representations derived by a person from environmental stimulation […]" (p. 384). Wertheimer (1985) also supports that "a students' representation is appropriate and satisfactory when

- the representation corresponds to the actual structure of the problem […];
- the representation is well-integrated in the sense that all of its components are appropriately interconnected […];
- the representation is well integrated with the problem solver's other knowledge […]" (p. 22, cited in Simon, 1986, p.249).

Moreover, cognitive researchers are investigating how these activities are processed from a psychological point of view and concretely in terms of how the students perceive the information on the computer screen, what parts of their brain are stimulated as they explore using different interaction techniques, and how they integrate and embody this information to their pre-existing knowledge. The questions posed here relate to the external stimulation delivered by new representational infrastructures. When a student reads a mathematical problem, information relating to the problem transits through the sensory register into their working memory. Sensory register is the unit where a stimulus is registered (Atkinson & Shiffrin, 1968). Working memory is the unit of the brain-memory "where the information is temporarily stored and processed" (Karadag, 2009, p.31). The use of a computing environment as dynamic geometry (DGS) facilitates the "dyna-linking"(Ainsworth, 1999a, p. 133). Furthermore, mental representations are stimulated in response to the problem and retrieved from their long-term memory, along with components of interrelated information from student's pre-existing knowledge. Johnson-Laird (1983) argues: *"to understand a physical system or a natural phenomenon one needs to have a mental model of this system that will allow […] the person who will build it to explain it and to predict about it" (p. 430).*

The next step is the incorporation of new information into the pre-existing structural units in the student's mind. In the words of Lester & Kehle (2003):

> "Successful problem solving in mathematics involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity (i.e. lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. (p. 510)

Schumann (2004) in his study "*Reconstructive Modelling inside Dynamic Geometry Systems*" developed a method for geometrical modelling on the basis of DGS. As he writes "Technically, tools must be capable of importing image files of such elements into DGS. The imported images can then be reconstructed by modelling making use of the adequate characteristics of DGS, which offer far wider options than conventional modelling tools" (p. 5).

To produce a mathematical model from a word problem in a DGS, you can combine a picture of reality with a diagram with concrete conceptual properties, to drawing the students' attention through interaction techniques to important properties which are essential for an investigation of the problem. This serves to reveal the theoretical object.

Schumann (2004) describes guidelines for creating a model in a DGS environment as follows (p.7):

- "Look for a moving object in your environment, in printed media or on the web which you think can be analyzed, reconstructed and simulated using the tools of geometry in the plane.
- Make a picture of that object using a digital camera, a camcorder, by scanning or by making a digital copy, and load the image files or the digitized video into your DGS.
- Analyze the picture/s of the moving object by drawing and measuring. Look for geometrical figures and rules. Observe the function of the moving object, or get information on its function. Attempt to find arguments for the rules and functions.
- Reconstruct a functioning model using the tools of DGS.
- Verify your reconstruction by means of simulations. Check whether the functionality of the reconstruction is sufficiently similar to the functionality of the original.

Publish your verified dynamic model, together with a description and the picture of the original object, e.g. on the web". (**Figure 5.14**)

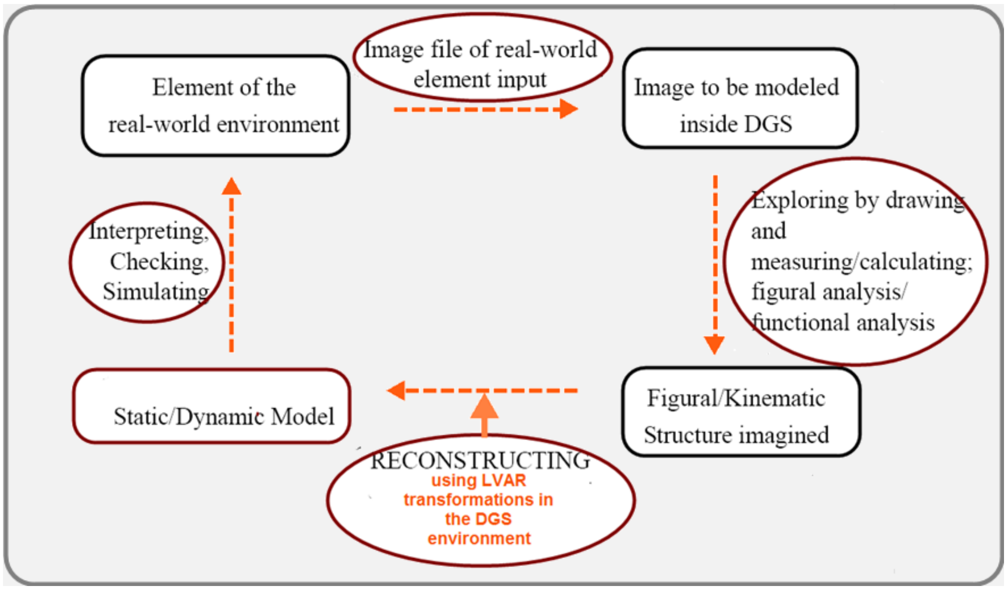


Figure 5.14. Schuman's (2004, p. 7) modelling process using LVARs (an adaptation for the current study)

In essence, the image conversion of the natural environment in the dynamic environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("a triangular island" for example) into a mental image, which is to say an internal representation recalling a *prototype image* (e.g., Hershkovitz, 1990) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. The student needs to explore the shape of the natural environment (e.g., properties of shapes such as its symmetry lines, etc.) and then construct the scale model. The digital image plays a supporting role in understanding the properties of shape but also can bring to the surface students' cognitive obstacles and, consequently, lead to errors. These errors are mainly due to their van Hiele level. As a result, students may not have the capacity to recognize the figure's properties, and, generally, to develop the solution with deductive reasoning.
For the design of activities and the modelling process in a DGS environment I always have in mind: "What would the individual have to know in order to be capable of doing *this* task without undertaking any learning, but given only some instructions?" (Battista, 2011, p. 515). For this I distinguish between real world problems modelled / simulated or not as follows (Patsiomitou, 2014):

- *Case A*: The problem is modelled in the dynamic environment. In the modeled dynamic representation, emphasis is given to the features associated with mathematics (e.g., the modeling of a kite can be done by constructing a rhomboid that emphasizes the verticality of the diagonals, etc.), rather than to other characteristics (e.g., the material, color, etc.). The students are able to experiment with the software tools on the digital image and to visualize the properties of the shapes that they are not able to perceive in the environment.
- *Case B*: The problem is not modelled in the dynamic environment, but the students are prompted to manage the image as if it was perceived in the natural environment. The students have to construct a simulation of the problem in a static, digital, or other physical means as a model of the natural environment. They also have to manage the (digital or not) image to gain intuition about the properties of the shape. According to Johnson-Laird (1983) the human beings understand the world through the representations of the world they create in their minds.

According to Schumann & Green (2000) "computer-aided teaching of mathematics, particularly in secondary education, plays a marginal role due to dissonance with the curriculum. This prevents a fruitful competition of old and new protocols in the treatment of mathematical problems" (p.338).

5.4. Learning Progression, Learning Trajectory and Teaching Cycle

5.4.1. Hypothetical Learning Trajectories or Hypothetical Learning Paths

Simon (1995) defined hypothetical learning trajectories as "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). A hypothetical learning trajectory is hypothetical “because […it] “is not knowable in advance” (Simon, 1995, p. 135). He used the metaphor of a sailor to explain the difference between a trajectory and a hypothetical learning trajectory:

> “You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your ‘hypothetical trajectory’.” (pp. 136-137)

In this thoughtful paragraph, I recognized my own experiences with my every year students in class. The way that my students interacted with the pre-prepared material (digital and otherwise) which I had planned for them, changed the whole path we followed, as I added paths to explain something that was not understood or helped students overcome their misconceptions by using a different path. This was the same feeling I had when I read how Clements & Sarama (2004) defined *learning trajectories* as “descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain” (p. 83).

Simon & Tzur (2004) define a learning trajectory as a way to describe in a constructivist framework “mechanism, reflection on task-effect relation, that is an elaboration of Piaget’s (2001) reflective abstraction and the ways such a mechanism can structure the use of each of several components of the hypothetical learning trajectory” (cited in Clements & Sarama, 2004, p.86).

Clements & Sarama (2004) in their article “*Learning Trajectories in Mathematics Education*” argue that “learning trajectories can have significance beyond curriculum development. There is evidence that superior teachers use a related conceptual structure. For example, in one study of a reform –based curriculum, the few teachers that had worth-while, in-depth discussions saw themselves not as moving through a curriculum but as helping students move through a progression or range of solution methods (Fuson, Carol & Drueck, 2000); that is, simultaneously using and modifying a type of hypothetical learning trajectory.” (p. 82) […] We believe that the notion of hypothetical learning trajectories is a unique and substantive contribution to the field. The construct differs from other models in that it involves self-reflexive constructivism and included the simultaneous consideration of mathematics, goals, models of children’s’ thinking, teachers’ and researchers’ models of children’s thinking sequences of instructional tasks, and the interaction of these at a detailed level of analyses of processes.” (p.87)

Clements & Sarama (2009) argue that a learning trajectory has three basic components:

- “The first part of a learning trajectory *is a mathematical goal.* Our goals are *the big ideas of mathematics*—clusters of concepts and skills that are mathematically central and coherent, consistent with children’s thinking, and generative of future learning” (p. 1).

The core of *“key” or “big” mathematical ideas* is incorporated into the formulation of the trajectory. Big ideas are “*the central, organizing ideas of mathematics—principles that define mathematical order* (Schifter & Fosnot, 1993, p. 35).”

- The second are *the levels of thought* through which a student passes during the experimental process, in order to develop understanding and skills. “Each [level is] more sophisticated than the last, which lead to achieving the mathematical goal. That is, the developmental progression describes a typical path children follow in developing understanding and skill about that mathematical topic” (p.2).
- The third part “consists of *set of instructional tasks* matched to each of the levels of thinking in the developmental progression” (p.2). Also, consist of the mathematical activities, the mathematical problems, the software files that mediate in the empirical construction of the understanding of meanings.

Fosnot (2003) in her article “Teaching and Learning in the 21th Century” gives examples of big ideas (p.9): (a) *Unitizing* requires that children use number to count not only objects, but also groups—and to count these both simultaneously […] (b) *Hierarchical inclusion:* Understanding that number nests like “russian dolls” […]”. Fosnot (2003) argues:

> “Through the centuries and across cultures as mathematical big ideas developed, the advances were often characterized by paradigmatic shifts in reasoning. That is because these structural shifts in thought

characterize the learning process in general. Thus, these ideas are "big" because they are critical ideas in mathematics itself and because they are big leaps in the development of the structure of children's reasoning" (p.9).

For example, number pi (π) is a mathematical abstract object but it can also be perceived as a result of a process (e.g., Patsiomitou, 2006f, 2007c, 2018a, pp.225-248) (**Figures 5.15a, b**). Specific examples from my experimental research using dynamic active representations have been analyzed in the methodology section of my study "*A dynamic active learning trajectory for the construction of number pi (π): transforming mathematics education*" (Patsiomitou, 2018a): (a) the construction of number pi as an approximation process (Patsiomitou, 2006c, 2007a, 2009). For this, I created in the Geometer's Sketchpad software the process of an inscribed or circumscribed n-gon in a circle with a view to using the tabularized measurements and calculations of a ratio in combination with the software's iteration process to lead the students to visualize the approximation process of number pi; (b) the construction of number pi through Riemann sums in a DGS environment (Patsiomitou, 2006c); (c) the construction of number pi by means of a real world problem (Patsiomitou, 2013b, 2016a, b). For this, I combined a digital visit to the Guggenheim museum in New York using the Google Earth software with dynamic representations of the Geometer's Sketchpad software (Patsiomitou, 2016a, b) and other digital web resources. My aim was the students to conceive the meaning of number pi as a limit using the iteration process of the Geometer's Sketchpad dynamic geometry software. Finally, the role the active representations play in the learning trajectory made me think of a way to define what a '*dynamic active learning trajectory*' is .

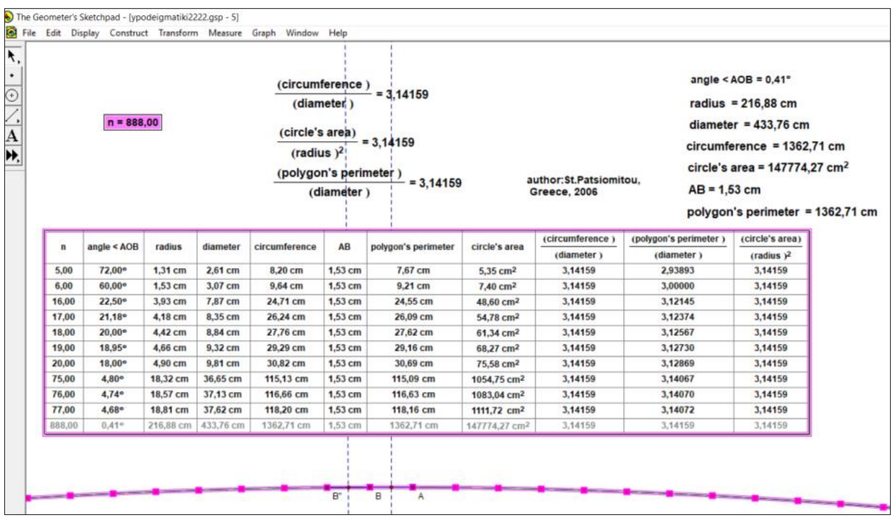


Figure 5.15a. Generating number pi through active representations (Patsiomitou, 2006f, 2007c, 2018a, p. 232)

As a mathematics teacher, I design instructional materials for my students (e.g., Patsiomitou, 2005a, 2006a, b, c, 2007a, 20008a, b, c, d), endeavouring to predict students thinking, or "imagining a route by which [the student] could have arrived (or could arrive) at a personal solution" (Gravemeijer & Terwel, 2000, p.780). This is in accordance to the "reinvention principle" (Freudenthal, 1973) or working in a DGS environment in accordance to the '*dynamic reinvention of knowledge'* principle (Patsiomitou, 2012a, b). Furthermore, "an individual's learning has some similarity to [the learning] that many of the students in the same class can benefit from the same mathematical task" (Simon, 1995, p. 135).

Furthermore these learning paths are dynamic, when instructional DG (Dynamic Geometry) activities are incorporated. Therefore, they could be defined as *Dynamic Hypothetical Learning Paths (DHLPs)* (see Patsiomitou, 2012b, 2014).

A Dynamic Hypothetical Learning Path (DHLP) can incorporate real-world problems or simulations of problems in the DGS environment that had been analyzed and designed in terms of (a) the students' van Hiele (vH) levels of thinking, starting from the lower vH levels to elicit higher vH levels, (b) their sequential conceptual content, and (c) the student's comprehension of the links between representations and mathematical meanings conceptually and procedurally. I have been designed and modified hypothetical learning trajectories /paths using DGS software (or combinations of software' environments), as a result of interactions with the students that participated, "adding the destinations that prior to [their] trip were unknown" (Simon, 1995, p.137). My study "*A* *dynamic active learning trajectory for the construction of number pi (π): transforming mathematics education*" (Patsiomitou, 2018a), reiterates the DHLPs I created since 2005 and use in class instruction for the construction of number pi (π) (indicative Figures 5.15a, b).

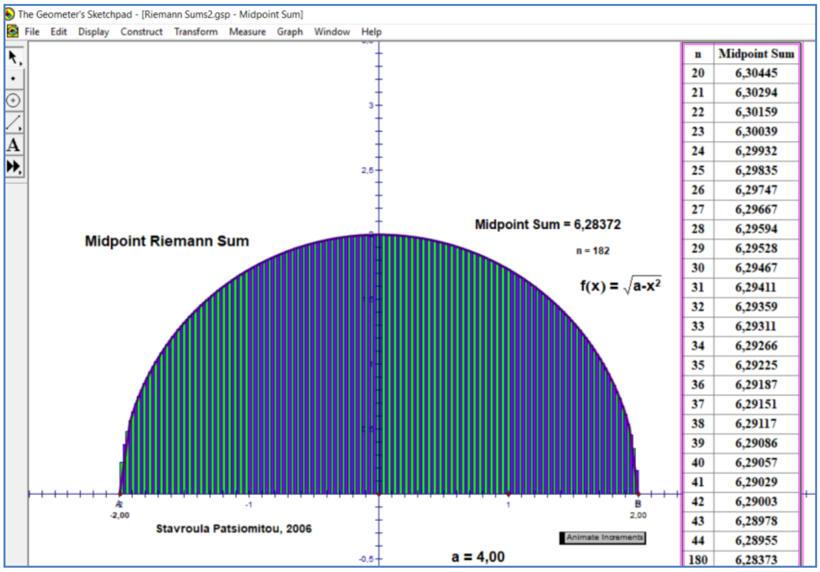


Figure 5.15b. Generating number pi through active representations (Patsiomitou, 2006f, 2007c, 2018a, p.238)

Points of departure for the anticipation of the *instrumental approach* though the dynamic trajectories were the questions (Patsiomitou, 2018a, p.227):

- How could a math lesson acquire interest for all students? Can external linking multiple representations captured by digital medium help students to link concepts and meanings across different disciplines such as geography, mathematics and history of mathematics?
- Do students understand the mathematical components of a mathematical meaning when they see real-world images?
- Can these linking images help students to recall important information which it is difficult to recall under other circumstances?
- What mathematical activities are most reflective of, and appropriate for, the essential development of students' logico-mathematical structures?
- How important is the role of a dynamic geometry software in reorganizing students' mental representations?
- How effective is the teaching and learning process that uses linking visual active representations to overcome cognitive or instrumental obstacles and develop students' understanding of mathematical concepts?
- How does instruction conducted through technological tools and explorations within a laboratory environment help high school math students to discover mathematics?

The learning trajectory was hypothetical at the beginning, as I had hypothesized "if and how [the students] would construct new interpretations, ideas, and strategies" (Fosnot, 2003, p. 10) and the path would follow as they worked on the problem. Moreover, the instructional design process was a synthesis of constructivism and discovery learning, as it was my intention: (a) the students to build on their previous knowledge, (b) the teaching and learning process would be supported through mathematical discourse and conceptual understanding and (c) the learning included students' discovery ("aha" expressions) and their dynamic reinvention of knowledge under investigation.

A hypothetical learning trajectory (HLT) is a cognitive tool based in social constructivism. Fosnot and Perry (1996/2005) in their study "*Constructivism: A Psychological Theory of Learning*" report some general principles of learning derived from constructivism, which as they stress "*is a theory about learning, not a description of teaching*". These principles are mentioned here briefly (Fosnot & Perry, 1996/2005)

- Learning is not the result of development; *learning is development*.[…]

- *Disequilibrium facilitates learning*. "Errors" need to be perceived as a result of learners' conceptions, and therefore not minimized or avoided.[…]
- *Reflective abstraction* is the driving force of learning. As meaning makers, humans seek to organize and generalize across experiences in a representational form. […]
- *Dialogue within a community engenders* further thinking. The classroom needs to be seen as a "community of discourse engaged in activity, reflection, and conversation" (Fosnot, 1989).[…]
- Learning is the result of activity and self-organization and proceeds towards the development of structures. As learners struggle to make meaning, progressive structural shifts in perspective are constructed—in a sense *"big ideas"* (Schifter & Fosnot, 1993.) These "big ideas" are learner constructed, central organizing principles that can be generalized across experiences, and that often require the undoing, or re-organizing of earlier conceptions. This process continues throughout development" (p. 22).

The teacher, however, "*expects the children to solve a problem in certain ways; in fact, expectations are different for different children*" (Fosnot, 2003, p. 10). Each new situation in class requires one or more decisions to be reconsidered in order to bring all the students closer to the predesigned goals. The focal point of interest, and subject under analysis, are the students' answers and the way in which they verbally formulate abstract meanings during the solution of a problem situation.

The instructional design process is designed in phases: what I did towards preparing the lesson before the instruction was delivered; what the organized topics were of the learning trajectory; what I predicted regarding the external stimulation delivered by new representational infrastructures in order to create successive stages in the transformation of previously learned material retrieved from the learner's memory etc. To design instruction, I have to establish a rationale for what has to be learned in order to be successful.

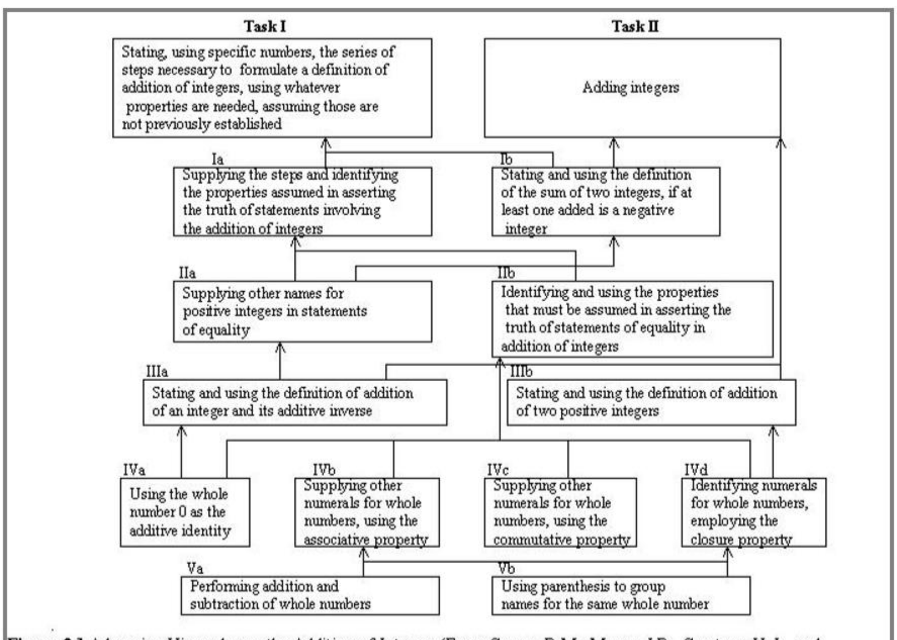


Figure 2.1 A learning Hierarchy on the Addition of Integers (From Gagne, R.M., Mayor, J.R., Garstens, H. L. and Paradise, N. E. (1962). Factors in acquiring knowledge of a mathematical task. Psychological Monographs, 76, No.526).

Figure 5.16. A learning hierarchy for the addition of integers (Gagné, Mayor, Garstens & Paradise, 1962 cited in Gagné, 1968, p. 65)

I speak of instruction rather than teaching, because "instruction may include events that are generated by a page of print, by a picture, by a television program, or by a combination of physical objects, among other things" (Gagne, Briggs & Wager, 1992, p.3), including instructional technology in an orchestration process, while the teacher plays an essential in selecting the events and sources as well as their subsequent planning and demonstration. On the other hand, instructional technology, as a "systematic application of theory and other organized knowledge to the task of instructional design" (Gagne, Briggs & Wager, 1992, p.20), has opened a window for students, teachers and researchers to discover and investigate mathematical meanings. In the **Figure 5.16** a learning hierarchy is presented on the *addition of integers* (Gagné, Mayor, Garstens & Paradise, 1962 cited

in Gagné, 1968, p. 65). Gagné (1962) "used the term "*learning hierarchy*" to refer to a set of specified intellectual capabilities having, according to theoretical considerations, an ordered relationship to each other" (cited in Gagné, 1968, p.64).
Moreover, in their article "*Learning Trajectories: Foundations for Effective, Research –Based Education*" in section "What, if anything, is "new" in the learning trajectories construct?", Clements & Sarama (2014) discuss what is new in learning trajectories, reporting the common characteristics the learning trajectories have with psychological and educational theories "for example, Bloom's taxonomy of educational objectives and Robert Gagne's conditions of learning and principles of instructional design, information-processing theories, information- processing models, developmental and cognitive science theories" (p.8-9). Among other things, the same authors support the following:

- "Learning trajectories include hierarchies of goals and competencies but do not limit them solely to sequences of skills as many of the earlier constructs did;
- They are not lists of everything students need to learn;
- They describe students level of thinking, not just their ability to carefully respond to a mathematics question;
- They have an interactionalist view of pedagogy;
- A single problem may be solved differently by students at different levels" (Clements & Sarama, 2014, p.9)

Table10-1 **Events of Instruction and Their Relation to Processes of Learning**

Instructional Event	Relation to Learning Process
1. Gaining attention	*Reception* of patterns of neural impulses
2. Informing learner of the objective	Activating a process of *executive control*
3. Stimulating recall of prerequisite learning	*Retrieval* of prior learning to working memory
4. Presenting the stimulus material	Emphasizing features for *selective perception*
5. Providing learning guidance	*Semantic encoding;* cues for *retrieval*
6. Eliciting the performance	Activating *response organization*
7. Providing feedback about performance correctness	Establishing *reinforcement*
8. Assessing the performance	Activating *retrieval;* making *reinforcement* possible
9. Enhancing retention and transfer	Providing cues and strategies for *retrieval*

Figure 5.17. Events of Instruction and their relation to the learning process (Gagné, Briggs & Wager, 1992. p.190) (adapted)

Gagné, Briggs & Wager (1992) proposed a systematic instructional design process "*The Events of Instruction and their relation to processes of learning*" (**Figure 5.17**), following a behaviorist approach for the learning process.
Even though I am a constructivist teacher, I find the "*gaining attention*" principle to be relevant to every moment of my teaching life. In class, there is nothing more important than gaining the attention of the students who think that mathematics is hard and not "nice". By giving them "beautiful mathematics" to construct, I "gain" their attention for what follows: constructing meanings.
Merill (2002) provides a conceptual framework for stating and relating the first principles of instruction. According to Merill (ibid.) learning is promoted when "(a) learners are engaged in solving real-world problems; (b) learners pre-existing knowledge is activated as a foundation for new knowledge; (c) new knowledge is demonstrated, applied and integrated into the learner's world" (pp.44-45). He created the following diagram (**Figure 5.18**) to represent his considerations regarding learning; I have adapted it for the needs of my study.

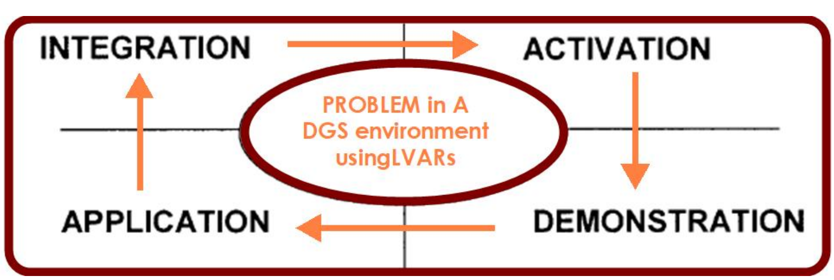


Figure 5.18. An adaptation for the current study of Merill's (2002, p. 45) phases for effective instruction

Wiggins and McTighe (2005, p.22) also provide a template for design questions addressed to teachers. For example: "What are the big ideas included in the activities? What misunderstandings can be predictable? What learning experiences and instruction will enable students to achieve the desired results?" Also, how will the instructional design help the students to reach the goals and the subgoals of the activity, hold their interest during the process, and provide opportunities to rethink and reflect on their understanding? The main problem in all design principles is according to Gravemeijer (2004) that

> "they take as their point of departure the sophisticated knowledge and strategies of experts to construe *learning hierarchies*. […] What is needed for reform mathematics education is a form of instructional design *supporting instruction that helps students to develop their current ways of reasoning* into more sophisticated ways of mathematical reasoning. For the instructional designer this implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing" (p.106).

Learning trajectories support the *guided reinvention* instruction. The teachers' task is to design a course "of action that fits anticipated student reactions" (Gravemeijer and Terwel, 2000, p.786). More precisely, the idea is this: the teaching material can be re-invented by students through continuous interaction with their schoolmates and the teacher in class. In this way the curriculum material can be reformed and the teachers can play a "substantial role [...] in shaping the curriculum experienced by students" (Remillard, 1999), whereas the teaching materials are "the primary vehicles used […] to stimulate curricular change [and] to change the nature of students' mathematics learning opportunities". The students "construct meaning for the mathematical concepts and procedures they are investigating and engage in meaningful problem-solving activities" (Fuson, Carrol and Drueck, 2000, p.277). Each new situation in class requires one or more decisions to be reconsidered in order to bring all the students closer to the predesigned goals. Freudenthal (1991) discussed of 'guided reinvention' to mention the kind of knowledge the students could acquire "as their own, personal knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p.786). On the other hand "the teachers should be given the opportunity [to their students to] build their own mathematical knowledge-store on the basis of such a learning process" (Gravemeijer & Terwel, 2000, p.786). Many researchers argue that working in a dynamic geometry environment allows students to reinvent their personal knowledge by interacting with the other members of the group or with the teacher (or the participating researcher). For example, Furringhetti & Paola (2003) support that "in this case, the reinvention is guided, […] by the use of the [dynamic geometry] environment". As I have investigated in previous studies the DGS environment affected students' *dynamic reinvention* of knowledge (Patsiomitou 2012a, b; 2014).

Papert (1984) in his study "*Microworlds: Transforming Education*" describes the experience of a little girl who discovered number "zero" as she played with a microworld. This was a crucial point for her understanding, as she understood that the command "S0" made the microworld stop moving. As Papert argues (1984, p. 81)

> "I think she was excited because she had discovered zero. They tell us in school that the Greek mathematicians, Pythagoras and Euclid and others, these incredibly inventive people, didn't know about zero. […] The fact that not every child discovers zero this way reflects an essential property of the learning process. No two people follow the same path of learnings, discoveries, and revelations. *You learn in the deepest way when something happens that makes you fall in love with a particular piece of knowledge*."

These words of Papert made me think of my own process with my students over the years teaching in class. My students loved particular pieces of knowledge with its active representations that made different students discover concepts in several different ways, at different times over the years. I also fell in love with the particular incidents, which have played an important role in my thinking process since then. The role the active representations play in a learning trajectory which, though it may take several different routes to reach it, has the same learning goal, made me think of a way to define what a *dynamic active learning trajectory* is, based on the previous definitions of Simon (1995) and Clement & Sarama (2004, 2014): *Dynamic Active Leaning trajectories* (Patsiomitou, 2018a, p. 244) are sequential instructional tasks and activities engaged in [with] a learning goal and designed [with dynamic active representations] to engender mental linking representations which help students develop their thinking in the specific math domain.

For me, a hypothetical learning trajectory which incorporates real world problems modeled in a DGS environment, a DHLP in other words is a crucial tool for the teaching and learning of mathematics.

If the teaching and learning of concepts through the use of real problems in a DGS environment is compared with the traditional approach, I conclude that, "the modelling perspective [using a DGS environment] offers major advantages. The process of modelling constitutes the bridge between mathematics as a set of tools for describing aspects of the real world, on the one hand, and mathematics as the analysis of abstract structures, on the other" (Corte, Verschaffel & Greer, 2000, p.71). Moreover, the intrinsic design of dynamic representational systems has essential impacts on the mental representations of the student, that is, the ways in which students construct their personal representations of meaning during the activity, whether these representations are directed at an individual student or in the student's collaborative environment with others. Accordingly, the conclusions can be used to analyze the potential of these tools for mathematics teaching and learning, to design new tools, and to better understand the ways in which these tools can be (instrumentally) decoded by teachers and students to be transformed into theoretical knowledge built through mediation.

5.4.2. What is a Learning Progression (LP)?

Duschl, Schweingruber, & Shouse (2007) define *learning progressions* as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (Duschl, Schweingruber, & Shouse, 2007, p.214).
Learning trajectories are "subsets of [a] learning progression […] as it requires developing and testing an entire series of learning [paths] that describe specifically how to move students toward conceptual understanding of the big idea[s] in [mathematics and particularly in geometry]" (Krajcik, Shin, Stevens & Short, 2009, p.27).

Table 1
Taxonomy of terms related to the process of developing, refining and empirically testing learning progressions.

Construct	Working Definition
Learning progression (LP)	1. Organizes the content of the discipline and describes a potential route towards more sophisticated knowledge. 2. Explicitly specifies the connections between ideas students need to build an integrated knowledge framework. 3. Links the content to appropriate phenomena or models that students should be able to explain 4. Provides potential instructional strategies and learning tasks to help students move from one level to the next
Hypothetical learning progression (HLP)	A learning progression developed based on the logic of the discipline ***and*** current learning research to describe a route for students to move from more naïve conceptions to a level of understanding closer to that of an expert.
Empirical progression (EP)	Developed from examining how students' ideas develop by the analysis of various assessment tasks. Used in the refinement and empirical testing of the HLP and HLT
Hypothetical learning trajectory (HLT)[a]	1. Addresses a specific learning goal 2. Specifies the means (e.g., contextualization, learning tasks, instruction) that will help students meet the learning goal 3. Includes potential student ideas and difficulties – Subset of a HLP – One or more HLT will describe how students can move from one point in the HLP to another – Provides *specific* instructional strategies and learning tasks for moving students from one point in the learning progression to another
Empirically-tested learning trajectory	A hypothetical learning trajectory that has been implemented in the classroom and evaluated through the results of learning and assessment tasks.
Empirically-tested learning progression	A sequence of empirically tested learning trajectories that were developed based on the HLP and together describe specifically how to move students toward conceptual understanding of the big idea in science over an extended period of time

[a] (Baroody, Cibulskis, Lai, & Li, 2004; Clement & Sarama, 2004; Simon, 1995)

Figure 5.19. A taxonomy on learning progressions (Stevens, Shin, & Krajcik, 2009, p. 2)

According to Stevens, Shin, & Krajcik (2009) *a learning progression* "must include not only an ordered description of how the important ideas can develop over time, but also: 1) possible instructional strategies and learning experiences that might help students move along the progression, 2) the difficulties students might have developing conceptual understanding based on current learning research, and 3) assessments that will define students' position on the progression" (Stevens, Shin, & Krajcik, 2009, p.3)
Also, in the Table 1 (**Figure 5.19**) Stevens, Shin, & Krajcik (2009) in their study "*Towards a model for the development of an empirically tested learning progression*" report a taxonomy of terms related to the process of developing, refining and empirically testing learning progressions.
A learning progression is also committed to the "notion of learning as an ongoing developmental progression. It is designed to help children continually build on, and revise their knowledge and abilities, starting from the initial conceptions about how the world works and curiosity about what they see around them" (National Research Council (NRC), 2010, p.2). Learning progressions has among others students' assessment as important

component that aids to "measure student understanding of the key concepts or practices and can track their developmental progress over time" (Corcoran, Mocher & Rogat, 2009, p.15).

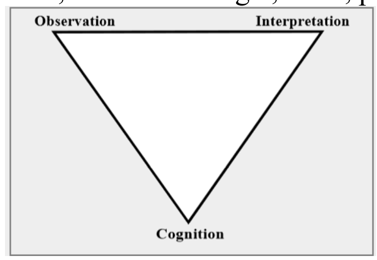


Figure 5.20. The assessment triangle (Pellegrino, Chudowsky, & Glaser, 2001, p. 44 cited in Shavelson et al., 2003, p. 3) (adapted)

A learning progression in mathematics (or other disciplines, e.g language, science) can be built upon the concept of the *Assessment Triangle* (Pellegrino, Chudowsky & Glaser, 2001) which "explicate three key elements underlying any assessment: (1) a model of student *cognition* and learning in the domain, (2) a set of beliefs about the kinds of *observations* that provide evidence of students competencies and (3) an *interpretation* process for making sense of evidence" (Pellegrino, Chudowsky & Glaser, 2001, p.44) (**Figure 5.20**).
Shavelson et al. (2003) clarify the elements of the *Assessment Triangle* as follows: The first element *cognition* "explains how students represent knowledge and develop competence" (Pellegrino, Chudowsky, & Glaser, 2001, p. 44, italics in original). The second element, *observations* are based on "[…] tasks or situations that prompt students to say, do, or create something to demonstrate knowledge and skills" (Pellegrino, Chudowsky, & Glaser, 2001, p. 47). And the third element is interpretation, which involves "… all the methods and tools used to reason from fallible observations" (Pellegrino, Chudowsky, & Glaser, 2001, p. 48).

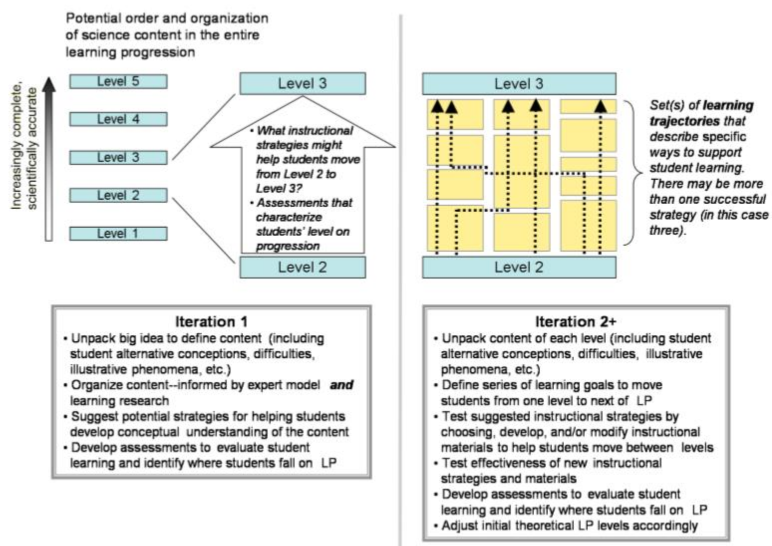


Figure 5.21. Model of a hypothetical learning progression and the process of development, refinement and empirical testing (Stevens, Shin, & Krajcik, 2009, p.4).

As Smith, Wiser, Anderson, & Krajcik (2006) argue the progress through learning progressions depends on instruction as well as the theory of van Hiele.
Krajcik, Shin, Stevens & Short, (2009, p.28), and similarly Stevens, Shin, & Krajcik (2009, p.4) illustrate the difference between *learning progressions* and *learning trajectories* in the **Figure 5.21**. As Stevens, Shin, & Krajcik (2009) argue "hypothetical learning trajectories (HLTs) [are] subsets of HLPs that describe specifically how to help students meet some or all of the learning goals that support students in moving from one HLP level to the next". Krajcik, Shin, Stevens & Short, (2009) consider that instructional sequences "provide the learning tasks and phenomena that students need to experience in order to build understanding of the learning goals" (p.20).
Besides Battista (2011) supports that a *learning progression* differs from a *learning trajectory* because it has not been designed "to test a curriculum, based on a fixed sequence of learning tasks in that curriculum. [Instead] it is

focusing on a formative assessment system that applies to many curricula […] based on many assessment tasks, not those in a fixed sequence" (p. 513).
Battista (2011) defines a learning trajectory as "a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning a topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence" (p.510). Moreover, Battista (ibid.) argues that "One critical difference between my definition of learning progressions and my definition of learning trajectories is that trajectories include descriptions of instruction, progressions do not" (p.512).
In my study "*Student's Learning Progression through Instrumental Decoding of Mathematical Ideas*" (Patsiomitou, 2014), I describe a learning progression comprised of a few DHLPs. Points of departure for the anticipation of the DHLPs were the questions:

- Do students understand the mathematical components of modeling when they see real-world environments' representations?
- What mental activities will the students develop when they participate in these learning [/ instructional] activities?
- What mathematical representations are most appropriate for student learning?
- How important is the role of a dynamic geometry program to reorganize students' mental representations?
- Does students' actual learning correspond with what was anticipated?
- How effective is the teaching and learning process using linking visual active representations (LVARs) to overcome cognitive obstacles and develop understanding of the mathematical concepts?
- Which mathematical competencies are developed through the DHLP?
- What is the need for students' learning and understanding in upper-class curriculum processes for innovative learning when new practices and ideas (i.e., fractal activities) are incorporated?

As a researcher, teacher-researcher, I know that the students encounter difficulties in order to understand the concepts in geometry. The connection between the "representation" and the "representing object" can create conflicts to students because they are not able to control the information that comes from the outside world (Mesquita, 1998). The question is how we can help students overcome the cognitive obstacles they face and what are these teaching situations which can provide the scaffolding to the next van Hiele level.
For the learning progression mentioned above, I developed the instructional activities based on an analysis of the results of my PhD thesis, with regard to students' evolution of understanding on instrumental decoding when they construct quadrilaterals. The complete study includes a detailed procedural analysis of the situations, the involved problems, in addition the problems' conceptual analysis, instrumental decoding and learning targets (e.g., different solving strategies, formulas or figure's decomposition). This incorporated the recognition and demonstration of transformations (e.g, recognition and drawing of symmetry lines or demonstration of reflections, translations and rotations) using multiple contexts (e.g., graphpapers, a computing environment). Furthermore, is described the recognition and utilization of properties that belong to a class of figures (or a subclass) and description of the characteristics of shapes and their relationships. We worked as a whole class, trying to develop a form of practice compatible with social constructivism (e.g., Wood & Yackel, 1990). I was actively involved with the children, encouraging small group cooperation both in and outside of class, without intently to show the process to complete the activity. I started the activity with a question; after the answers were given, I continued with sequential questions to clarify the explanations or to help students with the cognitive conflicts. Then, I asked the students to complete the task in the paper-pencil environment and collected their work to see the level of understanding from the correct answers. After the evaluation of the students' work, I continued with follow-up activities in the DGS environment to help the children reconstruct the solution methods. After the intervention with GSP activities, the paper-pencil work was repeated to see the difference in the students' learning and understanding of the concepts. Indicative of students' wrong representations will be presented and a short report made of their mistakes and misconceptions.

> *"The situations that children find problematic take a variety of forms and can include resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts and procedures, accounting for a surprising outcome (particularly when two alternative procedures lead to the same result), verbalizing their mathematical thinking, explaining or justifying a solution, resolving conflicting points of view, developing a framework that accommodates alternative*

solution methods, and formulating an explanation to clarify another child's solution attempt" (Cobb & Steffe, 1991, p.395)

In the Learning Progression I describe the following situations (Patsiomitou, 2014, p. 33):

Situation 1: The highlighted idea in mathematics, which is 'symmetry,' is interdisciplinary, connected with art and culture. The aim is to 'see' mathematics in any context.

Situation 2: The challenge is to connect the transformations in static and dynamic means conceptually and procedurally. *Instrumental decoding* of students' mathematical ideas plays a major role for the overcoming of cognitive and instrumental obstacles.

Situation 3: This situation aims to accomplish the figure's *symbol character*. The grid in the DGS environment provides a challenge for the experimentation. The important points in this situation are the students' methods of dealing with the questions: "Under what conditions does the rhombus become a square?" or "What are the similarities and differences between a kite and a square?"

Situation 4: The motivation for this situation is that: if students have a set of properties, to understand the kind of quadrilateral. This phase is very crucial for the students to acquire the competence to replace a figure with a set of properties that represent it and from these properties to construct the figure. In other words, the figure will acquire the *signal character*.

Situation 5: The recognition of differences and similarities between figures' symmetry properties demarcates the scope of this situation. The instructional process must focus at the understanding of the structuring process and not the learning of ready-made structures.

Situation 6: The development of structures in students' minds has been achieved with the synthesis of a more complex construction. The situations aim to develop the abstraction. Pythagorean Theorem's reconfigurations have been used as a tool for the development of students' instrumental decoding of a complex figure's anasynthesis. The 6th situation led students to think about self similarity, which is not included in high school curriculum.

Situation 7: Self-similarity, Pythagorean Theorem and the midpoint theorem are the mathematical backgrounds of this situation. Here is explained the rationale in the design process and the importance of linking visual active representations and instrumental decoding. The structures of fractals, [by applying the meaning of dynamic LVAR representations], aims that students (a) review most of theorems, (b) identify the potential weaknesses and cognitive obstacles that students face in their effort to understand the process, (c) develop the links between the virtual representations and the formulations with which students justify their construction, as a result of understanding the figures' transformations and symmetry, and (d) develop most of the competencies described in the beginning of the article and higher–order level skills (e.g., generalize patterns using recursion, use algebraic formulae and symbolic expressions to explain mathematical relationships, etc.) than those that they are able to develop through traditional mathematics. This is very important for their movement through vH levels.

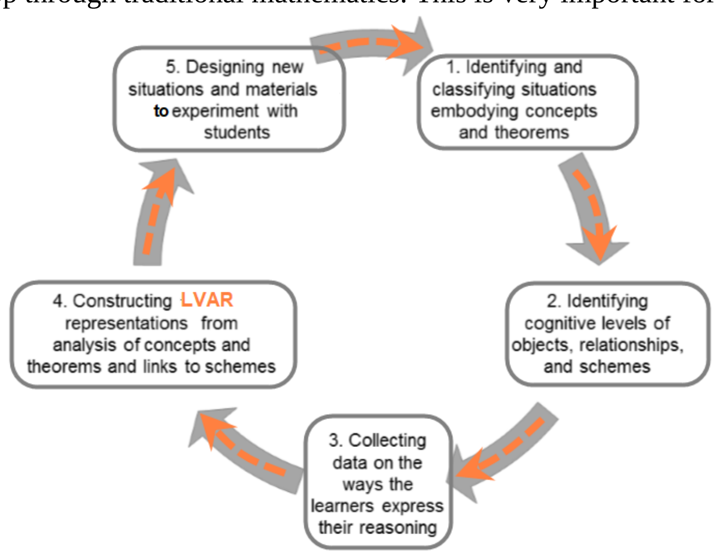


Figure 5.22. Vergnaud's (1988, p.149) approach for investigation in mathematics education cited in Long (2011, p.123) (An adaptation for the current study)

Vergnaud (1988, p. 149) proposes an approach in mathematics education (Figure 5.11), which involves (cited in Long, 2011, p. 123):

- *Identifying and classifying situations* […] which are at the cognitive level of the learner defining the conceptual domain.
- *Identifying levels of objects, relationships, and schemes* (concepts-in-action and theorems-in-action) currently employed by learners in engaging with the problem situations, spanning the cognitive domain.
- *Collecting data on the ways the learners articulate their reasoning*, identifying the links between the conceptual and cognitive domains as expressed by learners in the didactic domain.
- *Constructing symbolic representations* by observing and analysing the use of concepts and theorems (from the mathematical concept perspective) and schemes that learners use (from the psychological/cognitive perspective).
- *Designing new situations and materials* to experiment with students to inform the didactic perspective.

Vergnaud depicts this approach for investigation in mathematics education, which includes the steps presented in the **Figure 5.22** (Vergnaud, 1988, p.149 reported in Long, 2011, p.123).

> "A conceptual field is conceived as a set of problem situations, the solution of which requires mastery of several concepts […] (Vergnaud, 1988, p. 142 cited in Long, 2011, p.6). According to Long (2011) "The theory of conceptual fields has a mathematical framework, but in addition draws on a psychological perspective, notably the acquisition of concepts building on the work of Piaget, and the function of instruction building on the work of Vygotsky […] (p.4).

As teachers (or teacher- researchers) design teaching concepts and ways of interacting with their students, they increasingly feel the need to understand the minds of the students, looking for methods to lead their students to understand the concepts. Therefore, the determining factor is the teacher who decides on the objectives/aims of the teaching method and chooses the means for effective implementation of the objectives or of the educational process. The positive attitudes/behaviors of the teachers of mathematics with regard to mathematics, their positive position with regard to technology, and their interest in the students' understanding of the concepts, are the most important factors for the development of innovative applications in schools, in order to help students achieve "successively more sophisticated ways of thinking about a topic that can follow […]" (National Research Council, 2007, p. 214, cited in Battista, 2011, p. 508).

5.4.3. A DHLP for the Learning of Parallelograms

For my Ph.D thesis (Patsiomitou, 2012a), I designed a hypothetical learning path of both, DGS software and static means (i.e. constructions in paper and pencil environment, using ruler and compass) for the learning of quadrilaterals (i.e., a dynamic hypothetical learning path -DHLP- meaning a hypothetical learning path using dynamic geometry software) (Patsiomitou, 2012a, b), as I absolutely agree with what Isoda (2007) argues "*both computer and traditional technology are inevitable for mathematics teaching*". The path consisted of phases of learning. I developed material based on the van Hiele theory (Dina van Hiele in Fuys et al., 1984; Crowley, 1987). Through the DHLP, I tried to predict the thinking of the participating students as they responded to activities. I chose quadrilaterals because I had ascertained during previous research, which I conducted while writing my Master's thesis that students (in both, Primary and Secondary education) face many difficulties understanding their *conceptual and hierarchical structure*. This concrete research was the starting point for my creation of sequential activities in a DGS software environment, the Geometer's Sketchpad; I designed the DHLP to help the students of the experimental group to construct meanings and to develop argumentation and abstract processes. This led me to conclude that the way in which I designed the DGS problems and activities, creating semi-preconstructed dynamic visual "alive"- active linking representations impacted on the way in which students developed mental representations (e.g., Patsiomitou, 2007a, 2008 a, b, c, d, 2009a, b, c, 2010, 2011a, b, 2012a, b, 2013, 2014, 2018a, b). As I concluded later from my post-doctoral research, this conclusion can be extended to an individual student, a group of students or the whole class.
In my study "*A Linking Visual Active Representation DHLP for student's cognitive development*" (Patsiomitou, 2012b), I describe this 'dynamic' hypothetical learning path (DHLP) for the learning of the concept of parallelogram in geometry, which I "designed to engender those mental processes or actions [of students] hypothesized to move [them] through a developmental progression of levels of thinking" (Clements & Sarama, 2004, p.83).

The development of a concept is subjected to epistemological, historical and cognitive analysis. The aim is to identify how it evolved over the years. During my thorough investigation of the relative literature regarding quadrilaterals, I found research relating to:

- The effect of the constructional processes of quadrilaterals in a DGS software on to students' reasoning or the impact the processes involved in constructing quadrilaterals in a DGS software has on student's thinking (e.g., Mariotti, 1997, 2000; Vincent, 1998; Vincent & McCrae, 2001; Leung & Or, 2007)
- The development of different kinds of reasoning during the interaction with each other or with the DGS software problems on quadrilaterals (e.g., Hoyles, 1998; Arzarello et al., 1998; Hoyles & Healy, 1999; Mariotti, 2000; Hadas, Hershkowitz, & Schwarz, 2000; Marrades & Gutiérrez, 2000; Healy, 2000; Hölzl, 2001; Talmon & Yerushalmy, 2006)
- The solving of geometrical problems relating to quadrilaterals and modelled in a DGS software environment (e.g., Arzarello, Micheletti, Olivero & Robutti, 1998; Healy, 2000; Hadas, Hershkowitz, and Schwarz, 2000; Hanna, 2001; Mariotti, 2000; Jones, 2000; de Villiers, 2004a,b)
- How students think when they investigate problems that incorporate the meaning of symmetry in quadrilaterals (e.g., Arzarello, Micheletti, Olivero, & Robutti, 1998; Healy, 2000; Hadas, Hershkowitz & Schwarz, 2001; Mariotti, 2000; Jones, 2000; de Villiers, 2004a,b; Leikin, 2004; Jiang, 2002; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004a, b; Belfort & Guimarães, 2004; Graumman, 2005)
- The effect of making transformations of rotation and reflection using the DGS software (e.g., Edwards, 1991; Natsoulas, 2000; Hollebrands, 2003, 2004, 2006).

My study includes the following investigations (Patsiomitou, 2012b, p.58):

(a) *A detailed investigation of four phases* of the students of the experimental group that followed the DHLP. Investigation covered how every student of the experimental group developed his/her thinking, using a detailed analysis of their formulations and comparing the kind of representations they produced and the kinds of definitions and reasoning (i.e., inductive, abductive or deductive).

(b) *A detailed investigation of four evaluations of the students* of both groups (experimental, control groups) in a paper-pencil environment. This investigation covered how every student in both groups developed his/her thinking by comparing the milestones of their development moving through the van Hiele levels (i.e., the characteristics of every level as defined by Battista (2007) as they appeared in the paper-pencil tests). Moreover, I studied their ability to prove.

(c) *A comparison study between the students in both groups* (i.e., how the students in level 1 or level 2 of the experimental group developed the characteristics of each level and how members of the control group did the same).

The phases of the DHLP are interconnected in terms of: a) the conceptual context, b) the order in which the software's technological tools are introduced, and c) the increasing difficulty at both levels. The experimental process lasted approximately four months, from January to May, 2007. Firstly I examined student's level of geometric thought using the test developed by Usiskin (1982) which is in accordance to the van Hiele model using only the first twenty questions of the questionnaire. This description of the DHLP (Patsiomitou, 2012a, b) is a synthesis of an *instructional design process and a redesign process*, meaning a "systematic, self reflective spiral of planning, acting, observing and reflecting" (Steketee, 2004, p. 876).

(d) In the *instructional design process*, I describe how I predicted the hypothetical transitional understanding of the meaning of parallelograms and the students' way of thinking during the solution of the problems in combination with their actions in the software with the closest possible approach.

(e) In the *instructional redesign process*, I describe the procedures that demanded the addition of new tools, which helped the students of the experimental group overcome cognitive and instrumental obstacles that they faced during the research process.

Consequently, the description of the DHLP is separated into two sections for each phase (Patsiomitou, 2012b): (a) one which describes the aims of the DHLP as part of the general framework of the curriculum for the teaching and learning of geometry, and (b) a prediction process of the hypothetical interactions of the students with the tools, consequently an inductive way of thinking that has been supported by my previous observations. The DHLP is consisted of the following phases (Patsiomitou, 2012 a, b).

Phase A: Building and transforming quadrilaterals through Linking Visual Active Representations

The aim of the first phase of the research process was for the students to obtain the competence to build and transform linking structurally unmodified representations of parallelograms.

Problem 1: ***Construction of a parallelogram***: *construct a parallelogram if you know a straight line segment and a point on the screen.*

The transformation of the position of the point (-vertex of the parallelogram) through *theoretical dragging* leads to the transformation of the segment in order for the opposite sides to become congruent. The students *dynamically reinvent their understanding* through the process.

Problem 2: Construction of a rectangle. *Drag the vertex of the parallelogram you have constructed until it becomes a rectangle. Then, find a way to construct a robust construction of a rectangle.*

The rectangle is a fundamental meaning in parallelograms. Students are able to recognize the prototype image of the rectangle from the first classes of primary school. By this process, the students will construct the meaning of the rectangle as a specialization of the meaning of the parallelogram, incorporating the additional properties of the rectangle which will be dynamically reinvented.

Problem 3: Construction of a rhombus. *Join the opposite vertices of the parallelogram you constructed earlier. Drag one vertex until you construct a rhombus. What did you observe? Then, construct a robust rhombus.*

Design process: a) The students will shape the drawing of a rhombus by theoretically dragging the parallelogram so that the figure will obtain the property of the congruency of the sides and will match the mental prototype image the students have for the figure of the rhombus. b) A second intended activity will be for the students to theoretically drag the figure of the rhombus so that the isosceles triangles become equilaterals. The perception of the rhombus as a synthesis of two equilateral triangles may lead students to a cognitive conflict. c) A third intended activity will be to have the students build a robust construction of a rhombus. The cognitive task for the students is to connect the structure of the rhombus with the meaning of reflectional symmetry, and consequently see it as a reconfiguration (Duval, 1995) of the isosceles triangle. So, a new issue will arise: How can an isosceles triangle be constructed on screen?

Redesign process: At this point I introduced **a parametrical segment** (see Patsiomitou, 2008b, 2009) in order to help students to construct the congruent radius of the circles, or the congruent circles. Therefore, by using the parametrical segment to construct the circles and then by dragging its end points, the students would have the opportunity to link the process with the theory of geometry.

Problem 4: Construction of a square. *Construct a square with a free procedure.*

With the construction of a square the investigation of the students' understanding of the *hierarchical relationship* is aimed at (a) a specialized rectangle with additional properties (e.g., the congruency of its sides) and (b) a specialized rhombus with additional properties (e.g., the congruency of its angles).

Redesign process: This is a good point for the students to be introduced to the rotation of a segment. The students interact with an intermediary representation before seeing the final rotation of the object on screen.

The accomplishment of the first phase evoked a crucial issue for me: Can students use the figures' *secondary properties* to accomplish the construction of a parallelogram? By secondary properties are meant the properties of the figure's diagonals, which relate to the symmetry of the shape. This is in accordance with what Dina van Hiele (Fuys et al, 1984) argues, that "a student proves he possesses the structure of the analysis when he shows that he can manipulate the organizing principles. One of those organizing principles is symmetry" (p.184). For this, it is very important that the students follow the second phase.

The emphasis on construction using the Transform menu in GSP was shaped to facilitate the understanding of symmetries and strengthen the development of structures in the students' minds. This process can lead students to dynamically reinvent new ways of constructing parallelograms using DGS.

Phase B: Investigating and building figures through symmetry

In this phase the notion of symmetry ant their properties are introduced by using the transformations of the rotation and reflection of the software. The recognition/understanding of the symmetry of geometrical objects is the fundamental aim of this study, in accordance with van Hiele's theory. I separated the second phase into four subphases:

Part B1. The recognition—visualisation part of the second phase

Problem: Reflect point A (on a given line l) in order to construct its image, point A΄. Imagine that point A will approach the reflection line l (don't use the dragging mode of the software). Describe the movement of point A΄. Will it approach or move away from the reflection line? Then drag point A until it approaches the reflection line and check your previous formulation. What do you observe? Do you have to revise your previous statement? Give reasons.

Part B2. The perceptually componential analysis part of the second phase

Problem: Construct the axes of symmetry of rectangle. The students will face difficulties in understanding the meaning of axis of symmetry and how it differs from rotational symmetry, which is expressed with the misunderstanding of the roles that the secondary elements (for example, the medians of a triangle or the diagonals of a rectangle) play in the figures' symmetry. Another point is students' difficulty in distinguishing the difference between the meanings of "symmetry of an object with regard to an axis of symmetry" and the meaning of "symmetry lines of the shape."

Problem: Construct the axes of symmetry of rhombus. Then join the midpoints of the opposite sides with a segment and explain why it is an axis of symmetry or not. Then, drag the vertex of the rhombus to form a square.

Most students intuitively know that the axes of symmetry of a rhombus are its diagonals. This is a crucial point for the research process because the students have to overcome a cognitive obstacle: The segment that joins the midpoints of the opposite sides of the rhombus is not an axis of symmetry because this line is not perpendicular to the sides of the rhombus.

Problem: Construct a square' axes of symmetry. The students have to recognize/realize that the square concentrates all the properties that the previous figures did, with regard to its symmetry lines. This means the segment that joins the midpoints of the opposite sides of the square is a symmetry line, as are its diagonals, so the square concentrates all the properties of the rhombus and the rectangle with regard to symmetry lines. This means that the students can give hierarchy to the square as a rhombus or a rectangle and define it from its properties from the lines of symmetry.

Part B3. The informal componential analysis part of the second phase

Redesign process: The investigation of the meaning of rotational symmetry

The students' cognitive conflicts led me to redirect my study in order to include the investigation of the meaning of rotational symmetry. The students were confused about the two meanings and most students believed that the rotational symmetry of a point can be defined as a reflectional symmetry of the point.

In order to facilitate the process, I created a 'custom tool' that could apply the procedure of the rotation of a point by 180^{o}, appearing only as the final step of the rotation process (meaning the students could not see the entire intermediary steps of the rotation process) (an extended report regarding the use of the custom tool "symmetry" is in the Chapter IV, section 4.2.4).

Redesign process: The example and the counter-example of custom tool's use.

The difficulties that arose from the use of the custom tool made me use an example and a counter-example of its use. By example I mean, where the "custom tool" is helpful is in understanding that the rotation of every point of the circumference of a circle on its center (rotation of the circle around its center) results in the circumference of the same circle. By counter-example I mean that the rotation of an equilateral triangle at the intersection point of the perpendicular bisectors results in a different equilateral triangle (rotation by 180° of the original at the intersection point of the perpendicular bisectors)

Redesign process: The construction of the *structure of the bisected diagonals*

The students will construct the image of the segment CD by rotating it by 180° around H. There are several perspectives. From an *instrumental genesis* perspective, the students can construct an *instrumented action scheme* by using the custom tool "symmetry". Moreover they will be able to construct the meaning the "diagonals [of a parallelogram] are bisected /dichotomized". Consequently the procedure will help the students to recognize the figure of its properties, meaning the figure will acquire the *signal character*. This means that these representations are linked

- structurally as the dragging of any point does not modify the structure of the construction.
- conceptually through the meaning of the symmetry by center and the meaning of the intersected bisected diagonals.

Part B4. The formal componential analysis part of the second phase. The aim of this part of the third phase is for the students to construct a parallelogram with their starting point being their knowledge of the symmetry of the figure. I investigated whether the figures have acquired the *signal character* and if the students can justify their procedures theoretically. This is a very complex process since the students must have both *conceptual and procedural competence*, meaning the competence to instrumentally decode their mental representations of a set of properties with actions through the use of tools. This means, for example, to interpret the congruency with the circle tool and simultaneously bisect with the custom tool.

Furthermore, for them to construct the hierarchical categorization and definition of figures through their symmetrical properties and in accordance to their understanding.

Phase C: Investigation of problems aiming the students to construct the classification of quadrilaterals with regard to their diagonals. The students of all groups investigated several instances of Varignon's theorem occurring from the use of dragging (Patsiomitou, 2012a). Also they investigated several instances of Viviani's problem, in order to construct formal proofs and generalizations (still unpublished).

Varignon (1654-1722) proved that "*a parallelogram is formed when the midpoints of the sides of a convex quadrilateral are joined in order*". Varignon's proof was published in 1731 in "*Elemens de Mathematique"* (Oliver, 2001, p.316). Graumann (2005) in an extended and detailed description of the study of quadrilaterals classified the quadrilaterals with regard to their diagonals (**Figure 5.23a**). As he writes:

- "if we ask for all convex quadrilaterals whose middle-quadrilateral is a rectangle we will be lead to a new type of quadrilaterals namely those with *orthogonal diagonals*
- if we ask for all convex quadrilaterals whose middle-quadrilateral is a rhombus we will be lead to a new type of quadrilaterals namely those with *diagonals of equal length*
- if we ask for all convex quadrilaterals whose middle-quadrilateral is a square we will be lead to a new type of quadrilaterals namely those with *diagonals which are orthogonal and with equal length*"(p.194).

In this way he distinguished the quadrilaterals into separate categories: those whose diagonals are congruent, those whose diagonals are perpendicular and those whose diagonals have both properties. Moreover, he distinguished those quadrilaterals whose one diagonal intersects the other at an arbitrary point and whose one diagonal bisects the other diagonal.

Graumann (2005) continued the classification of the quadrilaterals by adding properties into each one of the above-mentioned categories until they had been led to a specialized figure such as a square whose diagonals are congruent and perpendicular.

In my study "*A Linking Visual Active Representation DHLP for student's cognitive development*" (Patsiomitou, 2012b), I describe a new classification of quadrilaterals, due to the different properties of the internal parallelograms (or middle-quadrilaterals) which are constructed if we join the midpoints of the external quadrilaterals (**Figure 5.23b**). This classification is reported in details in my study "*A Dynamic Teaching Cycle of Mathematics through Linking Visual Active Representations*"(Patsiomitou, 2015a, in Greek).

The internal quadrilateral (or *middle-quadrilateral* in the words of Graumann) is a parallelogram for every external quadrilateral. Graumann (2005) has represented this classification with a figure. I have constructed an adaptation of Graumann's figure (2005, p.194) by constructing the internal parallelogram, joining the midpoints of the sides of the external quadrilateral (Patsiomitou, 2012a, b, 2015a).

In this way, a new classification of quadrilaterals occurs. For example, *the parallelogram which is shaped from the midpoints of the sides of the quadrilaterals whose diagonals are perpendicular and bisected to each other is also a rectangle and in addition its sides are symmetrical with regard to the diagonals of the external quadrilateral.* (Patsiomitou, 2012a, b, 2015a)

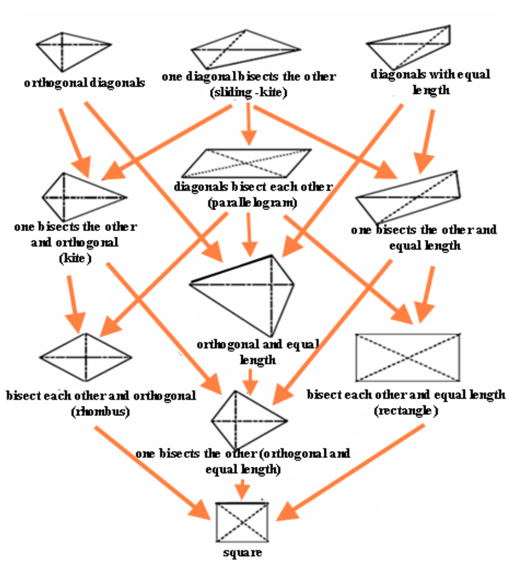


Figure 5.23a. Graumann's (2005, p. 194) "house of quadrilaterals" (adapted)

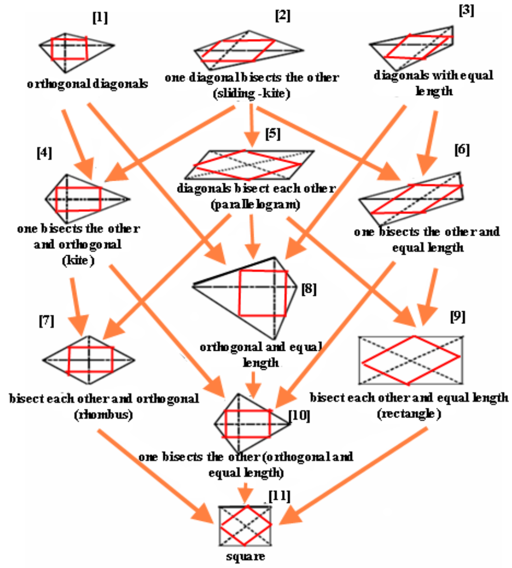


Figure 5.23b. An adaptation of Graumann's "house of quadrilaterals" to represent the internal quadrilaterals (Patsiomitou, 2012a, b,, 2015a)

I have constructed the **Table 5.3** below in which I have described the kind of the middle-parallelogram, which occurs in the internal section of the external quadrilateral.

TABLE 5.3. My proposal for the classification of the internally constructed quadrilaterals in Varignon's theorem, taking into account the non-convex quadrilaterals (Patsiomitou, 2012a, b, 2015a)

External quadrilateral (ABCD)	Internal quadrilateral (EFGH)
A quadrilateral whose diagonals are perpendiculars	[1] A rectangle
A Kite: one diagonal bisects the other and are perpendiculars	[4] A rectangle whose two parallel sides are symmetrical by the diagonal which is perpendicular to and bisects the other diagonal
A Rhombus: Each diagonal bisects the other and are perpendiculars	[7] A rectangle whose opposite sides are symmetrical by the diagonals of the external rhombus which are the axes of symmetry.
A quadrilateral whose one diagonal bisects the other, are perpendicular and congruent	A square whose two parallel sides are symmetrical by the diagonal which is perpendicular and bisects the other diagonal.
Sliding Kite : A quadrilateral whose one diagonal bisects the other	[2] A parallelogram
A Parallelogram: Diagonals bisect each other	[5] A parallelogram whose centre of symmetry coincides with the point at which the diagonals of the external parallelogram intersect.
A quadrilateral whose diagonals are perpendiculars and congruent.	[8] A square
A quadrilateral whose one diagonal bisects the other and are congruent	[10] A rhombus whose centre of symmetry coincides with the point at which the diagonal of the external quadrilateral (which bisects the other) also bisects the diagonals of rhombus.
A square	[11] A square whose centre of symmetry coincides with the center of symmetry of the external square.
A quadrilateral whose diagonals are congruent	[3] A rhombus
A quadrilateral whose one diagonal bisects the other and equal length	[6] A rhombus
A rectangle: A quadrilateral whose diagonals bisect each other and are congruent	[9] A rhombus that has its centre of symmetry at the point where the diagonals intersect, and whose diagonals are perpendicular lines to the sides of the rectangle
A non-convex quadrilateral whose diagonals are congruent	A rhombus
A non-convex quadrilateral whose diagonals are perpendiculars	A rectangle
A non-convex quadrilateral whose diagonals are congruent and perpendiculars	A square
....	....

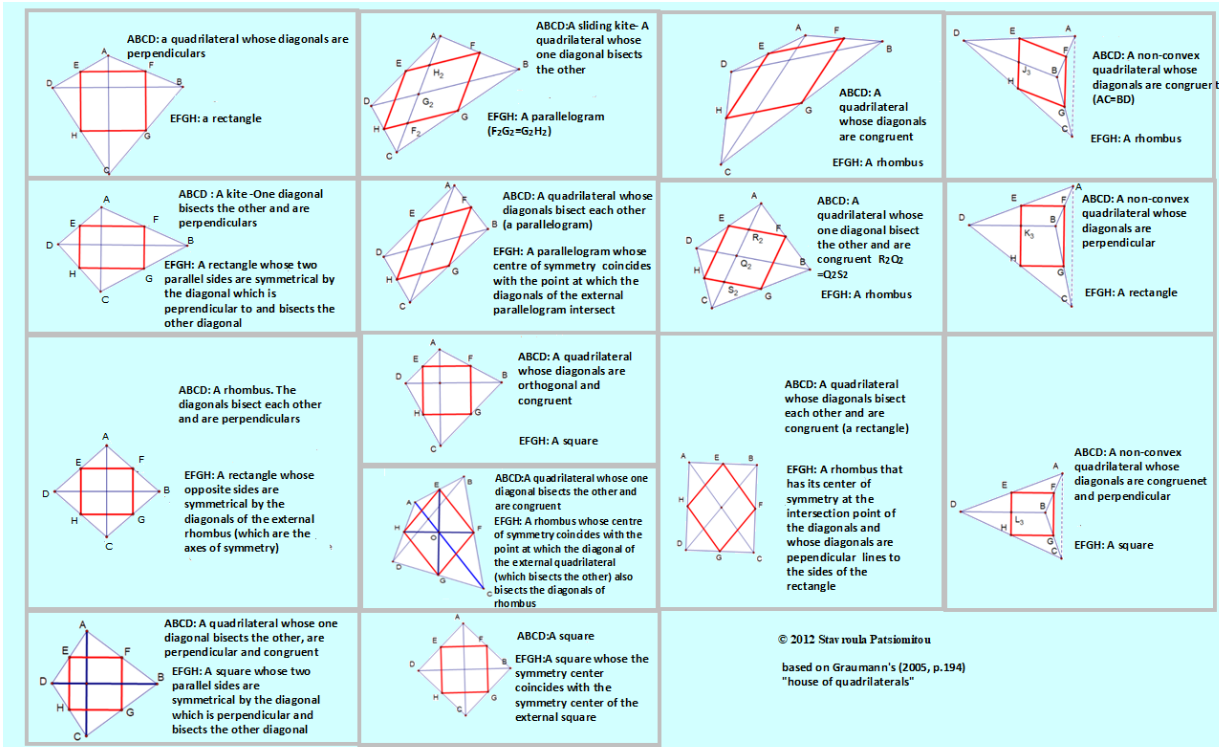


Figure 5.24. My proposal for the classification of the internally constructed quadrilaterals in Varignon's theorem, taking into account the non-convex quadrilaterals (Patsiomitou, 2012a, b, 2015a)

Consequently, the classification of a quadrilateral as a rhombus which occurs internally is not adequate with regard to the properties of the rectangle (the external quadrilateral). The classification of the rhombus as a quadrilateral whose diagonals are perpendiculars and are bisected accurately determines the parallelograms' shape, whose two sides are symmetrical as regards the diagonals of the kite.
In the **Figure 5.24** I explain the classification of the internally constructed quadrilaterals when we join the midpoints of the external quadrilaterals, taking into account the non-convex quadrilaterals.
Graumann (2005) gives examples of non-convex quadrilaterals, "such with one re-entrant angle, [or] with two sides cutting each other and also such with three co-linear vertices" (p. 191). Moreover, he states that

> "If we look out for the middle-quadrilaterals of nonconvex quadrilaterals we first can make the same considerations as above. Thus for all quadrilaterals, also the non-convex one, the middle-quadrilateral is a parallelogram, even for those in space. Also if we ask conversely for all non-convex quadrilaterals whose middle-quadrilateral is a rectangle, rhombus or square we are lead to the same conditions: diagonals with equal length, orthogonal diagonals or both conditions. Only by looking out for non-convex quadrilaterals with one of these conditions we have to make some new reflections" (p. 194-195).

The non-convex quadrilaterals in **Figure 5.24** belong to three main categories: those whose diagonals are perpendicular, those whose diagonals are congruent and those whose diagonals are both perpendicular and congruent. Of course, there are more instances of non-convex quadrilaterals.
For the development of the concept of parallelogram the notions of *specialization and generalization* (De Villier, 1994) have been taken into account. De Villier (1994) in his article "*The role and function of a hierarchical classification of quadrilaterals*" distinguishes between "two essentially different types of classification, namely *descriptive* (a posteriori) or *constructive* (a priori) classification, each of which can be either *hierarchical or partitional*" (p.13). According to De Villier (1994)

- "*hierarchical classification* is the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts" (p.11)
- "*partitional classification of concepts* is such a classification [which] the various subsets of concepts are considered to be disjoint from one another". (p.11)

In **Figure 5.25** De Villiers pictures a generalization and specialization of parallelograms "[emphasizing] that the generalization or specialization need not be hierarchical but could theoretically be partitional (although in actual practice this may be the exception rather than the rule)". (p.14).

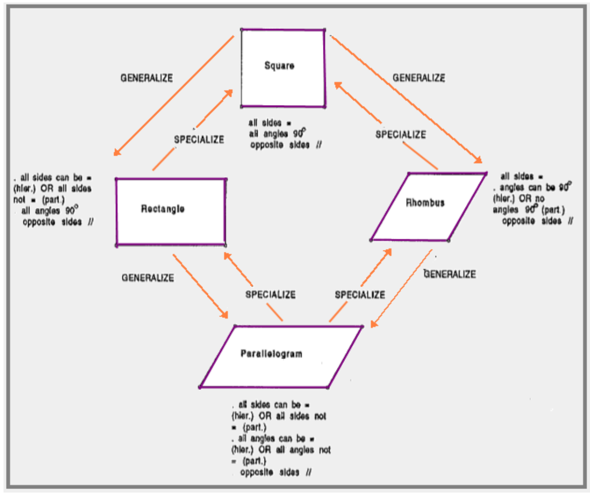


Figure 5.25. De Villiers' (1994, p. 13) *generalization and specialization* of parallelograms (adapted)

Phase D: The LVAR modes in correspondence to the learning phases are described

An extended study which describes the LVAR modes is included in my paper: "*Linking Visual Active Representations and the van Hiele model of geometrical thinking*" (Patsiomitou, 2008b), as well as the revised version of this paper: "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*" (Patsiomitou, 2010). I shall briefly discuss LVAR modes in the next sections.

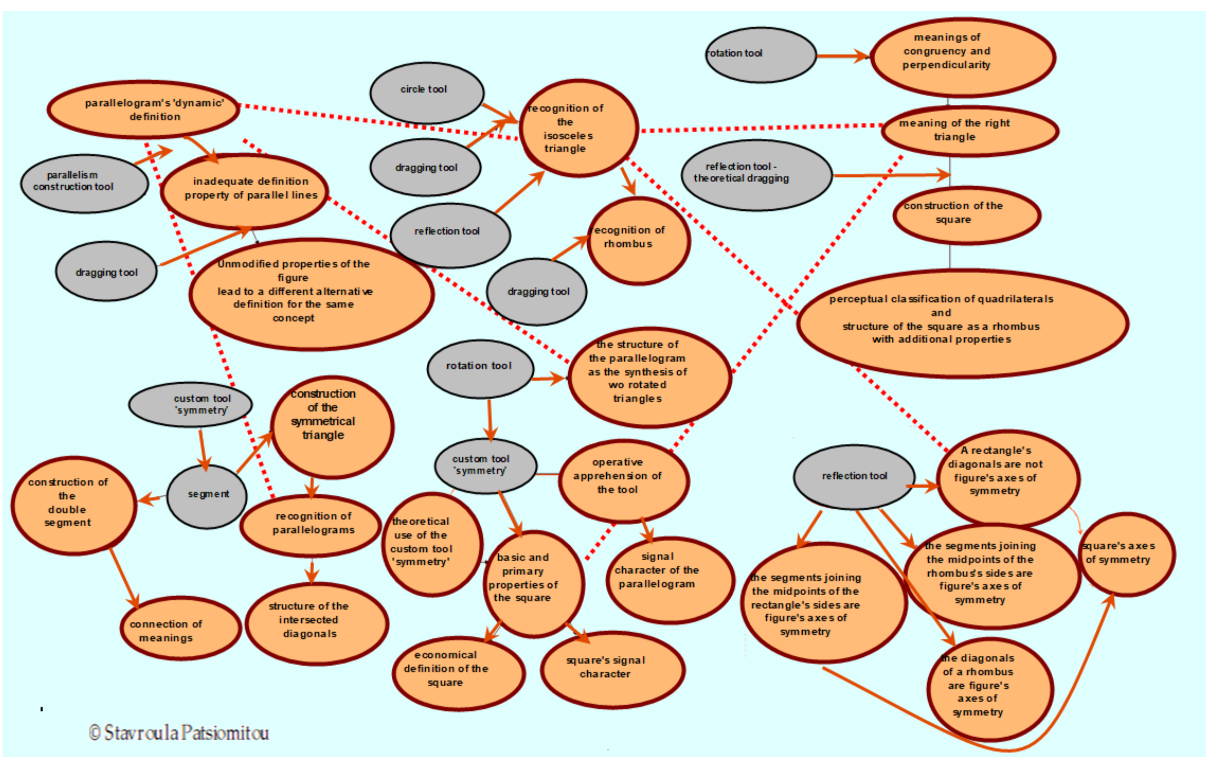


Figure 5.26. Cognitive Analysis of interactions with the tools**:** representing the instructional design process of the DHLP through LVARs using a pseudo-Toulmin model (Patsiomitou, 2014, p. 33)

The design and redesign of activities for the teaching and learning processes, with real problems or simulations of real-world problems through LVAR in the dynamic geometry software, and the results obtained from the research data (Patsiomitou, 2012 a, b), suggest that a student develops his/her abstractive competency when his/her cognitive structures are linked through representations that the student develops during the learning process.

> "*Apart from the aspect of anticipating the mental activities of the students, a key element of the notion of a hypothetical learning trajectory is that the hypothetical character of the learning trajectory is taken seriously. The teacher has to investigate whether the thinking of the students actually evolves as conjectured, and he or she has to revise or adjust the learning trajectory on the basis of his or her findings. In relation to this, Simon (1995) speaks of a mathematical teaching cycle. In a similar manner, Freudenthal (1973) speaks of thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment. If we accept this image of the role of the teacher in instruction that aims at helping students to invent some (to them) new mathematics, we may ask ourselves, what type of support should be offered to teachers. Apparently, we will have to aim at developing means of support that teachers can use in construing and revising hypothetical learning trajectories"* (Gravemejer, 2004, p.9).

The use of a computing environment such as dynamic geometry helps students to build 'a model of the meaning' (Thompson, 1987, p.85) and overcome the difficulties of translation between representations through the automatic translation or "dyna-linking" (Ainsworth, 1999a, p. 133), since [they] "encode causal, functional, structural, and semantic properties and relationships of a represented world – either abstract or concrete" (Sedig & Sumner, 2006, p.2). An implementation of the DHLP in school is described in my study "*Student's Learning Progression Through Instrumental Decoding of Mathematical Ideas*" (Patsiomitou, 2014), as I developed the instructional activities based on an analysis of the results of my PhD thesis, with regard to students' evolution of understanding on instrumental decoding when they construct quadrilaterals.

The whole action is *an innovative production* of a new approach to the educational process based on the theoretical underpinning of hypothetical learning trajectories. This innovation is introduced for the first time in the school of established practice, and thus, proposes the redevelopment / redesign of the everyday teaching practice by using LVARs, with proper interventions in school curriculum. Specifically, linked representations that the student is able to construct (Patsiomitou, 2012a, b):

5.4.4. The Mathematics Teaching Cycle

Simon (1995) introduced and developed the idea of the *Mathematics Teaching Cycle* and created a diagram (**Figure 5.27)** in order to represent the way that a hypothetical learning trajectory is an ongoing modification of three components:

- "the learning goal that defines the direction,
- the learning activities and
- the hypothetical learning process"(Simon, 1995, p. 136).

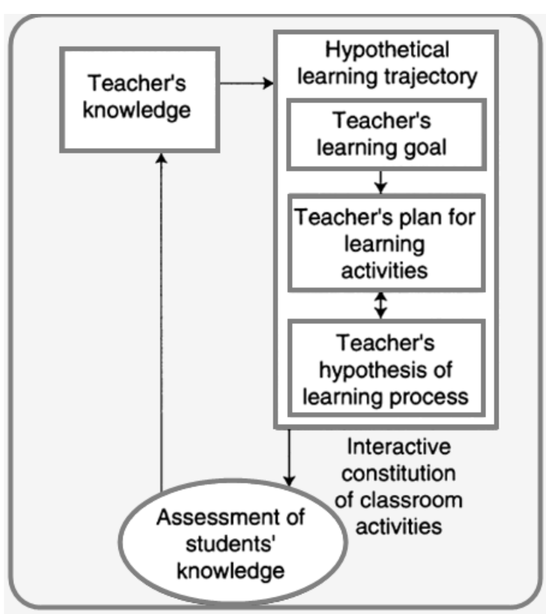


Figure 5.27. The Mathematics Teaching Cycle **(**Simon, 1995, p.136)*(adapted)*

The Mathematics Teaching Cycle portrays the relationship between the following areas of knowledge (Simon, 1995): "the teacher's knowledge of mathematics and his hypotheses about the students' understandings, several areas of teacher knowledge come into play, including the teacher's theories about mathematics teaching and learning; knowledge of learning with respect to the particular mathematical content; and knowledge of mathematical representations, materials, and activities" (p. 133).
Mathematics tasks are related to the teacher's mathematical and pedagogical knowledge. According to Simon (1995) "the ingredient necessary in order to initiate mathematics learning is *pedagogy*" (p. 115, italics in original manuscript). Furthermore, teacher's knowledge about effective mathematical pedagogy influences their instructional practices (e.g., Simon & Shifter, 1991; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).
The activities are "linked to [students'] curriculum and are being tested by collecting their work from classroom in which the curriculum is being used to specify the expected achievement levels; [the current] study tend to be longitudinal, with changes in student thinking and ability tracked over the course of […] a year." (Corcoran, Mocher & Rogat, 2009, p.33).
McGraw (2002, p.10) in her Ph.D thesis created an adaptation on Simon's (1995) teaching cycle, aiming to include the actual discussions with students that "occurred within the 'interaction with students', which influenced the "teacher's knowledge".

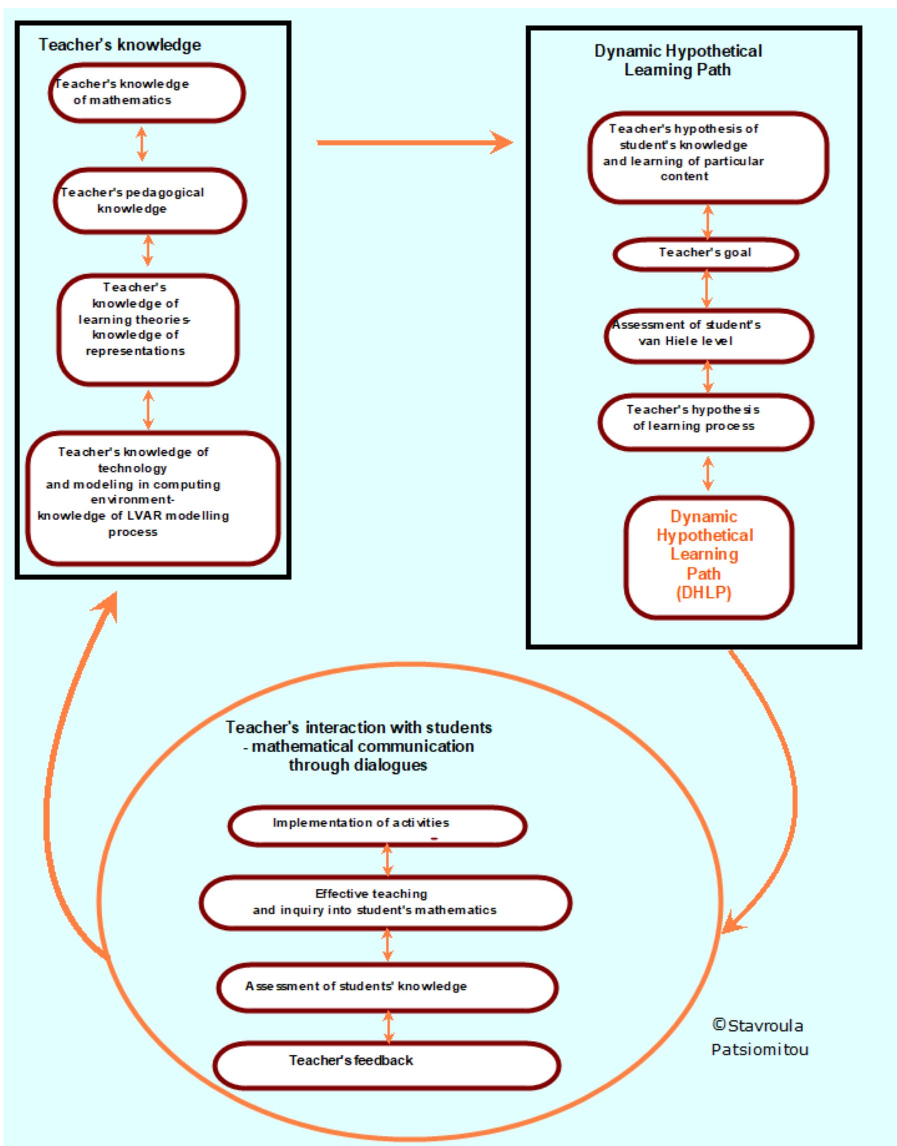


Figure 5.28. My proposal for the *Dynamic Mathematics Teaching Cycle, based on* Simon's (1995) *Mathematics Teaching Cycle* (Patsiomitou, 2014, p. 35)

The analysis of the teaching situations led me to create an iterative diagram (**Figure 5.28**) that is an adaptation of Simon's (1995, p.136) work, taking into account also the work of McGraw (2002, p.10). What has been examined is the use of technology in the teaching cycle, the influence of LVARs which plays an important role in the development of discussions, as well as students' vH level. The diagram aims to include the incorporation of technology practices in class. The results depend on the teacher's different types of knowledge [based on Schulman (1987) and Mishra and Koehler's (2006) framework of Technology, Pedagogy, and Content Knowledge (TPACK)], the students' backgrounds, external resources in the school environment, etc. The teacher's interaction with students and the mathematical communication through dialogues is accomplished in sequential situations: the implementation of activities, effective teaching and inquiry into students' mathematics, the assignment of students' knowledge, all of which leads to the teacher's feedback. These processes go on continually and can suggest adaptations in various domains of a teacher's knowledge, including in the following areas: mathematics, pedagogy, representations, technology, and modeling through LVAR representations.

5.5. What are LVARs?

Visual representation systems encourage students to interact with visually represented mathematical concepts and ideas. Since the 1980s, computers and technological tools have changed the way students, teachers and researchers think, allowing them to visualize mathematical ideas in ways that were previously impossible (Kaput, 1992; Balachef and Kaput, 1997). Technological tools can help the students focus their attention and translate between mathematical representations or interpret information received from a real world environment. Computers and technological tools have changed the context of mathematical activity, imbuing the instructional design process, and the relationship between math and other contexts, with new possibilities. Pea (1987) supports that these tools

> "help students develop the languages of mathematical thought by linking different representations of mathematical concepts, relationships, and processes.[…] The languages of mathematical thought, which become apparent in these different representations, include: (a) Natural language description of mathematical relations […]; (b) Equations composed of mathematical symbols […]; (c) Visual Cartesian coordinate graphs of functions in two and three dimensions; (d) Graphic representations of objects […]"(p.109)

Researchers investigated the influence of technology in mathematics education (e.g., Kaput, 1999), the influence of technology to problem solving, and to teaching and learning (e.g., Borba & Confrey, 1996; Schwartz & Yerushalmy, 1985). Moreover, Burger & Shaughnessy (1986) support that instruction in a successive sequence of increasing complexity has positive effects on students' development of thinking.

Visual Mathematical Representations (VMRs) are those representations, that in Sedig & Sumner's (2006, p.2) opinion:

> "encode these properties and relationships for a represented world consisting of mathematical structures or concepts (Cuoco & Curcio, 2001; Hitt, 2002; quoted in Sedig & Sumner, 2006) in providing a framework to help designers of mathematical cognitive tools in their selection and analysis of different interaction techniques as well as to foster the design of more innovative interactive mathematical tools".

Linking Visual Active Representations (LVARs) are Visual Mathematical Representations (VMRs) which are *dynamic, linking* and *active;* LVARs can help students in the proving process as I designed the LVAR modes in a successive sequence of increasing complexity.

The terms that I chose to define the concept of LVARs have been illustrated in previous studies (Patsiomitou, 2008a, b; Patsiomitou, 2010, p.2):

- The term '*linking*' was preferred to 'linked' because the former denotes something that can be linked, but is not necessarily linked at this moment.
- All DGS objects are necessarily '*visual*' representations of what they stand for.
- An '*active*' representation is a representation that causes action, motion or change because it is in operation, in effect or in progress. Dynamic representations can always *become active* if we cause an action on them, but they are not always pre-constructed.

LVARs always involve semi pre-constructed dynamic diagrams that can be linked and become active in accordance with the wishes of the user, meaning the user is not limited to "*actions pre-set by the sketch creator*" (Sinclair, 2001).

Namely, a representation can become *active* in a DGS environment as a student acts on it and decides the steps and techniques toward his aim, trying to address a problem.
The term "*active representation"* is considered in mindful processing of information in which students individually or in collaboration manipulate and interact with the objects and tools in the dynamic environment and construct their knowledge by reflecting on what they have created.
There is a lack of comprehension about how the *active representations* provided by the dynamic geometry software is elaborated or processed in the student's mind; whatever insights I can support are included in the results of my research (e.g. Patsiomitou, 2007a, b, 2008a, b, 2010, 2011a, b, 2012a, b, 2013a, b, 2014, 2016a, b; 2018a, b). We can gain insights through continuous recording of brain activity and interactivity with these representations. Other important considerations include: how these connections with the students' pre-existing information are achieved; how this connection is linked to the process of learning; how these active connections are modified; and hence the associations between different linking representations are stored in students' mind.

5.5.1. How did I design LVARS?

LVARs is a part of an *instructional design process* which I was designed in phases: for example, what I did towards preparing the lesson before the instruction was delivered; what the organized topics were of the learning trajectory; what I predicted regarding the external stimulation delivered by new representational infrastructures in order to create successive stages in the transformation of previously learned material retrieved from the learner's memory etc.
The *instructional design process* developed into a didactic experiment of action research (Kemmis & McTaggart, 1982; Schön, 1987). The qualitative study (Merriam, 1998) with a *quasi-experimental design* (Campbell & Stanley, 1963) was conducted in a public high school class in Athens Greece. For this study, the *constant comparative m*ethod was chosen in order to deduce a *grounded theory* (Strauss & Corbin, 1990). Concretely, during a didactic experiment with the support of Geometer's Sketchpad v4 dynamic geometry software, student participants followed a 4-phase Dynamic Hypothetical Learning Path (DHLP) (Patsiomitou, 2012 a, b) which I conceived and applied as part of my PhD thesis (**Figure 5.29**). The methodology used, consisted of the methodological forms of case studies of pairs of students, action research (Bogdan & Biklen, 1998) and "theory-building" (Eisenhardt, 2002) from case studies. I was responsible for the choice of activities, for session planning and for student assessment. The DHLP (aforementioned in section 5.4.3) is composed of four phases (Patsiomitou, 2010, p. 1): Phase 1– construction activities; Phase 2 – construction through symmetry/transformations activities; Phase 3 – the exploration of open-ended problems; and Phase 4 – building and transforming semi-predesigned Linking Visual Active Representations (LVARs). The four phases constitute a learning trajectory, hypothetical at the beginning of the process. The phases are interconnected in terms of: a) the conceptual context, b) the order in which the software's technological tools are introduced, and c) the increasing difficulty at both levels.
The learning path/trajectory was hypothetical at the beginning, as I had hypothesized "if and how [the students] would construct new interpretations, ideas, and strategies" (Fosnot, 2003, p. 10) and the path would follow as they worked on the problem. Moreover, the instructional design process was a synthesis of constructivism and discovery learning, as it was my intention: (a) the students to build on their previous knowledge, (b) the teaching and learning process would be supported through mathematical discourse and conceptual understanding and (c) the learning included students' discovery ("aha" expressions) and their dynamic reinvention of knowledge under investigation.
Cobb and Steffe (1983) encourage the idea of the researcher as teacher, arguing that: "the activity of exploring children's construction of mathematical knowledge must involve teaching". Steffe et at. (1983) describes constructivist teaching experiment as being "derived from Piaget's clinical interviews" and notes that: "A distinguishing characteristic of the technique is that the researcher acts as teacher. Being a participant in interactive communication with a child is necessary because there is no intention to investigate teaching a predetermined or accepted way of operating (p.177).
I shall report here what I also wrote in my study "*The development of students' geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment" (Patsiomitou, 2008a)*. I linked the sequential steps in a proof by linking sequential actions over multiple pages of the software or linked the steps in the representation of the problem in order to lead students to a cognitive linking of the used dynamic representations, based on the work of Kaput which supports that linking representations "*creates a whole that is more than the sum of its parts* […]" (Kaput, 1989). Furthermore, it creates a "temporal sequence of

the […] steps [representing] *the counterpart of the logic hierarchy* between the geometric properties of a figure" (Mariotti, 2002, p. 686). Students' understanding of meanings often led me to note the sequence of steps or stages through which they gathered information from the [computing] environment as stimuli. The information from the computer environment goes through a modification, linked to students' minds stored information (or is modified in the light of the information stored in their mind) so they can answer the teacher's questions or participate in a class discussion. The students possibly transform this stimulus into mental representations linked with similar pre-existing information in their minds. I say 'possibly', as I cannot see into my students' minds. What I can see is the following: those representations that could not be linked to previous information were rejected by the students, as they did not answer. What is important to investigate is the level of strength of these links or connections in the students' mind, which can illustrate how the learning of meanings was accomplished.

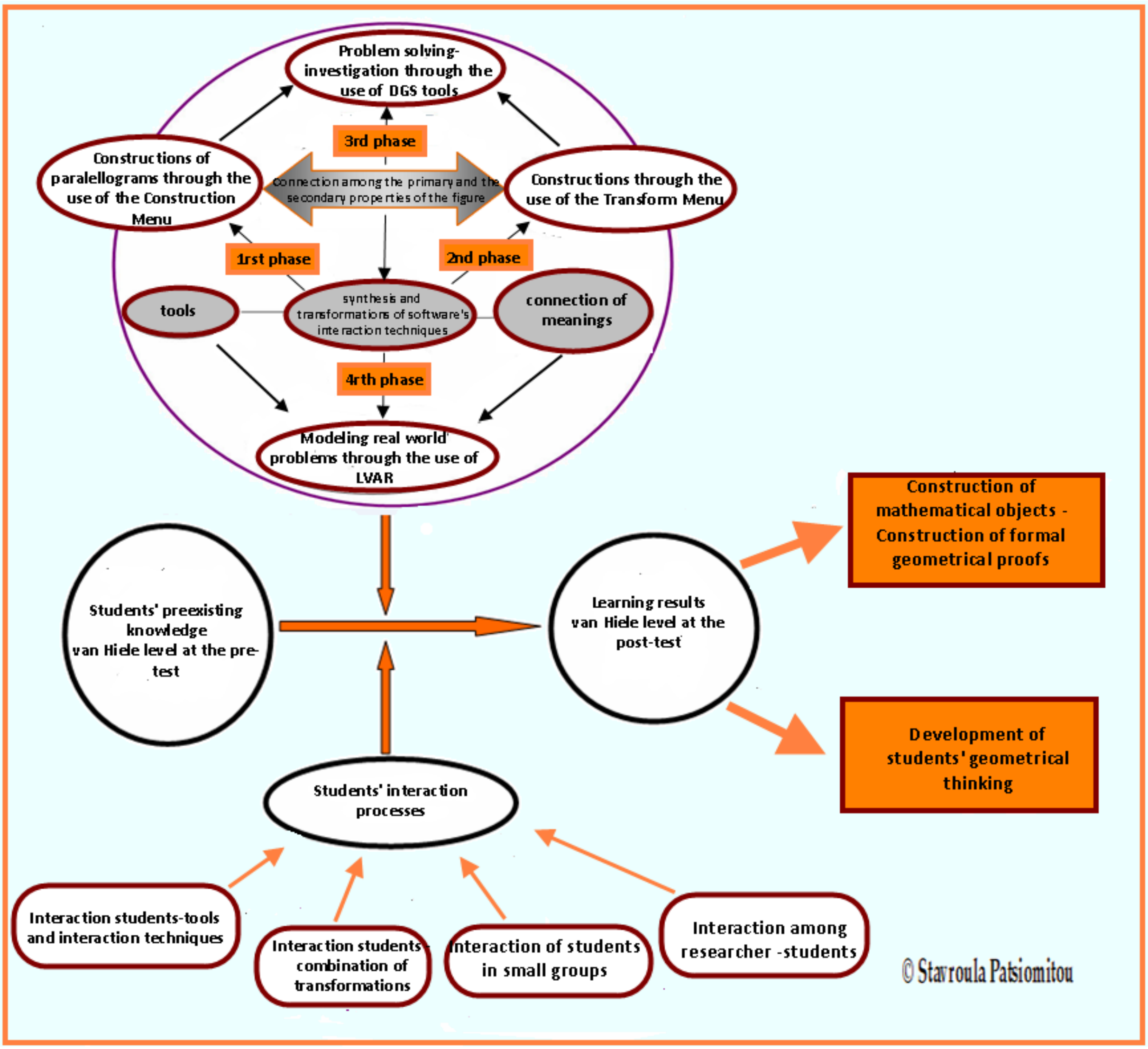


Figure 5.29: A diagram representing the instructional design process of the DHLP (Patsiomitou, 2012a, p. 456, in Greek)

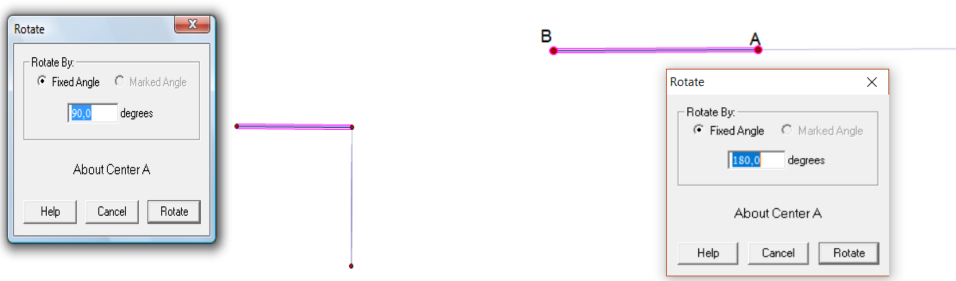
This concept of a design process accords with what several scholars have formulated in relation to the explicit linking of ideas through linking representations. For example, Thomas (2004, p.14) argues:

> "we might say that because conceptual ideas may be constructed from a number of representations it is a good idea for students to experience a number of these at the time they begin to learn the concept. In particular, the *explicit linking of ideas across representations* is very useful and important. Hence, teaching should seek to assist representational versatility by concurrently providing, and linking, a number of representations in each learning situation."

Reiterating what I have described at length in previous chapters of the current work, I will argue that, in order to develop an understanding of a meaning, the students have to create a transitional bridge between the external and internal representation (e.g, Kaput, 1999; Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001) of this meaning, through *instrumental decoding*. Moreover, students' visualization of an object may differ from their perception of it, while the important thing is to understand which mathematical concept or relationship is being represented.

To create a dynamic diagram during the *Linking Visual Active Representations (LVARs)* design process in the Geometer's Sketchpad environment, I used a diverse set of interaction techniques including "animating" a point on its path, 'tracing" a segment, "hiding and showing" action buttons, and "linking" or "presenting" action buttons, or a combination of interaction techniques (Patsiomitou, 2008a, b; 2010; 2012a, b), to achieve students' interaction. Sedig and Sumner (2006) have distinguished between *basic* and *task-based* interactions with visual mathematical representations (See also Sedig, & Liang, 2008). According to Sedig and Sumner (2006) "Benefits of animating VMRs include: attracting and directing attention to embedded detail, visualizing dynamic and transitional processes, supporting external cognition, increasing visual explicitness of encoded information, and facilitating perception of semantic and temporal transformations inherent in the VMR".

Sedig, Klawe, and Westrom (2001, cited in Sedig & Sumner, 2006) conducted an empirical study and they found that "*adding scaffolding to direct manipulation of representations of transformation geometry concepts significantly improved student learning*". Paraphrasing with what Sedig, Rowhani, & Liang (2005, p.422) support, I argue that "the interaction with [LVARs] in a computing environment has two aspects: the action upon a representation by the user through the intermediary of a human-computer interface, and the representation communicating back through some form of reaction or response." To illustrate this I captured two screenshots (**Figures 5.30a, b**), showing a pop-up menu of the rotate command (see also Patsiomitou, 2008a). When we press the "rotate" button the segment is transformed, by rotating it through 90 [or 180] degrees. We can also use a combination of action buttons for hiding/showing objects on screen, or a presentation button which repeats actions sequentially, and a link button which links the current page with the next one, thereby connecting the actions. The action upon the rotating command for the construction of transformation of an object has an important result: the reaction of the representation with a student's response "this angle is equal to 90 degrees, and the lines are perpendicular" or "this angle is equal to 180 degrees and the three points belong to same line.



Figures 5.30a, b. Screenshots showing the pop-up menu of the rotate command

The rotation command /technique "has been referred to as direct concept manipulation, as opposed to direct object manipulation" (p.35) because "if students are to focus on the concept of rotation, rather than focusing on the shape being rotated, they can directly interact with a visual representation of rotation" (Sedig et al., 2001 quoted in Sedig & Sumner, 2006).

LVARs in a geometry proving process allow students to act on and modify them using the full range of program features. I took into account two distinguished but interconnected options when designing the activities, both of which are based on classroom observations made over many years of my teaching geometrical proof to secondary-level students in class. Namely (Patsiomitou, 2008a, p. 358): *"linking the steps* in the constructional, transformational or explorative actions or processes in the software using interaction techniques" by "*linking the steps in the proof* via a sequence of pages or the same page in the DGS environment using interaction techniques". In other words, I took into account how to link *conceptual with procedural knowledge, or operational with structural understanding*. Both factors impact directly on how students are guided to the proof process through solving problems, and hence on how students are guided to abstract thought processes. Thus,

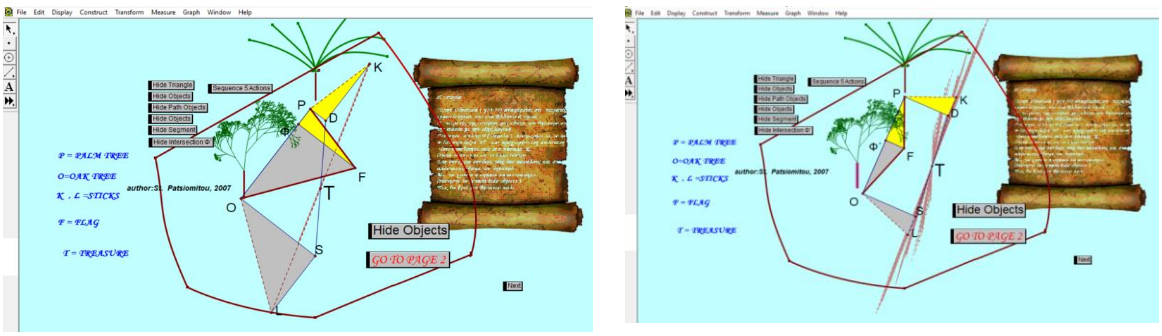
during the problem-solving process or when reproducing a theorem with a view to proving it, the teacher or students ask questions which help them construct the proof. Thus, a problem would be solved by breaking it down into a series of questions whose answers gradually distil the proof the students seek.
This process tends to remind us of the Socratic method ("maieftiki" in Greek) by which teachers ask questions designed to elicit the correct answer and reasoning processes. Socratic method is the method, which according to Weusijana, (2006) is "originated form Socrates' use of it in the Meno dialogue of Plato, but it has other names, such as the inquiry method of teaching (Collins, 1977).[…] The Socratic Method also encourages learners to generate their own answers to an educator's questions as opposed to being given the answers. Often such knowledge that is generated by learners instead of just being received is more easily recalled (Bobrow & Bower, 1969; Slamecka & Graf, 1978). Furthermore, the educator doing the questioning can also gain insight into how students are receiving instruction (Anderson, 1988)" (Baba Kofi Adam Cooper Puryear Weusijana, 2006, p. 26).
The Socratic Method is a dialectic method of inquiry. The questioning process thus helps students determine and extend their underlying knowledge. Freudenthal (1971, p.414) supports that "Socrates did not teach a readymade solution but the way of reinventing the solution." The same approach two millenia later was formulated by Comenius (quoted in Freudenthal, 1971): "The best way to teach an activity is to show it." According to Freudenthal "this is a socratic idea, though it involves more than a Socratic lesson. While Socrates taught his lesson, the slave listened, whereas Comenius will show the student an activity to explain afterwards what it means and finally to have it imitated by the student." Freudenthal supports that modern educators are likely to subscribe to a variation of Comenius' device while "The best way to teach an activity, is (not) to show it" but rather "*The best way to learn an activity, is to perform it*." Representations were the first empirical mode leading to the proving process in Ancient Greece, too (see, for example, Socrates and Meno) although the process observed in Euclid's "Elements" does not display a transition from visual representation to rigorous reasoning.
The LVAR process expands the "maieftiki" inquiry method as through LVARs, I investigated the ability of students to develop *deductive reasoning* during problem solving process, dynamically "*reinventing the subject matter*" (Freudenthal, 1971, p.416). More particularly, I investigated in how Geometer's Sketchpad, might contribute to developing students rigorous proof. The visual representations can proof only specialized cases, while the Euclidean proof can empower every case by reinforcing the initial visual proof. Through LVARs, the teacher can guide the students by means of elucidation or questions eliciting conclusions which form a step-by-step visual proof. The successive pages in the software also play a significant role, and can be seen as an active *"alive" dynamic section* (Patsiomitou, 2019a, b) in an e-book revealing the various stages in the proof. The sequence of increasingly sophisticated construction steps could thus correspond to the numbering of the action buttons which allows student to interact with the tool when they are motivated by their own thoughts or when they are encouraged to do so by their teacher in class.
Proof, proving and deductive reasoning in my opinion, are notions that are strongly connected to each other. What did I achieve through LVARs design process? According to Schoenfeld (1994b)

> "[...] proof becomes a natural means of exploration and communication [...and] proof will be a necessary component of the sense-making and discourse processes" (p.30).

I chose real problems that would stimulate the students' interest and induce them to use their knowledge of geometry to: a) visualize and conjecture, b) investigate, and c) prove. According to Kilpatrick (1987) problem formulating should be viewed not only as a goal of instruction but also as a means of instruction.
Here are two examples of half pre-designed steps to illustrate the software process (Patsiomitou, 2008):

- A) the straight line KL can be traced and every new position of point F produces a different trace of KL. This action corresponds to the question: "What would happen, do you think, if we changed the position of point F, where the flag is?" (**Figures 5.31a, b**). We then leave the students to answer by moving Point F. This produces a series of lines KL, all of which pass through Point T. "Trace" according to Jahn (2002) "emphasises a dynamic interpretation of the representation of a trajectory of a point […]".
- B) Pre-designed hide/show action button allow the user to hide objects (e.g., the flag and with it Point F), which allows students to experiment and investigate the subject in their own way. This action causes the lines which could lead to the treasure point T simultaneously disappearing. I consciously and intentionally left Point T on the screen, because if it disappeared, the students would be unable to visualize or investigate the problem.

This means that the process is directly linked to how the activities can scaffold the students to reach conclusions. Analyzing the logic of the design process, I should point out that I bore the following in mind when designing, constructing and implementing the activities: the process should be *active to* keep the students interested and promote *discovery* and *dynamic reinvention* of knowledge should be based on theories which deal with knowledge, teaching and the learning of Mathematics. Moreover, students should be conclude using Euclidean proof.



Figures 5.31a, b. Building LVAR modes (Patsiomitou, 2008a, b, 2010)

Yuen Lie Lim (2009) in her study "*A comparison of students' reflective thinking across different years in a problem-based learning environment*" argues that "Problem-based learning (PBL) is a constructivist approach to learning which is believed to promote *reflective thinking* in students" (p.1). She suggests that critical reflection may happen in stages. The four levels of reflective thinking are outlined in the following table (Mezirow, 1997, cited in Yuen Lie Lim, 2009, p.173).

Non-reflection	Habitual action	The learner engages in activity that is routinely and frequently conducted, with little conscious thought
	Understanding	The learner acts to comprehend and apply knowledge within contextual constraints, and without recognizing personal significance
	Reflection	The learner assesses the problem-solving process and uses this to make decisions about what is the best way to approach the problem, but without re-assessing assumptions on which beliefs are based
Reflection	Critical reflection	The learner evaluates ideas and actions in light of the assumptions which underlie them (i.e., the reasons for, and implications of, them)

Figure 5.32. Four levels of reflective thinking (Mezirow, 1997, cited in Yuen Lie Lim, 2009, p.3) (adapted)

According to Yuen Lie Lim (2009) "How do these levels of reflection relate to problem-based learning? For a start, […] the problem trigger ignites cognitive conflict, thereby providing the spark for reflective thought […]Finally, they would have to compare and evaluate various ideas and solutions" (p. 174).
Building on the above-reviewed theoretical background and new views, I introduced and defined the year 2008 the notions of *Linking Visual Active Representations* and *Reflective Visual Reaction* during a dynamic geometry problem solving session, directly connected with the design process in the software as follows (e.g., Patsiomitou, 2008a):

> **Linking Visual Active Representations** are the successive phases of the dynamic representations of the problem which link together the problem's constructional, transformed representational steps in order to reveal an ever increasing constructive complexity; since the representations build on what has come before, each one is more complex, and more integrated than in previous stages, due to the student's (or teacher's, in a half-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or exist in their mind).
> **Reflective Visual Reaction** is that reaction which is based on a reflective mode of thought, derived from interaction with LVAR in the software, thus complementing and adding to the student's pre-existing knowledge or facilitating comprehension and integration of new mathematical meanings.

Finally, “Linking Visual Active Representations” (LVARs) during a dynamic geometry problem solving session are defined as follows (e.g, Patsiomitou, 2012a, b, 2019a), incorporating the notion of *instrumental decoding* (Patsiomitou, 2011a, b, 2012 a, b) and the notion of *dynamic hypothetical learning path (DHLP)* (Patsiomitou, 2012a, b):

> **Linking Visual Active Representations** are the successive/consequential building steps in the dynamic representations of a problem or between problems, which repeat the same procedural steps or steps reversing a procedure in the same phase or between different phases of a *hypothetical learning trajectory*. LVARs reveal an increasing structural complexity by conceptually and structurally linking the transformational steps taken by the user (conducting anticipatory thought experiments) through the interaction techniques provided by the software as a result of his/her development of thinking and understanding of geometrical concepts, which are *instrumentally decoded* by the way s/he has visualized mentally what exist in his/her mind or a revision of it.
>
> **Reflective Visual Reaction** is the reaction based on a reflective mode of thought, derived from interaction with LVARs in the software.

My aims in developing LVAR (Patsiomitou, 2010) were to:

- enrich the existing curriculum with DGS–based problems that are adaptations and extensions of existing static activities, tasks and real-world situations;
- enrich students’ experiences with more effective presentation and interaction techniques better suited to the DGS environment than to other didactic materials;
- trigger students’ actual cognitions in geometry as well as their aesthetic and digital sense; and
- attract students to solve DGS problems designed to develop their mathematical understanding, deductive reasoning and formal Euclidean proof, either individually or in an orchestrated classroom process.

The first results from the use of LVARs have been described in my study “*The development of students’ geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment”* (Patsiomitou, 2008a) as well as in my study “*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*” (Patsiomitou, 2008b, 2010). Concretely:

LVARs motivated the students to answer rapidly and spontaneously. I asked the participated students sequential oral questions, which meant students did not have time to use paper and pencil. The classroom observations revealed that the same students did not always display the same spontaneous reflective reactions. The LVARs that spread over multiple pages helped the students to react instantaneously and to articulate their thoughts. The LVARs helped the students to operate in an auxiliary or complementary manner, assimilating or accommodating their prior knowledge, or as a confirmation of the student’s cognitive processes. The students’ RVR occurred at many points during the didactic experiment thanks to the use of interaction techniques. As a result the students constructed mental schemes for mathematical meanings and were “starting to develop longer sequences of statements” (De Villiers, 2004). LVARs helped the students form rigorous Euclidean proofs and they reached conclusions on the problem by correlating the theorems they already know. This is to say that LVARs assisted students to develop their van Hiele level.

The research has led me to conclude that LVARs have the following features:

- They appear in stages in dynamic linking illustrations, which help to recognise the connections between the objects in the diagrammatic representation and keep the students focused on the aim of the overall construction.
- They can be acted on and modified by students, allowing them to use the full range of program features, which renders them being ***“Alive”***.

These results were crucial, allowing me to continue the research and further investigate the effect that *Linking Visual Active Representation* modes (e.g., Patsiomitou, 2008b, 2010, 2012 a, b) have on students’ gradual acquisition of competence in the construction of rigorous proofs, as part of a problem solving process.

5.5.2. What are LVAR modes?

As I have written in my study “*Building LVAR (Linking Visual Active Representations) modes in a DGS environment“*(Patsiomitou, 2008b, 2010) the process of proving a problem or theorem consists of a series of steps which can function as responses, anticipating the questions posed explicitly or implicitly by teacher or student. This is what I had in mind when I designed the different LVAR modes in Sketchpad to link the proving process with envisaging arguments and combining "these arguments into a deductive chain that constitutes a sketch of the final proof" (Heinze, 2004, p.44)

Research has shown that even when working with static means, students start conjecturing when they face a problem situation. Depending on the tools with which they are provided and their interaction with the teacher or other students, they can develop elements of reasoning by "*developing specific competencies inherent in producing conjectures and proving the produced conjectures by taking elements of theoretical knowledge into account*" (Boero, 1999). Although the two phases—conjecture production and proof construction—cannot be separated and linearly sequenced, their component elements are described and reported by Boero (1999):

- *producing a conjecture* (which includes exploring the problem situation, identifying possible "regularities" and the conditions under which such regularities take place, identifying arguments for the plausibility of the produced conjecture);
- […] *exploring the content of the conjecture and the limits of its validity* (which includes heuristic, semantic (and even formal) elaborations about the links between hypothesis and thesis, identifying appropriate arguments for validation related to the reference theory, and envisaging possible links amongst them);
- *selecting and enchaining coherent, theoretical arguments into a deductive chain* […] ;
- *organizing the enchained arguments into a proof* that is acceptable according to current mathematical standards; and
- *approaching a formal proof* (or parts of the proof).

Heinze (2004, p.34) modified this sequence into five coding categories, of which the last three categories are:

- Phase 3 – *an explorative phase* based on the formulated conjecture and aimed at identifying appropriate arguments for the conjecture and a rough planning of a proof strategy, which can be divided into four subcategories: (a) referencing assumptions, (b) investigating assumptions, (c) collecting further information and (d) generating a proof idea;
- Phase 4 – *the combination of these* (verbal or written) arguments into a deductive chain that constitutes a sketch of the final proof; and
- Phase 5 – *the writing down of the chain* of arguments according to the standards of the mathematics classroom in question (including a retrospective overview of the proof process (Heinze, 2004, p. 35).

Stylianides (2007) in his study "*Proof and Proving in school Mathematics*" defines proof. As he states "*proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) [...]
2. It employs forms of reasoning (*modes of argumentation*) that are valid [...]
3. It is communicated with forms of expression (*modes of argument presentation*) that are appropriate [...] (p. 291).

In the following Table Stylianides (2007) "gives examples of the three components of a mathematical argument, mentioned in the definition of proof" (p. 207).

Table 1

Examples of the Three Components of a Mathematical Argument Mentioned in the Definition of Proof

Component of an argument	Examples
Set of accepted statements	Definitions, axioms, theorems, etc.
Modes of argumentation	Application of logical rules of inference (such as modus ponens and modus tollens), use of definitions to derive general statements, systematic enumeration of all cases to which a statement is reduced (given that their number is finite), construction of counterexamples, development of a reasoning that shows that acceptance of a statement leads to a contradiction, etc.
Modes of argument representation	Linguistic (e.g., oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc.

Figure 5.33. Examples of the three components of a mathematical argument, mentioned in the definition of proof (Stylianides, 2007, p. 207)

According to Schoenfeld (1994b) "*Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels*". (p.28)

A DGS environment like Sketchpad or Web Sketchpad is a perfect means to support the LVAR argumentation and proving process. I very often try to make a mental shift from an *observer's point of view* to an *actor's point of view* (Cobb, Yackel & Wood, 1992 in Gravemeijer, 2004) when designing activities, interchanging the predetermined student and teacher roles in my mind. By this, I mean that I place myself (as an observer) in the position of my students (as actors), trying to think as a student and responding to my own questions: How can a student perceive a mathematical meaning through a concrete learning path? Are the procedures sequential and the diagrams complementary? Do the activities help my students to recall preexisting structures? Clements & Sarama (2014) point out:

> "[…] When [the teachers] interact with the student, teachers also consider their own actions from the student's point of view. […]. Thus, the benefit for the teacher is to have a well-formed and specific set of expectations about students' ways of learning-a likely path that incorporates the big, worthwhile ideas" (p. 23).

Specifically, having observed that there are several ways of characterizing different connections/links between representations correlated with the use of different interaction techniques supported by the Sketchpad, I produced a characterization for these different modes of linking representations. The different LVAR modes can be built using a combination of different transformational processes and interaction techniques supported by the Sketchpad environment. The LVAR modes are described as follows (e.g., Patsiomitou, 2008b, pp. 169-174, Patsiomitou, 2010, p. 11-19):

- **Mode A-the inquiry/information mode**: In this mode of LVARs the students familiarize themselves with the field under investigation using the instantiated parts of the diagrams which lead them to discover a certain structure.

  The use of an action button "animation" (in combination with the trace command) for example, transforms the diagram into a "*diagram in motion*", reinforcing the formulation of conjectures since the stimulus received from the *hybrid-dynamic representation* leaves the properties of the figure unaltered despite the transformation it undergoes.
- **Mode B-the directed orientation mode:** The sequential constructional and transformational steps of LVARs are displayed as a global shape to which more elements and/or information are gradually added, rearranging, annotating and probing parts of the diagram when action buttons are directly manipulated. The steps in the construction of the diagrammatic reconstruction which are displayed by pressing the action buttons are linked to suitable questions and their answers.

  The process has the following advantages: during this process the students are led to cognitively connect additional, complementary, transformational reconstructions of the problem configuration and actions aimed at externalizing the student's thoughts by means of suitable chain questions which guide them towards the solution to the problem.
- **Mode C–the explicitation mode**: Transformations in increasingly complex linked dynamic and active representations of the same phase of the problem modify the on-screen configurations simultaneously. The figures on screen undergo a *metamorphosis* as a result of the manipulations.

  The student can observe a continuous flow on the screen because "*cause and effect are observed simultaneously*" (Sedig & Sumner, 2006, p.7).
- **Mode D–the free orientation mode**: LVARs can be displayed side by side on the same page of the software in an overview. The students can focus their observation on what extra information is presented in the next emerging iconic form of the representation. The emerging additional representations can be dragged independently. The students are led to a proof that confirms their initial reasoning, conjectures and exploratory processes.
- **Mode E–the integration mode:** Successive LVARs on different pages that are linked cognitively and not necessarily constructionally, compose the solution to the problem. The information with which they became familiar in the new network of evoked geometrical objects and their interrelationships is reviewed and summarized. The students have developed thinking processes and applied skills, developing a mathematical model to interpret the realistic problem.

A detailed description of the LVAR modes with screenshots of the sequential representations of two problems modelled in the software are presented in my study "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*" (Patsiomitou, 2010), a revised version of the study "*Linking Visual Active Representations and the van Hiele model of geometrical thinking*" (Patsiomitou, 2008b).
The papers also include examples of the LVAR modes correlated with excerpts from dialogues recorded during the research process in which I have identified students' arguments or conjectures and students deductive chains to construct their solutions to the problems presented to them.
For the needs of the current study I shall report examples of the LVAR modes of the modelling process of the problem *"A power plant is to be built to serve the needs of the cities of A (Athens), B (Patras) and C (Thessaloniki).Where should the power plant be located in order to use the least amount of high-voltage cable that will feed electricity to the three cities?"*(this is an adaptation of a similar problem cited in Olive, 2000).
In my studies screenshots of the sequential representations of two problems modelled in the software are presented and correlated with excerpts from dialogues recorded during the research process in which I have identified students' arguments or conjectures and students deductive chains to construct their solutions to the problems presented to them.

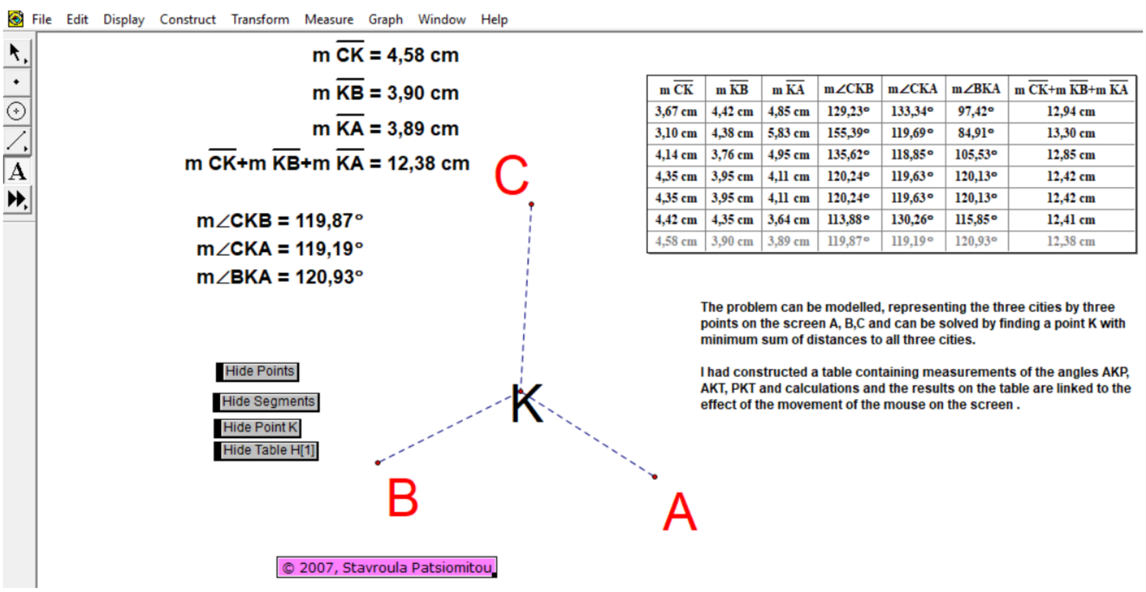


Figure 5.34a. Mode A, Sequential phases of experimentations with the dynamic objects (screenshot from the GSP file) (see also Patsiomitou, 2008b, 2010)

Mode A-the inquiry/information mode (Figure 5.34a): According to Patsiomitou (2008b, 2010): The students investigate the modifications made to the calculations of the segments to identify the different positions of point K. Changing the position of point K by dragging it is dynamically linked to the changes/ modifications in the resultant angles in the table and the upcoming modification to the sum of the segments. This process encourages students to observe that the minimal sum is observed when every angle is equal to 120°. The students are usually led to draw rough conclusions regarding the position of the point under investigation; for instance, that it is the circumcentre of the triangle ABC. The construction of the circumcentre and the measurements reveal cognitive conflicts in the students. The addition of a new line in the table for new measurements every time point K is dragged can lead students to posit inductive formulations which converge on the angles between the segments being 120 degrees. During this process, we have a reversible (bi-directional) transformation of a) the geometrical into an algebraic model, and b) the algebraic conclusions drawn from comparisons between on-screen dragging on the geometrical representation.

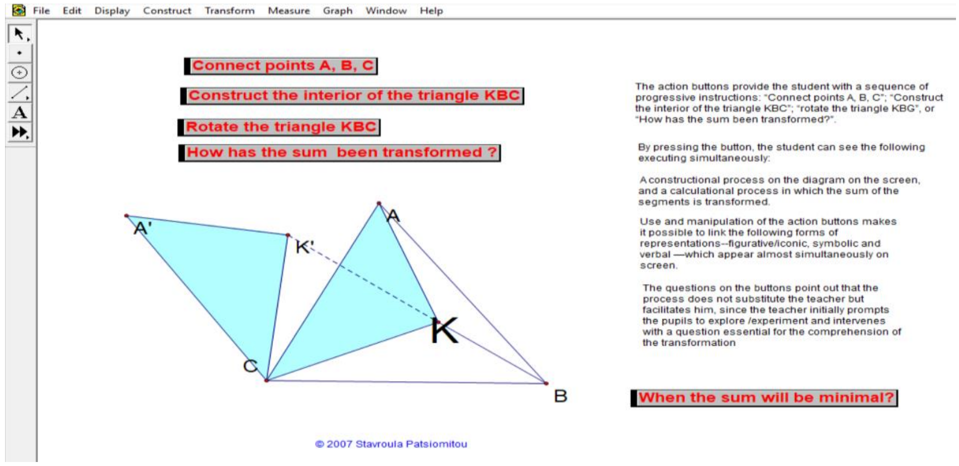


Figure 5.34b. Mode B- Sequential LVARs due to the action buttons (see also Patsiomitou, 2008b, 2010)

Mode B-the directed orientation mode (Figure 5.34b): According to Patsiomitou (2008b, 2010): By pressing the buttons, the student can see the following steps executed simultaneously: A constructional process on the on-screen diagram and a computational process in which the sum of the segments is transformed. Use and manipulation of the action buttons makes it possible to link the following forms of representations—figurative/iconic, symbolic and verbal—which appear almost simultaneously on screen. During this process, a geometrical object is transformed into a new geometrical object emanating from the rotation. This process leads to the transformation of the sum of the three segments AK, KK′ and K′B′ on a crooked line and is followed by a mental transformation. That is to say, the process begins into the *spatiographical* and leads to *the theoretical field*.

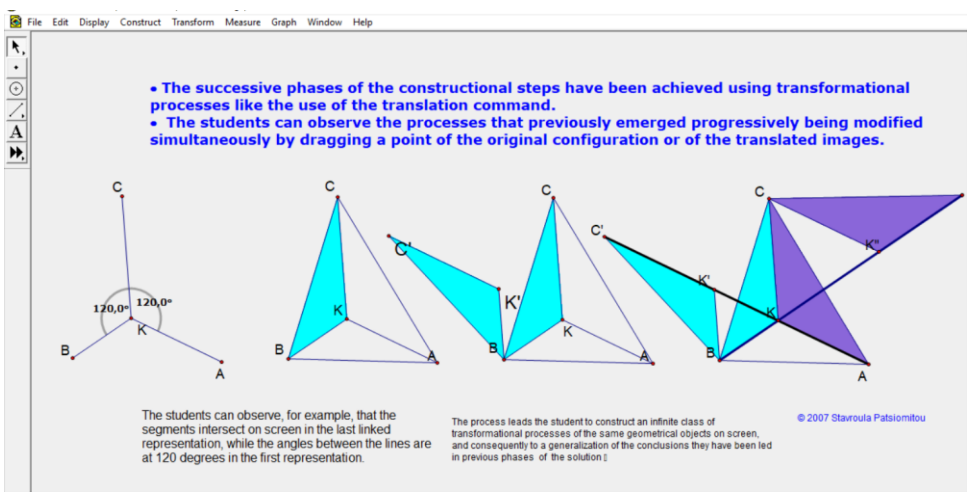


Figure 5.34c. Mode C-The transformed-translated phases of the LVARs (see also Patsiomitou, 2008b, 2010)

Mode C – the explicitation mode (Figure 5.34c). According to Patsiomitou (2008b, 2010):
I created the successive phases of the constructional steps, using the translation transformation process. By dragging a point of the original configuration or the translated images, the students can observe the processes that emerged previously being modified simultaneously. Students are able to directly assume or infer the properties and the *interrelationships* between figures from properties indicated on the diagram by conventional marks (for example the equality of angles, or the angles measurements). The process leads the student to construct an infinite class of transformational processes of the same geometrical object on screen; consequently leads to a *generalization* of the conclusions they have been led in previous phases of the solution.
Mode D –the free orientation mode (Figures 5.34d, e): According to Patsiomitou (2008b, 2010): Students are able to directly assume or infer the properties and the interrelationships between figures from properties indicated on the diagram by conventional marks (for example the equality of angles, or the angles measurements). It is essential that the student can display every step in the solution together on the same screen; only thus, can they

see the progressive changes globally. The students can use their creativity to pose open goals with multiple steps and alternative solutions, thereby extending their knowledge to what they have seen before. One could consider this, as the second part of directed orientation, in which the students learn to find their way through the network of relations assisted by their extant knowledge.

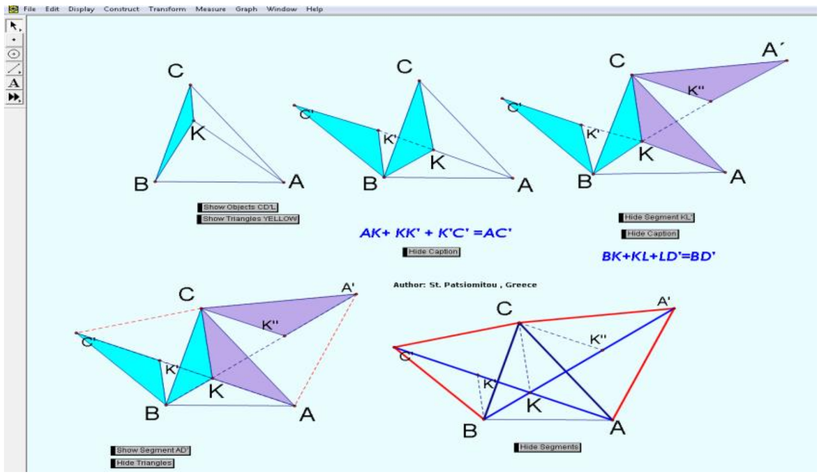


Figure 5.34d. Mode D, stages in the solution displayed side by side on the same page (see also Patsiomitou, 2008b, 2010)

For example, the proving process leads to a solution which requires the construction of the circumscribed circles of the equilateral triangles with a view to finding their intersection, which is the solution to the modelled problem".

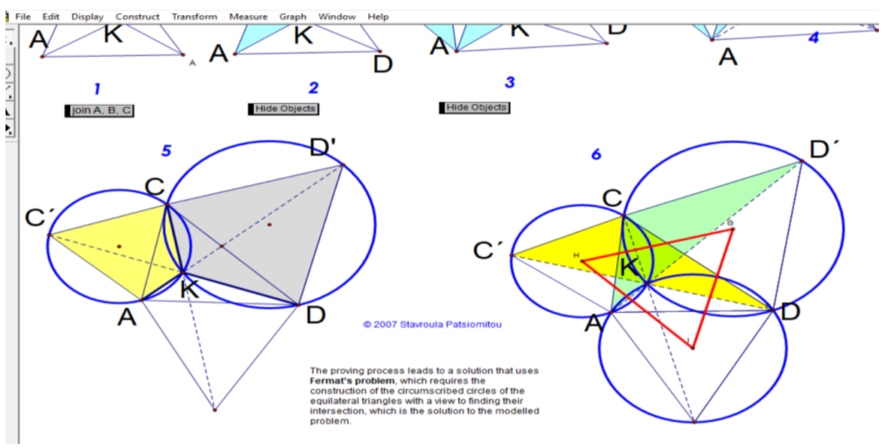


Figure 5.34e. Mode D, conceptual connection with Fermat's problem (see also Patsiomitou, 2008b, 2010)

Mode E –the integration mode (Figure 5.34f): According to Patsiomitou (2008b, 2010): The students are dynamically guided to reinvent an interpretation of the process in the modelled problem. At this stage, through dynamic reinvention can apply the custom-tool "*construction of the circumscribed circle*" to the sides of the equilateral triangles, so that the intersection point of the three circles gives the right place for point K, which is the solution and the interpretation of the solution to the real problem.

According to Patsiomitou (2008b, 2010) *"the building and transforming of the semi-predesigned LVAR leads the students to pass from a visual way of thinking to a theoretical geometrical one, or to pupils' mental transformations. Students use verbal formulations to exchange their ideas meaning that they transform their mental objects into a language mapping, corresponding to LVAR transformations on pages in the software.*

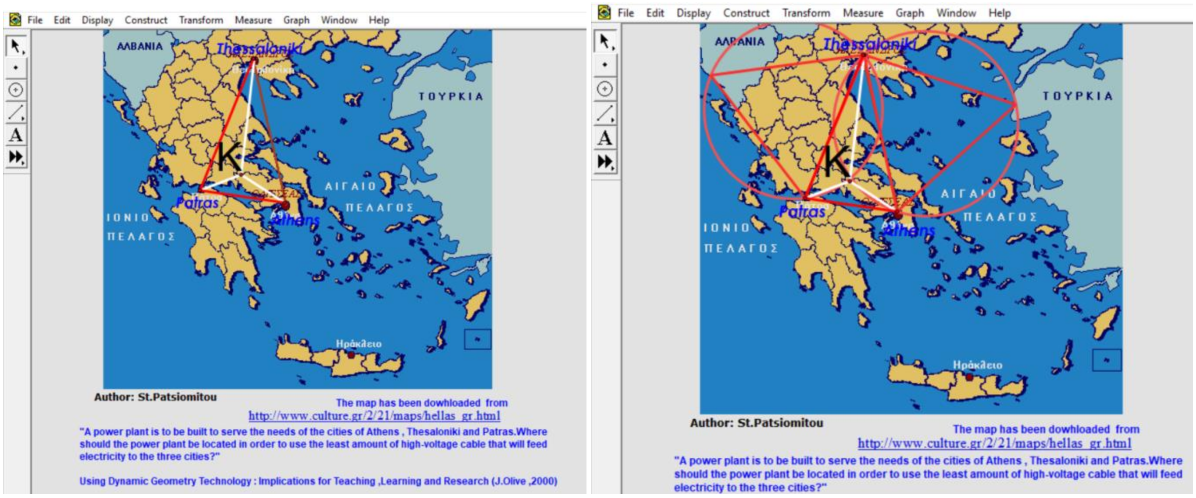


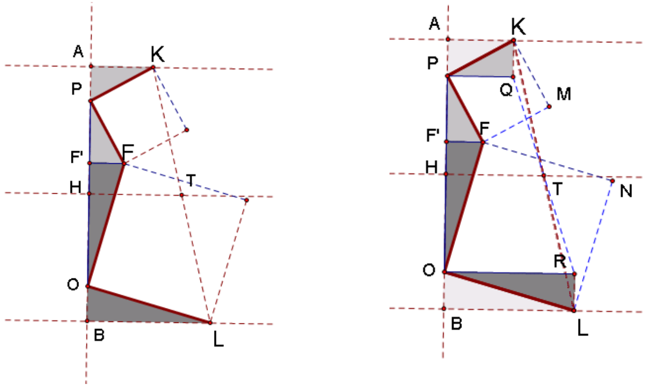
Figure 5.34f. Mode E, implementation of the solution to the real world problem (see also, Patsiomitou, 2008b, 2010)

In my study "*From Vecten's Theorem to Gamow's Problem: Building an Empirical Classification Model for Sequential Instructional Problems in Geometry*" (Patsiomitou, 2019a), I describe how I designed LVAR modes (A and C) incorporating the currently introduced notion of *hybrid objects* and *hybrid-dynamic diagrams*. Concretely in a trying to classify the different types of DGS problems I report how I transformed Vecten's theorem to a real-world situation in LVAR diagrams "*to gain students' interest and attention"* from the first moment (Gagné, Briggs, and Wager, 1992)**.**

Let me begin the story from the year 2007. In year 2007, I turned my investigations of Vecten's theorem to its known version as a real-world problem, created by Gamow (1948, reprinted 1988) through modeling it in Sketchpad DGS environment (e.g., Patsiomitou, 2008a, b) inspired by a work of Daniel Scher (2003), regarding the concrete problem. Daniel Scher (2003) designed the activity in multiple linked pages using Sketchpad v4. Previously, I also discussed the concrete problem with Professor Paris Pamfilos and Professor Constantinos Christou, when I was experimenting, using the Euclidraw Dynamic Geometry program (Web page [32]). Gamow's (1948, reprinted 1988) problem involving pirates and buried treasure seemed ideal for my students. I enhanced the problem with historical evidence from Homer, seeking thus to motivate my students to develop their interest in ancient history through geometry. Gamow's problem hinges on a treasure map found in an old man's attic. Here is the revision (Patsiomitou, 2008a, p. 357):

> "*In the Odyssey, Homer (c74-77) mentions that the pirates also raided Greek islands. The pirate in our story has buried his treasure on the Greek island of Thasos and noted its location on an old parchment. "You walk directly from the flag (point F) to the palm tree (point P), counting your paces as you walk. Then turn a quarter of a circle to the right and go to the same number of paces. When you reach the end, put a stick in the ground (point K). Return to the flag and walk directly to the oak tree (point O), again counting your paces and turning a quarter of a circle to the left and going the same number of paces. Put another stick in the ground (point L). The treasure is buried in the middle of the distance of the two sticks (point T)."*(**Figures5.35a, b5.36**). *After some years the flag was destroyed and the treasure could not be found through the location of the flag. Can you find the treasure now or is it impossible*?"

Many researchers have been attracted to the problem (e.g., De Villiers, 1999a). I considered the problem as particularly interesting because it allows three quite different approaches (Patsiomitou, 2008a, p. 366): (i) the so-called 'static' approach; (ii) a software-supported: 'dynamic' approach; and (iii) a paper and pencil 'dynamic' approach concerning dynamic methods in geometry, consisting of 'thinking in motion' in a paper pencil environment. In the current paper I shall describe how I designed mode A and mode C, trying to concentrate on two of its aspects: 1) linking actions with constructional steps in the software, and 2) linking the various visual steps in the proving process.



Figures 5.35a, b. Screenshots of sequential diagrams of Vecten's theorem in Sketchpad (Patsiomitou, 2008a; 2019a, p. 17)

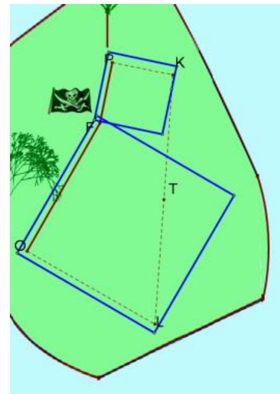
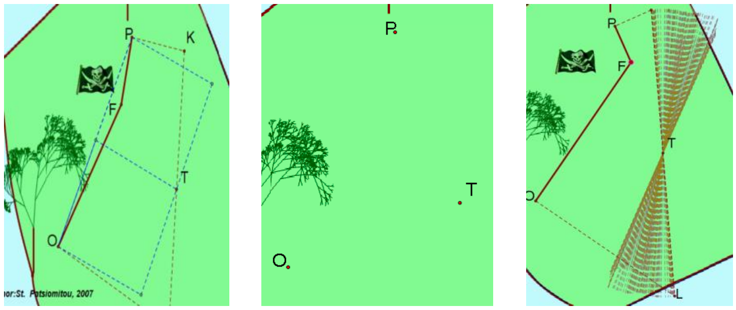


Figure 5.36. Screenshot of Gamow's problem in Sketchpad(Patsiomitou, 2008a; 2019a, p. 17)

Mode A: The synthesis of the dynamic representation incorporates an image that is a permanently annotated pictorial representation, a two-dimensional hybrid object representing the closed and curved polygonal island, annotated in green and two dynamic fractal trees placed on the island. The background (font) of the screen has been selected to be light-blue using the complex preferences pop-up menu, to give the impression of the sea around the island. The positions of the trees P, O are two points with zero degrees of freedom (**Figure 5.36**). According to the Geometer's Sketchpad reference manual (2001) "points are the fundamental building blocks of classical geometry, and geometric figures such as lines and circles are defined in terms of points" (p.11). Hollebrands, Laborde and Straeser (2008, p.165) identified the distinction between the three different kinds of points in a DGS environment: (a) a free point has 2 degrees of freedom, (b) a point on an object has 1 degree of freedom and (c) an intersection point has 0 degrees of freedom. Point F, which represents the position of a moveable flag, can move with two degrees of freedom and be dragged on screen. The rotation of the segments PF, FO to 90 degrees reorganizes the visual mathematical representations. Two new objects the segments PK, OL have been added on screen, the images of the PF, FO respectively. Point T (the treasure point), is the midpoint of the segment KL. It has constructed with zero degrees of freedom due to its dependence on the points K, L.
When students interact with the hybrid-dynamic diagram to create the rotations they interact with the intermediate representation of the pop-up menu for the selection of the rotation angle. The students can construct during instrumental genesis an *instrumented action scheme* of the perpendicularity and the congruence of the segments (PF and PK, OF and OL).The synthesis of the diagram leads to the following complex transformations (Patsiomitou, 2008a, 2012a):

- Rotation of the segments PF, FO to construct the points K, L (**Figures 5.37 a, b**). This portrays a rearrangement of the visual representation giving the students the opportunity to perceive the internal relations between the mathematical objects on screen. Point F can be dragged. This results in the transformation of the rotated segments, a **complex transformation of the dragging and rotation of a geometric object.**
- The hide–show action button for the points K, L, T also creates also a decomposition of the diagram. Concretely, an action button hides the point where the flag is located. The dependent objects have also been hidden (**Figures 5.37 b**).
- The dragging of KL on the screen creates traces of the segment, meaning a set of points through which the segment passes. In this case (**Figures 5.37c, 5.38**) the result is a **complex transformation of the dragging and tracing of a geometric object** (for example a point, a segment, or a line etc).



Figures 5.37 a, b, c. Screenshots from LVARs-mode A (before or after hiding the flag) (Patsiomitou, 2008a; 2019a, p. 17)

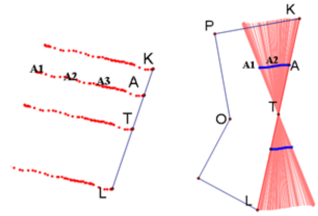


Figure 5.38. Screenshot of the combination of tracing & dragging the segment KL in Sketchpad (Patsiomitou, 2008a; 2019a, p. 17)

Subsequently, the images of KL (**Figure 5.38**) demonstrate the temporal positions of the segment as a correspondence of a point with its image. Every point of the initial constructed image has its correspondent image. Subsequently, we have a function f which corresponds to every point A on the segment KL to a point f(A) the image of point A, where the point A corresponds to point A1 to point A2, and then to point An with the n-dragging. We can thus see that the transformation of Point A is a 1–1 function to every dragging depended of the previous point–image. The set of 'A' images on screen created by the trace command is the set of points through which Point A passes. A point's dragging on screen results to the transformation of its position and simultaneously the appearance of tracing tracks on screen, which show the path that the point follows or the tracks that a line passes due to dragging transformations. This action results to the determination of a basic property of the diagram (or a property of the diagram that remains stable and unaltered) which cannot be directly perceived from the diagram. "Trace" according to Jahn (2002) "emphasizes a dynamic interpretation of the representation of a trajectory of a point … representing, at least implicitly, the image of a set of points for a certain application." (p. 79).

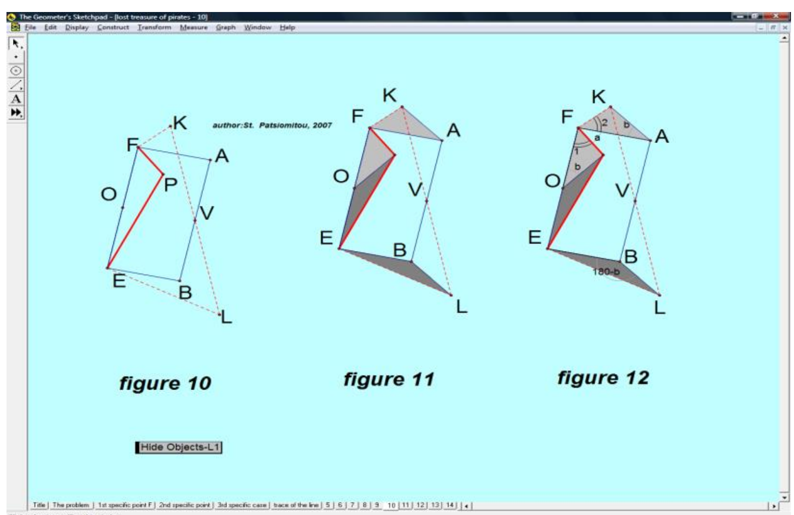


Figure 5.39a. Mode C- the reformulated RGNA problem (Patsiomitou, 2008a, p. 373)

Mode C started with a second problem, investigated by the students in a paper-pencil environment (using a reformulated RGNA problem) reported in Patsiomitou (2008a, p. 372):

> "*An archaeologist has an old map which explains the position of a clay pot: You walk directly from point P to point F (F, E are constant points) counting your paces as you go. Then turn right 90 degrees and walk the same number of paces from point F. When you reach the end, put a stick in the ground. Return to point P and walk directly to point E, again counting your paces and turning left 90 degrees and walking the same number of paces. Put another stick in the ground. The vessel (point V) is buried in the middle of the distance of the two sticks. Rejecting the procedure described above, the archaeologist did the following: starting from the midpoint of the segment FE, he followed the directions given on the map until he finally*

found the pot. a) Can you plot the shape according to the steps that archaeologist followed? And b) can you explain (using formal logic) why he was right?" (**Figure 5.39a**).

This is a complex phase. The dynamic diagrams are linked, using a translation transformation and every diagram on the right is a sequential successive and gradual procedure conducted on the previous one which is on the left. The translation gives to the dynamic representation the property to a simultaneous alteration of every dynamic object on them if we drag any point.

The synthesis of the dynamic LVARepresentation has the following design: Point P has two degrees of freedom and point O has 0 degrees of freedom. The screen background (light-blue colour) has been changed using Sketchpad's complex preferences dialogue in order to link it to the previous page. The experimental dragging of point V does not transform the rectangle's figure, which remains a hybrid object on screen. In order to solve the problem we have to follow the following analysis: we have to prove that V is the midpoint of KL, meaning we have to prove that KL and AB are dichotomized, or KA//=BL (Patsiomitou, 2008a, p.373).

From a diagram with congruent triangles which occurred from a rotation through 90 degrees (**Figure 5.39a**), the students can develop two subgoals (e.g., Patsiomitou, 2008b, 2010, 2012a, 2019b): firstly, proving that the sides are equal and, secondly--stemming from the segment's perpendicularity, proving that the sides are parallel. This is to say they have developed *a conceptual object*: the same objects act as parallel lines and as equal sides. (**Figure 5.39b**).

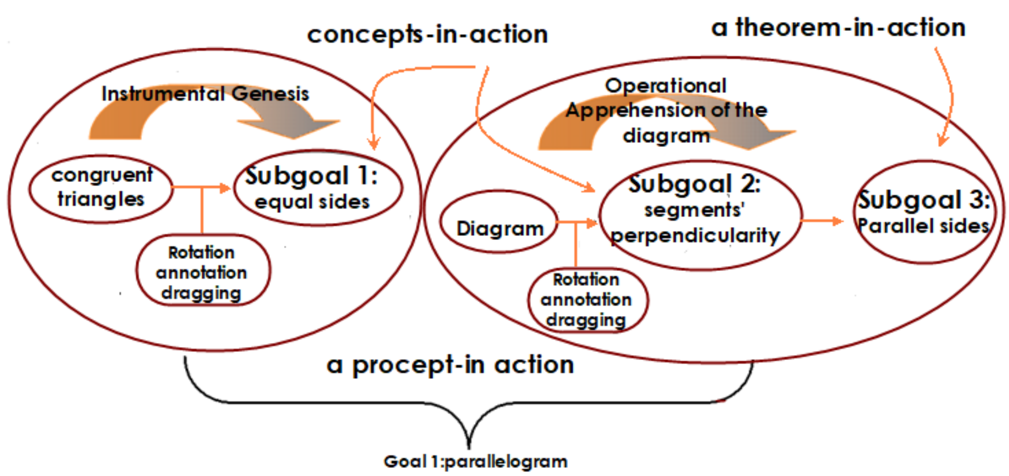


Figure 5.39b. Developing subgoals and goals (Patsiomitou, 2012a, 2019b)

According to Battista (2011) "Selecting/creating instructional tasks, adapting instruction to students' needs, […] require detailed, cognition-based knowledge of how students construct meanings for the specific mathematical topics targeted by instruction" (p.527).

For the student, solving a problem like Gramow's is like embarking on a journey into the unknown. They will meet conceptual obstacles along the way, and hence all manner of difficulties, but the benefits gained make the journey more than worthwhile, as the students emerges stronger from the experience. This is why mathematics educators need to take it on board that the journey is more important than the destination; that it is the process by which students arrive at an answer and the added sophistication they gain in their problem-solving, that raises their van Hiele level.

5.5.3. How can we solve a problem using LVARs?

Figure 5.38 depicts the problem-solution flow in a software environment, since the solution to the problem depends on the right decisions being made when the task's strategies are being designed, using LVAR modes, implementing different interaction techniques. Consequently, students can develop *conceptual transformations* during the process of dynamic geometry problem-solving as a means of achieving meaningful and deep learning and/or increasingly conceptual model building (e.g., Greeno, 1983; Mayer, 2000). The instrumental genesis theory helped me to interpret student behavior in the dynamic geometry environment and to observe the links between *procedural and conceptual components* within instrumentation schemes.

There were changes in the behavior of the students (Guin & Trouche, 1999, p.220) observed after their investigation in the dynamic geometry environment, taking into account their mathematical profile and the main features of the student's van Hiele level of geometrical thought claim. On the other hand, I observed that before the software sessions, the students in the experimental team didn't differ from the students of the same level in the control group with regard to representations of problems and reasoning.
Creating LVARs can make it easier for students to grasp the relevant concepts by allowing the continuous manipulation of mathematical objects in real time. The LVARs are embedded into a multi-page file, and their active "alive" functions lead students to solve the problem, while their construction can make geometry easier by being partly prepared by the teacher, which saves times. Teachers can thus improve their students' knowledge by eliciting mental schemes from them, which is to say the students can be guided to reach conclusions which form a step-by-step visual proof. There are studies (for example Bennett & Desforges, 1988) which caution that the problems used should be set in a familiar context and build on the students' extant knowledge. The aforementioned process described and analyzed, allowed me to reach conclusions relating to how students learn in a dynamic geometry environment with LVARs, which are comparable to an "alive" section in an e-book, and to how their level of reasoning develops—a phenomenon observed during the sessions.

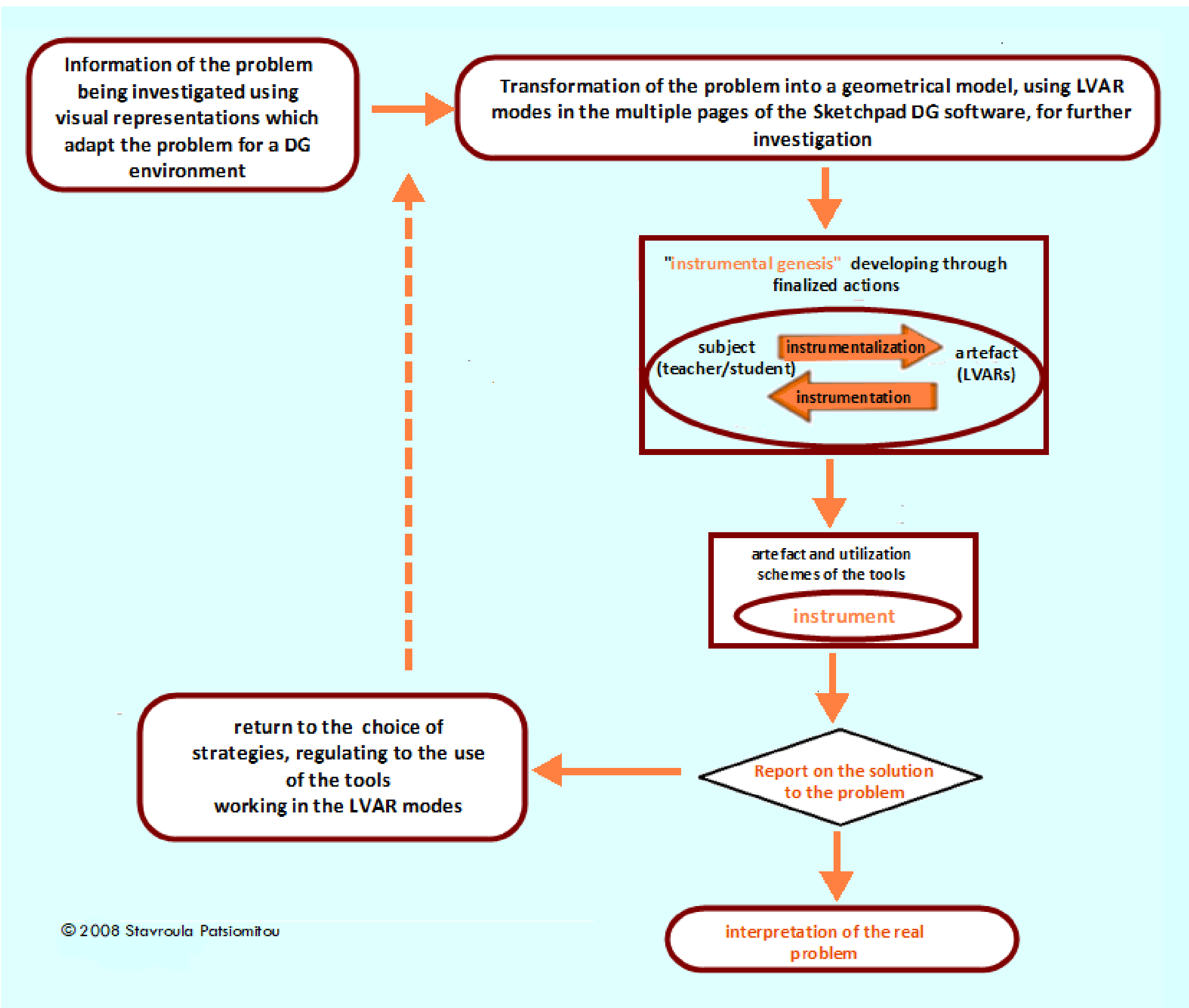


Figure 5.40: Interpretation of the problem solving in a dynamic geometry environment using LVARs (Patsiomitou, 2008a, p. 386) (modified)

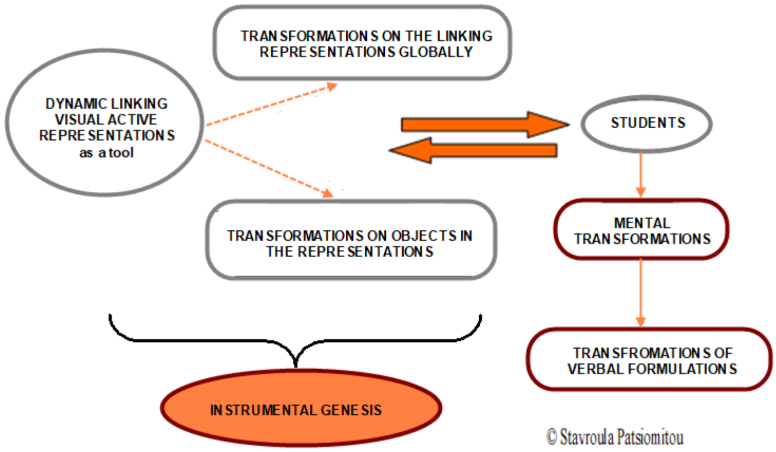


Figure 5.41. The transformations that occurred to students during their interaction with LVAR (Patsiomitou, 2010, p. 20) (modified)

Specifically, when students do not know how to go on, they return to a previous action, reconstructing or undoing a given step which doesn't seem to have helped, trying in this way to get feedback with regard to a future procedure relating to solving the problem. The LVARs allowed me to reach to a conclusion which depicts the problem-solution flow in a software environment, since the solution to the problem depends on the right decisions being made when the activity's strategies are being designed, on the right combination of LVAR, and on creating and choosing different interaction techniques. The instrumental genesis process via the transformations undergone by the LVARs and the utilization schemes the students draw up with a view to solving the problem lead to the construction of cognitive schemata which are developed through collaboration and constant interaction with the environment.

When the instrumental genesis occurs, transformations of linking representations globally or on the objects in the LVAR (i.e. artefacts or tools in the software) reflect on the assimilation or the accommodation of the situation by the subject. The students' development of geometrical thought takes place through the interaction with the LVAR in relation to the progressive adaptation of their schemes of use.

Therefore, it appears that the use of LVARs in the Sketchpad dynamic geometry environment proving process can organize the problem-solving situation using as tools the interaction techniques facilitated by the software, and the structuring and restructuring of the user's instrumental schemes it evokes as the activity unfolds. As the LVARs' composition changes, there is a transformation of the user's verbal formulations due to rules subjacent to the user's organized actions. Consequently, the scheme of use associated with the constructed instrument changes leads the students to pass from an empirical to a theoretical way of thinking or to students' mental transformations (**Figure 5.41**).

Mathematical properties can be described in terms of transformations which may be represented through several types of manipulative activities. In the case of modelling a problem in the DGS environment, this process can be achieved through interaction techniques in the software during the problem-solving process. Initially, the students perform actions upon semi-predesigned LVAR. But eventually when the LVARs as objects become distinct images, students are able to perform mental transformations upon these images in a cognitive operation which builds upon actions but goes beyond them. During the interaction with LVAR, two different developments occur simultaneously: One is *vision-spatial*, using processes on the screen to perform tasks (i.e. rotation) that are completed between a pre-image (the original figure before transformation) and an image (the corresponding figure after the transformation). The other is *conceptual*, using concepts (i.e. properties of figures, interrelationships between figures, theorems etc.) and verbalized thoughts. The process of proof is developed using verbal formulations and geometrical relationships which become conceptualized during the proving process. Students use verbal formulations to exchange their ideas. They transform their mental objects into a language mapping, corresponding to motion transformations on the sketch. *Semperasmatically*, actions on LVAR (or interaction with LVAR) leading to proofs also lead to the *development of geometrical thoughts*. Students can develop their level of thinking by proceeding through increasingly complex, sophisticated and integrated figures and visualizations to more complex linked representation of a problem, and thereby moving instantaneously between the successive Linking Visual Active Representations by means of their mental consideration and without returning to previous representations to reorganize their thoughts (Patsiomitou, 2008a, p. 388; Patsiomitou, 2008h, in Greek).

5.5.4. What is the Role of LVARs in Students' Thinking

Visualization of student's cognitive development during the research has been made by pointing out the main snapshots of their development in an Excel matrix. Concretely, I conceived and applied the following process: The first column of the matrix contains the tools and synthesis of tools that helped students to formulate an expression or a characteristic that could be an indication of their van Hiele level. Tools were categorized and every tool defined by a distinct code (Patsiomitou, 2012a, 2013a, p. 803). For example a few codes for the tools are described in the next table.

Table 5.4: Coding tools for the use of cognitive analysis (Patsiomitou, 2012a, 2013, p.803)

code	Tools
T1	for the point used by experimental dragging
T2	for the point used by theoretical dragging
T3	for the reflection tool
T8	for the circle tool
T9	for the rotation tool
T11	for the parametric tool
T12	for the custom tool "symmetry" (Patsiomitou, 2012a, p. 68)
T15	for the custom tool used with 'economy or catachresis'
T16	for the hide/show action button tool
T17	for the trace tool
T18	for the annotation tool

Furthermore, the characteristics of the van Hiele levels that appeared during analysis of students' dialogues led me to create an adaptation to Battista's (2007) categorization. Concretely, the first row of the Excel matrix contains characteristics of the van Hiele levels, each with a distinct code (Patsiomitou, 2012a, 2013a). For example, characteristics of level 1, 2.1 […] 3.4 were coded as follows: I0 for *cognitive conflicts* and I1 for *informal descriptions*, etc. described at the Table 1 below.

Table 5.5: An adaptation to Battista's (2007) categorization on van Hiele levels (Patsiomitou, 2012a, 2013a, p.804)

level 1	I0 for cognitive conflicts and I1 for incorrect and informal descriptions.
level 2.1	I2 for *dynamic perceptual definition* and I3 for the synthesis of formal and informal descriptions of students.
level 2.2	I4 for *incomplete definitions* and *incomplete* reasoning and I5 for inductive argumentation/concepts-in-action or theorems-in-action.
level 2.3	I6 for formal description and *non-economical definitions* and I7 for connections between meanings.
level 3.1	I8 for *economical definitions* and I9 for logical correlations between meanings.
level 3.2	I10 for structural analysis competence, I11 for abductive-deductive reasoning.
level 3.3	I12 for deductive argumentation, I13 for the *generic example* proof scheme.
level 3.4	I14 for *thought experiment* proof scheme, I15 for the competence of logical hierarchy.

Transformative reasoning, dynamic reinvention, and reflective visual reaction were coded as I16, I17, and I18, respectively. I posed the dialogue counting of the concrete team in which the student participated at the intersection of an intelligible parallel line, starting at the tool and moving to the horizontal axis, and an intelligible perpendicular line, starting from the van Hiele characteristic. The process was accomplished by tracing a crooked line through the counting so that the learning path of the student could be visualized; moreover, it would determine what tools affected to the student's movement at van Hiele levels. In my study "*Students learning paths as 'dynamic encephalographs' of their cognitive development*" (Patsiomitou, 2013), I present two examples of the visualization of students cognitive development through the use of tools. In the following example is illustrated how M7 developed her thinking through the LVAR process (**Figure 5.40)**, by pointing out the main snapshots of her development in an Excel matrix.

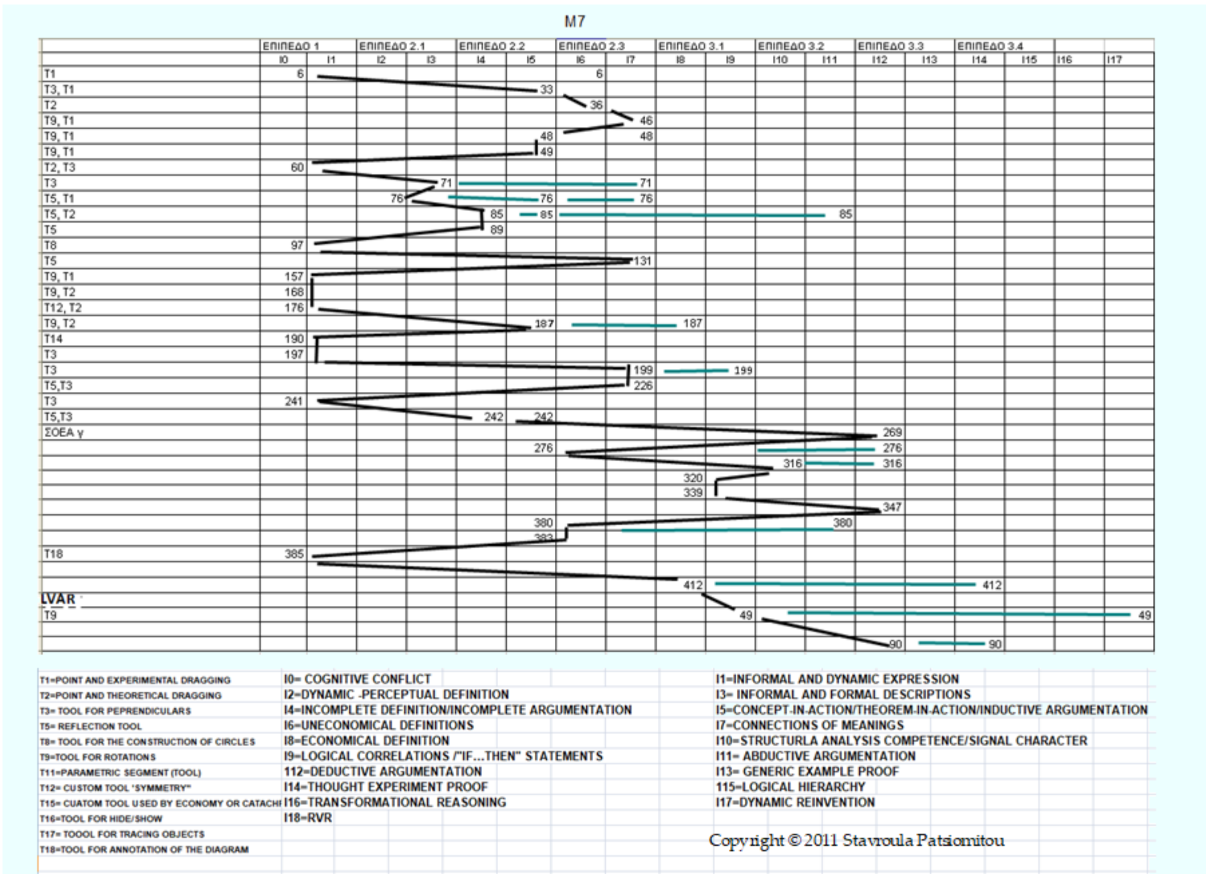


Figure 5.42. The development of M7 student's thinking in connection with the use of Sketchpad tools (Patsiomitou, 2012a)

5.6. Conclusions

Are the students able to grasp logical operations on abstract mathematical objects? What does it mean to obtain access to an abstract mathematical object or a mathematical entity? This assumption imposes a series of questions about the nature of the mathematical objects to which symbols are presumed to refer; for example, if we are not able to have access to mathematical objects, which processes could become mental objects whose aim is to reinforce students' cognitive development in mathematical thinking? Thus, we have to act or operate on external objects or on external representations of these objects or on their external symbols. This is in accordance with what Piaget (1970) stated about mathematical knowledge which can be abstracted either directly from objects or the external experiences we have in relation to the objects, or from operations that are mentally performed on objects.

The design and redesign of activities for the teaching and learning processes, with real problems through LVARs in the dynamic geometry software, and the results obtained from the research data (e.g., Patsiomitou, 2012 a, b), suggest that a student develops his/her abstract thinking when his/her cognitive structures are linked through *conceptual representations* that the student develops during the learning process. Moreover, the linking of sequential phases in a proof or actions over multiple pages or evolving steps in the representation of the problem leads to a cognitive linking of mental representations. A student can construct *linking active representations* (Patsiomitou, 2012a, b):

- *When s/he builds a representation* (for example, a figure) in order to create a unmodified construction, using software's interaction techniques by externalizing his/her mental approach or generally by transforming an external or internal representation to another representation in the same representational system or another one.
- *When s/he gets feedback from the theoretical dragging* (Patsiomitou, 2011a, b, 2012a, b, 2014) to mentally link figures' properties so that, because of the addition of properties, subsequent representations stem from earlier ones.
- *When s/he transforms representations* so that the subsequent representations stem from previous ones due to the addition of properties.
- *When s/he links* the developmental procedural aspects in a process of a *dynamic reinvention*
- *When s/he reverses mentally the procedure* in order to create the same figure in a phase of a dynamic hypothetical learning trajectory or between phases of the same dynamic hypothetical learning path.

The modeling of a problem in the dynamic environment can 'carry' any [mathematical] object to the classroom in two ways: through the use of digital images or through the use of their simulations. On the other hand, a technological tool is important as the design of artifacts can be generalized and replicated in any group of students, at different times and in any thematic framework (e.g., science, geography). Therefore, referring to LVAR is concluded in the following (Patsiomitou, 2012a, p. 498):

- How could this affect the students' understanding of the utilization of LVAR in the teaching and learning of other disciplines (e.g., physics or ancient Greek and history)? [or] Would students understand the obscure points of other disciplines, because of the interaction with the [appropriate] dynamic LVAR representations?
- Can the students develop their linking of the conceptual and procedural representations of these objects?

On the other hand, new cognitive tools are not included [or included in a very slow way] for the teaching of concepts. It is particularly important for the '*movement'* of a process by applying innovative practices to change the negative views that a large portion of teachers have regarding technology. This seems to focus on a lack of knowledge because of the phobias surrounding technological tools in the mathematics classroom, leading to an adherence to traditional teaching methods.

In general, the whole issue has to do with the way we perceive the world, the natural objects (unconscious), how we compare them mentally (consciously) with theoretical constructs of geometry in order to represent them and how we instrumental decode them using technology. Finally, it is important to continue teaching and research concepts in this vital field, through activities that involve children in the learning process, so using lining visual representations they will learn how to develop, interpret, and make sense of geometric concepts. This argument recognizes and underlines the force of *Kant's* argument (1929, "*Critique of Pure Reason*") that:

> There can be no doubt that all our knowledge begins with experience. For how should our faculty of knowledge be awakened into action did not objects affecting our senses partly of themselves *produce representations*, partly arouse the activity of our understanding to *compare these representations*, and, by *combining or separating them,* work up the raw material of the sensible impressions into that knowledge of objects which is entitled experience? [Because] "Understanding is the faculty of knowledge and […] *knowledge consists in the determinate relation of given representations to an object*".

In place of an epilogue: Are LVARs a new theory for teaching and learning?

As you set out for Ithaka *hope your road is a long one,* *full of adventure, full of discovery . . .[…]* *Keep Ithaka always in your mind* *Arriving there is what you're destined for.* *But don't hurry the journey at all.* *Better if it lasts for years,* *so you're old by the time you reach the island ,wealthy with all you've gained on the way,* *not expecting Ithaka to make you rich.[…]* From Ithaka by C.P. Cavafy	*Σα βγεις στον πηγαιμό για την Ιθάκη, να εύχεσαι νάναι μακρύς ο δρόμος, γεμάτος περιπέτειες, γεμάτος γνώσεις. […]* *Πάντα στον νου σου να' χεις την Ιθάκη. Το φθάσιμον εκεί είν' ο προορισμός σου.* *Αλλά μη βιάζεις το ταξίδι διόλου. Καλλίτερα χρόνια πολλά να διαρκέσει· και γέρος πια ν' αράξεις στο νησί, πλούσιος με όσα κέρδισες στον δρόμο, μη προσδοκώντας πλούτη να σε δώσει η Ιθάκη. […]* Κ.Π. «Ιθάκη» Καβάφη, Ποιήματα (1896-1933) http://www.kavafis.gr/poems/

The title of this monograph references the international bibliography relating to learning trajectories on Didactics of Mathematics. The basic idea was, starting from a general perspective on the didactics of mathematics, to provide a structured book for the teaching and learning of the didactics of mathematics incorporating LVAR. The spark for the conception of the meaning of *linking visual active representations* (LVAR) were the files I created in the Geometer's Sketchpad environment in 2005 and my research results. I grasped the meaning of LVAR when I was writing my Master thesis and created activities for the investigation of the meanings of limit and sequence.

The first international work on linking representations incorporating a few DGS activities of my Master Thesis, as well as the investigation and the experiment with these dynamic representations with primary and secondary-level students in a very condensed form had been submitted and accepted at the "*International Conference on Technology in Mathematics Teaching (ICTMT8)* in Hradec Králové: "*Fractals as a context of comprehension of the meanings of the sequence and the limit in a Dynamic Computer Software environment*."(Patsiomitou, 2007a). The linking diagrams can be transformed into "active [alive] diagrams", reinforcing the original image since the stimulus received from the visual representation leaves the properties of the figure unaltered despite the transformation it undergoes. The results in my study were very important for the teaching and learning of geometry.

However, I was still questioning whether the students' understanding had evoked from the active linking of the dynamic "alive" representations. In my PhD study, I introduced and investigated a hypothetical learning path for quadrilaterals which sought to raise the students van Hiele levels. The study of my PhD was conducted in a class at a public high school in Athens during the second term of the academic year (*from January-May 2007*), and involved 65 students aged 15-16. My curiosity led me to revisit the dialogues between myself and the participating students from the fourth phase of my didactic experiment. When I searched my field notes for words and phrases that would suggest my students' van Hiele level had moved, I found that the participated students answered my questions and used long complex, detailed, rigorous proofs. The first definition of the meaning of LVAR was accepted at three conferences during roughly the same period: An indicative expert of my study is presented in my paper "*The development of students' geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment*" (Patsiomitou, 2008a) and a condensed form of this paper has been accepted as Research Report included in the Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XX North American Chapter (Patsiomitou & Koleza, 2008). I presented also the paper "*The development of geometrical thinking through linking visual active representations*" (Patsiomitou & Koleza, 2009), on April 17, 2008 at the *5o Colloquium of Mathematics*, which had been contacted at the University of Crete. When I introduced and presented the meanings LVARs and RVR at the 5th Colloquium at the University of Crete, many important Professors (e.g., Professor Abraham Arcavi, Professor Michele Artique, Professor Nurit Hadas, Professor Hershkowitz Rina), were among others in the audience and gave me feedback. During the presentation, in which I felt under pressure to support my study, a Professor advised me to slow down: "Don't run Lina". I think I've been in a hurry all my life. I am a "runner".

My PhD started *June 2007*. In *December 2007*, my supervisor Professor Eugenia Koleza sent me a notification that she had applied/requested to be moved to another university.

During the "difficult" year 2008 the devotion to my study led me to write three more papers. The papers were accepted by the ATCM conference (Patsiomitou, 2008b, c, d). These are:

- "*Linking Visual Active Representations and the van Hiele model of geometrical thinking*" (Patsiomitou, 2008b);
- "*Do geometrical constructions affect students' algebraic expressions*?" (Patsiomitou, 2008c);
- "*Custom tools and the iteration process as the referent point for the construction of meanings in a DGS environment*" (Patsiomitou, 2008d).

The second and third papers incorporate preliminary research using LVARs. In the first paper, I presented the *LVAR modes* and snapshots of the research process. The above mentioned papers had been accepted to be published at the eJMT journal, after a peer review process. For this, an extended improved version of the paper including LVAR modes has been published, entitled as: "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*" (Patsiomitou, 2010).

The year 2008, my stress over the future of my Ph.D led me to write a monograph in Greek (up to 600 pages). It consists of 15 stand-alone chapters on a common theme: the development of structures through sequential activities in Sketchpad which I designed using linking visual active representations. In the monograph, I describe the way in which the included activities can be created, detailing the steps in their construction process and including illustrations. With the help of the concrete book, each teacher can create his own tools and activities by using the software's functions for the needs of his/her students. The book was approved by the Greek Pedagogical Institute and through the Greek Ministry of Education it had been sent to the libraries of the Experimental Model Secondary-level schools of Greece.

In *December 2008*, another professor (Assist. Professor Anastassios Emvalotis) agreed to replace my previous supervisor. Also, Professor Koleza notified me a letter of recommendation addressed to the University of Ioannina (Thu, 29 Jan 2009, 15:44:22); the official "restart" date of my Ph.D was *March 2009*.

I supported my PhD thesis *December of 2012*. I thank both supervisors from the heart, for the period of cooperation with everyone; their efforts to start and continue my PhD gave me the freedom I needed to pursue my inquiries and develop as a researcher. I would also like to express my gratitude to the following Professors (in alphabetical order): Anastasios Barkatsas, Constantinos Christou, Daniel Scher, Rudolf Straesser, Luc Trouche, as well as the software designer of the Geometer's Sketchpad software, Nicholas Jackiw. I thank them for the comments and the help, sent via e-mails over the years. I also greatly appreciate the comments via e-mail made by Professor Raymond Duval, Professor Usiskin and Professor Michael Shaughnessy. Moreover, I would like to thank all the referees for their comments to improve the quality of my studies. I deeply appreciate the time they dedicated to reading my papers and providing valuable feedback on them. It is important for me also to highlight how lucky I was to have listened the lectures of an excellent team of professors in the Mathematics Departments at the National and Kapodistrian University of Athens and the University of Cyprus in the frame of my Master in the Didactics and Methodology of Mathematics. They included the advisors on my Master's thesis, Professors Chronis Kynigos, Constantinos Christou and Theodosis Zachariades. My need to be constantly introducing and investigating new approaches in mathematics, following my own personal journey to knowledge, is largely due to my studies on this course. Thanks to them, I was already familiar with a substantial part of the extant theory relating to the subject of "Didactics of Mathematics" before I started my Ph.D. My experience as a teacher of Mathematics in secondary education was another very important factor in making me a successful "runner" in the Didactics of Mathematics.

In this volume, I have tried to include (an overview of) the current theoretical framework which helped me to arrive at key theoretical constructs including LVARs, RVR. For the need of my study I conceived also and defined the meanings**:** *instrumental decoding, theoretical and experimental dragging, instrumental obstacles* introduced in the paper "*Theoretical dragging: a non linguistic warrant leading to dynamic propositions*" (Patsiomitou, 2011a) presented at the PME35 Conference, at Ankara Turkey. The summer of 2009, I realized that the diagram created by Teppo (1991) did not cover the needs of my study. So I changed the diagram to cover the needs of my study. The concrete diagram is included in the published paper "*Secondary students' dynamic reinvention of geometric proof through the utilization of Linking Visual Active Representations*» (Patsiomitou, Barkatsas, and Emvalotis, 2010). The DHLP for quadrilaterals which I conceived and implemented in my PhD study is described in the paper "*A Linking Visual Active Representation DHLP for student's cognitive development*" (Patsiomitou, 2012b). I introduced also a pseudo-Toulmin for the needs of my studies.

In December 2012, I also wrote the work «*Students learning paths as 'dynamic encephalographs' of their cognitive development*» (Patsiomitou, 2013). The paper utilized all previous concepts for describing the phenomena of my study and further I formulated the meanings of: *dynamic (/ perceptual) definition and arbitrary economic definition*. In the same article I describe *an adaptation to Battista's (2007) categorization regarding the development of abstract processes* and the development of every skill in each van Hiele level (Patsiomitou, 2013, p. 804), as well as how it relates to the utilization of the software tools (different types of cognitive development of students at different stages of the study). All the above concepts arose from the need to interpret the results of the research process, to explain various phenomena explaining tool -student interactions mainly and the result of a thorough analysis of experimental procedures as immediate consequence of other concepts such as *digital proof* (Patsiomitou, 2006e, p.515) *parametric polygons* (Patsiomitou, 2007c, p.62) etc. which I had written and they had been accepted in earlier Greek scientific conferences. I have to mention that all the research study, the publications etc. occurred in parallel with my work at the secondary education as well as with the raising of my three children.

In brief, the theoretical constructs I conceived, introduced, applied and developed during the writing of my Ph.D. thesis and afterwards are as follows:

- *Linking Visual Active Representations* (LVAR) (Patsiomitou, 2008a, 2012a);
- *Reflective Visual Reaction* (Patsiomitou, 2008a, 2012a);
- *Theoretical and experimental dragging* (Patsiomitou, 2011a, b);
- *Instrumental decoding* (Patsiomitou, 2011a, b);
- *Dynamic point, dynamic segment, parametrical segment, dynamic meanings, dynamic proposals, instrumental obstacles* (Patsiomitou, 2011a, b);
- *Serial, verbal, place way, operational apprehension of the use of tools* (Patsiomitou, 2011a, b);
- *A pseudo-Toulmin's model* (Patsiomitou, 2011a, 2012b);
- *An adaptation of the Graumann's house of quadrilaterals* (Patsiomitou, 2012b);
- *A Dynamic Hypothetical Learning Path* (Patsiomitou, 2012b);
- *An adaptation of Battista (2007)'s categorization* regarding the development of students' abstract processes (Patsiomitou, 2012a, 2013a);
- *The Dynamic Teaching cycle* (Patsiomitou, 2012a, b, 2014);
- *The meaning of "alive" tool* (Patsiomitou, 2005a, 2018b);
- *Hybrid-dynamic objects, hybrid diagram, hybrid-dynamic diagram* (Patsiomitou, 2019a, b);
- *Dynamic object, Dynamic section* (Patsiomitou, 2019a, b);
- *A classification of dynamic problems* (Patsiomitou, 2019a).

Epilogue

Working towards a doctorate can seem like an odyssey, with Ithaca always far beyond the horizon. You' re sailing alone on the deep blue ocean water without chart or compass […] And as it was for Odysseus, it's not the arrival that matters, but the journey itself, discovering new routes, encountering obstacles and moonless nights which make the venture more exciting. You don't know which route to take, you could get lost along the way. There's no expectation of gold at the end of the rainbow… All you can think about is finally getting there. And when you finally do, every memory of the pain it took to get there magically vanishes.

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